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ILL POSEDNESS OF BINGHAM-TYPE MODELS FOR THE DOWNHILL FLOW OF A THIN FILM ON AN INCLINED PLANE

By

L. FUSI (*Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Viale Morgagni 67/a, 50134 Firenze, Italy*),

A. FARINA (*Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Viale Morgagni 67/a, 50134 Firenze, Italy*),

AND

F. ROSSO (*Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Viale Morgagni 67/a, 50134 Firenze, Italy*)

Abstract. In this paper we consider the flow of a thin layer of a Bingham-type material over an inclined plane with "small" tilt angle. A Bingham-type continuum is a material which behaves as a viscous fluid above a certain threshold (tied to the shear stress) and as a solid below such a threshold. We consider creeping flow and that the ratio between the thickness and the length of the layer is small, so that the lubrication approach is suitable. The unknowns of the model are the layer thickness, the position of the yield surface and the position of the advancing front. We first show that, though diverging in a neighborhood of the wetting front, the shear stress is integrable so that total dissipation is bounded. We then prove that the mathematical problem is inherently ill posed independently on the constitutive model selected for the solid domain. We therefore conclude that either the Bingham-type models are inappropriate to describe the thin film motion on an inclined surface or the lubrication technique fails in approximating such flows.

1. Introduction. In this paper we rigorously prove that the lubrication approximation for a Bingham-type flow over an inclined plane gives rise to ill-posed mathematical problems. By a Bingham-type material we mean a material that behaves as a linear viscous incompressible fluid when some invariant of the stress is larger than a critical value and as a solid (rigid, elastic, etc.; see e.g. [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15])

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E-mail address: fusi@math.unifi.it
E-mail address: farina@math.unifi.it
E-mail address: rosso@math.unifi.it

otherwise. We also refer the reader to [4], where interesting experimental investigations on these kinds of materials are reported.

After deriving the equations that govern the flow and after defining the appropriate boundary conditions (within the context of lubrication approximation), we show that the mathematical formulation (at the leading order terms) leads to a paradox in the vicinity of the advancing front. The interesting feature of our study is that the paradox is independent from the constitutive structure of the solid region, so that our results hold true for any kind of Bingham-like (or yield stress) materials.

This paradox is not new to the scientific community (see e.g. [2], [16]), but it is typically ascribed to the unboundedness of the stress in a neighborhood of the advancing front, which in turn is due to the classical no-slip condition on the bottom plane (see [3]). Actually we think that this explanation is not exhaustive. Indeed we prove (see Proposition 2) that even if locally the stress is unbounded, it is integrable and also the dissipation is integrable, indicating that a "weak" formulation of the problem may not highlight any paradox.

In this paper we present a different approach and we show that the paradox is essentially due to the assumption that the stress and the velocity are continuous across the yield surface in a neighborhood of the advancing front. We prove that this requirement leads to an evident contradiction, even if we consider a different scaling in the vicinity of the advancing front (see Section 5). This contradiction suggests that in the vicinity of the wetting front the material must "break down" (for instance with the solid part sliding over the fluid). Coupling these results with the ones presented in [15], we claim that, within the lubrication approximation framework, Bingham-type models are not suitable for describing thin films flowing on inclined planes (unless we are "far away" from the contact line and assume that the free surface of the fluid has constant height). Of course, it is likely that the full system of equations generated by the Bingham-type model does not exhibit any paradoxical behavior. In this case we conclude that the asymptotic technique is not applicable to the thin Bingham-type flows.

2. The physical model. Let us consider a two dimensional setting as the one depicted in Figure 1. In practice we consider a semi-infinite layer of material flowing down a plane with inclination α. We suppose that the material advances with velocity[1] $\dot{x}_a^*(t)$, so that the actual position of the advancing front is $x_a^*(t)$ (we assume that the initial position of the contact line is $x^* = 0$), where $x_a^*(t)$ is unknown. We assume that the characteristic longitudinal length spanned by the advancing front is L^* and we denote by H^* the transversal characteristic length. We make the assumption

$$\varepsilon = \frac{H^*}{L^*} \ll 1 \qquad \text{(lubrication approximation)}.$$

We denote by $y^* = h^*(x^*, t^*)$ the free surface (zero stress) and by $y^* = \sigma^*(x^*, t^*)$ the surface separating the unyielded (or solid) and yielded (or fluid) domains. We suppose that the velocity field can be written as

$$\boldsymbol{u}(x^*, y^*, t^*) = u_1^*(x^*, y^*, t^*)\boldsymbol{e}_x + u_2^*(x^*, y^*, t^*)\boldsymbol{e}_y$$

[1] Throughout this paper the superscript "$*$" denotes dimensional variables.

