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CONVERGENCE RATE TO THE SINGULAR SOLUTION FOR THE GELFAND EQUATION AND ITS STABILITY

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Abstract. We study the asymptotic behaviors of the solution to the Gelfand equation. The Gelfand equation appears in the kinetic theory of gravitational steady state and the theory of nonlinear diffusion. We present a convergence rate of the solutions of the Gelfand equation to the unique singular solution as r goes to infinity and prove asymptotic stability of the solution by considering the initial value problem for the Gelfand equation. To obtain the convergence rate and the point-wise stability estimate, we construct a uniform lower bound function and use the solution for the linearized Gelfand equation.

1. Introduction. In this paper, we study asymptotic behaviors of the solution to the Gelfand equation which is the radial solution of the quasilinear elliptic equation with exponential nonlinearity:

$$u'' + (n-1)\frac{u'}{r} = \kappa_n \exp(-u), \quad r \in \mathbb{R}^+, \quad (u(0), u'(0)) = (u_0, 0), \tag{1.1}$$

where κ_n is a positive constant and $n \in (2, \infty)$ is a real number which represents spatial dimension in the quasi-linear elliptic equation. In (1.1), $u''(r) + (n-1)\dfrac{u'(r)}{r}$ corresponds to the n-dimensional Laplacian operator with radial symmetry. In other words, the equation (1.1) is a special case of the Gelfand equation

$$\Delta_x u(x) = \kappa_n \exp(-u(x)), \quad x \in \mathbb{R}^n.$$

The Gelfand equation appears in several places in the mathematical sciences, for example, the theory of nonlinear diffusion [5,7,8] and the theory of thermal ignition of a chemically active mixture of gases [4]. The Gelfand equation also appears in the kinetic theory of gases. In kinetic theory, motions of gases can be described as dynamics of the one-particle distribution function $f(x, v, t)$, where x is the spatial variable and v and t are velocity and time respectively. For dilute gas, if we consider collisions of each particle of the

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gas and Coulomb's interaction as crucial factors of this dynamic, then its dynamic is described by the Vlasov-Poisson-Boltzmann system as follows:

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \varphi \cdot \nabla_v f = Q(f,f), \quad x, v \in \mathbb{R}^n,\ t > 0,$$
$$\Delta \varphi = \rho, \quad \rho = \int_{\mathbb{R}^n} f dv,$$

where φ is the self-consistent force potential and $Q(f,f)$ is the collision operator. We can represent $Q(f,f)$ explicitly as follows:

$$Q(f,f)(x,v,t) \equiv \frac{1}{Kn} \int_{\mathbb{R}^n \times \mathbf{S}_+^{n-1}} B(|v - v_*|, \theta)(f(v')f(v'_*) - f(v)f(v_*)) dv_* d\omega.$$

Here v' and v'_* represent the post-collided velocities from the pre-collided velocities v, v_*:

$$v' = v - [(v - v_*) \cdot \omega]\omega, \quad v'_* = v_* + [(v - v_*) \cdot \omega]\omega, \qquad \omega \in \mathbb{S}_+^{n-2}.$$

For the Vlasov-Poisson-Boltzmann (V-P-B) system, $f(x,v,t) = \exp(-c_0 v^2 - u(|x|))$ is a stationary classical solution of the gravitational V-P-B system with

$$\kappa_n = 2c_0(\pi/c_0)^{n/2}, \tag{1.2}$$

if $u(r)$ is the solution to the Gelfand equation (1.1). All radial symmetric solutions have only this form. For a detailed argument about this stationary solution, see [1, 2]. We refer to [3] for other issues related to stationary solutions of the V-P-B system. In [3], the authors prove that there exists a stationary solution with an external force.

Before we describe our main result, we briefly discuss the motivation of the paper. In [1], the authors consider dynamic instability in the L^p-norm for the V-P-B system. To obtain L^p-regularity, they investigate large time behavior of the solution $u(r)$ and obtain the convergence result of the solution to the unique singular solution of the Gelfand equation (1.1) by using log-scaling and geometric property of the solution $u(r)$ in the phase-plane (see Theorem 2.3). Therefore, it is natural to consider that how fast this solution $u(r)$ converges to the singular solution of the Gelfand equation and whether $u(r)$ is stable.

In this paper, we will use the following terminologies for convenience. For $n > 2$ in the Gelfand equation (1.1), we set

$$\text{Subcritical case} : n < 10, \quad \text{Critical case} : n = 10, \quad \text{Supercritical case} : n > 10.$$

The following are the main results of this paper. For each case with respect to $n > 2$, we have the following convergence rate in L^∞:

THEOREM 1.1 (Convergence rate). Let $u(r)$ be a solution to the Gelfand equation (1.1) and $\lambda = (n-2)(n-10)$. Then for the subcritical case, $u(r)$ converges to the singular solution $u_s(r;n)$ with $r^{-(n-2)/2}$-decay rate in L^∞; for the critical case and the supercritical case, it has $r^{-(n-2)/2} \log r$-decay rate and $r^{-(n-2-\sqrt{\lambda})/2}$-decay rate in L^∞, respectively. In other words, we have

$$|u(r) - u_s(r;n)| \le \frac{c(n,u)}{r^{\frac{n-2}{2}}} \qquad \text{(subcritical case)},$$
$$|u(r) - u_s(r;n)| \le \frac{c(n,u)}{r^{\frac{n-2}{2}}} \log r \qquad \text{(critical case)},$$

$$|u(r) - u_s(r;n)| \leq \frac{c(n,u)}{r^{\frac{n-2-\sqrt{\lambda}}{2}}} \qquad \text{(supercritical case)},$$

where $c(n,u)$ is a constant depending on n and u.

For the stability of the solution $u(r)$ to the Gelfand equation (1.1), we have the following result for each case:

THEOREM 1.2 (Stability). Assume that $\epsilon > 0$ is sufficiently small number and $\lambda = (n-2)(n-10)$. Let $u(r)$ and $\bar{u}(r)$ be the solutions to the Gelfand equation (1.1) with $|u(0) - \bar{u}(0)| = |u_0 - \bar{u}_0| < \delta < 1$ where δ depend on u_0 and ϵ. Then for $n > 2$, we have the following stability estimate:

$$\begin{aligned}
|u(r) - \bar{u}(r)| &\leq |u_0 - \bar{u}_0| \frac{C(u_0,\delta,n)}{(r+1)^{\frac{n-2}{2}-\epsilon}} && \text{(subcritical case)},\\
|u(r) - \bar{u}(r)| &\leq |u_0 - \bar{u}_0| \frac{C(u_0,\delta,n)}{(r+1)^{\frac{n-2}{2}-\epsilon}} \log(r+1) && \text{(critical case)},\\
|u(r) - \bar{u}(r)| &\leq |u_0 - \bar{u}_0| \frac{C(u_0,\delta,n)}{(r+1)^{\frac{n-2-\sqrt{\lambda}}{2}-\epsilon}} && \text{(supercritical case)},
\end{aligned}$$

where $C(u_0,\delta,n)$ is a constant depending on u_0, δ and n.

Next we briefly introduce our key idea to prove the above theorems. The main strategy is to construct a uniform lower bound function for the solution to the Gelfand equation and study the linearized Gelfand equation to obtain an implicit relation for the solution $u(r)$ to the Gelfand equation (1.1). By using this bound and implicit relation, we will obtain an a priori estimate related to the difference between the solution and the singular solution or two solutions: $u(r)$ and $\bar{u}(r)$. The important method to obtain an a priori estimate is to use the solution of the linearized Gelfand equation. There are three types of solutions of the linearized Gelfand equation with respect to $n > 2$, i.e., subcritical, critical and supercritical cases as defined before. For the subcritical case, the weight term in the inhomogeneous term of the implicit relation is bounded. Therefore, we can obtain an a priori estimate. However, for the critical case, the weight term in the inhomogeneous term is not bounded. Thus, we cannot apply the method used in the subcritical case to this case. To overcome this difficulty, we will use an order preserving property between the solutions to the Gelfand equation (1.1) for the critical case. By using the order preserving property, we can use the 'supercritical' linearized Gelfand equation for the critical case and we can avoid the unbounded weight term in the inhomogeneous term. For the supercritical case, we also use the order preserving property to obtain an a priori estimate of the subcritical case-type. The estimate implies the stability result by using the method in the subcritical case.

The rest of this paper is organized as follows. In Section 2, we will review the previous results related to the Gelfand equation and study the solution to the linearized Gelfand equation. In Section 3, we will prove the order preserving property between the solutions to the Gelfand equation. In Section 4, we will construct a uniform bound function for the solution to the Gelfand equation. In Section 5, we provide a convergence rate to

the singular solution. In Section 6, we will obtain asymptotic stability of the Gelfand equation. Finally, in Section 7, we conclude this paper with the summary.

2. Preliminaries. In this section, we review previous results related to the Gelfand equation and we provide elementary lemmata that will be used in the next sections.

2.1. *Existence of regular solutions and its convergence.* In this part, we study the existence of solutions to the Gelfand equation (1.1) and its large time behavior as r goes to infinity. We present the existence of regular solutions to the Gelfand equation. For the general discussion of existence theory, we refer to [9].

Consider the following second order ordinary differential equation with source term $S(u)$:

$$u'' + (n-1)\frac{u'}{r} = S(u), \quad r > 0, \quad (u(0), u'(0)) = (u_0, u_1), \tag{2.1}$$

where $S : \mathbb{R} \to \mathbb{R}$ is a continuous function, and boundary data at $r = 0$ has been chosen to guarantee the existence of regular solutions at $r = 0$. We next state the local and global existence of regular solutions to equation (2.1).

THEOREM 2.1 ([9]). (i) Suppose that the function S is locally Lipschitz. Then there exists $\delta > 0$ such that equation (2.1) has a unique solution $u \in \mathcal{C}^2([0, \delta))$, and this local solution satisfies a dichotomy: Either u is a global $\mathcal{C}^2$-solution or there exists $r_0 \in (0, \infty)$ satisfying

$$\limsup_{r \to r_0-} |u(r)| = \infty, \quad \limsup_{r \to r_0-} |u'(r)| = \infty.$$

(ii) Suppose S is locally Lipschitz and satisfies an additional integrability condition: There exists a $C < \infty$ such that

$$\int_{u_0}^{u} S(w)dw < C, \quad u \in (0, \infty).$$

Then there exists a unique global solution u satisfying $\sup_{r>0} |u'(r)| < \infty$.

Proof. (i) We refer to [9] for the proof.

