THE QUENCHING BEHAVIOR OF A SEMILINEAR HEAT EQUATION WITH A SINGULAR BOUNDARY OUTFLUX

BY

BURHAN SELCUK (Department of Computer Engineering, Karabuk University, Balıklar Kayası Mevkii, 78050, Turkey)

AND

NURI OZALP (Department of Mathematics, Ankara University, Besevler, 06100, Turkey)

Abstract. In this paper, we study the quenching behavior of the solution of a semilinear heat equation with a singular boundary outflux. We prove a finite-time quenching for the solution. Further, we show that quenching occurs on the boundary under certain conditions and we show that the time derivative blows up at a quenching point. Finally, we get a quenching rate and a lower bound for the quenching time.

1. Introduction. In this paper, we study the quenching behavior of solutions of the following semilinear heat equation with a singular boundary outflux:

\[
\begin{cases}
    u_t = u_{xx} + (1 - u)^p, & 0 < x < 1, \ 0 < t < T, \\
    u_x (0, t) = 0, \quad u_x (1, t) = -u^{-q}(1, t), & 0 < t < T, \\
    u(x, 0) = u_0 (x), & 0 \leq x \leq 1,
\end{cases}
\]

where \( p, q \) are positive constants and \( T \leq \infty \). The initial function \( u_0 : [0, 1] \to (0, 1) \) satisfies the compatibility conditions

\[
    u'_0 (0) = 0, \quad u'_0 (1) = -u_0^{-q}(1).
\]

Throughout this paper, we also assume that the initial function \( u_0 \) satisfies the inequalities

\[
    u_{xx}(x, 0) + (1 - u(x, 0))^{-p} \geq 0, \quad u_x(x, 0) \leq 0.
\]

Our main purpose is to examine the quenching behavior of the solutions of problem (1) having two singular heat sources. A solution \( u(x, t) \) of problem (1) is said to quench

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E-mail address: bselcuk@karabuk.edu.tr
E-mail address: nozalp@science.ankara.edu.tr

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if there exists a finite time $T$ such that

$$\lim_{t \to T^-} \max\{u(x,t) : 0 \leq x \leq 1\} \to 1 \text{ or } \lim_{t \to T^-} \min\{u(x,t) : 0 \leq x \leq 1\} \to 0.$$  

From now on, we denote the quenching time of problem (1) with $T$.

Since 1975, quenching problems with various boundary conditions have been studied extensively (cf. the surveys by Chan [12] and Kirk and Roberts [13] and [3, 16, 9, 11, 13, 15, 18]). In the literature, quenching problems have been less studied with two nonlinear heat sources. We give as examples two of these papers. Chan and Yuen [5] considered the problem

$$u_t = u_{xx}, \text{ in } \Omega,$$

$$u_x(0,t) = (1 - u(0,t))^{-p}, \ u_x(a,t) = (1 - u(a,t))^{-q}, \ 0 < t < T,$$

$$u(x,0) = u_0(x), \ 0 \leq u_0(x) < 1, \text{ in } \tilde{D},$$

where $a, p, q > 0, T \leq \infty, D = (0,a), \Omega = D \times (0,T)$. They showed that $x = a$ is the unique quenching point in finite time if $u_0$ is a lower solution, and that $u_t$ blows up at quenching. Further, they obtained criteria for nonquenching and quenching by using the positive steady states. Zhi and Mu [19] considered the problem

$$u_t = u_{xx} + (1 - u)^{-p}, \ 0 < x < 0, \ 0 < t < T,$$

$$u_x(0,t) = u^{-q}(0,t), \ u_x(1,t) = 0, \ 0 < t < T,$$

$$u(x,0) = u_0(x), \ 0 < u_0(x) < 1, \ 0 \leq x \leq 1,$$

where $p, q > 0$ and $T \leq \infty$. They showed that $x = 0$ is the unique quenching point in finite time if $u_0$ satisfies $u_0''(x) + (1 - u_0(x))^{-p} \leq 0$ and $u_0'(0) \geq 0$. Further, they obtained the quenching rate estimate which is $(T - t)^{1/2(q+1)}$ if $T$ denotes the quenching time.

Here in this paper, a quenching problem with two types of singularity terms, namely, a source term $(1 - u)^{-p}$ and the boundary outflux term $-u^{-q}$, is considered. In Section 2, we first show that quenching occurs in finite time under condition (2). Then, we show that the only quenching point is $x = 0$ under conditions (2) and (3). Further, we show that $u_t$ blows up at quenching time. In Section 3, we get a quenching rate and a lower bound for quenching time.

2. Quenching on the boundary and blow-up of $u_t$.

Remark 1. We assume that the conditions (2) and (3) are proper. Namely, we can easily construct such an initial function satisfying (2), (3) and compatibility conditions. Let $u_0(x) = 0.9 - \frac{2}{3}x^{4.5}$. For example, for $p = 9$ and $q = \log_{30/7} 3$, $u_0(x)$ satisfies (2), (3) and compatibility conditions.

