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GLOBAL EXISTENCE FOR SCHRÖDINGER–DEBYE SYSTEM FOR INITIAL DATA WITH INFINITE L^2-NORM

By

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Abstract. In this paper we study global-in-time existence for the Cauchy problem associated to the Schrödinger–Debye system for a class of initial data with infinite L^2-norm, namely weak-L^p spaces. This model appears in nonlinear optics as a perturbation of the classical nonlinear Schrödinger equation (NLS). Our results exhibit differences between both models in that setting, e.g. the Debye perturbation imposes restrictions in the spatial dimension. We also analyze the asymptotic stability of the solutions.

1. Introduction. We consider the Cauchy problem for the Schrödinger–Debye (S-D) system, which consists of the coupled equations

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta_x u = uv, & t \geq 0,\ x \in \mathbb{R}^n, \\ \mu\partial_t v + v = \lambda|u|^p, & \mu > 0,\ \lambda = \pm 1, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & \end{cases} \tag{1.1}$$

where u is a complex-valued function and v is a real-valued function. For the case $p = 2$, these equations appear in the context of nonlinear optics modeling interactions of an electromagnetic wave with a nonresonant medium, where the material response time μ is relevant. More precisely, u denotes the envelope of a light wave that goes through media whose response is nonresonant, and v is a change induced in its refraction index with a slight delay μ. Similar to the physical theory of the nonlinear Schrödinger equation,

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for the system (1.1) the parameters $\lambda = -1$ and $\lambda = 1$ model focusing and defocusing situations, respectively. See Newell and Moloney [12] for a more complete discussion of this model. For the sake of simplicity, throughout this paper, v and v_0 will be abusively named as delay and initial delay, respectively.

The L^2-norm (*mass*) of u is invariant for the system (1.1), namely

$$\int_{\mathbb{R}^n} |u(x,t)|^2 dx = \int_{\mathbb{R}^n} |u_0(x)|^2 dx. \tag{1.2}$$

It is not known the existence of other conservation law for this system, but in the physical case ($p = 2$) we have the pseudo-Hamiltonian

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Big(|\nabla u|^2 + \lambda|u|^4 - \lambda\mu^2|\partial_t v|^2\Big)dx = 2\lambda\mu\int_{-\infty}^{\infty}|\partial_t v|^2 dx. \tag{1.3}$$

Local and global-in-time well-posedness for the Cauchy problem (1.1) in the continuous ($x \in \mathbb{R}^n$) and periodic ($x \in \mathbb{T}^n$) contexts have been developed in the works [1, 4, 5, 7, 8] with initial data (u_0, v_0) in classical Sobolev spaces $H^{s_1} \times H^{s_2}$ with positive indexes and in particular with finite L^2-norm. All results from these references only consider the dimensions $n = 1, 2$, or 3. Specifically, the authors of [8] studied the cases $n = 1$ and $p = 2$, and they have obtained global well-posedness in the Sobolev spaces

$$H^s(\mathbb{R}) \times H^s(\mathbb{R}), \text{ with } -3/14 < s < 0. \tag{1.4}$$

This result shows that, from the point of view of global-in-time existence, the system (1.1) does not have the same behavior as the associated NLS

$$i\partial_t u + \tfrac{1}{2}\Delta u = \lambda u\,|u|^p\,, \tag{1.5}$$

because, for $n = 1$ and $p = 2$, the flux of (1.5) is not uniformly continuous on bounded sets of Sobolev spaces with $s < 0$. More recently, for the corresponding L^2-critical case ($n = 2$ and $p = 2$), the authors of [9] obtained global solutions in the energy space $H^1 \times L^2$ for arbitrary data by means of carefully handling the pseudo-Hamiltonian (1.3). This is another difference with respect to the associated 2D cubic nonlinear Schrödinger equation, where the formation of singularity in H^1 occurs at finite time.

The above facts indicate that Debye perturbation $(1.1)_2$ tends to smoothing out solutions in the framework of Sobolev spaces. An interesting aspect of the work [8] is that solutions may have infinite L^2-norm since the authors considered negative indexes $s < 0$; however, this is in the one-dimensional case. Therefore, in high dimensions $n > 1$, a natural question is to ask whether there is another environment allowing solutions outside the L^2-space, in which the models (1.1) and (1.5) exhibit different features.

Motivated by that, we study (1.1) in weak-L^r ($L^{(r,\infty)}$) by considering higher dimensions. More precisely, we show existence of global-in-time solutions for (1.1) in the class (see Theorem 2.1)

$$\sup_{t>0} t^{\alpha}\|u\|_{L^{(p+2,\infty)}} < \infty \qquad \text{and} \qquad \sup_{t>0} t^{\beta}\|v\|_{L^{(\frac{p+2}{p},\infty)}} < \infty, \tag{1.6}$$

where

$$\alpha = \frac{1}{p} - \frac{n}{2(p+2)} \qquad \text{and} \qquad \beta = p\alpha. \tag{1.7}$$

A result about L^p-regularity of solutions is also proved. The initial condition (u_0, v_0) is taken in a such way that

$$\sup_{t>0} t^{\alpha} \|S(t)u_0\|_{L^{(p+2,\infty)}} < \infty \qquad \text{and} \qquad v_0 \in L^{(\frac{p+2}{p},\infty)}, \tag{1.8}$$

where $S(t) = e^{it\Delta}$ stands for the free unitary Schrödinger group. The class (1.8) allows us to consider singular-homogeneous initial conditions that do not belong to any $L^r(\mathbb{R}^n)$, and it contains data blowing up at finitely many points. For p in a given interval, these initial data do not belong to $L^2_{loc}(\mathbb{R}^n) \times L^2_{loc}(\mathbb{R}^n)$, that is, they do not have finite local L^2-norm (see Remark 2.2 (a)). Moreover, there is no inclusion relation between the classes (1.4) and (1.8), even when $p = 2$.