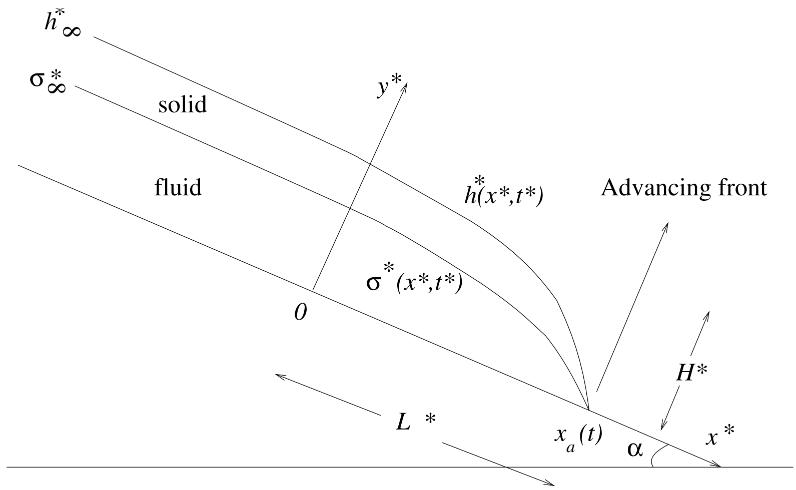

FIG. 1. Sketch of the system. The tilt angle is "small", i.e. $\tan\alpha = \mathcal{O}(\varepsilon)$.

and that the stress is given by

$$\mathbf{T}^* = -P^*\mathbf{I} + \mathbf{S}^*,$$

where P^* is the mean normal stress (pressure) and $\mathbf{S}^*$ denotes the deviatoric part. We introduce the invariants

$$II^*_{\mathbf{S}} = \sqrt{\frac{1}{2}\ \mathrm{tr}\ (\mathbf{S}^{*\mathbf{2}})}, \qquad II^*_{\mathbf{D}} = \sqrt{\frac{1}{2}\ \mathrm{tr}\ (\mathbf{D}^{*\mathbf{2}})},$$

where $\mathbf{D}^* = \frac{1}{2}\left[\mathbf{L}^* + \mathbf{L}^{*\mathbf{T}}\right]$ and $\mathbf{L}^* = \nabla \boldsymbol{u}^*$. The Bingham-type model requires

$$\begin{cases} \mathbf{S}^* = \left(2\eta^* + \dfrac{\tau_o^*}{II^*_{\mathbf{D}}}\right)\mathbf{D}^* & \text{when} \quad II^*_{\mathbf{S}} \geq \tau_o^*, \\ \mathbf{S}^* = \mathbf{S}^*_{\text{solid}} & \text{when} \quad II^*_{\mathbf{S}} \leq \tau_o^*, \end{cases}$$

where η^* is the viscosity in the fluid region, τ_o^* is the critical stress and $\mathbf{S}^*_{\text{solid}}$ represents the generic constitutive tensor of the solid region (rigid, elastic, etc.). In the hypothesis of creeping flow the balance of linear momentum yields

$$\begin{cases} \rho^* g^* \sin\alpha - P^*_x + (S^*_{11})_x + (S^*_{12})_y = 0, \\ -\rho^* g^* \cos\alpha - P^*_y + (S^*_{12})_x + (S^*_{22})_y = 0, \end{cases} \tag{2.1}$$

where ρ^* is the density (assumed to be the same in the liquid and solid regions) and g^* is gravity. Mass balance is given by

$$u^*_{1x} + u^*_{2y} = 0. \tag{2.2}$$

Following [15] we rescale the problem by setting

$$x^* = L^* x, \quad x^*_a = L^* x_a, \quad y^* = \varepsilon L^* y, \quad u^*_1 = U^* u_1, \quad u^*_2 = \varepsilon U^* u_2, \tag{2.3}$$

$$\mathbf{S}^* = \left(\frac{\eta^* U^*}{H^*}\right)\mathbf{S}, \quad P^* = \left(\frac{\eta^* U^* L^*}{H^{*2}}\right)P, \qquad \mathbf{D}^* = \left(\frac{U^*}{H^*}\right)\mathbf{D}, \tag{2.4}$$

$$II^*_{\mathbf{D}} = \left(\frac{U^*}{H^*}\right)II_{\mathbf{D}}, \quad II^*_{\mathbf{S}} = \left(\frac{\eta^* U^*}{H^*}\right)II_{\mathbf{S}}, \quad \sigma^* = H^*\sigma, \quad h^* = H^* h, \tag{2.5}$$

where U^* is the characteristic longitudinal velocity (see e.g. [1])

$$U^* = \frac{{H^*}^2 \rho^* g^* \sin\alpha}{\eta^*}. \tag{2.6}$$

We also introduce the Bingham number

$$\mathsf{Bi} = \frac{\tau_o^* H^*}{\eta^* U^*},$$

so that the dimensionless expression of the fluid stress is

$$\mathbf{S} = \left[2 + \frac{\mathsf{Bi}}{II_{\mathbf{D}}}\right]\mathbf{D}. \tag{2.7}$$

REMARK 2.1. Exploiting (2.6) the Reynolds number characterizing the flow is

$$\mathsf{Re} = \frac{{U^*}^2}{H^* g^* \sin\alpha}.$$

Hence the hypothesis of creeping flow, $\mathsf{Re} \ll 1$, yields $U^* \ll \sqrt{H^* g^* \sin\alpha}$; i.e. the characteristic velocity is by far smaller than the velocity of a free falling body on the inclined plane.