(ii) For the proof of global existence, it suffices to show that u' is uniformly bounded by (i). For this, we multiply the equation (2.1) by u' to get

$$\frac{1}{2}|u'(r)|^2 + (n-1)\int_0^r \frac{|u'(\zeta)|^2}{\zeta} d\zeta = \int_0^r u'(\zeta)S(u(\zeta))d\zeta = \int_{u_0}^{u(r)} S(\zeta)d\zeta \le C.$$

Hence the local solution obtained in Theorem 2.1 is actually a global solution. □

REMARK 2.2. (i) For local existence of the solution to the ODE (1.1), we refer to [1].

(ii) Note that the source term $\kappa_n e^{-u}$ in the ODE (1.1) is locally Lipschitz and satisfies

$$\int_{u_0}^{u} \kappa_n e^{-\eta} d\eta = \kappa_n(e^{-u_0} - e^{-u}) < \kappa_n e^{-u_0}, \qquad \text{for all } u. \tag{2.2}$$

Hence the result (ii) in Theorem 2.1 leads to the global existence of $\mathcal{C}^2$-smooth solutions to the ODE (1.1).

(iii) This equation is a second-order ordinary differential equation. Therefore, it needs the initial data $u(0) = u_0$ and $u'(0) = u_1$. However, u_1 should be zero because of the well-posedness and structure of this equation.

(iv) Note that for $n > 2$, the function defined by

$$u_s(r;n) := 2\log r + \log \frac{\kappa_n}{2(n-2)} \tag{2.3}$$

is the singular solution to the ODE (1.1), i.e., it satisfies

$$u'' + (n-1)\frac{u'}{r} = \kappa_n e^{-u}, \qquad \lim_{r\to 0^+} u(r) = -\infty.$$

For the large time behavior, authors in [1] showed that the solution to the ODE (1.1) converges to the singular solution (2.3) as r goes to infinity. They obtained the following theorem by using log-scaling and phase-space analysis methods:

THEOREM 2.3 ([1]). *(Asymptotic behavior at infinity).* For $n > 2$, let $u = u(r)$ be a global regular solution to the ODE (1.1). Then for any initial value $u(0) = u_0$, $u = u(r)$ approaches to the singular solution (2.3) as $r \to \infty$, i.e.,

$$\lim_{r\to\infty} |u(r) - u_s(r;n)| = 0.$$

We can also find the following result in their proof of Theorem 2.3.

PROPOSITION 2.1 ([1]). Let $u = u(r)$ be a solution to the Gelfand equation in Theorem 2.3 and $u_s(r;n)$ be the singular solution. Then for $n > 2$, we have

$$\lim_{r\to\infty} |r(u'(r) - u_s'(r;n))| = 0.$$

REMARK 2.4. In the original proof of Theorem 2.3, the authors obtained the convergence result but not the convergence rate.

2.2. *A linearized Gelfand operator.* In this part, we provide the solution of the linearized equation related to the Gelfand equation (1.1). Consider the following equation:

$$\begin{aligned} &v'' + \frac{\alpha v'}{r} + \frac{\beta v}{r^2} = 0, \quad r \in [k,\infty), \\ &v(k) = v_k,\ v'(k) = v_k', \end{aligned} \tag{2.4}$$

where α and β are real numbers. We substitute the variable r into t by using the following transform:

$$r = \exp t, \quad \log r = t$$

to obtain the following autonomous equation:

$$\frac{d^2}{dt^2}v(e^t) + (\alpha-1)\frac{d}{dt}v(e^t) + \beta v(e^t) = 0, \quad t \in [\log k, \infty).$$

By the above transform, we can obtain the following lemmata. The proofs of these lemmata are straightforward by the above argument, and so we omit the proofs.

LEMMA 2.5. Consider the following ODE:

$$v'' + \alpha\frac{v'}{r} + \beta\frac{v}{r^2} = 0, \quad r \in [k, \infty),$$
$$v(k) = v_k, \quad v'(k) = v'_k,$$

where k is a positive real number.

(i) For $\lambda = (\alpha-1)^2 - 4\beta > 0$, we have the solution $v(r)$ of the above equation as follows:

$$v(r) = \frac{(\alpha-1+\sqrt{\lambda})v_k + 2kv'_k}{2\sqrt{\lambda}}\Big(\frac{k}{r}\Big)^{\frac{\alpha-1-\sqrt{\lambda}}{2}} - \frac{(\alpha-1-\sqrt{\lambda})v_k + 2kv'_k}{2\sqrt{\lambda}}\Big(\frac{k}{r}\Big)^{\frac{\alpha-1+\sqrt{\lambda}}{2}}.$$

(ii) For $\lambda = (\alpha-1)^2 - 4\beta = 0$, we also have

$$v(r) = \big(v_k + \frac{\alpha-1}{2}v_k \log\frac{r}{k} + kv'_k \log\frac{r}{k}\big)\Big(\frac{k}{r}\Big)^{\frac{\alpha-1}{2}}.$$

(iii) For $\lambda = (\alpha-1)^2 - 4\beta < 0$,

$$v(r) = \Big[v_k \cos(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{k}) + \frac{(\alpha-1)v_k + 2kv'_k}{\sqrt{-\lambda}}\sin(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{k})\Big]\Big(\frac{k}{r}\Big)^{\frac{\alpha-1}{2}}.$$

For the nonhomogeneous equation, we also obtain the following result:

LEMMA 2.6. Consider the nonhomogeneous equation as follows with zero initial data:

$$v'' + \alpha\frac{v'}{r} + \beta\frac{v}{r^2} = f(r), \quad r \in [k, \infty),$$
$$v(k) = 0, \quad v'(k) = 0,$$

where k is a positive real number and $f(r)$ is a given real valued function.

(i) For $\lambda = (\alpha-1)^2 - 4\beta > 0$, we have

$$v(r) = \frac{r}{\sqrt{\lambda}}\int_k^r f(\tau)\Big[\Big(\frac{\tau}{r}\Big)^{\frac{1+\alpha-\sqrt{\lambda}}{2}} - \Big(\frac{\tau}{r}\Big)^{\frac{1+\alpha+\sqrt{\lambda}}{2}}\Big]d\tau.$$

(ii) For $\lambda = (\alpha-1)^2 - 4\beta = 0$,

$$v(r) = r\int_k^r f(\tau)\Big(\frac{\tau}{r}\Big)^{\frac{1+\alpha}{2}}\log\frac{r}{\tau}d\tau.$$

(iii) For $\lambda = (\alpha-1)^2 - 4\beta < 0$,

$$v(r) = \frac{2r}{\sqrt{-\lambda}}\int_k^r f(\tau)\Big(\frac{\tau}{r}\Big)^{\frac{\alpha+1}{2}}\sin(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{\tau})d\tau.$$

REMARK 2.7. (i) There is a characteristic property of the solution formula for each region of λ in Lemmas 2.5 and 2.6. For $\lambda > 0$ and $\lambda = 0$, i.e., the supercritical and critical cases, the weight term

$$\Big(\frac{\tau}{r}\Big)^{\frac{-\sqrt{\lambda}}{2}} - \Big(\frac{\tau}{r}\Big)^{\frac{+\sqrt{\lambda}}{2}} \quad \text{and} \quad \log\frac{r}{\tau}$$

are positive but not uniformly bounded. For the subcritical case ($\lambda < 0$), the weight term $\sin(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{\tau})$ can have both signs but is uniformly bounded.

(ii) We will use the characteristic property of each formula to obtain an a priori estimate for a convergence rate and the stability of the Gelfand equation by choosing α and β appropriately.

3. Order preserving property of the Gelfand equation for the critical and supercritical cases, $n \geq 10$. In this section, we will study an order preserving property of the Gelfand equation (1.1). Consider a family of the solutions to the Gelfand equation (1.1). For each $u(0) = u_0$, there exists a global solution $u(r)$ by Section 2. There is an interesting property of these global solutions: the singular solution $u_s(r;n)$ defined in (2.3) is the minimum solution among the solutions of the Gelfand equation (1.1). For a detailed statement, see the following proposition.

PROPOSITION 3.1 ([1]). For the critical and supercritical cases, i.e., $n \geq 10$, the solution $u = u(r)$ of the Gelfand equation and the singular solution $u_s(r;n)$ in Theorem 2.3 and Remark 2.2 has the following property:

$$u(r) > u_s(r;n).$$

Proof. The proof of Proposition 3.1 can be found in [1]. We omit its proof. □

REMARK 3.1. The above proposition means that for the critical and supercritical cases, all solutions $u(r)$ are greater than the singular solution $u_s(r;n)$ for $r > 0$. Since $u_s(r;n)$ is also the solution to the Gelfand equation with $u_s(0;n) = -\infty$, we can interpret this phenomena as the fact that there is the order preserving property between a solution $u(r)$ and the singular solution $u_s(r;n)$.

We can extend the result between a solution $u(r)$ and the singular solution $u_s(r;n)$ in the above proposition to between two solutions $u(r)$ and $\bar{u}(r)$ as follows:

PROPOSITION 3.2. Let $u = u(r)$ and $\bar{u} = \bar{u}(r)$ be solutions to the Gelfand equation (1.1). For the critical and supercritical cases, i.e., $n \geq 10$, we have the following property:

$$u(0) > \bar{u}(0) \text{ implies } \quad u(r) > \bar{u}(r), \quad \text{for all } r > 0.$$

Similarly, $u(0) < \bar{u}(0)$ implies that $u(r) < \bar{u}(r)$, for all $r > 0$.