The approach employed here follows the same spirit of the one used in [3, 6] (see also [10]), where the authors proved global-in-time existence results for the corresponding NLS equation for all dimensions $n \geq 1$ by considering small data in L^r and weak-L^r, respectively. However, our study shows that the presence of the delay provokes two differences in comparison with NLS equations, namely the restriction on the dimension $n < 6$ (see Remark 2.2 (b)) and the possibility of considering large data v_0 after making a dynamical rescaling over the delay-parameter μ. Indeed, if (u, v) is a solution of (1.1), then

$$(\tilde{u}, \tilde{v}) = \Big(\mu^{1/p} u(\mu^{1/2}x, \mu t),\, \mu v(\mu^{1/2}x, \mu t)\Big)$$

is a solution of (1.1) by considering $\mu = 1$, that is,

$$\begin{cases} i\partial_t \tilde{u} + \frac{1}{2}\Delta_x \tilde{u} = \tilde{u}\tilde{v}, \quad t > 0,\ x \in \mathbb{R}^n, \\ \partial_t \tilde{v} + \tilde{v} = \lambda |\tilde{u}|^p, \\ \tilde{u}(x, 0) = \mu^{1/p} u_0(\mu^{1/2}x), \quad \tilde{v}(x, 0) = \mu v_0(\mu^{1/2}x). \end{cases} \tag{1.9}$$

Then, the smallness assumption on data v_0 can be removed by taking $\mu > 0$ small enough and using Theorem 2.1 in the case $\mu = 1$.

Moreover, we also analyze the asymptotic stability of these global solutions and prove that some types of initial perturbations vanish as $t \to \infty$ (see Theorem 2.3). This result shows a distinct influence of the initial envelope of light u_0 and initial delay v_0 on the long-time behavior of the solutions (see Remark 2.4 (d)).

The paper is organized as follows. In Section 2 we review some basic properties about weak-L^p spaces and state our results, which are proved in Section 3.

2. Functional setting and results. In this section we prove the existence of global solutions for system (1.1) including homogeneous data. We start by recalling that $f \in L^{(r,\infty)}$(weak-L^r) if only if $\|f\|^*_{(r,\infty)} = \sup_{t>0} t\lambda_f^{1/r}(t) < \infty$, where

$$\lambda_f(t) = \big|\{x \in \mathbb{R}^n;\ |f(x)| > t\}\big|$$

is the distribution function of f and $|A|$ denotes the Lebesgue measure of a set A. The quantity $\|\cdot\|^*_{(r,\infty)}$ is not a norm, but, for $1 < r \leq \infty$, $L^{(r,\infty)}$ endowed with the norm

$\|\cdot\|_{(r,\infty)} = \sup_{t>0} t^{1/r} f^{**}(t)$ is a Banach space, where f^{**} is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds \qquad \text{and} \qquad f^*(t) = \inf\{s>0 : \ \lambda_f(s) \le t\}.$$

Also, $\|\cdot\|^*_{(r,\infty)} \le \|\cdot\|_{(r,\infty)} \le \frac{r}{r-1}\|\cdot\|^*_{(r,\infty)}$ which implies that $\|\cdot\|^*_{(r,\infty)}$ and $\|\cdot\|_{(r,\infty)}$ induce the same topology on $L^{(r,\infty)}$. The Hölder inequality works well in the framework of $L^{(r,\infty)}$-spaces, namely

$$\|fg\|_{(r,\infty)} \le \frac{r}{r-1}\|f\|_{(r_1,\infty)}\|g\|_{(r_2,\infty)} \ , \tag{2.1}$$

for $1 < r_1, r_2 < \infty$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. For a deeper discussion about weak-L^p we refer the reader to [2].

Let $\alpha = \frac{1}{p} - \frac{n}{2(p+2)}$ and $\beta = p\alpha$. We define $\mathcal{M}_p^0$ as the space of all pairs $(u_0, v_0) \in (\mathcal{S}'(\mathbb{R}^n), L^{(\frac{p+2}{p},\infty)})$ such that the norm

$$\|(u_0,v_0)\|_p^0 = \max\left\{\left(\sup_{t>0} t^{\alpha}\|S(t)u_0\|_{(p+2,\infty)}\right), \|v_0\|_{(\frac{p+2}{p},\infty)}\right\} < \infty.$$

Also, $\mathcal{M}_p$ denotes the space of Bochner measurable functions $(u,v) : \mathbb{R} \longrightarrow L^{(p+2,\infty)} \times L^{(\frac{p+2}{p},\infty)}$ such that the norm $\|(u,v)\|_{\mathcal{M}_p} = \max\{\|u\|_{\infty,\alpha}, \|v\|_{\infty,\beta}\}$ is finite, where

$$\|u\|_{\infty,\alpha} = \sup_{t>0} t^{\alpha}\|u(\cdot,t)\|_{(p+2,\infty)} \text{ and } \quad \|v\|_{\infty,\beta} = \sup_{t>0} t^{\beta}\|v(\cdot,t)\|_{(\frac{p+2}{p},\infty)}.$$

Recall that p_0 denotes the positive root of the equation $np^2 + (n-2)p - 4 = 0$, that is,

$$p_0 = \frac{\sqrt{n^2+12n+4} - (n-2)}{2n}. \tag{2.2}$$

In what follows, we state our global-in-time existence theorem with weak data.

Theorem 2.1. Let $1 \le n < 6$, $\max\{p_0, 1\} < p < \frac{4}{n-2}$ and $(u_0, v_0) \in \mathcal{M}_p^0$.

(i) (Existence) There exists $\varepsilon > 0$ such that if

$$\sup_{t>0} t^{\alpha}\|S(t)u_0\|_{(p+2,\infty)} \le \varepsilon \qquad \text{and} \qquad \|v_0\|_{(\frac{p+2}{p},\infty)} \le \varepsilon, \tag{2.3}$$

then the system (1.1) has a global solution $(u,v) \in \mathcal{M}_p$ which is the unique one in the closed ball of $\mathcal{M}_p$ centered at the origin and with radius 2ε.