REMARK 2.2. It is easy to check that

$$\underbrace{\left(\frac{\eta^* U^*}{H^*}\right)}_{\text{Characteristic viscous stress}} = \underbrace{\left(\frac{\eta^* U^* L^*}{H^{*2}}\right)}_{\text{Characteristic pressure}} \varepsilon,$$

indicating that the viscous characteristic stress is much smaller than the pressure drop (i.e. the motion is essentially driven by gravity).

2.1. *Nondimensional formulation.* Adopting the scaling introduced in the previous section, equations (2.1), (2.2) become

$$\begin{cases} 1 - P_x + \varepsilon(S_{11})_x + (S_{12})_y = 0, \\ \\ -\theta - P_y + \varepsilon^2(S_{12})_x + \varepsilon(S_{22})_y = 0, \end{cases} \tag{2.8}$$

$$u_{1x} + u_{2y} = 0, \tag{2.9}$$

where

$$\theta = \frac{\varepsilon}{\tan\alpha}.$$

We have

$$II_{\mathbf{D}} = \sqrt{\varepsilon^2 u_{1x}^2 + \frac{1}{4}\left(u_{1y} + \varepsilon^2 u_{2x}\right)^2}, \qquad II_{\mathbf{S}} = \sqrt{S_{12}^2 + \frac{1}{2}\left(S_{11}^2 + S_{22}^2\right)} \tag{2.10}$$

and

$$\mathbf{D}=\begin{bmatrix} \varepsilon u_{1x} & \frac{1}{2}\left(u_{1y}+\varepsilon^2 u_{2x}\right) \\ \frac{1}{2}\left(u_{1y}+\varepsilon^2 u_{2x}\right) & \varepsilon u_{2y} \end{bmatrix}.$$

2.2. *Boundary conditions.* We write the boundary conditions on $y=\sigma$, $y=h$, and on $y=0$ without entering into the details of their derivation (we refer the reader to [14] and [15]).

On σ we assume:

- continuity of the velocity

$$[\![u_1]\!] = [\![u_2]\!] = 0; \tag{2.11}$$

- Von Mises condition $II_{\mathbf{S}} = \mathsf{Bi}$;
- continuity of the normal and tangential stress

$$\begin{cases} -[\![P]\!](1+\varepsilon^2\sigma_x^2) - 2\varepsilon^2 [\![S_{12}]\!]\sigma_x + \varepsilon^2 [\![S_{11}]\!]\sigma_x^2 + \varepsilon [\![S_{22}]\!] = 0, \\ [\![S_{12}]\!](1-\varepsilon^2\sigma_x^2) - \varepsilon h_x \left([\![S_{22}]\!] - [\![S_{11}]\!]\right) = 0. \end{cases} \tag{2.12}$$

On $y=h$ we impose:

- zero stress (i.e. free surface condition)

$$\begin{cases} - P \,(1+\varepsilon^2 h_x^2) - 2\varepsilon^2 \, S_{12} \, h_x + \varepsilon^2 \, S_{11} \, h_x^2 + \varepsilon \, S_{22} \, = 0, \\ S_{12}(1-\varepsilon^2 h_x^2) - \varepsilon h_x \left(\, S_{22} \, - \, S_{11} \, \right) = 0; \end{cases} \tag{2.13}$$

- material surface condition

$$h_t + u_1 h_x = u_2, \tag{2.14}$$

where both u_1, and u_2 are evaluated on $y=h$.

On $y=0$, we assume

- $\boldsymbol{u}(x,0,t)=0$, i.e. no slip and impermeable wall.

Exploiting (2.9) and the fact that $u_2\,(x,0,t)=0$, equation (2.14) can be rewritten as

$$h_t + \chi_x = 0, \tag{2.15}$$

with $\chi = \int_0^h u_1 dy$ referred to as the local instantaneous flow rate.

