QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXXIII, NUMBER 3
SEPTEMBER 2015, PAGES 511–523
http://dx.doi.org/10.1090/qam/1390
Article electronically published on June 12, 2015

STOKES FLOW APPLIED TO THE SEDIMENTATION OF A RED BLOOD CELL

By

M. HADJINICOLAOU (*School of Science and Technology, Hellenic Open University, 11 Sahtouri str., GR-26 222 Patras, Greece*),

G. KAMVYSSAS (*Department of Mechanical Engineering, TEI of Western Greece, 1, M. Alexandrou str., Koukouli, GR-26 334 Patras, Greece*),

AND

E. PROTOPAPAS (*School of Science and Technology, Hellenic Open University, 11 Sahtouri str., GR-26 222 Patras, Greece*)

Abstract. The sedimentation of a red blood cell (RBC) through the blood plasma is described as the translation of a rigid inverted prolate spheroid through a quiescent unbounded viscous fluid. The inverted spheroid is moving with constant velocity along its axis of symmetry. The physical characteristics of the RBC and the blood plasma allow us to consider this problem as a Stokes flow problem. The Kelvin inversion method and the concept of semiseparation of variables for the Stokes operators, which we used for solving the Stokes flow past an RBC, are also employed here. The stream function is then expressed as a series expansion in terms of Gegenbauer functions of even order. Through this, important hydrodynamic quantities such as the drag force, the drag coefficient and the terminal settling velocity of the RBC are calculated. The celebrated Stokes formula for the drag force exerted on a sphere is now expanded in order to account for the shape deformation of an RBC. Sample streamlines are depicted showing the dependence of these quantities on the geometrical characteristics of the RBC and also of any inverted prolate spheroid. The obtained results seem to be directly applicable in medical tests such as the Erythrocyte Sedimentation Rate (ESR).

1. Introduction. A theoretical investigation of the blood flow aims, in a general sense, for a better understanding of its overall hydrodynamic properties and the resulting rheological behaviour. Needless to say this knowledge allows for more accurate

Received October 15, 2013 and, in revised form, November 30, 2013.
2010 *Mathematics Subject Classification.* Primary 35Q35, 76D07, 92C05.
Key words and phrases. Stokes flow, red blood cell (RBC), blood plasma flow, mathematical model, Kelvin inversion, stream function, drag force, drag coefficient, terminal settling velocity, sedimentation.
E-mail address: hadjinicolaou@eap.gr
E-mail address: greg@teiwest.gr
E-mail address: lprotopapas@eap.gr

predictions and effective treatment of several diseases. Human blood is considered as a Newtonian fluid which incorporates three kinds of cells suspended in it, namely, the red blood cells (RBCs), the white blood cells and the platelets.

Many studies have dealt with the modelling of the flow through blood vessels, in which the RBCs are moving together with the plasma, with due attention to the shape deformation of the RBCs. The majority of these studies are concerned with numerical simulations [1], [2], [3], [4], [5], but certain analytical treatments have been applied as well [6].

Another category of studies are motivated by the modelling of processes such as the sedimentation of RBCs, dealing with the relative motion of RBCs with respect to the blood plasma. Of particular interest is the haematological test called the Erythrocyte Sedimentation Rate (ESR), which is clinically used either for the diagnosis of some infectious, or autoimmune, or malignant processes and inflammatory conditions or as a tool for the prognosis of several diseases such as prostate cancer, coronary artery disease, stroke, etc. [7], [8], [9], and is based on the calculation of the rate at which the RBCs fall in vitro. In some of these cases, ESR is also employed for monitoring the response to the therapy.

Oka [7] proposed a physical theory for the erythrocyte sedimentation that takes into account the aggregation of the erythrocytes. He demonstrated the dependence of the sedimentation velocity on the ultimate size of the aggregation and on the retardation time. Reuben and Shannon [8] looked into the factors that are influencing the ESR and they showed that in non-healthy situations, the RBCs aggregate in arrays along a single axis, which is perpendicular to the plane of the cell. He also showed that if the terminal velocity U_∞ is known, the concentration of the RBCs can be derived by solving an ordinary differential equation. Furthermore, one can also calculate the sedimentation curve [7] or the instant when all the RBCs have reached the bottom of the tube [10]. Yamaikina and Ivashkevich [9] studied the erythrocyte sedimentation in capillaries. They built a mathematical model, employing only a rheological parameter and the hydrodynamic radius, through which they assessed their experimental data.

In the present work we study the onset of sedimentation of an RBC through the blood plasma as a result of its having a higher density than the blood plasma itself. The RBC is primarily treated geometrically as a biconcave disk [3], [4], [5], which, at rest, has a major diameter [11] of about $8\mu m$ and thickness of at least $2\mu m$. Its mathematical representation is given by an inverted prolate spheroid moving along its axis of symmetry within a viscous fluid being at rest at infinity. The shape of the RBC remains unchanged. The process is assumed to be reduced to a Stokes flow problem caused by the translation of a rigid inverted prolate spheroid within a quiescent unbounded viscous fluid. This assumption is justified by taking into account the physical properties of the blood plasma and the geometrical characteristics of the RBC [12]. Furthermore, Stokes flow appears in the flow in capillaries [13].