(ii) (L^p-regularity) Moreover, if, in addition, $\sup_{t>0} t^{\alpha}\|S(t)u_0\|_{p+2}$ and $\|v_0\|_{\frac{p+2}{p}}$ are small enough, then the previous solution (u,v) is a Bochner measurable function $(u,v) : \mathbb{R} \longrightarrow L^{p+2} \times L^{\frac{p+2}{p}}$ satisfying

$$\sup_{t>0} t^{\alpha}\|u(\cdot,t)\|_{p+2} < \infty \qquad \text{and} \qquad \sup_{t>0} t^{\beta}\|v(\cdot,t)\|_{\frac{p+2}{p}} < \infty. \tag{2.4}$$

Remark 2.2. Some remarks on Theorem 2.1 follow.

(a) ***Homogeneous and singular data.*** Theorem 2.1 allows us to consider homogeneous and singular initial data. Let $P_m(x)$ and $Q_m(x)$ be homogeneous polynomials of degree m, with Q_m harmonic. A simple computation shows that

$$v_0 = \varepsilon_1 P_m(x)\,|x|^{-\frac{pn}{p+2}-m} \in L^{(\frac{p+2}{p},\infty)}$$

and, by [6, Prop. 3.7-3.9], $u_0 = \varepsilon_2 Q_m(x)\,|x|^{-\frac{2}{p}-m}$ satisfies

$$\sup_{t>0} t^{\alpha}\|S(t)u_0\|_{(p+2,\infty)} = \|S(1)u_0\|_{(p+2,\infty)} \leq \varepsilon_2 \|S(1)\left(Q_m(x)\,|x|^{-\frac{2}{p}-m}\right)\|_{p+2} < \infty.$$

Notice that, despite $u_0, v_0 \to 0$ as $|x| \to \infty$, $u_0, v_0 \notin L^r(\mathbb{R}^n)$ for all $1 \leq r \leq \infty$. We also can take (u_0, v_0) blowing up at finitely many points. For instance

$$u_0 = \sum_{j=1}^{N_1} \gamma_j P_{m,j}(x - x_j)\,|x - x_j|^{-\frac{2}{p}-m},$$

$$v_0 = \sum_{j=1}^{N_2} \eta_j Q_{m,j}(x - \tilde{x}_j)\,|x - \tilde{x}_j|^{-\frac{pn}{p+2}-m},$$

where $x_j, \tilde{x}_j \in \mathbb{R}^n$, $P_{m,j}$ is as P_m, $Q_{m,j}$ is as Q_m, and γ_j, η_j are small enough. Observe that if either $p \leq \frac{4}{n}$ or $p \geq 2$, then $(u_0, v_0) \notin L^2_{loc}(\mathbb{R}^n) \times L^2_{loc}(\mathbb{R}^n)$, that is, the data does not have finite local L^2-norm.

(b) ***Delay influence on dimension n.*** Making a comparison with the Schrödinger equation (1.5), the presence of the delay v imposes a further restriction on n, namely $1 < \frac{4}{n-2} \Leftrightarrow n < 6$, which is a necessary condition for the term $|u|^p$ be locally Lipschitz continuous. This fact was used to prove the contraction property of the map $\Phi(u, v)$ defined by (3.4)–(3.5) (see (3.11)). We observe that for (1.5) this restriction is not necessary because the nonlinearity $u\,|u|^p$ is locally Lipschitz continuous when $p > 0$.

In view of the definition of the space $\mathcal{M}_p$, the solution (u, v) provided by item (i) of Theorem 2.1 satisfies $\|u(\cdot, t)\|_{(p+2,\infty)} = O(t^{-\alpha})$ and $\|v(\cdot, t)\|_{(\frac{p+2}{p},\infty)} = O(t^{-\beta})$ as $t \to \infty$. Among others, the next result proves that certain perturbations of (u, v) are negligible, for large value of t, with respect to the quantities $t^{\alpha+h}\|\cdot\|_{(p+2,\infty)}$ and $t^{\beta+h}\|\cdot\|_{(\frac{p+2}{p},\infty)}$, for $0 \leq h < 1 - p\alpha$.

Theorem 2.3 (Asymptotic stability). *Let $0 \leq h < 1 - p\alpha$ and assume the hypotheses of Theorem 2.1. Let $(u,\, v)$ and $(\widetilde{u},\, \widetilde{v})$ be two global solutions of the system (1.1) obtained through Theorem 2.1, with respective initial conditions (u_0, v_0) and $(\widetilde{u}_0, \widetilde{v}_0)$ (reducing the $\mathcal{M}^0_p$-norm of the data if necessary). Then*

$$\lim_{t\to\infty} t^{\alpha+h}\,\|S(t)(u_0 - \widetilde{u}_0)\|_{(p+2,\infty)} = 0 \tag{2.5}$$

if and only if

$$\lim_{t\to\infty} t^{\alpha+h}\,\|u(\cdot, t) - \widetilde{u}(\cdot, t)\|_{(p+2,\infty)} = \lim_{t\to\infty} t^{\beta+h}\,\|v(\cdot, t) - \widetilde{v}(\cdot, t)\|_{(\frac{p+2}{p},\infty)} = 0. \tag{2.6}$$

Moreover, the same conclusion holds true if we change the norm $\|\cdot\|_{(r,\infty)}$ by its strong version $\|\cdot\|_r$.

Remark 2.4. Now we make some comments on Theorem 2.3 and its proof.