Remark 2.3. On the free surface the zero stress condition entails $\left(\frac{1}{\varepsilon}P\mathbf{I}+\mathbf{S}\right)\boldsymbol{n}_h = 0$, where $\boldsymbol{n}_h$ is the outward normal and P, $\mathbf{S}$, are evaluated on h. So, wherever the surface is regular, $\det\left(\frac{1}{\varepsilon}P\mathbf{I}+\mathbf{S}\right)=0$. The latter entails

$$0 = \det\left(\frac{1}{\varepsilon}P\mathbf{I}+\mathbf{S}\right) = \det(\mathbf{S}) - \underbrace{\mathrm{tr}\,(\mathbf{S})}_{=0}\frac{P}{\varepsilon} + \left(\frac{P}{\varepsilon}\right)^2 \quad\Rightarrow\quad \det\mathbf{S} = -\left(\frac{P}{\varepsilon}\right)^2,$$

namely $S_{12}^2 - S_{11}S_{22} = -\left(\frac{P}{\varepsilon}\right)^2$. Hence, recalling (2.10), we get

$$2II_{\mathbf{S}}^2 = 2S_{12}^2 + S_{11}^2 + S_{22}^2 = \underbrace{(S_{11}+S_{22})^2}_{(\mathrm{tr}\mathbf{S})^2=0} - 2\left(\frac{P}{\varepsilon}\right)^2,$$

that is, $|\,P|_h\,| = \varepsilon\, II_{\mathbf{S}}|_h$.

3. Zero order approximation. We now focus on the zero order approximation considering the case $\theta = \mathcal{O}(1)$, which corresponds to a "small" tilt angle. In this case we get

$$\begin{cases} (S_{12}^{(0)})_y = P_x^{(0)} - 1, \\ P_y^{(0)} + \theta = 0, \end{cases} \qquad \begin{cases} P^{(0)}|_h = 0, \\ (S_{12}^{(0)})|_h = 0, \end{cases} \qquad \begin{cases} [\![\, P^{(0)} \,]\!]_\sigma = 0, \\ [\![\, S_{12}^{(0)} \,]\!]_\sigma = 0. \end{cases} \tag{3.1}$$

Integration yields

$$S_{12}^{(0)} = (\theta h_x - 1)(y-h), \tag{3.2}$$

so that, exploiting the Von Mises criterion, we obtain

$$\mathsf{Bi} = (\theta h_x - 1)(\sigma - h). \tag{3.3}$$

Moreover, since at the zero order $S_{12}^{(0)} = u_{1y}^{(0)} + \mathsf{Bi}$, we get

$$u_1^{(0)}(x,y,t) = (\theta h_x - 1)\left[\frac{(y-\sigma)^2}{2} - \frac{\sigma^2}{2}\right]. \tag{3.4}$$

The velocity $u_1^{(0)}$ in the solid region clearly depends on the mathematical structure of the tensor $\mathbf{S}$ in the solid domain. The kinematical condition (2.15) yields

$$h_t + \underbrace{\left[\int_0^\sigma u_1^{(0)} dy + \int_\sigma^h u_{1_{\text{solid}}}^{(0)} dy\right]_x}_{\chi} = 0,$$

where $u_{1_{\text{solid}}}^{(0)}$ is the longitudinal component of the velocity in the solid region. Easy calculations show that the above reduces to

$$h_t - \left[(\theta h_x - 1)\frac{\sigma^3}{3}\right]_x = -\left[\int_\sigma^h u_{1_{\text{solid}}}^{(0)} dy\right]_x. \tag{3.5}$$

3.1. *Constraints on the derivatives of* $u_1^{(0)}$. The first constraint is simply $u_{1y}^{(0)}|_\sigma = 0$ (see formula (3.4)). The second is $u_{1x}^{(0)}|_\sigma = 0$. Indeed we can prove the following.

PROPOSITION 1. If the mathematical problem (3.1) admits a solution and all derivatives are bounded on σ, then $u_{1x}^{(0)}|_\sigma = 0$.

Proof. Recalling (2.10) and recalling that $u_{1y}^{(0)} = 0$ on $y = \sigma$ we have

$$II_{\mathbf{D}}|_\sigma = \varepsilon\sqrt{\left[(u_{1x}^{(0)})^2 + \frac{1}{4}(u_{1y}^{(1)})^2\right]} + \mathcal{O}(\varepsilon^2).$$

The nondimensional constitutive equation in the viscous domain is

$$II_{\mathbf{D}}\mathbf{S} - (2II_{\mathbf{D}} + \mathsf{Bi})\mathbf{D} = 0, \tag{3.6}$$

which holds for $II_{\mathbf{S}} \geq \mathsf{Bi}$. Multiplying (3.6) by $II_{\mathbf{D}}$ we get

$$II^2_{\mathbf{D}}\mathbf{S} = (2II^2_{\mathbf{D}} +^2 \mathsf{Bi} II_{\mathbf{D}})\mathbf{D}. \tag{3.7}$$