Remark 2. If $u_0$ satisfies (3), then we get $u_x < 0$ in $(0,1] \times (0,T)$ by the maximum principle. Thus we get $u(0,t) = \max_{0 \leq x \leq 1} u(x,t)$.

Lemma 1. If $u_0$ satisfies (2), then $u_t(x,t) \geq 0$ in $[0,1] \times [0,T)$.
Proof. We give the proof by utilizing Lemma 3.1 in [10]. Let \( v = u_t(x, t) \). Then \( v(x, t) \) satisfies
\[
\begin{align*}
    v_t &= v_{xx} + p(1 - u)^{-p-1}v, \quad 0 < x < 1, \quad 0 < t < T, \\
    v_x(0, t) &= 0, \quad v_x(1, t) = qu^{-q-1}(1, t)v(1, t), \quad 0 < t < T, \\
    v(x, 0) &= u_{xx}(x, 0) + (1 - u(x, 0))^{-p} \geq 0, \quad 0 \leq x \leq 1.
\end{align*}
\]
For any fixed \( \tau \in (0, T) \), let
\[
\begin{align*}
    L &= \max_{0 \leq x \leq 1, \ 0 \leq t \leq \tau} \left( \frac{1}{2} qu^{-q-1}(x, t) \right), \\
    M &= 2L + 4L^2 + \max_{0 \leq x \leq 1, \ 0 \leq t \leq \tau} \left( p(1 - u(x, t))^{-p-1} \right).
\end{align*}
\]
Set \( w(x, t) = e^{-Mt-Lx^2}v(x, t) \). Then \( w \) satisfies
\[
\begin{align*}
    w_t &= w_{xx} + 4Lxw_x + cw, \quad 0 < x < 1, \quad 0 < t \leq \tau, \\
    w_x(0, t) &= 0, \quad w_x(1, t) = d(t)w(1, t), \quad 0 < t \leq \tau, \\
    w(x, 0) &\geq 0, \quad 0 \leq x \leq 1,
\end{align*}
\]
where
\[
c = c(x, t) = 4L^2(x^2 - 1) + p(1 - u(x, t))^{-p-1} - \max_{0 \leq x \leq 1, \ 0 \leq t \leq \tau} \left( p(1 - u(x, t))^{-p-1} \right) \leq 0
\]
and
\[
d(t) = -\max_{0 \leq x \leq 1, \ 0 \leq t \leq \tau} \left( qu^{-q-1}(x, t) \right) + qu^{-q-1}(1, t) \leq 0.
\]
By the maximum principle and the Hopf lemma, we obtain that \( w \geq 0 \) in \([0, 1] \times [0, \tau]\). Thus, \( u_t(x, t) \geq 0 \) in \([0, 1] \times [0, T]\).

**Theorem 1.** If \( u_0 \) satisfies (2), then there exists a finite time \( T \) such that the solution \( u \) of problem (1) quenches at time \( T \).

**Proof.** Assume that \( u_0 \) satisfies (2). Then we get
\[
\omega = -u^{-q}(1, 0) + \int_0^1 (1 - u(x, 0))^{-p} \, dx > 0.
\]
Introduce a mass function: \( m(t) = \int_0^1 (1 - u(x, t)) \, dx, \quad 0 < t < T \). Then
\[
m'(t) = u^{-q}(1, t) - \int_0^1 (1 - u(x, t))^{-p} \, dx \leq -\omega,
\]
by Lemma 1. Thus, \( m(t) \leq m(0) - \omega t \), which means that \( m(T_0) = 0 \) for some \( T_0(0 < T \leq T_0) \), which means that \( u \) quenches in a finite time.

**Theorem 2.** If \( u_0 \) satisfies (2) and (3), then \( x = 0 \) is the only quenching point.

**Proof.** Define
\[
J(x, t) = u_x + \varepsilon (b_2 - x) \quad \text{in} \ [b_1, b_2] \times [\tau, T),
\]
where \( b_2 \in (0, 1], \ b_1 \in (0, b_2), \ \tau \in [0, T) \) and \( \varepsilon \) is a positive constant to be specified later. Then, \( J(x, t) \) satisfies
\[
J_t - J_{xx} = p(1 - u)^{-p-1}u_x < 0 \quad \text{in} \ (b_1, b_2) \times [\tau, T),
\]
since \( u_x(x,t) < 0 \) in \((0,1] \times [0,T)\). Thus, \( J(x,t) \) cannot attain a positive interior maximum by the maximum principle. Further, if \( \varepsilon \) is small enough, \( J(x,\tau) < 0 \) since \( u_x(x,t) < 0 \) in \((0,1] \times [0,T)\). Furthermore, if \( \varepsilon \) is small enough,