(a) Step 1 of the proof of Theorem 2.3 actually gives us more information, namely

$$\sup_{t>0} t^{\alpha+h}\|u(\cdot, t)\|_{(p+2,\infty)} < \infty \qquad \text{and} \qquad \sup_{t>0} t^{\beta+h}\|v(\cdot, t)\|_{(\frac{p+2}{p},\infty)} < \infty, \tag{2.7}$$

provided that $\|(u_0,v_0)\|_p^0$ is small enough and

$$\limsup_{t\to\infty} t^{\alpha+h}\,\|S(t)u_0\|_{(p+2,\infty)}<\infty.$$

Since $h\geq 0$, these two last constraints imply that

$$\sup_{t>0} t^{\alpha+h}\,\|S(t)u_0\|_{(p+2,\infty)}<\infty. \tag{2.8}$$

In fact the proof of (2.7) requires only (2.8) and a smallness assumption on $\|(u_0,v_0)\|_p^0$. A smallness condition on $\sup_{t>0} t^{\alpha+h}\,\|S(t)u_0\|_{(p+2,\infty)}$ is not needed.

(b) By taking $\widetilde{u}_0=0$ and $\widetilde{u}=0$, a consequence of Theorem 2.3 is

$$\|u(\cdot,t)\|_{(p+2,\infty)}=o(t^{-(\alpha+h)}) \qquad \text{and} \qquad \|v(\cdot,t)\|_{(\frac{p+2}{p},\infty)}=o(t^{-(\alpha+h)}),$$

provided $\|S(t)u_0\|_{(p+2,\infty)}=o(t^{-(\alpha+h)})$ as $t\to\infty$. It is interesting to observe that no further restrictions is required on delay v_0.

Moreover, note that $\frac{1}{p}-\frac{n(p+1)}{2(p+2)}<0$ when $p>p_0$. Thus, if $u_0-\widetilde{u}_0\in L^{(\frac{p+2}{p+1},\infty)}$ and $0\leq h<\frac{n(p+1)}{2(p+2)}-\frac{1}{p}$, then (2.5) holds true. Indeed,

$$\begin{aligned}0&\leq \lim_{t\to\infty} t^{\alpha+h}\,\|S(t)(u_0-\widetilde{u}_0)\|_{(p+2,\infty)}\\ &\leq C_S\left(\lim_{t\to\infty} t^{\frac{1}{p}-\frac{n(p+1)}{2(p+2)}+h}\right)\|u_0-\widetilde{u}_0\|_{(\frac{p+2}{p+1},\infty)}=0.\end{aligned}$$

(c) ***Basin of attraction.*** Let (u,v) be the solution obtained through Theorem 2.1 with initial data $(u_0,v_0)\in\mathcal{M}_p^0$. Take $\psi_0=u_0+\varphi$ and $\omega_0=v_0+\phi\in L^{(\frac{p+2}{p+1},\infty)}$ small enough with $\varphi\in\mathcal{S}(\mathbb{R}^n)$, and consider the solution (ψ,ω) corresponding to $(\psi_0,\omega_0)\in\mathcal{M}_p^0$. Since $\psi_0-u_0=\varphi$ satisfies (2.5) with $h=0$, then

$$\lim_{t\to\infty} t^{\alpha}\,\|\psi(\cdot,t)-u(\cdot,t)\|_{(p+2,\infty)}=\lim_{t\to\infty} t^{\beta}\,\|\omega(\cdot,t)-v(\cdot,t)\|_{(\frac{p+2}{p},\infty)}=0. \tag{2.9}$$

Therefore, considering smooth perturbations, one obtains a basin of attraction in the sense of (2.9) around each initial data $(u_0,v_0)\in\mathcal{M}_p^0$. In fact, this basin is characterized by all perturbations φ that satisfy (2.5).

(d) ***Distinct influence of $\boldsymbol{u_0}$ and $\boldsymbol{v_0}$.*** Let u_0 and v_0 be as in item (c). For $\varphi=\frac{\widetilde{\varepsilon}}{|x|^{2/p}}(\notin\mathcal{S}(\mathbb{R}^n))$, we have that

$$\lim_{t\to\infty} t^{\alpha}\left\|S(t)\frac{\widetilde{\varepsilon}}{|x|^{2/p}}\right\|_{(p+2,\infty)}=\left\|S(1)\frac{\widetilde{\varepsilon}}{|x|^{2/p}}\right\|_{(p+2,\infty)}\neq 0,$$

and then it follows from Theorem 2.3 (with $\phi=0$ or not) that

$$\lim_{t\to\infty} t^{\alpha}\,\|\psi(\cdot,t)-u(\cdot,t)\|_{(p+2,\infty)}\neq 0.$$

In other words, the disturbance $\psi(\cdot,t)-u(\cdot,t)$ persists at long times. Notice that in Theorem 2.3 we have not assumed any restriction of type (2.5) on $v_0-\widetilde{v}_0$. So, taking $\varphi=0$ in the previous item, we can see that the effect of any initial-delay small perturbation $\phi\in L^{(\frac{p+2}{p+1},\infty)}$ vanishes as $t\to\infty$. This reveals a distinct influence of the initial conditions u_0 and v_0 on the long-time behavior of solutions.

3. Proofs of the main results. Here we show the proof of our main results.

3.1. *Proof of Theorem* 2.1. The proof of our results is based on the following basic properties of the Schrödinger group (see [6, 10]):

$$\|S(t)\varphi\|_{(p+2,\infty)} \le C_S|t|^{-\frac{np}{2(p+2)}}\|\varphi\|_{(\frac{p+2}{p+1},\infty)}, \tag{3.1}$$

$$\|S(t)\varphi\|_{p+2} \le \widetilde{C}_S|t|^{-\frac{np}{2(p+2)}}\|\varphi\|_{\frac{p+2}{p+1}}, \tag{3.2}$$

where $\|\cdot\|_r$ denotes the norm of the Lebesgue space L^r and $C_S, \widetilde{C}_S$ are positive constants that depend only on p. Moreover, the following elementary fact will be useful in the proofs. If $a, b \ge 0$ with $\max\{a,b\} < 1$, then the Beta function $B(\cdot,\cdot)$ verifies

$$B(1-a, 1-b) = \int_0^1 \frac{dx}{(1-x)^a\, x^b} < \infty. \tag{3.3}$$

We begin by proving the existence of solutions.