Focusing on the component $S_{12}|_{y=\sigma}$ and recalling that $S_{12}|_{y=\sigma} = \mathsf{Bi} + \mathcal{O}(\varepsilon)$, we get (on $y = \sigma$)

$$\left\{\varepsilon^2\left[(u^{(0)}_{1x})^2 + \frac{1}{4}(u^{(1)}_{1y})^2\right] + \mathcal{O}(\varepsilon^3)\right\}\cdot\{\mathsf{Bi} + \mathcal{O}(\varepsilon)\} = \frac{1}{2}\left\{\varepsilon u^{(1)}_{1y} + \mathcal{O}(\varepsilon^2)\right\}$$

$$\cdot\left\{2\varepsilon^2\left[(u^{(0)}_{1x})^2 + \frac{1}{4}(u^{(1)}_{1y})^2\right] + \mathcal{O}(\varepsilon^3) + \mathsf{Bi}\varepsilon\sqrt{\left[(u^{(0)}_{1x})^2 + \frac{1}{4}(u^{(1)}_{1y})^2\right] + \mathcal{O}(\varepsilon^2)}\right\}.$$

At the leading order (that is at order ε^2) we obtain

$$\left[(u^{(0)}_{1x})^2 + \frac{1}{4}(u^{(1)}_{1y})^2\right] = \frac{1}{2}u^{(1)}_{1y}\sqrt{\left[(u^{(0)}_{1x})^2 + \frac{1}{4}(u^{(1)}_{1y})^2\right]}, \quad \text{on} \quad y = \sigma.$$

The above holds true if, on $y = \sigma$, $\left[(u^{(0)}_{1x})^2 + \frac{1}{4}(u^{(1)}_{1y})^2\right] = 0$ or if $u^{(1)}_{1y} = 0$. In both cases we end up with $u^{(0)}_{1x}\Big|_{\sigma} = 0$. □

Now, by (3.4), Proposition 1 implies

$$\left[(\theta h_x - 1)\sigma^2\right]_x = 0. \tag{3.8}$$

4. The mathematical problem. The mathematical problem for h and σ is now given by (3.3), (3.5), (3.8); that is,

$$\begin{cases} \mathsf{Bi} = (\theta h_x - 1)(\sigma - h), & -\infty < x < x_a(t), \quad t > 0, \\ h_t - \left[(\theta h_x - 1)\dfrac{\sigma^3}{3}\right]_x = -\left[\int_\sigma^h u^{(0)}_{1_{\text{solid}}}dy\right]_x, & -\infty < x < x_a(t), \quad t > 0, \\ \left[(\theta h_x - 1)\sigma^2\right]_x = 0, & -\infty < x < x_a(t), \quad t > 0, \end{cases} \tag{4.1}$$

where the function $x_a(t)$ is unknown. We consider the "boundary conditions"(as depicted in Figure 1)

$$\lim_{x\to\to-\infty} h\,(x,t) = h_\infty, \qquad \lim_{x\to\to-\infty} \sigma\,(x,t) = \sigma_\infty, \tag{4.2}$$

such that

$$h_\infty - \sigma_\infty = \mathsf{Bi}\ .$$

The above conditions correspond, from the physical point of view, to an unperturbed reservoir aloof from the advancing front. Hence the mathematical problem consists of finding three functions $h(x,t)$, $\sigma(x,t)$ and $x_a(t)$ fulfilling (4.1) and (4.2). Actually we show that in the vicinity of the advancing front $x_a(t)$, system (4.1), (4.2) leads to a mathematical paradox. From $(4.1)_3$

$$(\theta h_x - 1)\sigma^2 = -f^2(t), \tag{4.3}$$

where $f^2(t)$ can be evaluated letting $x \to -\infty$, i.e. $f^2(t) = \sigma_\infty^2$. Hence

$$f^2(t) = (h_\infty - \mathsf{Bi})^2 = \sigma_\infty^2 \qquad \underset{(4.3)}{\Rightarrow} \qquad \theta h_x - 1 = -\frac{(h_\infty - \mathsf{Bi})^2}{\sigma^2}. \tag{4.4}$$

Before proceeding further we prove the following.

PROPOSITION 2. If (4.3) holds up to $x = x_a(t)$, then $\lim_{x\to x_a(t)} |u_{1y}^{(0)}(x,0)| = \infty$, with $|u_{1y}^{(0)}|$ integrable in a neighborhood of $(x_a(t), 0)$.

Proof. From (3.4) we have $u_{1y}^{(0)}|_{y=0} = -\sigma(\theta h_x - 1)$. Then, exploiting (4.4),

$$(\mathsf{Bi} - h_\infty)^2 = \sigma u_{1y}^{(0)}|_{y=0}, \quad \forall\, x < x_a(t).$$

Therefore, since $\lim_{x\to x_a(t)} \sigma = 0$, we obtain $\lim_{x\to x_a(t)} u_{1y}^{(0)}|_{y=0} = +\infty$, which means that $S_{12}^{(0)}$ is unbounded as $x \to x_a(t)$. If, on the other hand, we consider

$$\begin{aligned} \int_0^{x_a(t)} \int_0^{\sigma(x,t)} S_{12}^{(0)} dy\, dx &= \int_0^{x_a(t)} \left(\int_0^{\sigma(x,t)} u_{1y}^{(0)}(\xi, y, t) dy \right) dx \\ &= -\int_0^{x_a(t)} (\theta h_x - 1) \frac{\sigma^2}{2} d\xi \underset{(4.4)}{=} \frac{(\mathsf{Bi} - h_\infty)^2}{2} x_a(t), \end{aligned}$$

we find that the stress (and hence the dissipation) is integrable in the fluid domain. ☐