The solution of the problem is based on the methods and the results that were developed in [6], where the blood plasma was considered to flow around a stationary RBC. These methods are the semiseparable spectral decomposition of the Stokes flow operators in the spheroidal geometry and the Kelvin transformation for the Stokes flow.

In fluid mechanics, Stokes flow is considered to be the slow motion of a small particle through a viscous fluid when the inertial forces are neglected, that is to say, for small Reynolds number. Stokes flow has been studied extensively in the literature because of the important role it plays in fluid dynamics. G. Stokes [14], [15] introduced a scalar function ψ, the so-called Stokes stream function, and through this he derived representations for the velocity and the pressure field in the case of a viscous, incompressible, axisymmetric creeping flow. As a consequence, the stream function ψ has to satisfy a fourth order partial differential equation ($E^4\psi = 0$) which is linear, but it has non-constant coefficients [16]. Due to the very complicated form that this equation takes in the various curvilinear systems, a complete form of the stream function has been obtained only for the spherical geometry [16] and for the prolate and the oblate spheroidal ones [17], [18]. Recently Hadjinicolaou and Protopapas [19] derived the eigenfunctions of the Stokes operator and developed an accurate method for the calculation of the generalized eigenfunctions of the Stokes operator in the modified inverted prolate coordinate system, providing a new analytical expression for the stream function in this particular system.

The Stokes stream and bistream operators for the axisymmetric flow are the partial differential operators of the second and fourth order E^2 and E^4, respectively. The stream functions that belong to the $ker\{E^2\}$ represent irrotational flows, while the rotational flows can be described through functions that belong to the $ker\{E^4\}$ but don't belong to the $ker\{E^2\}$. It has been shown in [17] that in spheroidal coordinates the solutions of the equation $E^4\psi = 0$ are in semiseparable form and are given in particular products of Gegenbauer functions. The general solution is then represented as a series expansion in terms of the eigenfunctions of the E^2 and the generalized eigenfunctions of the E^2.

Kelvin inversion, or the method of reciprocal radii, is based on a domain transformation which, in the case of $\mathbb{R}^3$, is an inversion with respect to a sphere. Early in 1845, Lord Kelvin, inspired by Green's ideas [20], presented this inversion method as a method for solving boundary value problems for partial differential equations. According to this, the solution of a boundary value problem in an interior domain can be obtained from the solution of the 'analogous' problem in the exterior domain and vice versa. Dassios and Kleinman [21], [22] used it in acoustic scattering problems, while Baganis and Hadjinicolaou [23], [24], through Kelvin's inversion, derived, for the first time, analytical solutions of exterior Dirichlet and Neumann problems for a non-convex domain. Dassios [25] investigated the application of the Kelvin inversion to the Stokes operators. Some key results regarding the application of the Kelvin transformation in medical problems are given in [26]. The concept of semiseparation was first introduced in [17] in order to express the general solution for the Stokes flow in prolate spheroidal coordinates, and since then it has been applied to studies in chemical engineering such as flow and transport phenomena in particle systems, fluidised beds, etc. [18].

We further exploit the applicability of the obtained theoretical results to a medical application related to the microscale perspective of the blood flow. Specifically, we study the onset of sedimentation of an RBC through the blood plasma as a result of its higher density compared to the one of the blood plasma. The obtained results seem to be suitable for medical applications such as the clinical haematological test called Erythrocyte Sedimentation Rate (ESR). They may also be employed in engineering applications and

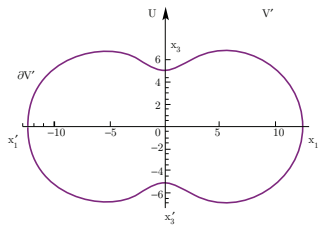

FIG. 1. Translating an RBC

the study of transport processes. Besides, they provide a basis for numerical implementation [27], [28] and the assessment of existing numerical results. The structure of the manuscript is as follows. In Section 2 we derive the stream function that describes the flow around the RBC by using appropriately the series expansion describing the stream function for the Stokes flow past a rigid and stationary inverted prolate spheroid. A numerical investigation is presented that reveals the significance of the first two terms of the series. In Section 3 we derive the drag force, the drag coefficient and the terminal velocity. The obtained results are discussed in Section 4.

2. Translation of the RBC. We consider a solid RBC moving through an unbounded quiescent fluid V', with constant velocity U, parallel to the x_3-axis in the positive direction. The RBC is considered to be an inverted prolate spheroid having its centre at the origin of a Cartesian coordinate system (x_1, x_2, x_3), as shown in Figure 1.