Proof. For the proof of this proposition, we will use the similar method as in the proof of Proposition 3.1 (see [1]). We define a new variable t as follows:

$$t = \log r, \ r = e^t \quad \text{(log-scaling)}.$$

For the notational convention, we define $V(t)$ as follows:

$$V(t) = u(e^t) - \bar{u}(e^t).$$

We will consider the trajectory of $(V(t), V'(t)) \in \mathbb{R}^2$ with respect to t later. By definition of V, we can easily calculate V' and V'' as follows:

$$V'(t) = \frac{dV(t)}{dt} = e^t u'(e^t) - e^t \bar{u}'(e^t)$$

and

$$V''(t) = \frac{d^2V(t)}{dt^2} \quad = \quad e^t u'(e^t) - e^t \bar{u}'(e^t) + e^{2t} u''(e^t) - e^{2t} \bar{u}''(e^t).$$

By the above two relations, we can obtain a second order ordinary differential equation for V as follows:

$$\begin{aligned} V''(t)+(n-2)V'(t) &= e^{2t}u''(e^t)-e^{2t}\bar{u}''(e^t)+(n-1)e^t u'(e^t)-(n-1)e^t\bar{u}'(e^t) \\ &= e^{2t}\kappa_n e^{-u(e^t)}-e^{2t}\kappa_n e^{-\bar{u}(e^t)} \\ &= e^{2t-\bar{u}(e^t)}\kappa_n(e^{-u(e^t)+\bar{u}(e^t)}-1) \\ &= 2(n-2)e^{u_s(e^t;n)-\bar{u}(e^t)}(e^{-V(t)}-1), \quad \text{by definition of } u_s(r;n). \end{aligned}$$

We denote that $K(t) := e^{u_s(e^t;n)-\bar{u}(e^t)}$. Then by Proposition 3.1, we have

$$0 < K(t) < 1, \quad \text{for all } t \in \mathbb{R}. \tag{3.1}$$

Therefore, we have the following ordinary differential equation for $V(t)$:

$$V''(t)+(n-2)V'(t) = 2(n-2)K(t)(e^{-V(t)}-1).$$

We rewrite the above equation as a system of the first order ODEs as follows:

$$\begin{aligned} V'(t) &= Q(t), \\ Q'(t) &= -(n-2)Q(t)+2(n-2)K(t)(e^{-V(t)}-1), \end{aligned} \tag{3.2}$$

subject to

$$\lim_{t\to-\infty} V(t) = \lim_{t\to-\infty} u(e^t)-\bar{u}(e^t) = u(0)-\bar{u}(0) > 0$$

and

$$\lim_{t\to-\infty} V'(t) = \lim_{t\to-\infty} e^t u'(e^t)-e^t\bar{u}'(e^t) = 0.$$

We now investigate several properties of the trajectory $(V(t), Q(t))$. Consider the trajectory $(V(t), Q(t))$ in phase plane $\mathbb{R}^2$. Then we have

$$\lim_{t\to-\infty}(V(t), Q(t)) = (u(0)-\bar{u}(0), 0), \quad \text{where } u(0)-\bar{u}(0) > 0. \tag{3.3}$$

Suppose that $V(t) > 0$ and $Q(t) > 0$. Then this assumption implies

$$\frac{dV(t)}{dt} = Q(t) > 0 \quad \text{and} \quad \frac{dQ(t)}{dt} = -(n-2)Q(t)+2(n-2)K(t)(e^{-V(t)}-1) < 0, \tag{3.4}$$

because $0 < K(t) < 1$ and $n \geq 10$. Moreover, $V(t) > 0$ and $Q(t) < 0$ imply that

$$\frac{dV(t)}{dt} = Q(t) < 0. \tag{3.5}$$

We will use proof by contradiction. Suppose that there is an $r_1 \in \mathbb{R}$ such that $u(r_1) - \bar{u}(r_1) \leq 0$. By (3.3) and continuity, there exists $r_0 \in \mathbb{R}$ satisfying $V(\log r_0) := u(r_0) - \bar{u}(r_0) = 0$. We denote the smallest element among them by $r_0 = e^{T_0}$. By (3.4), it is impossible that $V(T) = 0, Q(T) > 0$. Therefore, the only possible case is $V(T) = 0$, $Q(T) \leq 0$.

Suppose that $V(T) = 0$, $Q(T) < 0$. Let us take any $\mu > 0$. Then there might be several intersection points in the phase-plane $\mathbb{R}^2$ between the line $Q = -\mu V$ and the trajectory $(V(t), Q(t))$ in $(-\infty, T]$. We set $(V(T_{-1}), Q(T_{-1})) = (V_\mu, Q_\mu)$ to be the intersection point that has the smallest V value.

By the above, we have

$$
\begin{aligned}
-\mu \le \left.\frac{dQ}{dV}\right|_{V=V_\mu} &= -(n-2) + 2(n-2)K(t)\left.\frac{e^{-V}-1}{Q}\right|_{V=V_\mu} && \text{by (3.2)} \\
&= -(n-2) + 2(n-2)K(t)\frac{e^{-V_\mu}-1}{-\mu V_\mu} \\
&= -(n-2) + \frac{2(n-2)}{\mu}K(t)\frac{e^{-V_\mu}-1}{-V_\mu} \\
&< -(n-2) + \frac{2(n-2)}{\mu} && \text{by } V_\mu > 0 \text{ and (3.1).}
\end{aligned}
\tag{3.6}
$$

By (3.6), we can obtain that

$$\mu^2 - (n-2)\mu + 2(n-2) > 0, \quad \text{for all } \mu > 0.$$

This implies $(n-2)^2 - 8(n-2) < 0$, but we already assume the critical and supercritical cases, i.e., $n \ge 10$, and this implies $(n-2)^2 - 8(n-2) \ge 0$. This gives a contradiction.

The only remaining case is $V(T) = 0, Q(T) = 0$. Then it is contradictory to the uniqueness of the original ODE system. Therefore, there is no $r_1 \in \mathbb{R}$ such that $u(r_1) - \bar{u}(r_1) \le 0$. This completes the proof. □

REMARK 3.2. The above three properties (3.3), (3.4) and (3.5) mean that the trajectory $(V(t), Q(t))$ starts at $(u(0) - \bar{u}(0), 0)$ with $u(0) - \bar{u}(0) > 0$ and swirls in a clockwise direction around origin $(0, 0)$.

4. Uniform lower bound of global solutions. In this section, we will prove that all solutions u to the Gelfand equation have uniform lower bound regardless of initial data $u(0) = u_0$. We state the main result of this part as follows:

PROPOSITION 4.1. Let $u(r)$ be a solution to the ODE (1.1) with $n > 2$. Then $u(r)$ satisfies the following lower bound estimate:

$$u(r) \ge 2\log r + \log \kappa_n + C_b(n), \quad r > 0, \tag{4.1}$$

where $C_b(n)$ is a constant depending on n and κ_n is a constant defined in (1.2).

To prove this proposition, we need several steps. We will investigate a linearized solution operator of this nonlinear equation. Consider the following second order ODE with a source term $F(r)$:

$$
\begin{aligned}
&X''(r) + \frac{n-1}{r}X'(r) + \left(\frac{n-2}{2}\right)^2\frac{1}{r^2}X(r) = -\frac{(10-n)(n-2)}{4r^2} + \frac{(n-2)^2}{4r^2}F(r), \\
&(X(k), X'(k)) = (0, X_1), \quad r > k.
\end{aligned}
\tag{4.2}
$$

By Lemmas 2.5 and 2.6, it is easy to see that $X(r)$ in (4.2) satisfies the following relation:

$$\begin{aligned} X(r) &= \left(\frac{r}{k}\right)^{1-n/2} kX_1 \log\frac{r}{k} - \frac{r^{1-n/2}}{4}\int_k^r (10-n)(n-2)\tau^{\frac{n}{2}-2}\log\left(\frac{r}{\tau}\right)d\tau \\ &\quad + \frac{r^{1-n/2}}{4}\int_k^r (n-2)^2 F(r)\tau^{\frac{n}{2}-2}\log\left(\frac{r}{\tau}\right)d\tau \\ &= \left(\frac{r}{k}\right)^{1-n/2} kX_1 \log\frac{r}{k} - (10-n)(n-2)\frac{4-(\frac{r}{k})^{1-n/2}(4+2(n-2)\log\frac{r}{k})}{4(n-2)^2} \\ &\quad + \frac{r^{1-n/2}}{4}\int_k^r (n-2)^2 F(r)\tau^{\frac{n}{2}-2}\log\left(\frac{r}{\tau}\right)d\tau. \end{aligned} \tag{4.3}$$

The solution (4.3) and the following lower bound of $r \cdot X'(r)$ will be used in the proof of Proposition 4.1.

LEMMA 4.1. Let $u = u(r)$ be a radial solution to the ODE (1.1) and

$$X(r) = u(r) - 2\log r + C,$$

where C is a constant. Then we have

$$r \cdot X'(r) > -2 \quad \text{for} \quad r > 0.$$

Proof. By the definition of $u(r)$, we have

$$(r^{n-1}u')' = r^{n-1}\kappa_n e^{-u}. \tag{4.4}$$

We integrate (4.4) to obtain the following relation:

$$u'(r) = \frac{1}{r^{n-1}}\int_0^r \tau^{n-1}\kappa_n e^{-u(\tau)}d\tau.$$

Therefore, we can use the monotonicity of u, i.e., $u' > 0$, to find that

$$X'(r) = u'(r) - \frac{2}{r} > -\frac{2}{r}.$$

□

REMARK 4.2. $u(r)$ in Proposition 4.1 satisfies the following estimate:

$$u(r) \geq \max\{u_0, 2\log s + C(n,\kappa_n)\},$$

because $u(r)$ is an increasing function.

To prove Proposition 4.1, we will take a proper mode of the linear operator with the singular solution $u_s(r;n)$. In the sequel, we will obtain a uniform lower bound by using (4.3). The following lemma is a crucial part of the proof for Proposition 4.1.

LEMMA 4.3. Let $u = u(r)$ be a radial solution to the ODE (1.1). We define $U_s(r)$ as follows:

$$U_s(r) := u_s(r;n) + \log\frac{8}{n-2} = 2\log r + \log\kappa_n + 2\log\frac{2}{n-2},$$

where $u_s(r;n)$ is the singular solution defined by (2.3). Suppose that there is a point $k > 0$ such that $u(k) = u_s(k)$. Then we have the following lower bound of $u(r)$:

$$u(r) > U_s(r) + C(n), \quad r > k,$$

where $C(n)$ is a constant depending on n.

Proof. Note that u and U_s satisfy

$$u''(r)+\frac{n-1}{r}u'(r)=\kappa_n e^{-u(r)} \text{ and } \quad U_s''(r)+\frac{n-1}{r}U_s'(r)=\frac{8}{n-2}\kappa_n e^{-U_s(r)}, \text{respectively.}$$

We denote that $v(r):=u(r)-U_s(r)$. Then the function $v(r)$ satisfies

$$\begin{aligned} v''(r)+\frac{n-1}{r}v'(r) &= \kappa_n e^{-u(r)}-\frac{8}{n-2}\kappa_n e^{-U_s(r)} \\ &= \frac{8}{n-2}\kappa_n e^{-U_s(r)}\Big(\frac{n-2}{8}e^{-u(r)+U_s(r)}-1\Big) \\ &= \frac{2(n-2)}{r^2}\Big(\frac{n-2}{8}e^{-v(r)}-1\Big). \end{aligned}$$

On the other hand, it follows from the assumption of this lemma and Lemma 4.1 that $v(k)=0$ and $v'(k)>-2/k$. Hence $v(r)$ satisfies the following IVP:

$$\begin{aligned} &v''(r)+\frac{n-1}{r}v'(r)=\frac{2(n-2)}{r^2}\Big(\frac{n-2}{8}e^{-v(r)}-1\Big), \quad r>k, \\ &v(k)=0, \qquad v'(k)=u'(k)-U_s'(k). \end{aligned} \tag{4.5}$$

We now claim that $v(r)$ satisfies the inequality

$$v(r)>C(n) \quad \text{for } r>k,$$

where $C(n)$ is a constant depending on n.