Combining (4.3) and $(4.1)_1$ one gets

$$h = \sigma + \frac{\mathsf{Bi}}{f^2}\sigma^2, \tag{4.5}$$

where $f^2 = \sigma_\infty^2$. Differentiating the above w.r.t. x, one can easily find that

$$\theta h_x - 1 = \theta \sigma_x \left(1 + 2\sigma \frac{\mathsf{Bi}}{f^2}\right) - 1,$$

which plugged into (4.3) yields

$$\sigma_x \frac{2\mathsf{Bi}\sigma^3 + \sigma^2 f^2}{\sigma^2 - f^2} = \frac{f^2}{\theta}$$

or equivalently

$$\sigma_x \left[2\sigma\mathsf{Bi} + f^2 + \frac{2\mathsf{Bi}\sigma f^2 + f^4}{\sigma^2 - f^2} \right] = \frac{f^2}{\theta}. \tag{4.6}$$

Integration of (4.6) between x_a and x gives

$$\frac{\sigma^2 \mathsf{Bi}}{f^2} + \sigma + \mathsf{Bi} \ln \left| \left(\frac{\sigma}{f}\right)^2 - 1 \right| - \frac{f}{2} \ln \left| \frac{\sigma}{f} - 1 \right| + \frac{f}{2} \ln \left| \frac{\sigma}{f} + 1 \right| = \frac{x - x_a(t)}{\theta}, \tag{4.7}$$

where we have exploited the fact that $\sigma(x_a(t), t) = 0$ (see Figure 1). In a neighborhood of the advancing front $x = x_a(t)$, $\sigma \ll 1$, so that we can expand the logarithmic terms in (4.7) getting

$$\sigma \approx \frac{x - x_a(t)}{\theta},$$

and because of (4.5)

$$h \approx \frac{x - x_a(t)}{\theta} + \frac{\mathsf{Bi}}{f^2}\left(\frac{x - x_a(t)}{\theta}\right)^2.$$

Therefore

$$(\theta h_x - 1) \approx \frac{2\mathsf{Bi}}{f^2\theta}(x - x_a(t)), \qquad (\sigma - h) = -\frac{\mathsf{Bi}}{f^2}\left[\frac{x - x_a(t)}{\theta}\right]^2,$$

yielding

$$-\frac{2\mathsf{Bi}^2}{f^4}\left[\frac{x - x_a(t)}{\theta}\right]^3 = \mathsf{Bi},$$

which is an obvious contradiction. Hence the model, independently of the constitutive relation of the solid region, breaks down.

This result seems to suggest that the reason for the paradox has to be sought in the scaling, which could be incorrect in the vicinity of the wetting front. Indeed, considering L^* as the characteristic longitudinal length entails that the free surface h^* decreases from h^*_∞ to 0 in a space interval whose length is $\mathcal{O}(L^*)$. In the next section, we remove this hypothesis and consider, in a neighborhood of the advancing front, a different length scale.

5. A different scaling approach. In this section we assume that the stress free surface h^* goes to zero in an interval whose depth is of the order of δ^*, with $\delta^* \ll H^*$. In practice we assume that $h^*(x^*, t^*)$ falls abruptly to the bottom surface in a $\mathcal{O}(\delta^*)$ neighborhood of $x^*_a(t^*)$ (see Figure 2). In the proximity of $(x^*_a, 0)$ we rescale the system as in (2.3)-(2.6), with the exception of

$$x^* = \delta^* x, \qquad u_1^* = \varepsilon U^* u_1, \qquad u_2^* = U^* u_2,$$

and where we assume $\varepsilon = \dfrac{\delta^*}{H^*}$.

Remark 5.1. The scaling for the velocity comes from the following observation: the time taken by a particle to reach the bottom surface is H^*/U^*. At the same time the particle moves in the longitudinal direction only up to a distance $\mathcal{O}(\delta^*)$. Hence, if the characteristic velocity along y^* is U^*, then the velocity along x^* must be εU^*.