Let V' be the exterior fluid domain, $\partial V'$ denote the surface of the RBC, E'^2 be the Stokes operator, $E'^4 = E'^2 \circ E'^2$ be the Stokes bistream operator, $\mathbf{r}'$ be the position vector and $\psi_s(\mathbf{r}')$ be the stream function. The problem at hand in any axisymmetric system is defined as

$$E'^{\,4}\psi_s\left(\mathbf{r}'\right) = 0, \quad \boldsymbol{r'} \in V', \tag{2.1}$$

$$\psi_s\left(\boldsymbol{r'}\right) + \frac{1}{2}\varpi'^2 U = 0, \quad \boldsymbol{r'} \in \partial V', \tag{2.2}$$

$$\frac{\partial \psi_s\left(\mathbf{r}'\right)}{\partial n} = -\frac{1}{2}\frac{\partial}{\partial n}\varpi'^2 U, \quad \boldsymbol{r'} \in \partial V', \tag{2.3}$$

$$\frac{\psi_s}{r'^2} \to 0, \quad r' \to +\infty, \tag{2.4}$$

where ϖ' is the cylindrical coordinate. Relation (2.2) denotes that there is no relative tangential velocity component on the surface of the RBC, relation (2.3) implies that the RBC is impenetrable, and relation (2.4) expresses that the blood plasma extends to infinity where it is at rest.

The problem (2.1) to (2.4), translated to the inverted prolate spheroidal coordinate system (τ', ζ'), reads

$$E'^{\,4}\psi_s\left(\tau', \zeta'\right) = 0, \quad (\tau', \zeta') \in V', \tag{2.5}$$

$$\psi_s\left(\tau', \zeta'\right) = -\frac{b^4(1-\zeta'^2)(\tau^2-1)}{2c^2(\tau^2+\zeta'^2-1)}U, \quad (\tau', \zeta') \in \partial V', \tag{2.6}$$

$$\frac{\partial \psi_s}{\partial \tau'} = -U \frac{\partial}{\partial \tau'} \frac{b^4(1-\zeta^2)(\tau'^2-1)}{2c^2(\tau'^2+\zeta^2-1)^2}, \quad (\tau', \zeta') \in \partial V', \tag{2.7}$$

$$\frac{\psi_s}{r'^2} \to 0, \quad r' \to +\infty, \tag{2.8}$$

where $\psi_s(\tau', \zeta')$ is the stream function in the inverted system, $r' = c\sqrt{\tau'^2 + \zeta'^2 - 1}$ and $(\tau', \zeta') \in \partial V'$ denotes the surface of the RBC.

It is known [16] that once the stream function $\psi_a(\tau', \zeta')$ for the Stokes flow past a stationary solid has been obtained, the stream function $\psi_s(\tau', \zeta')$ for the translating solid is given through the relation

$$\psi_s(\tau', \zeta') = \psi_\alpha(\tau', \zeta') - \psi_\infty(\tau', \zeta'), \tag{2.9}$$

where $\psi_\infty(\tau', \zeta')$ is the stream function for the unperturbed flow at infinity. In the case of a uniform flow with constant velocity U it becomes

$$\psi_\infty(\tau', \zeta') = \frac{1}{2} \frac{b^4(1-\zeta^2)(\tau^2-1)}{c^2(\tau^2+\zeta^2-1)^2} U, \tag{2.10}$$

where (τ', ζ') is the prolate spheroidal coordinate system [30], $b > 0$ is the radius of the sphere of inversion and $c > 0$ is the semifocal distance.

The blood plasma flow around a solid RBC has been studied in [6], and the stream function $\psi_a(\tau', \zeta')$ was obtained by employing Kelvin's inversion method [25] and the concept of the semiseparation of variables [17]. Accordingly, the stream function for the Stokes flow problem in the interior of a prolate spheroid is

$$\psi(\tau, \zeta) = \sum_{n=1}^{\infty} g_{2n}(\tau) G_{2n}(\zeta). \tag{2.11}$$

Using results from [25] we derive the stream function

$$\psi_\alpha(\tau', \zeta') = \frac{b^3}{r^3} \psi(\tau, \zeta) \tag{2.12}$$

for the blood plasma flow past an inverted prolate spheroid, which represents the RBC. This is given through a series expansion in terms of Gegenbauer functions of the first and the second kind [29], as follows:

$$\psi_\alpha(\tau', \zeta') = \frac{b^3}{c^3 \sqrt{\tau^2+\zeta^2-1}^3} \sum_{n=1}^{\infty} g_{2n}(\tau) G_{2n}(\zeta), \tag{2.13}$$

where

$$g_2(\tau) = A_2 G_2(\tau) - \frac{9bcU}{5} H_2(\tau) - bcU G_1(\tau) + E_2 G_4(\tau) - \frac{6bcU}{5} H_4(\tau), \tag{2.14}$$

$$\begin{aligned} g_{2n}(\tau) = A_{2n} G_{2n}(\tau) &+ \frac{bcU}{2}(-w_{n-1} e_{n-1}^2 - w_n d_n^2) H_{2n}(\tau) \\ &+ \frac{bcU}{2} w_{n-1} e_{n-1} d_{n-1} H_{2n-2}(\tau) \\ + E_{2n} G_{2n+2}(\tau) &+ \frac{bcU}{2} w_n d_n e_n H_{2n+2}(\tau), \ n \geq 2, \end{aligned} \tag{2.15}$$