The proof of the claim. For the lower bound, we rewrite equation (4.5) by adding the following term:

$$\Big(\frac{n-2}{2}\Big)^2\frac{v(r)}{r^2}+\frac{2(n-2)}{r^2}-\frac{2(n-2)^2}{8r^2}$$

as

$$\begin{aligned} &v''(r)+(n-1)\frac{v'(r)}{r}+\Big(\frac{n-2}{2}\Big)^2\frac{v(r)}{r^2}+\frac{(10-n)(n-2)}{4r^2} \\ &\quad=\frac{(n-2)^2}{4r^2}\Big(e^{-v(r)}+v(r)-1\Big), \quad r>k, \\ &v(k)=0, \qquad v'(k)=u'(k)-U_s'(k). \end{aligned} \tag{4.6}$$

We use formula (4.3) to find an implicit representation for $v(r)$ in the ODE (4.6) as follows:

$$\begin{aligned} v(r) \;=\; &\Big(\frac{r}{k}\Big)^{1-n/2}kv'(k)\log\frac{r}{k}-(10-n)(n-2)\frac{4-(\frac{r}{k})^{1-n/2}(4+2(n-2)\log\frac{r}{k})}{4(n-2)^2} \\ &+\frac{r^{1-n/2}}{4}\int_k^r(n-2)^2(e^{-v(\tau)}+v(\tau)-1)\tau^{\frac{n}{2}-2}\log\Big(\frac{r}{\tau}\Big)d\tau. \end{aligned}$$

We now use Lemma 4.1 and the elementary estimate

$$kv'(k)>-2 \quad \text{and} \quad e^{-x}+x-1>0, \quad x\neq 0$$

to find

$$v(r)>-2\Big(\frac{r}{k}\Big)^{1-n/2}\log\frac{r}{k}-(10-n)(n-2)\frac{4-(\frac{r}{k})^{1-n/2}(4+2(n-2)\log\frac{r}{k})}{4(n-2)^2}. \tag{4.7}$$

For $2 < n \leq 6$, since $r/k > 1$ and

$$\left(\frac{r}{k}\right)^{1-n/2}\left(-2\log\frac{r}{k} + (10-n)\frac{(4+2(n-2)\log\frac{r}{k})}{4(n-2)}\right)$$

is strictly decreasing function for $r/k > 1$, in (4.7), we can conclude that

$$v(r) > -\frac{10-n}{n-2}.$$

For $n > 6$, since $\left(\frac{r}{k}\right)^{1-n/2}\left(-2\log\frac{r}{k} + (10-n)\frac{(4+2(n-2)\log\frac{r}{k})}{4(n-2)}\right)$ has minimum value $-\frac{n-6}{n-2}\exp\left(\frac{-4}{n-6}\right)$ at $\frac{r}{k} = \exp\frac{8}{(2-n)(6-n)}$, we have the following inequality:

$$v(r) > -\frac{10-n}{n-2} - \frac{n-6}{n-2}\exp\left(\frac{-4}{n-6}\right).$$

This completes the proof of the claim and we prove this lemma. □

Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. By the definition of $U_s(r)$, we know that $-\infty = \lim_{r\to 0+} U_s(r) < u(0)$. Therefore, there are only two possibilities as follows:

(i) $U_s(r)$ is always below $u(r)$ for all $r \geq 0$, i.e., $U_s(r) < u(r), \quad r \geq 0$.

(ii) There is a point $k > 0$ such that $U_s(k) = u(k)$.

We consider case (i). It is obvious that $u(r) \geq 2\log r + C(n)$. For case (ii), by Lemma 4.3, we have $u(r) \geq 2\log r + C(n)$. Thus, we complete the proof of Proposition 4.1. □

5. Convergence rate to the singular solution. In this section, we will obtain a convergence rate to the singular solution for the solution to the Gelfand equation (1.1). Consider a solution $u(r)$ to the Gelfand equations and the singular solution $u_s(r;n)$ defined in (2.3). For the reader's convenience, we recall that the Gelfand equation, i.e., $u(r)$ and $u_s(r;n)$, satisfy the following equations:

$$u'' + (n-1)\frac{u'}{r} = \kappa_n e^{-u}, \quad u(0) = u_0, \tag{5.1}$$

and

$$u_s'' + (n-1)\frac{u_s'}{r} = \kappa_n e^{-u_s}, \quad u_s(r;n) = \log\frac{r^2 k_n}{2(n-2)}. \tag{5.2}$$

We subtract two equations (5.1) and (5.2) to obtain the following equation for $v(r) = u(r) - u_s(r;n)$:

$$v'' + (n-1)\frac{v'}{r} = \quad \kappa_n(e^{-u} - e^{-u_s}) = \frac{2(n-2)}{r^2}(e^{-v} - 1). \tag{5.3}$$

As mentioned before, the main strategy to obtain an a priori estimate is to use the linearized Gelfand equation. If we assume that $|v| \ll 1$, then we have $(e^{-v} - 1) \approx -v$. Therefore, we can obtain the linearized equation of (5.3). To obtain decay rate of $v(r)$, we will use this linearized equation of (5.3). In the following lemma, we will recall a property of the solution of the linearized equation in Section 2.

LEMMA 5.1. For $n > 2$, let $v(r)$ be a solution to the following equation:

$$\begin{aligned} &v'' + (n-1)\frac{v'}{r} + \frac{2(n-2)v}{r^2} = f(r), \quad r > k, \\ &v(k) = v_k, \quad v'(k) = v'_k, \end{aligned} \tag{5.4}$$

where v_k and v'_k are fixed constants. Then $v(r)$ satisfies the following formulas:

(i) For $2 < n < 10$, we have

$$\begin{aligned} v(r) \quad = \quad & \left[v_k \cos(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{k}) + \frac{(n-2)v_k + 2kv'_k}{\sqrt{-\lambda}}\sin(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{k}) \right] \left(\frac{k}{r}\right)^{\frac{n-2}{2}} \\ & + \frac{2}{r^{\frac{n-2}{2}}\sqrt{-\lambda}} \int_k^r f(\tau)\tau^{\frac{n}{2}} \sin(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{\tau})d\tau. \end{aligned}$$

(ii) For $n = 10$,

$$v(r) \quad = \quad \left(v_k + \frac{n-2}{2}v_k \log\frac{r}{k} + kv'_k \log\frac{r}{k}\right)\left(\frac{k}{r}\right)^{\frac{n-2}{2}} + \frac{\log r}{r^{\frac{n-2}{2}}} \int_k^r f(\tau)\tau^{\frac{n}{2}}(1 - \frac{\log\tau}{\log r})d\tau.$$

(iii) For $n > 10$,

$$\begin{aligned} v(r) \quad = \quad & \frac{(n-2+\sqrt{\lambda})v_k + 2kv'_k}{2\sqrt{\lambda}}\left(\frac{k}{r}\right)^{\frac{n-2-\sqrt{\lambda}}{2}} - \frac{(n-2-\sqrt{\lambda})v_k + 2kv'_k}{2\sqrt{\lambda}}\left(\frac{k}{r}\right)^{\frac{n-2+\sqrt{\lambda}}{2}} \\ & + \frac{r}{\sqrt{\lambda}} \int_k^r f(\tau)\left[\left(\frac{\tau}{r}\right)^{\frac{n-\sqrt{\lambda}}{2}} - \left(\frac{\tau}{r}\right)^{\frac{n+\sqrt{\lambda}}{2}}\right]d\tau. \end{aligned}$$

Here $\lambda = (n-2)^2 - 8(n-2)$.

Proof. By Lemmas 2.5 and 2.6, we can easily verify this lemma. □

As we study in Lemma 5.1 and the previous sections, each solution $v(r)$ to the linearized equations of the subcritical, critical and supercritical cases has a different property. Therefore, we can expect that the proof of each case will be different. To prove Theorem 1.1, we will separate it into three cases, i.e., the subcritical, critical and supercritical cases.

The proof of Theorem 1.1. For the reader's convenience, we recall that

$$v(r) = u(r) - u_s(r; n),$$

and the following relation for v:

$$v'' + \frac{(n-1)v'}{r} = \frac{2(n-2)}{r^2}(e^{-v} - 1). \tag{5.5}$$

We now add $\dfrac{2(n-2)v}{r^2}$ to each side of the above equation to obtain

$$v'' + \frac{(n-1)v'}{r} + \frac{2(n-2)v}{r^2} = \frac{2(n-2)}{r^2}(e^{-v} + v - 1). \tag{5.6}$$

∘ *Subcritical case* $(2 < n < 10)$*:* Assume that $2 < n < 10$. By Lemma 5.1, we have an implicit relation for v as follows:

$$\begin{aligned} v(r) &= \left[v_k \cos(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{k}) + \frac{(n-2)v_k + 2kv_k'}{\sqrt{-\lambda}}\sin(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{k})\right]\left(\frac{k}{r}\right)^{\frac{n-2}{2}} \\ &\quad + \frac{2}{r^{\frac{n-2}{2}}\sqrt{-\lambda}}\int_k^r \frac{2(n-2)}{\tau^2}(e^{-v(\tau)} + v(\tau) - 1)\tau^{\frac{n}{2}}\sin(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{\tau})d\tau, \end{aligned}$$

where $\lambda = (n-2)^2 - 8(n-2)$. Therefore, we have the following representation of $r^{\frac{n-2}{2}}v(r)$:

$$\begin{aligned} r^{\frac{n-2}{2}}v(r) &= \left[v_k \cos(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{k}) + \frac{(n-2)v_k + 2kv_k'}{\sqrt{-\lambda}}\sin(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{k})\right]k^{\frac{n-2}{2}} \\ &\quad + \frac{2}{\sqrt{-\lambda}}\int_k^r \frac{2(n-2)}{\tau^2}(e^{-v(\tau)} + v(\tau) - 1)\tau^{\frac{n}{2}}\sin(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{\tau})d\tau. \end{aligned}$$

This implies that

$$\begin{aligned} |r^{\frac{n-2}{2}}v(r)| &\le C(n)\big(|v_k| + k|v_k'|\big)k^{\frac{n-2}{2}} + C(n)\int_k^r \frac{1}{\tau^2}|e^{-v(\tau)} + v(\tau) - 1|\tau^{\frac{n}{2}}d\tau \\ &= C(n)\big(|v_k| + k|v_k'|\big)k^{\frac{n-2}{2}} \\ &\quad + C(n)\int_k^r \frac{1}{\tau}\left|\frac{e^{-v(\tau)} + v(\tau) - 1}{v(\tau)}\right||\tau^{\frac{n-2}{2}}v(\tau)|d\tau. \end{aligned} \tag{5.7}$$