It is easy to check that with this new scaling problem (2.1)-(2.2) can be rewritten as

$$\begin{cases} \varepsilon - P_x + \varepsilon(S_{11})_x + \varepsilon(S_{12})_y = 0, \\ -\theta - \varepsilon P_y + (S_{12})_x + \varepsilon(S_{22})_y = 0, \end{cases} \tag{5.1}$$

$$u_{1x} + u_{1y} = 0. \tag{5.2}$$

The deviatoric part of the fluid Cauchy stress is still given by (2.7), with

$$\mathbf{D} = \begin{bmatrix} \dfrac{u_{1x}}{\varepsilon} & \dfrac{1}{2}\left(u_{1y} + \dfrac{u_{2x}}{\varepsilon^2}\right) \\ \dfrac{1}{2}\left(u_{1y} + \dfrac{u_{2x}}{\varepsilon^2}\right) & \dfrac{u_{2y}}{\varepsilon} \end{bmatrix}$$

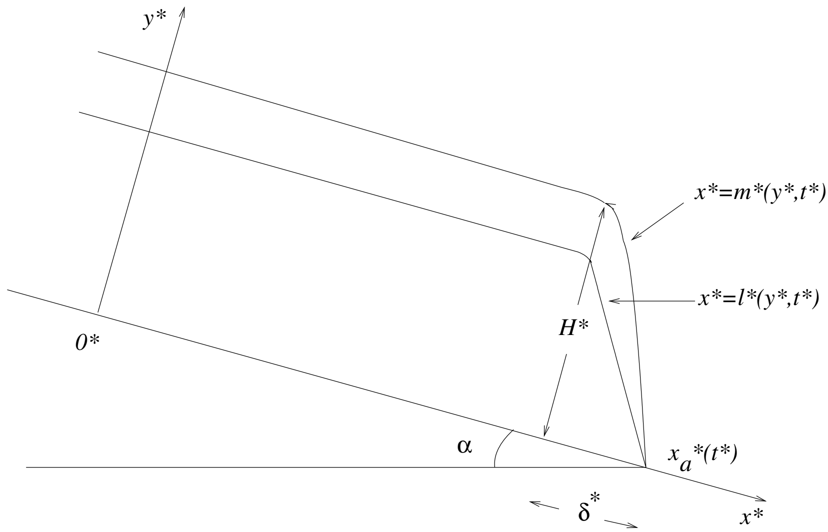

FIG. 2. Sketch of the system.

and

$$II_{\mathbf{D}} = \frac{1}{\varepsilon^2}\sqrt{\varepsilon^2 u_{1x}^2 + \frac{1}{4}\left(u_{1y}\varepsilon^2 + u_{2x}\right)^2}.$$

Hence, the stress components are

$$S_{11} = \left[2 + \frac{\mathsf{Bi}}{II_{\mathbf{D}}}\right]\frac{u_{1x}}{\varepsilon}, \qquad S_{22} = \left[2 + \frac{\mathsf{Bi}}{II_{\mathbf{D}}}\right]\frac{u_{2y}}{\varepsilon},$$

$$S_{12} = \frac{1}{2}\left[2 + \frac{\mathsf{Bi}}{II_{\mathbf{D}}}\right]\left(u_{1y} + \frac{u_{2x}}{\varepsilon^2}\right).$$

Next we consider the yield surface and the free surface[2] as functions of (y,t) as depicted in Figure 2. In practice we denote the yield surface and the free surface as $x^* = l^*(y^*, t^*)$ and $x^* = m^*(y^*, t^*)$, respectively, so that, in nondimensional variables, we can write $x = l(y,t)$ and $x = m(y,t)$. Proceeding as in Section 2.2 we get

$$[\![\, u_1 \,]\!] = [\![\, u_2 \,]\!] = 0, \tag{5.3}$$

$$\begin{cases} -[\![\, P \,]\!](1+\varepsilon^2 l_y^2) - 2\varepsilon l_y [\![\, S_{12} \,]\!] + \varepsilon^2 l_y^2 [\![\, S_{22} \,]\!] + [\![\, S_{11} \,]\!] = 0, \\ [\![\, S_{12} \,]\!](1-\varepsilon^2 l_y^2) - \varepsilon l_y \left([\![\, S_{22} \,]\!] - [\![\, S_{11} \,]\!]\right) = 0, \end{cases} \tag{5.4}$$

on $x = l(y,t)$, whereas on $x = m(y,t)$ we impose

$$\begin{cases} -\, P\,(1+\varepsilon^2 m_y^2) - 2\varepsilon m_y\, S_{12} \,+ \varepsilon^2 m_y^2\, S_{22} \,+ S_{11} = 0, \\ S_{12}(1-\varepsilon^2 m_y^2) + \varepsilon m_y\,(\, S_{11} \,-\, S_{22}\,) = 0. \end{cases} \tag{5.5}$$

The free surface $y = m\,(x,t)$ fulfills the usual kinematical condition.

[2]Recall that the yield surface and the free surface are unknown.