$$A_2 = bcU \frac{\begin{vmatrix} \frac{9}{5}H_2(\tau_0) + G_1(\tau_0) + \frac{6}{5}H_4(\tau_0) & G_4(\tau_0) \\ \frac{9}{5}H_2'(\tau_0) + G_1'(\tau_0) + \frac{6}{5}H_4'(\tau_0) & G_4'(\tau_0) \end{vmatrix}}{\begin{vmatrix} G_2(\tau_0) & G_4(\tau_0) \\ G_2'(\tau_0) & G_4'(\tau_0) \end{vmatrix}}, \tag{2.16}$$

$$E_2 = bcU \frac{\begin{vmatrix} G_2(\tau_0) & \frac{9}{5}H_2(\tau_0) + G_1(\tau_0) + \frac{6}{5}H_4(\tau_0) \\ G_2'(\tau_0) & \frac{9}{5}H_2'(\tau_0) + G_1'(\tau_0) + \frac{6}{5}H_4'(\tau_0) \end{vmatrix}}{\begin{vmatrix} G_2(\tau_0) & G_4(\tau_0) \\ G_2'(\tau_0) & G_4'(\tau_0) \end{vmatrix}}, \tag{2.17}$$

$$A_{2n} = \frac{bcU}{2} \frac{\begin{vmatrix} (w_{n-1}e_{n-1}^2 + w_n d_n^2)H_{2n}(\tau_0) - w_{n-1}e_{n-1}d_{n-1}H_{2n-2}(\tau_0) - w_n d_n e_n H_{2n+2}(\tau_0) & G_{2n+2}(\tau_0) \\ (w_{n-1}e_{n-1}^2 + w_n d_n^2)H_{2n}'(\tau_0) - w_{n-1}e_{n-1}d_{n-1}H_{2n-2}'(\tau_0) - w_n d_n e_n H_{2n+2}'(\tau_0) & G_{2n+2}'(\tau_0) \end{vmatrix}}{\begin{vmatrix} G_{2n}(\tau_0) & G_{2n+2}(\tau_0) \\ G_{2n}'(\tau_0) & G_{2n+2}'(\tau_0) \end{vmatrix}}, \tag{2.18}$$

$$E_{2n} = \frac{bcU}{2} \frac{\begin{vmatrix} G_{2n}(\tau_0) & (w_{n-1}e_{n-1}^2 + w_n d_n^2)H_{2n}(\tau_0) - w_{n-1}e_{n-1}d_{n-1}H_{2n-2}(\tau_0) - w_n d_n e_n H_{2n+2}(\tau_0) \\ G_{2n}'(\tau_0) & (w_{n-1}e_{n-1}^2 + w_n d_n^2)H_{2n}'(\tau_0) - w_{n-1}e_{n-1}d_{n-1}H_{2n-2}'(\tau_0) - w_n d_n e_n H_{2n+2}'(\tau_0) \end{vmatrix}}{\begin{vmatrix} G_{2n}(\tau_0) & G_{2n+2}(\tau_0) \\ G_{2n}'(\tau_0) & G_{2n+2}'(\tau_0) \end{vmatrix}}, \tag{2.19}$$

$$w_n = \frac{(-1)^n(4n+1)(2n)!}{2^{2n}(n!)^2}, \tag{2.20}$$

$$e_n = \frac{2(2n+1)(n+1)}{4n+1}, \tag{2.21}$$

$$d_n = \frac{2n(2n-1)}{4n+1}, \tag{2.22}$$

and $\tau = \tau_0$ denotes the surface of the prolate spheroid. Then, by employing the relations (2.9), (2.10), (2.13) to (2.22) and taking into account that

$$\begin{aligned} \frac{1}{2}\frac{bc(1-\zeta^2)(\tau^2-1)}{\sqrt{\tau^2+\zeta^2-1}}U = \frac{bcU}{2}\left[-2G_1(\tau) - \frac{18}{5}H_2(\tau) - \frac{12}{5}H_4(\tau)\right]G_2(\zeta) \\ + \frac{bcU}{2}\sum_{n=2}^{\infty} w_{n-1}e_{n-1}d_{n-1}H_{2n-2}(\tau)G_{2n}(\zeta) \\ + \frac{bcU}{2}\sum_{n=2}^{\infty}(-w_{n-1}e_{n-1}^2 - w_n d_n^2)H_{2n}(\tau)G_{2n}(\zeta) \\ + \frac{bcU}{2}\sum_{n=2}^{\infty} w_n d_n e_n H_{2n+2}(\tau)G_{2n}(\zeta), \end{aligned} \tag{2.23}$$

we finally arrive at

$$\psi_s(\tau', \zeta') = \frac{b^3}{c^3\sqrt{\tau^2+\zeta^2-1}^3}\sum_{n=1}^{\infty}\left[\mathrm{A}_{2n}G_{2n}(\tau) + \mathrm{E}_{2n}G_{2n+2}(\tau)\right]G_{2n}(\zeta), \tag{2.24}$$

where the coefficients A_{2n}, E_{2n} are defined by (2.16) to (2.22) and the prolate spheroidal coordinates (τ, ζ) are related to the inverted prolate spheroidal coordinates (τ', ζ')