We can assume that $v(r)$ is sufficiently small on $[k, \infty)$ if we take k to be a large number by Theorem 2.3. We will take k large to satisfy that, for $r \in [k, \infty)$,

$$\left|\frac{e^{-v(r)} + v(r) - 1}{v(r)}\right| < 2v(r) \ll \frac{n-2}{16C(n)},$$

where $C(n)$ is the constant in (5.7). This implies that

$$|r^{\frac{n-2}{2}}v(r)| \le C(n)\big(|v_k| + k|v_k'|\big)k^{\frac{n-2}{2}} + \frac{n-2}{16}\int_k^r \frac{1}{\tau}|\tau^{\frac{n-2}{2}}v(\tau)|d\tau.$$

By the Gronwall's lemma, we have the following estimate:

$$|r^{\frac{n-2}{2}}v(r)| \le C(n)\big(|v_k| + k|v_k'|\big)k^{\frac{n-2}{2}}\left(\frac{r}{k}\right)^{\frac{n-2}{16}}. \tag{5.8}$$

By (5.7) and (5.8), we have

$$\begin{aligned} |r^{\frac{n-2}{2}}v(r)| &\le C(n)\big(|v_k| + k|v_k'|\big)k^{\frac{n-2}{2}} + C(n)\int_k^r \frac{1}{\tau}\left|\frac{e^{-v(\tau)} + v(\tau) - 1}{v(\tau)}\right||\tau^{\frac{n-2}{2}}v(\tau)|d\tau \\ &\le C(n)\big(|v_k| + k|v_k'|\big)k^{\frac{n-2}{2}} + C(n)\int_k^r \frac{1}{\tau}|2v(\tau)||\tau^{\frac{n-2}{2}}v(\tau)|d\tau \\ &\le C(n)\big(|v_k| + k|v_k'|\big)k^{\frac{n-2}{2}}\left[1 + 2C(n)^2\big(|v_k| + k|v_k'|\big)k^{\frac{n-2}{2}}\int_k^r \tau^{-\frac{n-2}{2}-1}\tau^{\frac{n-2}{8}}d\tau\right] \\ &\le C(n,k)\big(|v_k| + k|v_k'|\big)k^{\frac{n-2}{2}}, \end{aligned}$$

where $C(n,k)$ is a constant depending on n and k. Thus, we have the following convergence rate for the subcritical case:

$$|u(r)-u_s(r;n)| = |v(r)| \le C(n,k)\frac{(|v_k|+k|v_k'|)k^{\frac{n-2}{2}}}{r^{\frac{n-2}{2}}} \quad \text{on } r\in[k,\infty).$$

∘ *Critical case* ($n=10$)*:* Assume that $n=10$. Because of the $\log r$ term in Lemma 5.1 for $n=10$, we cannot use the method in the subcritical case ($2<n<10$). To avoid this unboundedness of the weighted term, we will use a kind of order preserving property between the solution and the singular solution $u_s(r;n)$ (see Section 3). This property allows us to use the supercritical linearized equation. Recall (5.6) and subtract $\frac{v}{r^2}$ from each side; then we have the following equation:

$$v''+\frac{(n-1)v'}{r}+\frac{2(n-2)v-v}{r^2}=\frac{2(n-2)}{r^2}(e^{-v}+v-1)-\frac{v}{r^2}.$$

By the formula in Lemmas 2.5 and 2.6 with $\alpha=9$, $\beta=15$ and $\lambda=4>0$, we have that

$$\begin{aligned} v(r) =& \frac{10v_k+2kv_k'}{4}\Big(\frac{k}{r}\Big)^3-\frac{6v_k+2kv_k'}{4}\Big(\frac{k}{r}\Big)^5 \\ &+\frac{r}{2}\int_k^r\left(\frac{16}{\tau^2}(e^{-v}+v-1)-\frac{v}{r^2}\right)\Big[\Big(\frac{\tau}{r}\Big)^4-\Big(\frac{\tau}{r}\Big)^6\Big]d\tau. \end{aligned} \tag{5.9}$$

Because we assume that $n=10$, we have $u(r)-u_s(r;n)>0$, i.e., $v(r)>0$ for all $r>0$ by Proposition 3.1.

Since the weight term in (5.9) is positive, i.e., $\Big(\frac{\tau}{r}\Big)^4-\Big(\frac{\tau}{r}\Big)^6>0$ for $\tau<r$, we have the following inequality:

$$0>\frac{r}{2}\int_k^r -\frac{v}{\tau^2}\Big[\Big(\frac{\tau}{r}\Big)^4-\Big(\frac{\tau}{r}\Big)^6\Big]d\tau. \tag{5.10}$$

Therefore, we have the following estimate:

$$\begin{aligned} 0<v(r) &= \frac{10v_k+2kv_k'}{4}\Big(\frac{k}{r}\Big)^3-\frac{6v_k+2kv_k'}{4}\Big(\frac{k}{r}\Big)^5 \\ &\quad+\frac{r}{2}\int_k^r\left(\frac{16}{\tau^2}(e^{-v}+v-1)-\frac{v}{\tau^2}\right)\Big[\Big(\frac{\tau}{r}\Big)^4-\Big(\frac{\tau}{r}\Big)^6\Big]d\tau \\ &\le \frac{10v_k+2kv_k'}{4}\Big(\frac{k}{r}\Big)^3-\frac{6v_k+2kv_k'}{4}\Big(\frac{k}{r}\Big)^5 \\ &\quad+\frac{r}{2}\int_k^r\frac{16}{\tau^2}(e^{-v}+v-1)\Big[\Big(\frac{\tau}{r}\Big)^4-\Big(\frac{\tau}{r}\Big)^6\Big]d\tau, \qquad \text{by (5.10).} \end{aligned}$$

The above inequality implies the following relation for $r^3v(r)$:

$$|r^3v(r)| \le C(|v_k|+k|v_k'|)k^3+C\int_k^r\frac{1}{\tau}\left|\frac{e^{-v}+v-1}{v}\right|v\tau^3d\tau.$$

We can apply the method in the subcritical case ($2<n<10$) to the above relation for $r^3v(r)$. Thus, we have the following estimate:

$$|v(r)|\le C\frac{(|v_k|+k|v_k'|)k^3}{r^3} \quad \text{on } r\in[k,\infty).$$

If we plug the above result into the formula in Lemma 5.1 with $f = \dfrac{2(n-2)}{r^2}(e^{-v}+v-1)$ and $\lambda = 0$, then we can obtain the following convergence rate for the critical case:

$$|u(r) - u_s(r;n)| = |v(r)| \quad \leq \quad C(n,k)(|v_k| + k|v_k'|)(\log \frac{r}{k} + 1)\Big(\frac{k}{r}\Big)^4.$$

∘ *Supercritical case* ($n > 10$)*:* Assume that $n > 10$. Then $v(r)$ satisfies the following equation:

$$v'' + \frac{(n-1)v'}{r} + \frac{2(n-2)v}{r^2} = \frac{2(n-2)}{r^2}(e^{-v} + v - 1).$$

We use the formula in Lemma 5.1 with $f(r) = \dfrac{2(n-2)}{r^2}(e^{-v} + v - 1)$ to obtain the following relation:

$$\begin{aligned} v(r) \quad = \quad & \frac{(n-2+\sqrt{\lambda})v_k + 2kv_k'}{2\sqrt{\lambda}}\Big(\frac{k}{r}\Big)^{\frac{n-2-\sqrt{\lambda}}{2}} - \frac{(n-2-\sqrt{\lambda})v_k + 2kv_k'}{2\sqrt{\lambda}}\Big(\frac{k}{r}\Big)^{\frac{n-2+\sqrt{\lambda}}{2}} \\ & + \frac{r}{\sqrt{\lambda}}\int_k^r \frac{2(n-2)}{\tau^2}(e^{-v} + v - 1)\Big[\Big(\frac{\tau}{r}\Big)^{\frac{n-\sqrt{\lambda}}{2}} - \Big(\frac{\tau}{r}\Big)^{\frac{n+\sqrt{\lambda}}{2}}\Big]d\tau. \end{aligned} \tag{5.11}$$

We have $u(r) - u_s(r;n) > 0$, i.e., $v(r) > 0$ for all $r > 0$ by Proposition 3.1 and $n > 10$. By (5.11) and $v(r) > 0$, we have that

$$\begin{aligned} 0 < v(r) \quad = \quad & \frac{(n-2+\sqrt{\lambda})v_k + 2kv_k'}{2\sqrt{\lambda}}\Big(\frac{k}{r}\Big)^{\frac{n-2-\sqrt{\lambda}}{2}} - \frac{(n-2-\sqrt{\lambda})v_k + 2kv_k'}{2\sqrt{\lambda}}\Big(\frac{k}{r}\Big)^{\frac{n-2+\sqrt{\lambda}}{2}} \\ & + \frac{r}{\sqrt{\lambda}}\int_k^r \frac{2(n-2)}{\tau^2}(e^{-v} + v - 1)\Big[\Big(\frac{\tau}{r}\Big)^{\frac{n-\sqrt{\lambda}}{2}} - \Big(\frac{\tau}{r}\Big)^{\frac{n+\sqrt{\lambda}}{2}}\Big]d\tau \\ \leq \quad & \frac{(n-2+\sqrt{\lambda})|v_k| + 4|kv_k'|}{2\sqrt{\lambda}}\Big(\frac{k}{r}\Big)^{\frac{n-2-\sqrt{\lambda}}{2}} \\ & + \frac{r}{\sqrt{\lambda}}\int_k^r \frac{2(n-2)}{\tau^2}(e^{-v} + v - 1)\Big(\frac{\tau}{r}\Big)^{\frac{n-\sqrt{\lambda}}{2}} d\tau. \end{aligned}$$

This implies the following inequality:

$$|v(r)| \quad \leq \quad C(n)(|v_k| + k|v_k'|)\Big(\frac{k}{r}\Big)^{\frac{n-2-\sqrt{\lambda}}{2}} + C(n)\int_k^r \frac{1}{\tau}|e^{-v} + v - 1|\Big(\frac{\tau}{r}\Big)^{\frac{n-2-\sqrt{\lambda}}{2}} d\tau.$$

Therefore, we can apply the same method as in the subcritical case ($2 < n < 10$) to the above relation to obtain the following estimate:

$$|u(r) - u_s(r;n)| = |v(r)| \leq C(n,k)\frac{(|v_k| + k|v_k'|)k^{\frac{n-2-\sqrt{\lambda}}{2}}}{r^{\frac{n-2-\sqrt{\lambda}}{2}}} \quad \text{on } r \in [k, \infty).$$

Thus, we have obtained the convergence rate of the solution of the Gelfand equation to the singular solution for three cases and we complete the proof of Theorem 1.1.

Remark 5.2. We note that for $n > 10$, we have $n - 2 - \sqrt{\lambda} > 3$. It implies that the decay rate in the above equation is greater than $3/2$.