5.1. *The zero order approximation.* In the viscous region the zero order approximation of problem (5.1)-(5.2) yields

$$\begin{cases} u^{(0)}_{2xx} = 0, \\ \\ u^{(0)}_{2xy} = 0, \end{cases}$$

which, in turn, implies $u^{(0)}_{2x} = u^{(0)}_{2x}(t)$. From (5.4) we see that, at the leading order, $u^{(0)}_{2x}|_{x=l} = 0$ (assuming $\mathbf{S}$ bounded on the solid part), so that

$$u^{(0)}_2 = f(y,t), \tag{5.6}$$

with $f(y,t)$ unknown. From the continuity equation

$$u^{(0)}_1 = -f_y(y,t)x + g(y,t),$$

with $g(y,t)$ unknown. Now we observe that assuming the Von Mises criterion $II_{\mathbf{S}} = \mathsf{Bi}$ on $x = l$, we get to a contradiction. Indeed, from the constitutive relation

$$II_{\mathbf{D}} II_{\mathbf{S}} = (2II_{\mathbf{D}} + \mathsf{Bi}) II_{\mathbf{D}},$$

we find $II_{\mathbf{D}} = 0$ on $x = l$ and

$$II^2_{\mathbf{S}} = \left[S^2_{12} + \frac{1}{2}(S^2_{11} + S^2_{22}) \right].$$

Hence

$$II^2_{\mathbf{D}} II^2_{\mathbf{S}} = [\ 2II_{\mathbf{D}} + \mathsf{Bi}\ ]^2 \cdot \left\{ \frac{1}{2} \left[\left(\frac{u_{1x}}{\varepsilon} \right)^2 + \left(\frac{u_{2y}}{\varepsilon} \right)^2 \right] + \frac{1}{4\varepsilon^4} \left[\varepsilon^2 u_{1y} + u_{2x} \right]^2 \right\}.$$

Therefore, on $x = l$, we get $II_{\mathbf{D}} = 0$ and $II_{\mathbf{S}} = \mathsf{Bi}$, so that

$$\left\{ \frac{1}{2} \left[\left(\frac{u_{1x}}{\varepsilon} \right)^2 + \left(\frac{u_{2y}}{\varepsilon} \right)^2 \right] + \frac{1}{4\varepsilon^4} \left[\varepsilon^2 u_{1y} + u_{2x} \right]^2 \right\} = 0,$$

entailing $u_{1x} = u_{2y} = 0$ and $\left[\varepsilon^2 u_{1y} + u_{2x} \right] = 0$. Hence $u^{(0)}_{2x} = 0$ (which we have already found in (5.6)) and $u^{(0)}_{1x} = u^{(0)}_{2y} = 0$, on $x = l$. This clearly means that $f_y(y,t) \equiv 0$ and

$$\begin{cases} u^{(0)}_2 = f(t), \\ \\ u^{(0)}_1 = g(y,t). \end{cases}$$

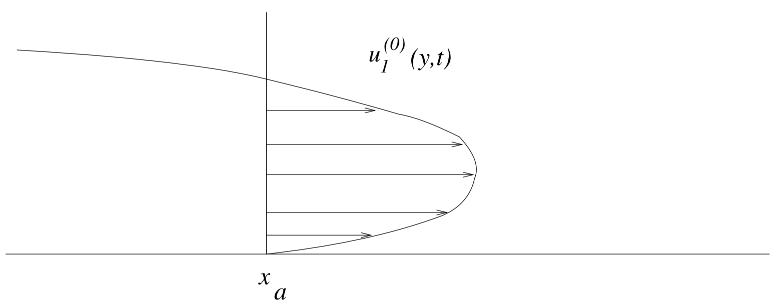

FIG. 3. Material flow.

The no-slip condition implies $f(t) = 0$ so that $u_2^{(0)} = 0$ and $u_1^{(0)} = g(y,t)$, with $g(0,t) = 0$. This last result is an obvious contradiction, since it would imply that the contact line does not move and the flow could be like the one depicted in Figure 3.

6. Conclusions. We have studied the flow of a Bingham-type material down an inclined plane in the lubrication approximation, analyzing the case $\tan\alpha = \mathcal{O}(\varepsilon)$. We have formulated a mathematical problem for the three unknowns of the problem: the free surface $h(x,t)$, the yield surface $\sigma(x,t)$, and the position of the advancing front $x_a(t)$. The problem, though seemingly solvable (three equations for three unknowns), gives rise to an evident paradox: the advancing front cannot move. The same result is obtained also considering a thin layer located on the wetting front and rescaling x, y variables accordingly.

We believe that the main inconsistency of the problem does not lie in the diverging stress on $x_a(t)$ (a well known feature of many triple point models), but in the continuity of the stress across the yield surface. Indeed, we show that in a neighborhood of x_a, the stress is integrable (and thus dissipation is bounded).

Interestingly the analysis developed is independent from the specific constitutive model of the solid region. In a certain sense, we could say that the mathematical paradox originates from yield surface, i.e. from the "abrupt switch" in the constitutive model. We believe that the inconsistencies can be overcome by adopting models such as Papanastasiou (see e.g. [18]), since they do not have a clear separation between the domains.

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