Existence. Consider the ball

$$B_p(a) = \{(u,v) \in \mathcal{M}_p; \|u\|_{\infty,\alpha} \le a \text{ and } \|v\|_{\infty,\beta} \le a\}$$

and the metric $d[\cdot,\cdot]$ in $\mathcal{M}_p$

$$d[(u,v),(\tilde{u},\tilde{v})] = \max\left(\|u-\tilde{u}\|_{\infty,\alpha}\,,\ \|v-\tilde{v}\|_{\infty,\beta}\right).$$

In order to apply a fixed-point argument, we will show that the mapping $\Phi(u,v) := (\Phi_1(u,v), \Phi_2(u,v))$, defined by

$$\Phi_1(u,v) := S(t)u_0 - i\int_0^t S(t-s)u(\cdot,s)v(\cdot,s)ds \tag{3.4}$$

$$\Phi_2(u,v) := e^{-\frac{t}{\mu}}v_0 + \tfrac{\lambda}{\mu}\int_0^t e^{-\frac{t-s}{\mu}}|u(\cdot,s)|^p ds, \tag{3.5}$$

is a contraction on the metric space $(B_p(a), d)$, for a suitable value of a.

Computing $\|\cdot\|_{\infty,\alpha}$ and $\|\cdot\|_{\infty,\beta}$ in (3.4) and (3.5), we obtain

$$\|\Phi_1(u,v)\|_{\infty,\alpha} \le \sup_{t>0} t^\alpha \|S(t)u_0\|_{(p+2,\infty)} + \sup_{t>0} t^\alpha \int_0^t \|S(t-s)uv(\cdot,s)\|_{(p+2,\infty)}ds, \tag{3.6}$$

$$\|\Phi_2(u,v)\|_{\infty,\beta} \le \sup_{t>0} t^\beta \|e^{-\frac{t}{\mu}}v_0\|_{(\frac{p+2}{p},\infty)} + \tfrac{1}{\mu}\sup_{t>0} t^\beta \int_0^t \|e^{-\frac{t-s}{\mu}}|u(\cdot,s)|^p\|_{(\frac{p+2}{p},\infty)}ds. \tag{3.7}$$

Now we estimate the nonlinearities

$$N_1(u,v) = \int_0^t \|S(t-s)uv(\cdot,s)\|_{(p+2,\infty)}ds$$

$$N_2(u,v) = \tfrac{1}{\mu}\int_0^t \|e^{-\frac{t-s}{\mu}}|u(\cdot,s)|^p\|_{(\frac{p+2}{p},\infty)}ds.$$

By using (3.1), Hölder's inequality (2.1), and (3.3), it follows that

$$\begin{aligned}
N_1(u,v) &\leq C\int_0^t (t-s)^{-\frac{np}{2(p+2)}}\|uv\|_{(\frac{p+2}{p+1},\infty)}ds \\
&\leq C\int_0^t (t-s)^{-\frac{np}{2(p+2)}}\|u\|_{(p+2,\infty)}\|v\|_{(\frac{p+2}{p},\infty)}ds \\
&\leq \|u\|_{\infty,\alpha}\|v\|_{\infty,\beta}C\int_0^t (t-s)^{-\frac{np}{2(p+2)}}s^{-(\alpha+\beta)}ds \\
&\leq Ct^{-\frac{np}{2(p+2)}-(\alpha+\beta)+1}C\int_0^1 (1-s)^{-\frac{np}{2(p+2)}}s^{-(\alpha+\beta)}ds\|u\|_{\infty,\alpha}\|v\|_{\infty,\beta} \\
&= K_1 t^{-\alpha}\|u\|_{\infty,\alpha}\|v\|_{\infty,\beta}
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
N_2(u,v) &\leq \tfrac{1}{\mu}\int_0^t \|e^{-\frac{t-s}{\mu}}|u(\cdot,s)|^p\|_{(\frac{p+2}{p},\infty)}ds \leq \tfrac{1}{\mu}\int_0^t e^{-\frac{t-s}{\mu}}\|\,|u|\,(\cdot,s)\|^p_{(p+2,\infty)}ds \\
&\leq \|u\|^p_{\infty,\alpha}\tfrac{1}{\mu}\left(\int_0^{t/2} e^{-\frac{t-s}{\mu}}s^{-p\alpha}ds + \int_{t/2}^t e^{-\frac{t-s}{\mu}}s^{-p\alpha}ds\right) \\
&\leq \|u\|^p_{\infty,\alpha}\tfrac{1}{\mu}\left(e^{-\frac{t}{2\mu}}\int_0^{t/2} s^{-\beta}ds + 2^\beta t^{-\beta}\int_{t/2}^t e^{-\frac{t-s}{\mu}}ds\right) \\
&= \|u\|^p_{\infty,\alpha}\left(\frac{1}{1-\beta}\tfrac{1}{\mu}e^{-\frac{t}{2\mu}}t\, t^{-\beta} + 2^\beta t^{-\beta}(1-e^{-\frac{t}{2\mu}})\right) \\
&\leq K_2 t^{-\beta}\|u\|^p_{\infty,\alpha}.
\end{aligned} \tag{3.9}$$

Assuming that $(u,v)\in B_p(2\varepsilon)$ with $0<4K_1\varepsilon<1$ and $0<2^pK_2p\varepsilon^{p-1}<1$, we get

$$\begin{aligned}
d[(\Phi_1(u,v),\Phi_2(u,v)),(0,0)] &\leq \max\left(\|\Phi_1(u,v)\|_{\infty,\alpha}\,,\ \|\Phi_1(u,v)\|_{\infty,\beta}\right) \\
&\leq \max\left(\varepsilon+4K_1\varepsilon\,\varepsilon,\ \varepsilon+2^{p-1}K_2\varepsilon^{p-1}2\varepsilon\right) \\
&\leq \max\left(\varepsilon+\,\varepsilon,\ \varepsilon+\varepsilon\right)\leq 2\varepsilon,
\end{aligned} \tag{3.10}$$

and so $\Phi(B_p(2\varepsilon))\subset B_p(2\varepsilon)$. Also, by recalling the inequality

$$||u|^p-|\widetilde{u}|^p|\leq C\,|u-\widetilde{u}|\,(|u|^{p-1}+|\widetilde{u}|^{p-1})\quad\text{for}\quad p>1, \tag{3.11}$$

a similar argument yields

$$d[\Phi(u,v),\Phi(\widetilde{u},\widetilde{v})]\leq \max(4K_1\varepsilon\,,2^pK_2p\varepsilon^{p-1})d[(u,v),(\tilde{u},\tilde{v})],$$

and therefore Φ is a contraction in $B_p(2\varepsilon)$. Now, an application of the Banach fixed-point theorem concludes the proof of the existence statement.