\[
J(b_1, t) = u_x(b_1, t) + \varepsilon (b_2 - b_1) < 0,
J(b_2, t) = u_x(b_2, t) < 0,
\]

for \( t \in (\tau, T) \). By the maximum principle, we obtain that \( J(x,t) < 0 \), i.e., \( u_x < -\varepsilon (b_2 - x) \) for \((x,t) \in [b_1, b_2] \times [\tau, T)\). Integrating this with respect to \( x \) from \( b_1 \) to \( b_2 \), we have

\[
u(b_2, t) - u(b_1, t) = \varepsilon(b_2 - b_1)^2 \frac{\varepsilon(b_1 - b_2)^2}{2} < 1 - \varepsilon(b_2 - b_1)^2 < 1.
\]

So \( u \) does not quench in \((0,1]\). The theorem is proved. \( \square \)

**Theorem 3.** If \( p \geq 1 \), then \( u_t \) blows up at the quenching point \( x = 0 \).

**Proof.** Suppose that \( u_t \) is bounded on \([0,1] \times [0,T)\). Then, there exists a positive constant \( M \) such that \( u_t < M \). That is,

\[
u_{xx} + (1 - u)^p < M.
\]

Multiplying this inequality by \( u_x \), and integrating with respect to \( x \) from 0 to \( x \), we have

\[
\ln|1 - u(0,t)| > \frac{-1}{2} u_x^2 + \ln|1 - u(x,t)| + M [u(x,t) - u(0,t)]
\]

for \( p = 1 \) and

\[
\frac{(1 - u(0,t))^{p+1}}{-p+1} > \frac{-1}{2} u_x^2 + \frac{(1 - u(x,t))^{p+1}}{-p+1} + M [u(x,t) - u(0,t)]
\]

for \( p \neq 1 \). We have, as \( t \rightarrow T^- \) and \( p \geq 1 \), that the left-hand side tends to negative infinity, while the right-hand side is finite. This contradiction shows that \( u_t \) blows up at the quenching point \( x = 0 \). \( \square \)

**3. A quenching rate and a lower bound for the quenching time.** In this section, we get a quenching rate and a lower bound for the quenching time. Throughout this section, we assume that

\[
u_x(x,0) \leq -xu^{-q}(x,0), 0 \leq x \leq 1, \quad (4)
\]

\[
u_t(0,t) = u_{xx}(0,t) + (1 - u(0,t))^{-p}, 0 < t < T. \quad (5)
\]

**Theorem 4.** If \( u_0 \) satisfies (2), (3), (4) and (5), then there exists a positive constant \( C_1 \) such that

\[
u(0,t) \geq 1 - C_1 (T - t)^{1/(p+1)},
\]

for \( t \) sufficiently close to \( T \).

**Proof.** Define \( J(x,t) = u_x + xu^{-q} \) in \([0,1] \times [0,T)\). Then, \( J(x,t) \) satisfies

\[
J_t - J_{xx} = [p(1-u)^{-p-1} + 2qu^{-q-1}] u_x - qu^{-q-1}(1-u)^{-p} - q(q+1) xu^{-q-2} u_x^2,
\]
since \( u_x < 0 \), \( J(x,t) \) cannot attain a positive interior maximum. On the other hand, \( J(x,0) \leq 0 \) by (4) and

\[
J(0,t) = 0, \quad J(1,t) = 0,
\]

for \( t \in (0,T) \). By the maximum principle, we obtain that \( J(x,t) \leq 0 \) for \( (x,t) \in [0,1] \times [0,T] \). Therefore

\[
J_x(0,t) = \lim_{h \to 0^+} \frac{J(h,t) - J(0,t)}{h} = \lim_{h \to 0^+} \frac{J(h,t)}{h} \leq 0.
\]

From (5), we get

\[
J_x(0,t) = u_{xx}(0,t) + u^{-q}(0,t)
\]

\[
= u_t(0,t) - (1-u(0,t))^{-p} + u^{-q}(0,t) \leq 0
\]

and

\[
u_t(0,t) \leq (1-u(0,t))^{-p}.
\]

Integrating for \( t \) from \( t \) to \( T \) we get

\[
u(0,t) \geq 1 - C_1(T-t)^{1/(p+1)},
\]

where \( C_1 = (p+1)^{1/(p+1)} \). \( \square \)

**Remark 3.** We can calculate a lower bound for the quenching time. From Theorem 4, a lower bound is \((1-u_0(0))^{p+1}/(p+1)\) for quenching time \( T \). If we choose, as in Remark 1, \( u_0(x) = 0.9 - \frac{2}{3}x^{4.5} \), then we have \( T = 10^{-11} \) for \( p = 9 \).

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**References**