6. Stability. In this section, we will present global stability of the solution to the Gelfand equation. As mentioned before, the main strategy of the proof in this part is similar to that given in Section 5. However, the difference with Section 5 is that we cannot use the smallness of $v(r)$ by $k \to \infty$ and the order preserving property between the solution $u(r)$ of the Gelfand equation and the singular solution $u_s(r;n)$, i.e., Proposition 3.1. In this part, we will use local stability and the order preserving property in Proposition 3.2 as the alternative of the smallness of $v(r)$ and Proposition 3.1 respectively.

Consider a solution $u(r)$ to the Gelfand equation with initial $u(0) = u_0$ as follows:

$$u'' + (n-1)\frac{u'}{r} = \kappa_n e^{-u}, \quad u(0) = u_0. \tag{6.1}$$

To establish a stability estimate, we consider a family of solutions to the Gelfand equation with different initial data with $|u(0) - \bar{u}(0)| \ll 1$, i.e.,

$$\bar{u}'' + (n-1)\frac{\bar{u}'}{r} = \kappa_n e^{-\bar{u}}, \quad \bar{u}(0) = \bar{u}_0 \quad \text{with } |u_0 - \bar{u}_0| < \delta < 1. \tag{6.2}$$

The range of δ in (6.2) will be determined later. We subtract two equations for u and $\bar{u}$, i.e., (6.1) and (6.2), to obtain the following equation for $v(r) = u(r) - \bar{u}(r)$:

$$v'' + (n-1)\frac{v'}{r} = \kappa_n e^{-u} - \kappa_n e^{-\bar{u}}, \quad v(0) = u_0 - \bar{u}_0. \tag{6.3}$$

Lemma 6.1 (Local stability). Let $u(r)$ and $\bar{u}(r)$ be the solutions to the Gelfand equation (1.1). Then we have the following local stability estimates:

$$|u(r) - \bar{u}(r)| \quad \leq \quad |u_0 - \bar{u}_0| r^{C(n,u_0)} \tag{6.4}$$

and

$$|u'(r) - \bar{u}'(r)| \quad \leq \quad |u_0 - \bar{u}_0| r^{C(n,u_0)}, \tag{6.5}$$

where $C(n, u_0)$ is a constant depending on n, u_0.

Proof. We multiply each side of (6.3) by r^{n-1} and integrate each side on $[0, r)$. Then we have the following integral equation:

$$\begin{aligned} &\int_0^r s^{n-1} v''(s) ds + (n-1)\int_0^r s^{n-1}\frac{v'(s)}{s} ds = \int_0^r s^{n-1}(\kappa_n e^{-u(s)} - \kappa_n e^{-\bar{u}(s)}) ds, \\ &v(0) = u_0 - \bar{u}_0. \end{aligned}$$

By integration by parts, the above equation implies that

$$\begin{aligned} r^{n-1}v'(r) - \int_0^r (n-1)s^{n-2}v'(s)ds + (n-1)\int_0^r s^{n-1}\frac{v'(s)}{s}ds \\ = \int_0^r s^{n-1}(\kappa_n e^{-u(s)} - \kappa_n e^{-\bar{u}(s)})ds, \\ v(0) = u_0 - \bar{u}_0. \end{aligned} \tag{6.6}$$

We divide (6.6) by r^{n-1} and integrate each side one more time on $[0, r]$ to obtain that

$$v(r) = u_0 - \bar{u}_0 + \int_0^r \frac{1}{t^{n-1}} \int_0^t s^{n-1}(\kappa_n e^{-u(s)} - \kappa_n e^{-\bar{u}(s)}) ds dt.$$

By the Fubini theorem, we have the following relation for $v(r)$:

$$\begin{aligned} v(r) &= u_0 - \bar{u}_0 + \int_0^r \int_s^r \frac{1}{t^{n-1}} s^{n-1}(\kappa_n e^{-u(s)} - \kappa_n e^{-\bar{u}(s)}) dt ds \\ &= u_0 - \bar{u}_0 + \int_0^r \frac{1}{n-2}\Big(\frac{1}{s^{n-2}} - \frac{1}{r^{n-2}}\Big) s^{n-1}(\kappa_n e^{-u(s)} - \kappa_n e^{-\bar{u}(s)}) ds. \end{aligned}$$

To obtain a local stability estimate for $v(r) = u(r) - \bar{u}(r)$, we will use the above implicit expression for v. The above relation implies that

$$\begin{aligned} |v(r)| &= \left| u_0 - \bar{u}_0 + \int_0^r \frac{1}{n-2}\Big(\frac{1}{s^{n-2}} - \frac{1}{r^{n-2}}\Big) s^{n-1}(\kappa_n e^{-u(s)} - \kappa_n e^{-\bar{u}(s)}) ds \right| \\ &\le |u_0 - \bar{u}_0| + \int_0^r \left| \frac{1}{n-2}\Big(\frac{1}{s^{n-2}} - \frac{1}{r^{n-2}}\Big) s^{n-1}(\kappa_n e^{-u(s)} - \kappa_n e^{-\bar{u}(s)}) \right| ds \\ &\le |u_0 - \bar{u}_0| + \frac{1}{n-2}\int_0^r \left| \Big(\frac{1}{s^{n-2}} - \frac{1}{r^{n-2}}\Big) s^{n-1} \right| |\kappa_n e^{-u(s)} - \kappa_n e^{-\bar{u}(s)}| ds. \end{aligned}$$

We can divide the interval $[0, \infty)$ into two cases: $v(s) \ge 0$ and $v(s) < 0$.

For the $s \in \{v(s) = u(s) - \bar{u}(s) \ge 0\}$ case, we can obtain the following estimate:

$$\begin{aligned} |\kappa_n e^{-u(s)} - \kappa_n e^{-\bar{u}(s)}| = |\kappa_n| \left| \frac{e^{-v(s)} - 1}{v(s)} \right| \frac{|v(s)|}{e^{\bar{u}(s)}} &\le |\kappa_n| \frac{|v(s)|}{e^{\bar{u}(s)}} \\ &\le \frac{|\kappa_n||v(s)|}{\exp\max\{\bar{u}_0, 2\log s + C(n)\}} \\ &\le \frac{|\kappa_n||v(s)|}{\exp\max\{u_0 - 1, 2\log s + C(n)\}}. \end{aligned} \tag{6.7}$$

The above inequalities hold because $|\frac{e^{-x}-1}{x}| \le 1$ for $x \ge 0$ and

$$\bar{u}(s) \ge \max\{\bar{u}_0, 2\log s + C(n)\}$$

by Proposition 4.1 and $|u_0 - \bar{u}_0| < \delta < 1$.

For the $s \in \{v(s) = u(s) - \bar{u}(s) < 0\}$ case, we also obtain the following estimate in the same way:

$$\begin{aligned} |\kappa_n e^{-u(s)} - \kappa_n e^{-\bar{u}(s)}| &\le \frac{|\kappa_n||v(s)|}{\exp\max\{u_0, 2\log s + C(n)\}} \\ &\le \frac{|\kappa_n||v(s)|}{\exp\max\{u_0 - 1, 2\log s + C(n)\}}. \end{aligned} \tag{6.8}$$

By (6.7) and (6.8), we can obtain the following a priori estimate of $|v|$:

$$|v(r)| \le |u_0 - \bar{u}_0| + \frac{1}{n-2}\int_0^r \Big(\frac{1}{s^{n-2}} - \frac{1}{r^{n-2}}\Big) s^{n-1} \frac{|\kappa_n||v(s)|}{\exp\max\{u_0 - 1, 2\log s + C(n)\}} ds.$$

By Gronwall's lemma, this a priori estimate implies that

$$|v(r)| \le |u_0 - \bar{u}_0| r^{C(n,u_0)}.$$

Since we have

$$v'(r) = \frac{1}{t^{n-1}} \int_0^t s^{n-1}(\kappa_n e^{-u(s)} - \kappa_n e^{-\bar{u}(s)}) ds$$

and by estimate (6.7), (6.8) and (6.4), we have the following estimate for the derivative of v:

$$|v'(r)| \quad \leq \quad |u_0 - \bar{u}_0| r^{C(n,u_0)}.$$

Therefore, we obtained the local stability estimate for the Gelfand equation, i.e., (6.4) and (6.5). □

In the rest of this section, we will obtain the global stability estimate of the Gelfand equation on $[k, \infty)$ by a similar manner as in Section 5.

PROPOSITION 6.1. For the subcritical case ($2 < n < 10$), we have

$$|u(r) - \bar{u}(r)| \leq |u_0 - \bar{u}_0| \frac{C(u_0, \delta, n)}{(r+1)^{\frac{n-2}{2} - \epsilon}},$$

where $C(u_0, \delta, n)$ is a constant depending on u_0, δ and n.

Proof. In this part, we will take any fixed constant $\epsilon \in (0, \frac{n-2}{2})$.

We recall the subtraction of two Gelfand equations, i.e., the equation for $v(r)$:

$$v'' + (n-1)\frac{v'}{r} = \kappa_n e^{-u} - \kappa_n e^{-\bar{u}}, \quad v(k) = u_k - \bar{u}_k. \tag{6.9}$$

We use the following notation for convenience:

$$u(k) = u_k, \ \bar{u}(k) = \bar{u}_k.$$

We add the $2(n-2)v/r^2$ term to each side of (6.9). Then we have the following equation:

$$v'' + (n-1)\frac{v'}{r} + \frac{2(n-2)}{r^2} v = \kappa_n e^{-u} - \kappa_n e^{-\bar{u}} + \frac{2(n-2)}{r^2} v, \quad v(k) = u_k - \bar{u}_k. \tag{6.10}$$

Then the subcritical assumption ($2 < n < 10$) implies $\lambda = (n-2)^2 - 8(n-2) < 0$. By $\lambda < 0$ and Lemma 5.1, we have the following relation for $v(r)$ with $f(r) = \kappa_n e^{-u} - \kappa_n e^{-\bar{u}} + \dfrac{2(n-2)}{r^2} v$:

$$\begin{aligned} v(r) = & \left[v_k \cos(\frac{\sqrt{-\lambda}}{2} \log \frac{r}{k}) + \frac{(n-2)v_k + 2kv_k'}{\sqrt{-\lambda}} \sin(\frac{\sqrt{-\lambda}}{2} \log \frac{r}{k}) \right] \left(\frac{k}{r}\right)^{\frac{n-2}{2}} \\ & + \frac{2}{r^{\frac{n-2}{2}} \sqrt{-\lambda}} \int_k^r \left(\kappa_n e^{-u(\tau)} - \kappa_n e^{-\bar{u}(\tau)} + \frac{2(n-2)}{\tau^2} v(\tau) \right) \tau^{\frac{n}{2}} \sin(\frac{\sqrt{-\lambda}}{2} \log \frac{r}{\tau}) d\tau. \end{aligned} \tag{6.11}$$