Regularity. From the fixed-point argument, we know that the previous solution is the limit in $B_p(2\varepsilon)$ of the Picard sequence $\{(u_m,v_m)\}_{m\geq 1}$

$$u_1(\cdot,t)=S(t)u_0\quad\text{and}\quad u_{m+1}(\cdot,t)=S(t)u_0-i\int_0^t S(t-s)u_m(\cdot,s)v_m(\cdot,s)ds, \tag{3.12}$$

$$v_1(\cdot,t)=e^{-\frac{t}{\mu}}v_0\quad\text{and}\quad v_{m+1}(\cdot,t)=e^{-\frac{t}{\mu}}v_0+\tfrac{\lambda}{\mu}\int_0^t e^{-\frac{t-s}{\mu}}|u_m(\cdot,s)|^pds. \tag{3.13}$$

In order to prove the regularity (2.4), it is sufficient to show that

$$\sup_{t>0} t^{\alpha}\|u_m(\cdot,t)\|_{p+2} \leq C \text{ and } \sup_{t>0} t^{\beta}\|v_m(\cdot,t)\|_{\frac{p+2}{p}} \leq C, \text{ for all } m \geq 1. \tag{3.14}$$

To this end, we proceed as in the proof of the inequalities (3.6)–(3.10), but this time we employ (3.2) and the Hölder inequality in L^r-spaces, instead of (3.1) and (2.1), in order to bound (3.12)–(3.13) as

$$\sup_{t>0} t^{\alpha}\|u_{m+1}(\cdot,t)\|_{p+2} \leq \sup_{t>0} t^{\alpha}\|S(t)u_0\|_{p+2} + \widetilde{K}_1 \sup_{t>0} t^{\alpha}\|u_m(\cdot,t)\|_{p+2} \sup_{t>0} t^{\beta}\|v_m(\cdot,t)\|_{\frac{p+2}{p}}, \tag{3.15}$$

$$\sup_{t>0} t^{\beta}\|v_{m+1}(\cdot,t)\|_{\frac{p+2}{p}} \leq \|v_0\|_{\frac{p+2}{p}} + \widetilde{K}_2 \left(\sup_{t>0} t^{\beta}\|u_m(\cdot,t)\|_{\frac{p+2}{p}}\right)^p. \tag{3.16}$$

Let us denote $U_m = \sup_{t>0} t^{\alpha}\|u_m(\cdot,t)\|_{p+2}$ and $V_m = \sup_{t>0} t^{\beta}\|v_m(\cdot,t)\|_{\frac{p+2}{p}}$, for $m \geq 2$, with $U_1 = \sup_{t>0} t^{\alpha}\|S(t)u_0\|_{p+2}$ and $V_1 = \|v_0\|_{\frac{p+2}{p}}$. Notice that (3.15)–(3.16) can be rewritten as the following system of recurrence inequalities

$$U_{m+1} \leq U_1 + \widetilde{K}_1 U_m V_m \qquad \text{and} \qquad V_{m+1} \leq V_1 + \widetilde{K}_2 (U_m)^p, \quad m \geq 1. \tag{3.17}$$

It is known that systems of inequalities such as (3.17) can be solved provided that U_1,V_1 are small enough (see, for instance, [11]). More precisely, if U_1,$V_1 \leq \widetilde{\varepsilon}$, with $\widetilde{\varepsilon}$ chosen in a such way that $4\widetilde{K}_1\widetilde{\varepsilon} < 1$ and $2^p\widetilde{K}_2\widetilde{\varepsilon}^{p-1} < 1$, then one can find $C > 0$ such that $U_m, V_m \leq C$, for all $m \geq 1$.

3.2. *Proof of Theorem* 2.3. The proof is divided into three steps.

First step. We start by showing that

$$\sup_{t>0} t^{\alpha+h}\left\|u - \widetilde{u}\right\|_{(p+2,\infty)} < \infty \qquad \text{and} \qquad \sup_{t>0} t^{\beta+h}\left\|v - \widetilde{v}\right\|_{(\frac{p+2}{p},\infty)} < \infty. \tag{3.18}$$

Let $\{(u_m, v_m)\}_{m\geq 1}$ and $\{(\widetilde{u}_m, \widetilde{v}_m)\}_{m\geq 1}$ be the Picard sequences converging to (u, v) and $(\widetilde{u}, \widetilde{v})$, respectively (see (3.12)–(3.13)). We estimate