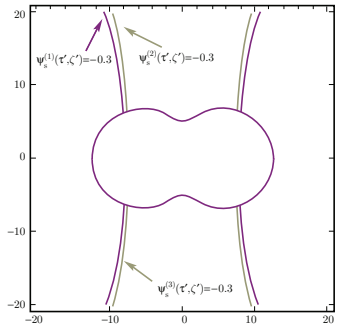

FIG. 2. Streamlines in the plane $x_2 = 0$

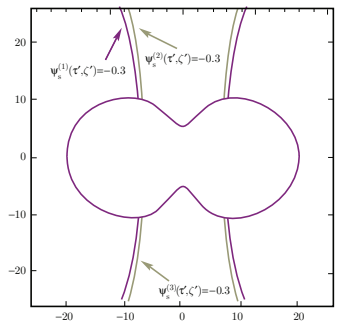

FIG. 3. Streamlines in the plane $x_2 = 0$

through the expressions

$$\zeta^2 = \frac{b^4 + c^2 r'^2 - \sqrt{\left(b^4 + c^2 r'^2\right)^2 - 4b^4 c^4 \tau'^2 \zeta'^2}}{2c^2 r'^2}, \tag{2.25}$$

$$\tau^2 = \frac{2c^2 b^4 \tau'^2 \zeta'^2}{r'^2 \left[b^4 + c^2 r'^2 - \sqrt{\left(b^4 + c^2 r'^2\right)^2 - 4b^4 c^4 \tau'^2 \zeta'^2}\right]}. \tag{2.26}$$

In the next graphs, we demonstrate sample streamlines, assuming $U = 0.01$ and radius of the sphere of inversion $b = 5$, for various values of the aspect ratio k (length /thickness). Due to the fast convergence of the series, we use only the first term or we add the second one. In order to demonstrate this result we define

$$\psi_s^{(1)}(\tau', \zeta') = \frac{b^3}{c^3 \sqrt{\tau^2 + \zeta^2 - 1}^3} \left[\mathrm{A}_2 G_2(\tau) + \mathrm{E}_2 G_4(\tau)\right] G_2(\zeta), \tag{2.27}$$

$$\psi_s^{(2)}(\tau', \zeta') = \psi_s^{(1)}(\tau', \zeta') + \frac{b^3}{c^3 \sqrt{\tau^2 + \zeta^2 - 1}^3} \left[\mathrm{A}_4 G_4(\tau) + \mathrm{E}_4 G_6(\tau)\right] G_4(\zeta), \tag{2.28}$$

and

$$\psi_s^{(3)}(\tau', \zeta') = \psi_s^{(2)}(\tau', \zeta') + \frac{b^3}{c^3 \sqrt{\tau^2 + \zeta^2 - 1}^3} \left[\mathrm{A}_6 G_6(\tau) + \mathrm{E}_6 G_8(\tau)\right] G_6(\zeta), \tag{2.29}$$

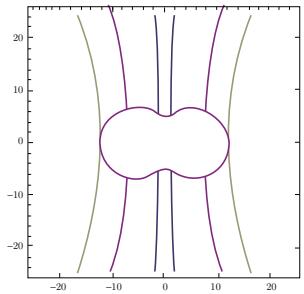

FIG. 4. Streamlines in the plane $x_2 = 0$ with $k = \frac{5}{2}$

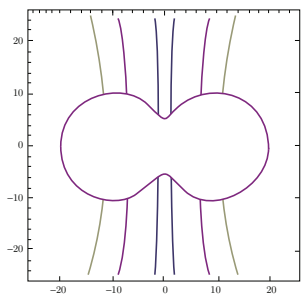

FIG. 5. Streamlines in the plane $x_2 = 0$ with $k = 4$

where the coefficients of the Gegenbauer functions are calculated by applying appropriately the boundary conditions and using orthogonality arguments.

In Figures 2 and 3 we depict the streamlines $\psi_s^{(1)}, \psi_s^{(2)}, \psi_s^{(3)}$ for the same value $\psi_s^{(1)} = \psi_s^{(2)} = \psi_s^{(3)} = -0.3$ and for two different values of the axis ratio, i.e., $k = 2.5$ and $k = 4$ respectively. From both cases we may deduce that the first term provides the qualitative behavior, while the second and third terms of the series provide quantitative corrections.

To this end, in Figures 4 and 5 we illustrate indicatively, for $k = 2.5$ and $k = 4$, sample streamlines when the stream function has successively the values $-0.01, -0.3, -0.7$ as they emanate from the surface of the inverted prolate spheroid towards infinity, and by employing only the first term of the series.

3. Drag force, drag coefficient, terminal settling velocity. An important quantity for the study of the sedimentation of RBCs is the so-called terminal settling velocity. When a particle is moving within a viscous fluid under the influence of gravity, it ultimately gets a uniform velocity due to counterbalance of the gravitational and the hydrodynamic forces. It is then falling at this constant speed called the terminal settling velocity [16]. The force exerted by the fluid on a body can be calculated as the integral of the stress dyadic over the entire surface of the body. Due to the symmetry of the flow, only the force acting parallel to the axis of revolution is significant, and provided that

the fluid is at rest at infinity, it is given [16], [31] by the relation

$$F_z = 8\pi\mu \lim_{r\to+\infty} \frac{r\psi_s}{\varpi'^2}, \tag{3.1}$$

where μ is the shear viscosity and

$$\varpi' = \frac{b^2\sqrt{1-\zeta^2}\sqrt{\tau^2-1}}{c(\tau^2+\zeta^2-1)} \tag{3.2}$$

is the radial cylindrical coordinate in the inverted prolate coordinate system.