We multiply each side of (6.11) by $r^{\frac{n-2}{2}}$ to get the following relation for $r^{\frac{n-2}{2}} v(r)$:

$$\begin{aligned} r^{\frac{n-2}{2}} v(r) \quad = \quad & \left[v_k \cos(\frac{\sqrt{-\lambda}}{2} \log \frac{r}{k}) + \frac{(n-2)v_k + 2kv_k'}{\sqrt{-\lambda}} \sin(\frac{\sqrt{-\lambda}}{2} \log \frac{r}{k}) \right] k^{\frac{n-2}{2}} \\ & + 2\sqrt{-\lambda} \int_k^r (\kappa_n e^{-u(\tau)} - \kappa_n e^{-\bar{u}(\tau)} + \frac{2(n-2)}{\tau^2} v(\tau)) \tau^{\frac{n}{2}} \sin(\frac{\sqrt{-\lambda}}{2} \log \frac{r}{\tau}) d\tau \end{aligned}$$

$$
\begin{aligned}
&= \left[v_k \cos(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{k}) + \frac{(n-2)v_k + 2kv_k'}{\sqrt{-\lambda}}\sin(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{k})\right]k^{\frac{n-2}{2}} \\
&\quad +2\sqrt{-\lambda}\int_k^r \Big(\kappa_n e^{-u(\tau)} - \kappa_n e^{-\bar{u}(\tau)} + \frac{\kappa_n}{e^{u(\tau)}}v(\tau)\Big)\tau^{\frac{n}{2}}\sin(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{\tau})d\tau \\
&\quad +2\sqrt{-\lambda}\int_k^r \Big(\frac{2(n-2)}{\tau^2}v(\tau) - \frac{\kappa_n}{e^{u(\tau)}}v(\tau)\Big)\tau^{\frac{n}{2}}\sin(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{\tau})d\tau \\
&=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}
$$

Before considering the $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$ terms, we will determine k. We take $k > 1$ sufficiently large so that

$$
4\sqrt{-\lambda}(n-2)\Big|1 - \frac{1}{e^{u-u_s}}\Big| \le \frac{\epsilon}{2}, \quad \text{for all } r > k > 1. \tag{6.12}
$$

By Theorem 2.3, it is possible for the above choice of k.

Consider the $\mathcal{I}_1$ term. On the fixed time k, we can control $v_k = u_k - \bar{u}_k$ and $v_k' = u_k' - \bar{u}'_k$ by the initial data $u_0 - \bar{u}_0$ because of the local estimate of the previous part, i.e., we have

$$
|\mathcal{I}_1| \le C(u_0, k, n)|u_0 - \bar{u}_0|, \tag{6.13}
$$

where $C(u_0, k, n)$ is a positive constant depending on u_0, k and n.

We now restrict the range of $|u_0 - \bar{u}_0|$ as follows:

$$
|u_0 - \bar{u}_0| < \delta =: \frac{\epsilon}{8e^{-C_b(n)}\max(\sqrt{-\lambda}, 1)C(u_0, k, n)}. \tag{6.14}
$$

For the $\mathcal{I}_2$ term, we can obtain the following estimate:

$$
\begin{aligned}
|\mathcal{I}_2| &= \left|2\sqrt{-\lambda}\int_k^r \Big(\kappa_n e^{-u(\tau)} - \kappa_n e^{-\bar{u}(\tau)} + \frac{\kappa_n}{e^{u(\tau)}}v(\tau)\Big)\tau^{\frac{n}{2}}\sin(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{\tau})d\tau\right| \\
&\le 2\kappa_n\sqrt{-\lambda}\left|\int_k^r \Big(e^{-u(\tau)} - e^{-\bar{u}(\tau)} + \frac{1}{e^{u(\tau)}}v(\tau)\Big)\tau^{\frac{n}{2}}d\tau\right| \\
&= 2\kappa_n\sqrt{-\lambda}\left|\int_k^r \frac{1}{e^{u(\tau)}}\frac{1 - e^{v(\tau)} + v(\tau)}{v(\tau)}v(\tau)\tau^{\frac{n}{2}}d\tau\right| \\
&= 2\kappa_n\sqrt{-\lambda}\left|\int_k^r \frac{\tau^2}{e^{u(\tau)}}\frac{1 - e^{v(\tau)} + v(\tau)}{v(\tau)}v(\tau)\tau^{\frac{n-2}{2}}\tau^{-1}d\tau\right| \\
&\le 2\kappa_n\max(\sqrt{-\lambda}, 1)\left|\int_k^r \frac{\tau^2}{e^{u(\tau)}}\frac{1 - e^{v(\tau)} + v(\tau)}{v(\tau)}v(\tau)\tau^{\frac{n-2}{2}}\tau^{-1}d\tau\right|.
\end{aligned}
$$

By Proposition 4.1, we have $\left|\frac{\tau^2}{e^{u(\tau)}}\right| \le \frac{1}{\kappa_n e^{-C_b(n)}}$. This implies that

$$
|\mathcal{I}_2| \le 2e^{-C_b(n)}\max(\sqrt{-\lambda}, 1)\int_k^r \left|\frac{1 - e^{v(\tau)} + v(\tau)}{v(\tau)}\right||v(\tau)\tau^{\frac{n-2}{2}}|\frac{1}{\tau}d\tau. \tag{6.15}
$$

Claim: $|v| < \dfrac{\epsilon}{8e^{-C_b(n)}\max(\sqrt{-\lambda}, 1)}$ for all $r > k$.

Proof. By (6.13) and (6.14), we have

$$
|v(k)| < \frac{\epsilon}{8e^{-C_b(n)}\max(\sqrt{-\lambda}, 1)}.
$$

Therefore, there is a maximal constant $T > k$ such that $|v| < \dfrac{\epsilon}{8e^{-C_b(n)}\max(\sqrt{-\lambda},1)}$ for $r \in (k,T)$, and we have

$$\left|\frac{1-e^v+v}{v}\right| < 2|v| < \frac{\epsilon}{4e^{-C_b(n)}\max(\sqrt{-\lambda},1)}. \tag{6.16}$$

By (6.16), the above relation (6.15) implies that

$$|\mathcal{I}_2| \le \frac{\epsilon}{2}\int_k^r |v(\tau)\tau^{\frac{n-2}{2}}|\frac{1}{\tau}d\tau.$$

For the $\mathcal{I}_3$ term, we have the following estimate:

$$\begin{aligned}
|\mathcal{I}_3| &= \left|2\sqrt{-\lambda}\int_k^r \Big(\frac{2(n-2)}{\tau^2}v(\tau) - \frac{\kappa_n}{e^{u(\tau)}}v(\tau)\Big)\tau^{\frac{n}{2}}\sin(\frac{\sqrt{-\lambda}}{2}\log\frac{r}{\tau})d\tau\right| \\
&\le 2\sqrt{-\lambda}\int_k^r \left|\Big(\frac{2(n-2)}{\tau^2}v(\tau) - \frac{\kappa_n}{e^{u(\tau)}}v(\tau)\Big)\tau^{\frac{n}{2}}\right|d\tau \\
&\le 2\sqrt{-\lambda}\int_k^r \left|2(n-2) - \frac{\kappa_n r^2}{e^{u(\tau)}}\right||v(\tau)\tau^{\frac{n-2}{2}}|\frac{1}{\tau}d\tau \\
&\le 4\sqrt{-\lambda}(n-2)\int_k^r \left|1-\frac{1}{e^{u-u_s}}\right||v(\tau)\tau^{\frac{n-2}{2}}|\frac{1}{\tau}d\tau.
\end{aligned}$$

We already take k sufficiently large so that $4\sqrt{-\lambda}(n-2)\left|1-\dfrac{1}{e^{u-u_s}}\right| \le \dfrac{\epsilon}{2}$, and this implies that

$$|\mathcal{I}_3| \le \frac{\epsilon}{2}\int_k^r |v(\tau)\tau^{\frac{n-2}{2}}|\frac{1}{\tau}d\tau.$$

Therefore, we have the relation of $|r^{\frac{n-2}{2}}v(r)|$ as follows:

$$|r^{\frac{n-2}{2}}v(r)| \le C(u_0,k,n)|u_0-\bar{u}_0| + \epsilon\int_k^r |v(\tau)\tau^{\frac{n-2}{2}}|\frac{1}{\tau}d\tau.$$

Thus, by Granwall's lemma, we have

$$|r^{\frac{n-2}{2}}v(r)| \le C(u_0,k,n)|u_0-\bar{u}_0|\left(\frac{r}{k}\right)^{\epsilon}. \tag{6.17}$$

We already determine size of $\delta > 0$ as follows:

$$|u_0-\bar{u}_0| < \delta = \frac{\epsilon}{8e^{-C_b(n)}\max(\sqrt{-\lambda},1)C(u_0,k,n)}.$$

This implies that for all $r \in (k,T)$,

$$|v(r)| \le \frac{\epsilon}{8e^{-C_b(n)}\max(\sqrt{-\lambda},1)}\frac{1}{k^{\epsilon}r^{\frac{n-2}{2}-\epsilon}} < \frac{\epsilon}{8e^{-C_b(n)}\max(\sqrt{-\lambda},1)}$$

by $r > k > 1$ and $\frac{n-2}{2} > \epsilon > 0$.

The above leads to the fact that $T = \infty$ and we prove the claim. □

Therefore, by (6.17), we have the following result for the subcritical case:

$$|u(r)-\bar{u}(r)| \le |u_0-\bar{u}_0|\frac{C(u_0,\delta,n)}{(r+1)^{\frac{n-2}{2}-\epsilon}}, \quad \text{for } |u_0-\bar{u}_0| < \delta.$$

□

PROPOSITION 6.2. For the critical case ($n = 10$), we have

$$|u(r) - \bar{u}(r)| \le |u_0 - \bar{u}_0| \frac{C(u_0, \delta, n)}{(r+1)^{\frac{n-2}{2}-\epsilon}} \log(r+1),$$

where $C(u_0, \delta, n)$ is a constant depending on u_0, δ and n.