$$\begin{aligned}
t^{\alpha+h}&\left\|u_{m+1}(t) - \widetilde{u}_{m+1}(t)\right\|_{(p+2,\infty)} \leq \sup_{t>0} t^{\alpha+h}\left\|S(t)(u_0 - \widetilde{u}_0)\right\|_{(p+2,\infty)} \\
&+ \left\{C_S \int_0^1 (1-s)^{-\frac{np}{2(p+2)}} s^{-(\alpha+\beta)-h} ds\right\} \\
&\times \left\{\|v_m\|_{\infty,\beta} \sup_{t>0} t^{\alpha+h}\left\|u_m - \widetilde{u}_m\right\|_{(p+2,\infty)} + \|\widetilde{u}_m\|_{\infty,\alpha} \sup_{t>0} t^{\beta+h}\left\|v_m - \widetilde{v}_m\right\|_{(\frac{p+2}{p},\infty)}\right\}
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
t^{\beta+h}&\left\|v_m(t) - \widetilde{v}_m(t)\right\|_{(\frac{p+2}{p},\infty)} \leq \left\|v_0 - \widetilde{v}_0\right\|_{(\frac{p+2}{p},\infty)} \\
&+ \tfrac{\lambda}{\mu}p \int_0^t e^{-\frac{t-s}{\mu}}\left\|u_m - \widetilde{u}_m\right\|_{(p+2,\infty)} (\|u_m\|_{(p+2,\infty)}^{p-1} + \|\widetilde{u}_m\|_{(p+2,\infty)}^{p-1}) ds \\
\leq& \left\|v_0 - \widetilde{v}_0\right\|_{(\frac{p+2}{p},\infty)} \\
&+ \tfrac{\lambda}{\mu}p2^p\varepsilon^{p-1} \int_0^1 te^{-(\frac{1-s}{\mu})t} s^{-p\alpha-h} ds \cdot \sup_{t>0} t^{\alpha+h}\left\|u_m - \widetilde{u}_m\right\|_{(p+2,\infty)}.
\end{aligned} \tag{3.20}$$

Denote $U_m = \sup_{t>0} t^{\alpha+h} \|u_m - \widetilde{u}_m\|_{(p+2,\infty)}$ and $V_m = \sup_{t>0} t^{\beta+h} \|v_m - \widetilde{v}_m\|_{(\frac{p+2}{p},\infty)}$, for all $m \geq 2$. Observe that (3.19)–(3.20) imply the recursive inequality

$$U_{m+1} + V_{m+1} \leq \{U_1 + V_1\} + \max\{\varepsilon K_1^*, \varepsilon^{p-1} K_2^*\} \{U_m + V_m\},$$

where $U_1 + V_1 = \sup_{t>0} t^{\alpha+h} \|S(t)(u_0 - \widetilde{u}_0)\|_{(p+2,\infty)} + \|v_0 - \widetilde{v}_0\|_{(\frac{p+2}{p},\infty)} < \infty$ (by hypotheses) and K_1^*, K_2^* depend only on h, α, p, μ, n. By taking $\varepsilon > 0$ small enough such that $L_\varepsilon = \max\{\varepsilon K_1^*, \varepsilon^{p-1} K_2^*\} < 1$, we obtain that the sequences U_m,V_m are bounded and, by standard arguments, (3.18) holds.

Second step. We first prove that (2.5) implies (2.6). To this end, we subtract the equations satisfied by (u, v) and $(\widetilde{u}, \widetilde{v})$, and afterward take the norms $t^{\alpha+h} \|\cdot\|_{(p+2,\infty)}$ and $t^{\beta+h} \|\cdot\|_{(\frac{p+2}{p},\infty)}$ to obtain the inequalities

$$\begin{aligned} t^{\alpha+h} \|u(t) - \widetilde{u}(t)\|_{(p+2,\infty)} &\leq \sup_{t>0} t^{\alpha+h} \|S(t)(u_0 - \widetilde{u}_0)\|_{(p+2,\infty)} \\ &\quad + t^{\alpha+h} \left\| \int_0^t S(t-s)[(u - \widetilde{u})v] ds \right\|_{(p+2,\infty)} \\ &\quad + t^{\alpha+h} \left\| \int_0^t S(t-s)[\widetilde{u}(v - \widetilde{v})] ds \right\|_{(p+2,\infty)} \\ &:= I_0(t) + I_1(t) + I_2(t) \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} t^{\beta+h} \|v(t) - \widetilde{v}(t)\|_{(\frac{p+2}{p},\infty)} &\leq \left\| e^{-\frac{t}{\mu}} (v_0 - \widetilde{v}_0) \right\|_{(\frac{p+2}{p},\infty)} \\ &\quad + \tfrac{\lambda}{\mu} \left\| \int_0^t e^{-\frac{t-s}{\mu}} \left(|u(s)|^p - |\widetilde{u}(s)|^p\right) ds \right\|_{(\frac{p+2}{p},\infty)} \\ &:= J_0(t) + J_1(t). \end{aligned} \tag{3.22}$$

Since $(u, v) \in B_p(2\varepsilon)$, we employ the change of variable $s \longmapsto ts$ and estimate I_1 as

$$\begin{aligned} I_1(t) &\leq t^{\alpha+h} C_S \int_0^t (t-s)^{-\frac{np}{2(p+2)}} s^{-(\alpha+\beta)-h} (s^\beta \|v(s)\|_{(\frac{p+2}{p},\infty)}) s^{\alpha+h} \|u(s) - \widetilde{u}(s)\|_{(p+2,\infty)} ds \\ &\leq 2\varepsilon C_S \int_0^1 (1-s)^{-\frac{np}{2(p+2)}} s^{-(\alpha+\beta)-h} (ts)^{\alpha+h} \|u(ts) - \widetilde{u}(ts)\|_{(p+2,\infty)} ds. \end{aligned} \tag{3.23}$$

For I_2, a similar computation yields

$$I_2(t) \leq 2\varepsilon C_S \int_0^1 (1-s)^{-\frac{np}{2(p+2)}} s^{-(\alpha+\beta)-h} (ts)^{\beta+h} \|v(ts) - \widetilde{v}(ts)\|_{(\frac{p+2}{p},\infty)} ds. \tag{3.24}$$

Before dealing with $J_1(t)$, we remember that $\sup_{t>0} te^{-at} = \frac{1}{a} e^{-1}$.