Using (2.24) and (2.30) we arrive at

$$F_z = -\frac{4\pi\mu b}{c^2}\left[\frac{3}{2}bcUQ_0(\tau_0) + \sum_{n=2}^{\infty}(A_{2n}+E_{2n})(-1)^{n+1}\frac{1\cdot 3\cdot \ldots \cdot(2n-3)}{2\cdot 4\cdot \ldots \cdot(2n)}\right] \tag{3.3}$$

or

$$F_z = -\frac{4\pi\mu bU}{c^2}\left[\frac{3}{2}bcQ_0(\tau_0) + \sum_{n=2}^{\infty}\left(\overline{A_{2n}}+\overline{E_{2n}}\right)(-1)^{n+1}\frac{1\cdot 3\cdot \ldots \cdot(2n-3)}{2\cdot 4\cdot \ldots \cdot(2n)}\right], \tag{3.4}$$

where

$$Q_0(x) = \frac{1}{2}\ln\frac{x+1}{x-1} \tag{3.5}$$

is the zeroth order Legendre function of the second kind and

$$A_{2n} = \overline{A_{2n}}U, \tag{3.6}$$

$$E_{2n} = \overline{E_{2n}}U. \tag{3.7}$$

The drag coefficient is

$$C_D = \frac{-8\mu(\tau_0^2-1)}{\rho U^2 b^3}\left[\frac{3}{4}bcU\ln\frac{\tau_0+1}{\tau_0-1} + \sum_{n=2}^{\infty}(A_{2n}+E_{2n})(-1)^{n+1}\frac{1\cdot 3\cdot \ldots \cdot(2n-3)}{2\cdot 4\cdot \ldots \cdot(2n)}\right], \tag{3.8}$$

where all the coefficients have been calculated previously through (2.16) to (2.22).

Denoting by F_G the buoyant force, the terminal velocity U_∞ of the RBC is defined [16] as the velocity of the RBC when

$$F_z = F_G. \tag{3.9}$$

We know that

$$F_G = (\rho'-\rho)gV_{RBC}, \tag{3.10}$$

where ρ' is the density of the RBC, ρ is the density of the blood plasma, g is the gravitational constant and V_{RBC} is the volume of the RBC, so using the relations (3.4), (3.9), (3.10) we get

$$U_\infty = -\frac{c^2(\rho'-\rho)gV_{RBC}}{4\pi\mu b\left[\frac{3}{2}bcQ_0(\tau_0) + \sum\limits_{n=2}^{\infty}\left(\overline{A_{2n}}+\overline{E_{2n}}\right)(-1)^{n+1}\frac{1\cdot 3\cdot \ldots \cdot(2n-3)}{2\cdot 4\cdot \ldots \cdot(2n)}\right]}, \tag{3.11}$$

with

$$V_{RBC} = \frac{b^6\pi(2\alpha^2+3\beta^2)}{6\alpha^2\beta^3} + \frac{\beta b^6\pi}{4\alpha^3 c}\ln\frac{\alpha+c}{\alpha-c}, \tag{3.12}$$

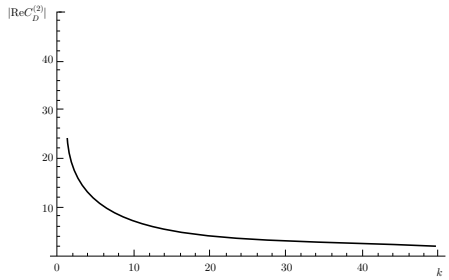

FIG. 6. Plot of $|ReC_D^{(2)}|$ with respect to k

where α is the length of the large semiaxis, β is the length of the small semiaxis and c stands for the semifocal distance of the original prolate spheroid with

$$\alpha^2 = \beta^2 + c^2. \tag{3.13}$$

Since we know that

$$\alpha = c\tau_0 \tag{3.14}$$

and

$$\beta = c\sqrt{\tau_0^2 - 1}, \tag{3.15}$$

we arrive at

$$\tau_0^2 = \frac{\dfrac{a^2}{b^2}}{\dfrac{a^2}{b^2} - 1}, \tag{3.16}$$

while, by recalling that $k > 1$ is the axial to radial aspect ratio of the semiaxes, from (3.16) we get

$$\tau_0 = \frac{k}{\sqrt{k^2 - 1}} > 1. \tag{3.17}$$

In order to depict qualitative results regarding the behavior of the drag coefficient C_D with respect to the variations of the aspect ratio k, we use indicatively only the first term of the series expansion of the C_D given in (3.8), and we get

$$Re\ C_D^{(2)} = -12\sqrt{\tau_0^2 - 1}\, ln \frac{\tau_0 + 1}{\tau_0 - 1}, \tag{3.18}$$

and by using (3.17) we find that

$$Re\ C_D^{(2)} = \frac{-24}{\sqrt{k^2 - 1}} ln\left(k + \sqrt{k^2 - 1}\right), \tag{3.19}$$

which shows that the first term of the drag force and the drag coefficient are given through a logarithmic function of the aspect ratio, which is in accordance with the spherical results provided in the literature [16]. Taking the limit of $|ReC_D^{(2)}|$ when $k \to 1$ we find that it is 24, which is the value of ReC_D in the sphere. In Figure 6 we can see the monotonical decreasing of the $|ReC_D^{(2)}|$ as the axes ratio changes vary from 1 to 10.