Proof. Assume that $n = 10$. Due to a similar reason as in the proof of convergence rate for the critical case in Section 5, we cannot use the method in the subcritical case ($2 < n < 10$). The strategy is similar to the proof of convergence rate for the critical case in Section 5. Recall (6.10) and subtract $\frac{v}{r^2}$ from each side to obtain the following equation:

$$v'' + (n-1)\frac{v'}{r} + \frac{2(n-2) - v}{r^2} v = \kappa_n e^{-u} - \kappa_n e^{-\bar{u}} + \frac{2(n-2)}{r^2} v - \frac{v}{r^2}.$$

By the formula in Lemmas 2.5 and 2.6 with $\alpha = 9$, $\beta = 15$ and $\lambda = 4 > 0$, we have the relation of $v(r)$ as follows:

$$\begin{aligned} v(r) = {} & \frac{10v_k + 2kv_k'}{4}\Big(\frac{k}{r}\Big)^3 - \frac{6v_k + 2kv_k'}{4}\Big(\frac{k}{r}\Big)^5 \\ & + \frac{r}{2}\int_k^r \left(\kappa_n e^{-u} - \kappa_n e^{-\bar{u}} + \frac{2(n-2)}{\tau^2} v - \frac{v}{\tau^2}\right)\Big[\Big(\frac{\tau}{r}\Big)^4 - \Big(\frac{\tau}{r}\Big)^6\Big] d\tau. \end{aligned} \tag{6.18}$$

Assume that $v(0) > 0$. Since the $v(0) < 0$ case is exactly the same as the $v(0) > 0$ case, we will only consider the $v(0) > 0$ case. For the critical and supercritical cases ($n \ge 10$), $v(0) > 0$ implies that $v(r) > 0$ for all $r > 0$ by Proposition 3.2, i.e., the order preserving property. Since the weight term in (6.18) is positive, i.e., $\Big(\frac{\tau}{r}\Big)^4 - \Big(\frac{\tau}{r}\Big)^6 > 0$ for $\tau < r$, we have the following inequality:

$$0 > \frac{r}{2}\int_k^r -\frac{v}{\tau^2}\Big[\Big(\frac{\tau}{r}\Big)^4 - \Big(\frac{\tau}{r}\Big)^6\Big] d\tau. \tag{6.19}$$

By (6.18) and (6.19), we can obtain the following inequality for $v(r)$:

$$\begin{aligned} 0 < v(r) = {} & \frac{10v_k + 2kv_k'}{4}\Big(\frac{k}{r}\Big)^3 - \frac{6v_k + 2kv_k'}{4}\Big(\frac{k}{r}\Big)^5 \\ & + \frac{r}{2}\int_k^r \left(\kappa_n e^{-u} - \kappa_n e^{-\bar{u}} + \frac{2(n-2)}{\tau^2} v\right)\Big[\Big(\frac{\tau}{r}\Big)^4 - \Big(\frac{\tau}{r}\Big)^6\Big] d\tau \\ \le {} & \frac{10|v_k| + 4|kv_k'|}{4}\Big(\frac{k}{r}\Big)^3 \\ & + \frac{r}{2}\int_k^r \left(\kappa_n e^{-u} - \kappa_n e^{-\bar{u}} + \frac{\kappa_n}{e^{\bar{u}(\tau)}} v(\tau)\right)\Big[\Big(\frac{\tau}{r}\Big)^4 - \Big(\frac{\tau}{r}\Big)^6\Big] d\tau \\ & + \frac{r}{2}\int_k^r \left(-\frac{\kappa_n}{e^{\bar{u}(\tau)}} v(\tau) + \frac{2(n-2)}{\tau^2} v\right)\Big[\Big(\frac{\tau}{r}\Big)^4 - \Big(\frac{\tau}{r}\Big)^6\Big] d\tau \\ =: {} & I_1 + I_2 + I_3 \\ & \Big(\text{for the } v(0) < 0 \text{ case, we replace } \frac{\kappa_n}{e^{\bar{u}(\tau)}} v(\tau) \text{ by } \frac{\kappa_n}{e^{u(\tau)}} v(\tau)\Big). \end{aligned} \tag{6.20}$$

Since $\bar{u}(r) - u_s(r) > 0,\ v(r) > 0$ for all $r > 0$, we have

$$\begin{aligned} I_2 &= \int_k^r \left(\kappa_n e^{-u} - \kappa_n e^{-\bar{u}} + \frac{\kappa_n}{e^{\bar{u}(\tau)}} v(\tau)\right)\left[\left(\frac{\tau}{r}\right)^4 - \left(\frac{\tau}{r}\right)^6\right] d\tau \\ &= \int_k^r \kappa_n e^{-\bar{u}}\left(e^{-v} - 1 + v(\tau)\right)\left[\left(\frac{\tau}{r}\right)^4 - \left(\frac{\tau}{r}\right)^6\right] d\tau \qquad (6.21) \\ &\le \int_k^r \kappa_n e^{-\bar{u}}\left(e^{-v} - 1 + v(\tau)\right)\left(\frac{\tau}{r}\right)^4 d\tau \end{aligned}$$

and

$$\begin{aligned} I_3 &= \int_k^r \left(-\frac{\kappa_n}{e^{\bar{u}(\tau)}} v(\tau) + \frac{2(n-2)}{\tau^2} v\right)\left[\left(\frac{\tau}{r}\right)^4 - \left(\frac{\tau}{r}\right)^6\right] d\tau \\ &= \int_k^r \frac{2(n-2)}{\tau^2} v\left(1 - \frac{1}{e^{\bar{u}(\tau) - u_s(\tau)}}\right)\left[\left(\frac{\tau}{r}\right)^4 - \left(\frac{\tau}{r}\right)^6\right] d\tau \qquad (6.22) \\ &\le \int_k^r \frac{2(n-2)}{\tau^2} v\left(1 - \frac{1}{e^{\bar{u}(\tau) - u_s(\tau)}}\right)\left(\frac{\tau}{r}\right)^4 d\tau. \end{aligned}$$

By (6.20), (6.21) and (6.22), we have the following estimate:

$$0 < v(r) \quad \le \quad \frac{10|v_k| + 4|kv_k'|}{4}\left(\frac{k}{r}\right)^3 + \frac{r}{2}\int_k^r \left(\kappa_n e^{-u} - \kappa_n e^{-\bar{u}} + \frac{2(n-2)}{\tau^2} v\right)\left(\frac{\tau}{r}\right)^4 d\tau.$$

The above inequality implies the following relation of $|r^3 v(r)|$:

$$\begin{aligned} |r^3 v(r)| &\le C(|v_k| + k|v_k'|)k^3 + C\left|\int_k^r \left(\kappa_n e^{-u} - \kappa_n e^{-\bar{u}} + \frac{2(n-2)}{\tau^2} v\right)\tau^4 d\tau\right| \\ &\le C(|v_k| + k|v_k'|)k^3 + C\left|\int_k^r \left(\kappa_n e^{-u} - \kappa_n e^{-\bar{u}} + \frac{\kappa_n}{e^{u(\tau)}} v(\tau)\right)\tau^4 d\tau\right| \\ &\quad + C\left|\int_k^r \left(-\frac{\kappa_n}{e^{u(\tau)}} v(\tau) + \frac{2(n-2)}{\tau^2} v\right) v\tau^4 d\tau\right| \\ &\le C(|v_k| + k|v_k'|)k^3 + C(n)\int_k^r \left|\frac{1 - e^v + v(\tau)}{v}\right| |v\tau^3| \frac{1}{\tau} d\tau \\ &\quad + C(n)\int_k^r \left|1 - \frac{1}{e^{u - u_s}}\right| |v\tau^3| \frac{1}{\tau} d\tau. \end{aligned}$$

Therefore, we can apply the same method as in the subcritical case to obtain the following estimate:

$$|u(r) - \bar{u}(r)| \le |u_0 - \bar{u}_0| \frac{C(u_0, \delta)}{(r+1)^{3-\epsilon}}, \quad \text{for } |u_0 - \bar{u}_0| < \delta.$$

If we plug the above result into the formula in Lemma 5.1 with $f = \kappa_n e^{-u} - \kappa_n e^{-\bar{u}} + \dfrac{2(n-2)}{\tau^2} v$ and $\lambda = 0$, then we can obtain the following global stability estimate:

$$|u(r) - \bar{u}(r)| \le |u_0 - \bar{u}_0| \frac{C(u_0, \delta)}{(r+1)^{4-\epsilon}}, \quad \text{for } |u_0 - \bar{u}_0| < \delta.$$

□

Proposition 6.3. For the supercritical case ($n > 10$), we have

$$|u(r) - \bar{u}(r)| \le |u_0 - \bar{u}_0| \frac{C(u_0, \delta, n)}{(r+1)^{\frac{n-2-\sqrt{\lambda}}{2} - \epsilon}},$$

where $C(u_0, \delta, n)$ is a constant depending on u_0, δ and n.

Proof. We recall (6.10):

$$v'' + (n-1)\frac{v'}{r} + \frac{2(n-2)}{r^2}v = \kappa_n e^{-u} - \kappa_n e^{-\bar{u}} + \frac{2(n-2)}{r^2}v, \quad v(k) = u_k - \bar{u}_k.$$

For the supercritical case ($n > 10$), we have $\lambda = (n-2)^2 - 8(n-2) > 0$. Therefore, by Lemma 5.1, we have the following relation for $v(r)$:

$$\begin{aligned} v(r) = {} & \frac{(n-2+\sqrt{\lambda})v_k + 2kv_k'}{2\sqrt{\lambda}}\Big(\frac{k}{r}\Big)^{\frac{n-2-\sqrt{\lambda}}{2}} - \frac{(n-2-\sqrt{\lambda})v_k + 2kv_k'}{2\sqrt{\lambda}}\Big(\frac{k}{r}\Big)^{\frac{n-2+\sqrt{\lambda}}{2}} \\ & + \frac{r}{\sqrt{\lambda}}\int_k^r \Big(\kappa_n e^{-u(\tau)} - \kappa_n e^{-\bar{u}(\tau)} + \frac{2(n-2)}{\tau^2}v(\tau)\Big)\Big[\Big(\frac{\tau}{r}\Big)^{\frac{n-\sqrt{\lambda}}{2}} - \Big(\frac{\tau}{r}\Big)^{\frac{n+\sqrt{\lambda}}{2}}\Big]d\tau. \end{aligned}$$

The above relation and Proposition 3.2, i.e., the order preserving property, allow us to use the same method as in the critical case ($n = 10$) in this section for the supercritical case. Thus, we present the result as follows:

$$|u(r) - \bar{u}(r)| \le |u_0 - \bar{u}_0| \frac{C(u_0, \delta, n)}{(r+1)^{\frac{n-2-\sqrt{\lambda}}{2} - \epsilon}}, \quad \text{for } |u_0 - \bar{u}_0| < \delta.$$

□

Proof of Theorem 1.2. By Proposition 6.1, 6.2 and 6.3, we prove Theorem 1.2.

7. Conclusion. In this paper, we obtained the convergence rate of the solution of the Gelfand equation to its singular solution and the stability estimate. The main tool to deal with the exponential nonlinearity term in the Gelfand equation was to investigate the solution for its linearized equation. For the convergence rate of the solution, we first obtained the uniform lower bound. To construct the uniform lower bound, we added an appropriate additional term and we can find the linear operator and control the nonlinear term to establish the estimate for this lower bound. For a convergence rate and pointwise stability estimate, we also use the linear operator of the Gelfand equation and this lower bound of the solution with an a priori assumption. Especially for critical case, we use a kind of order preserving property between the solutions to the Gelfand equation.

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