$$\begin{aligned} J_1(t) &\leq \tfrac{\lambda}{\mu} p t^{\beta+h} \int_0^t e^{-\frac{t-s}{\mu}} \left\| |u - \widetilde{u}| \left(|u|^{p-1} + |\widetilde{u}|^{p-1}\right) \right\|_{(\frac{p+2}{p},\infty)} ds \\ &\leq \tfrac{\lambda}{\mu} p t^{\beta+h} \int_0^t e^{-\frac{t-s}{\mu}} s^{-p\alpha-h} [s^{\alpha(p-1)} (\|u\|_{(p+2,\infty)}^{p-1} + \|\widetilde{u}\|_{(p+2,\infty)}^{p-1})] s^{\alpha+h} \|u - \widetilde{u}\|_{(p+2,\infty)} ds \\ &:= H(t), \end{aligned} \tag{3.25}$$

where we have used the Hölder inequality in the last estimate. Next we split the integral within $H(t)$ into two parts and estimate

$$\leq \tfrac{\lambda}{\mu} p 2^p \varepsilon^{p-1} t^{\beta+h} \left(\int_0^{t/2} e^{-(\frac{t-s}{\mu})} s^{-\beta-h} ds \sup_{t>0} t^{\alpha+h} \|u(t) - \widetilde{u}(t)\|_{(p+2,\infty)} \right.$$
$$\left. + \, t^{\beta+h} \int_{t/2}^{t} e^{-(\frac{t-s}{\mu})} s^{-\beta-h} s^{\alpha+h} \|u(s) - \widetilde{u}(s)\|_{(p+2,\infty)} \, ds \right)$$

$$\leq \lambda p 2^p \varepsilon^{p-1} \left(\frac{1}{\mu} \cdot t e^{-(\frac{1}{2\mu})t} \cdot \frac{(1/2)^{1-\beta-h}}{(1-\beta-h)} \cdot \sup_{t>0} t^{\alpha+h} \|u(t) - \widetilde{u}(t)\|_{(p+2,\infty)} \right.$$
$$\left. + \, 2^{\beta+h} \int_{1/2}^{1} \frac{1}{\mu} t e^{-(\frac{1-s}{\mu})t} ds \sup_{t/2<s<t} s^{\alpha+h} \|u(s) - \widetilde{u}(s)\|_{(p+2,\infty)} \right)$$

$$:= H_1(t) + H_2(t). \tag{3.26}$$

By (3.18) and elementary properties of $\limsup\limits_{t\to\infty}$, we get

$$\limsup_{t\to\infty} H_1(t) \leq C \limsup_{t\to\infty} t e^{-\frac{1}{2\mu}t} = 0, \tag{3.27}$$

$$\begin{aligned} \limsup_{t\to\infty} H_2(t) &\leq \lambda p 2^p \varepsilon^{p-1} 2^{\beta+h} \limsup_{t\to\infty} \left\{ (1 - e^{-\frac{t}{2\mu}}) \sup_{t/2<s<t} s^{\alpha+h} \|u(s) - \widetilde{u}(s)\|_{(p+2,\infty)} \right\} \\ &\leq \lambda p 2^p \varepsilon^{p-1} 2^{\beta+h} \limsup_{t\to\infty} t^{\alpha+h} \|u(t) - \widetilde{u}(t)\|_{(p+2,\infty)} . \end{aligned} \tag{3.28}$$

Now, define

$$A := \limsup_{t\to\infty} t^{\alpha+h} \|u(t) - \widetilde{u}(t)\|_{(\rho+2,\infty)} + \limsup_{t\to\infty} t^{\beta+h} \|v(t) - \widetilde{v}(t)\|_{(\frac{p+2}{p},\infty)} < \infty.$$

So, taking $\limsup\limits_{t\to\infty}$ in (3.21) and (3.22) and using the bounds (3.23)–(3.28), we obtain

$$\begin{aligned} A &\leq \limsup_{t\to\infty} (I_0(t) + J_0(t)) + \max\left\{ \limsup_{t\to\infty} [I_1(t) + I_2(t)] \ , \ \limsup_{t\to\infty} [H_1(t) + H_2(t)] \right\} \\ &\leq 0 + \max\left\{ 2\varepsilon C_S \int_0^1 (1-s)^{-\frac{np}{2(p+2)}} s^{-(\alpha+\beta)-h} ds, \ \lambda p 2^p \varepsilon^{p-1} 2^{\beta+h} \right\} A \\ &:= \Gamma(\varepsilon) A. \end{aligned} \tag{3.29}$$

Choosing $\varepsilon > 0$ small enough such that $\Gamma(\varepsilon) < 1$, we get that $A = 0$, which implies (2.6).

Third step. In order to prove the reciprocal assertion, we proceed as in the proof above by changing the place of $t^{\alpha+h} \|u(t) - \widetilde{u}(t)\|_{(p+2,\infty)}$ and $t^{\alpha+h} \|S(t)(u_0 - \widetilde{u}_0)\|_{(p+2,\infty)}$

by the other one. In this case, instead of (3.29) we get

$$\begin{aligned}
&\limsup_{t\to\infty} t^{\alpha+h}\left\|S(t)(u_0-\widetilde{u}_0)\right\|_{(p+2,\infty)}\\
&\leq \limsup_{t\to\infty}[t^{\alpha+h}\left\|u(t)-\widetilde{u}(t)\right\|_{(p+2,\infty)}]\\
&+\max\left\{\limsup_{t\to\infty}[I_1(t)+I_2(t)]\ ,\ \ \limsup_{t\to\infty}[H_1(t)+H_2(t)]\right\}\\
&\leq 0+\max\left\{2\varepsilon C_S\int_0^1(1-s)^{-\frac{np}{2(p+2)}}s^{-(\alpha+\beta)-h}ds,\ \ \lambda p2^p\varepsilon^{p-1}2^{\beta+h}\right\}A\\
&=0,
\end{aligned}$$

because $A=0$ by hypothesis.

The strong version of the results follows in an entirely parallel way to the previous proof by taking the quantities $t^{\alpha+h}\left\|\cdot\right\|_{p+2}$ and $t^{\beta+h}\left\|\cdot\right\|_{\frac{p+2}{p}}$ in place of $t^{\alpha+h}\left\|\cdot\right\|_{(p+2,\infty)}$ and $t^{\beta+h}\left\|\cdot\right\|_{(\frac{p+2}{p},\infty)}$, respectively. The proof is finished.

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