4. Discussion. In the present work, we derive an analytic solution for the creeping motion of an RBC within the blood plasma, which is modelled as the creeping motion of an inverted prolate spheroid within a Newtonian fluid being at rest at infinity. This assumption is thoroughly discussed and justified in the introduction.

We employed the stream function for the Stokes flow around a stationary inverted prolate spheroid and we took into account the term representing the unperturbed flow field at infinity. Through this procedure, we obtain the stream function for the relative motion of the RBC within the fluid which is given as a series expansion of Gegenbauer functions of even order.

Reduction to the spherical geometry is obtained by setting the aspect ratio of the spheroidal axes close to the value one, where the already known spherical solution [16] is regained. Furthermore, by using the stream function expansion, we further calculate important hydrodynamic quantities, such as the drag force exerted by the fluid on the surface of the inverted spheroid, the dimensionless quantity: drag coefficient and the limiting value of the velocity, which is called the terminal settling velocity. The obtained expressions expand the well known Stokes formula for a sphere to the case of a non-convex body, described by an inverted prolate spheroid. Rewriting the obtained expressions with respect to the aspect ratio of the axes of the prolate spheroid that is inverted, we may reduce our results to the spherical ones. The dependence of the stream function, the drag force and the drag coefficient on the geometrical characteristics of the inverted prolate spheroid has also been investigated for various values of the aspect ratio of the spheroid. We provide this way, ready to use hydrodynamic expressions for RBCs or for any inverted prolate spheroid having various geometrical characteristics.

It turns out that when we adopt an almost spherical description for the RBC, the drag force exerted by the fluid on the RBC gets smaller values compared to those it gets when departing from the spherical shape, where we observe that it increases monotonically. This result indicates the significance of an RBC resembling an inverted spheroid rather than employing other sphere-like shapes.

Sample streamlines have also been plotted, depicting the flow field around the translating RBC using indicatively the first term. One gets similar flow behaviour by using more terms of the series expansion. The obtained analytical expansions are expected to be useful in medical applications, such as the calculation of the sedimentation rate (ESR) of RBCs, the study of the aggregation of RBCs the drug delivery, the study of transport processes and also for deriving more accurate predictions concerning the blood flow behaviour.

Acknowledgments. This research has been co-financed by the European Union (European Social Fund - ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: ARCHIMEDES III.

References

[1] K. Tsubota, S. Wada, and T. Yamaguchi, *Particle Method for Computer Simulation of Red Blood Cell Motion in Blood Flow*, Computer Methods and Programs in Biomedicine **83** (2006), 139-146.

[2] T. Zien, *Hydrodynamics of Bolus Flow—An Analytic Approach to Blood Flow in Capillaries*, Bulletin of Mathematical Biophysics **31** (1969), 681-694.

[3] P. Bagchi, *Mesoscale Simulation of Blood Flow in Small Vessels*, Biophysical Journal **92** (2007), 1858-1877.

[4] H. Noguchi and G. Gompper, *Shape Transitions of Fluid Vesicles and Red Blood Cells in Capillary Flows*, Proc. Natl. Acad. Sci. USA **102** (2005), 14159-14164.

[5] J. L. McWhirter, H. Noguchi and G. Gompper, *Flow-Inducted Clustering and Alignment of Red Blood Cells in Microchannels*, Proc. Natl. Acad. Sci. USA **106** (2009), 6039-6043.

[6] G. Dassios, M. Hadjinicolaou, and E. Protopapas, *Blood plasma flow past a red blood cell: mathematical modelling and analytical treatment*, Math. Methods Appl. Sci. **35** (2012), no. 13, 1547–1563, DOI 10.1002/mma.2540. MR2957516

[7] S. Oka, *A physical theory of erythrocyte sedimentation*, Biorheology **22** (1985), 315-321.

[8] A. J. Reuben and A. G. Shannon, *Some problems in the mathematical modelling of erythrocyte sedimentation*, IMA J. Math. Appl. Med. Biol. **7** (1990), no. 3, 145–156. MR1085599 (92e:92011)

[9] V. I. Yamaikina and V. Ivashkevich, *A mathematical model of erythrocyte sedimentation in capillaries*, J. Eng. Phys. Thermophys. **72** (1999), 55-61.

[10] J. Tan, B.-D. Lee, L. Polo-Parada and S. Sengupta, *Kinetically limited differential centrifugation as an inexpensive and readily available alternative to centrifugal elutriation*, BioTechniques **53** (2002), no. 2, 104-108.

[11] Y. Sugii, R. Okuda, K. Okamoto, and H. Madarame, *Velocity Measurement of Both Red Blood Cells and Plasma of in vitro Blood Flow using High-Speed Micro PIV Technique*, Institute of Physics Publishing Meas. Sci. Technol. **16** (2005), 1126-1130.

[12] J. A. Davis, D. W. Inglis, K. J. Morton, D. A. Lawrence, L. R. Huang, S. Y. Chou, J. C. Sturm, and R. H. Austin, *Deterministic Hydrodynamics: Taking Blood Apart*, PNAS **103** (2003), no 40, 14779-14784.

[13] R. Lima, T. Ishikawa, T. Imai et al., *Blood Flow Behavior in Microchannels: Past, Current and Future Trends*, Chem. Biomed. Eng. (2012), 513-547.

[14] G. G. Stokes, *On the Theories of the Internal Friction of Fluids in Motion and the Equilibrium and Motion of Elastic Solids*, Trans. Camp. Phil. Soc. **8** (1845), 287-319.

[15] G. G. Stokes, *On the Effect of the Internal Friction of Fluids on the Motion of Pendulums*, Trans. Camp. Phil. Soc. **8** (1845), 8-106.

[16] J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics*, Kluwer Academic Publishers, 1991.

[17] G. Dassios, M. Hadjinicolaou, and A. C. Payatakes, *Generalized eigenfunctions and complete semiseparable solutions for Stokes flow in spheroidal coordinates*, Quart. Appl. Math. **52** (1994), no. 1, 157–191. MR1262325 (95d:76034)

[18] G. Dassios, M. Hadjinicolaou, F. A. Coutelieris and A. C. Payatakes, *Stokes Flow in Spheroidal Particle-in-cell Models with Happel and Kuwabara Boundary Conditions*, International Journal of Engineering Science **33** (1995), 1465-1490.

[19] M. Hadjinicolaou and E. Protopapas, *On the R-semiseparation of the Stokes bi-stream operator in inverted prolate spheroidal geometry*, Math. Methods Appl. Sci. **37** (2014), no. 2, 207–211, DOI 10.1002/mma.2841. MR3153095

[20] G. Green, *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*. Nottingham: Private Publication, 1928. Also in Mathematical Papers of the Late George Green (N. M. Ferrers, ed.), London: Macmillan, 1871.

[21] George Dassios and R. E. Kleinman, *On Kelvin inversion and low-frequency scattering*, SIAM Rev. **31** (1989), no. 4, 565–585, DOI 10.1137/1031126. MR1025482 (90i:35073)

[22] George Dassios and Ralph E. Kleinman, *On the capacity and Rayleigh scattering for a class of nonconvex bodies*, Quart. J. Mech. Appl. Math. **42** (1989), no. 3, 467–475, DOI 10.1093/qjmam/42.3.467. MR1018522 (90h:35061)

[23] G. Baganis and M. Hadjinicolaou, *Analytic solution of an exterior Dirichlet problem in a non-convex domain*, IMA J. Appl. Math. **74** (2009), no. 5, 668–684, DOI 10.1093/imamat/hxp023. MR2549954 (2011a:35124)

[24] G. Baganis and M. Hadjinicolaou, *Analytic solution of an exterior Neumann problem in a non-convex domain*, Math. Methods Appl. Sci. **33** (2010), no. 17, 2067–2075, DOI 10.1002/mma.1316. MR2762318 (2011m:35053)

[25] George Dassios, *The Kelvin transformation in potential theory and Stokes flow*, IMA J. Appl. Math. **74** (2009), no. 3, 427–438, DOI 10.1093/imamat/hxn027. MR2507298 (2010h:76052)

[26] G. Baganis, G. Dassios, M. Hadjinicolaou, and E. Protopapas, *The Kelvin transformation as a tool for analyzing problems in medicine and technology*, Math. Methods Appl. Sci. **37** (2014), no. 2, 194–199, DOI 10.1002/mma.2903. MR3153093

[27] M. R. Ismailov, A. N. Shevchuk, H. Khusanov, *Mathematical model describing erythrocyte sedimentation rate. Implications for blood viscocity changes in traumatic shock and crush syndrome*, Biomedical Engineering OnLine, 2005, DOI 10.1186/1475-925X-4-24.

[28] K. Cha, F. E. Brown, and W. D. Wilmore, *A new bioelectrical impedance method for measurement of the erythrocyte sedimentation rate*, Physiol. Meas. **15** (1994), 499-508.

[29] N. N. Lebedev, *Special functions and their applications*, Dover Publications Inc., New York, 1972. Revised edition, translated from the Russian and edited by Richard A. Silverman; unabridged and corrected republication. MR0350075 (50 #2568)

[30] P. Moon and D. E. Spencer, *Field theory handbook,* Including coordinate systems, differential equations and their solutions, 2nd ed., Springer-Verlag, Berlin, 1988. MR947546 (89i:00026)

[31] L. E. Payne and E. H. Pell, *The Stokes flow problem for a class of axially symmetric bodies*, Fluid Mech. **7** (1960), 529. MR0115471 (22 #6272)