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GLOBAL EXISTENCE AND ENERGY DECAY OF A NONDISSIPATIVE CAUCHY VISCOELASTIC PROBLEM

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Abstract. A viscoelastic Cauchy problem subjected to a nonlinear source term is investigated. The memory term in the system involves a kernel which is regular, as is usually the case, but the system is not dissipative and is considered in the whole space. We prove global existence and nonexistence results. In the case of global existence, we show that solutions go to zero in a polynomial manner as time goes to infinity under some conditions on the source.

1. Introduction. In this work we are concerned with the following Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = h(t)|u|^{p-2}u, \quad x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where u_0, u_1 are initial data, g is the memory kernel and $h(t)$ is a positive nonincreasing function to be specified later.

This type of evolution equation arises in the study of strain solitary waves [1,2] and in the theory of viscoelasticity (see [3,4]). The memory term, represented by the convolution term in the equation, expresses the fact that the stress at any instant t depends on the past history of strains which the material has undergone from time 0 up to t. It is known that this term causes a kind of damping. But this damping is not strong enough to compete with the source term. For that reason, it is essential to have a function $h(t)$ that annihilates that source or reduces its effect. Taking this role of $h(t)$ into account makes the problem more challenging, as the whole system will not necessarily be dissipative.

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The wave equation in $\mathbb{R}^n$ with a weak damping $-u_t$, a strong damping Δu_t or a nonlinear damping $h(u_t)$ has been investigated by a number of researchers (see [5]–[10]). Many results on the asymptotic behavior of solutions exist in the literature. In contrast, the same equation with the much weaker viscoelastic damping is not well-studied.

In [11], Hrusa and Nohel treated the one-dimensional nonlinear viscoelastic equation

$$u_{tt} = (\phi(u_x(x,t)))_x - \int_0^t a'(t-s)\,(\psi(u_x(x,s)))_x\,ds, \quad x \in \mathbb{R},\ t > 0, \tag{2}$$

and proved, under reasonable conditions on ϕ, ψ and a smallness condition on the initial data, the existence of a unique global classical solution. They also established an asymptotic result, but no rate of decay was given. Dassios and Zafiropoulus [12] showed, for the same kernel, that the decay is of order $t^{-3/2}$ if the material is occupying the whole space $\mathbb{R}^3$. Rivera [13] extended the result of Dassios and Zafiropoulus to $\mathbb{R}^n$. Precisely, he showed that if the kernel is decaying exponentially, then the solution decays exponentially for a material occupying bounded domains, whereas the decay is of the order $t^{-n/2}$ for material occupying the whole n-dimensional space.

For nonexistence and formation of singularities, we quote the early work of Dafermos [14] in 1985. Kafini and Messaoudi in [15] considered the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(x,s)ds + u_t = |u|^{p-2}u, \quad x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{3}$$

and showed that if the initial energy is negative and

$$\int_0^\infty g(s)ds < \frac{2p-4}{2p-3}, \qquad \int_{\mathbb{R}^n} u_0 u_1 dx \geq 0,$$

then the solution blows up in finite time. In [16], the same authors proved a blow-up result for a coupled system of the form

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(x,s)ds = f_1(u,v), \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ v_{tt} - \Delta v + \displaystyle\int_0^t h(t-s)\Delta v(x,s)ds = f_2(u,v), \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{4}$$

In the absence of the source term, the same authors [17] established a decay result for a viscoelastic Cauchy problem. They showed that for compactly supported initial data and for an exponentially decaying relaxation function, the decay of the first energy of solution is polynomial. The finite-speed propagation is used to compensate for the lack of Poincaré's inequality in $\mathbb{R}^n$.

Assumptions on g like

$$1-\int_0^\infty g(s)ds = l > 0 \tag{5}$$

and, for $a > 0$,

$$g'(t) \le -ag^p(t), \quad 1 \le p < 3/2, \qquad t \ge 0, \tag{6}$$

were encountered in most of the earlier works on linear viscoelastic wave equations on bounded domains (see [18]–[20]). Lately, some papers have appeared with alternative conditions (see [21]–[23]). For instance, Furati and Tatar [21] proved that relaxation functions g such that $g(t)e^{\alpha t}$ and $g'(t)e^{\alpha t}$ have small L^1-norms also lead to an exponential decay. Conditions like (6) were not imposed. In particular, g' is not necessarily always negative. Messaoudi and Tatar [24] improved some earlier results concerning the exponential decay. They showed that the weak dissipation induced by the convolution term is enough to drive the system to rest with an exponential rate. Precisely, they established their result under the conditions

$$g'(t) \le 0 \quad \text{and} \quad \int_0^\infty g(t)e^{\alpha t}dt < +\infty \text{ for some large } \alpha > 0. \tag{7}$$

In [25], Tatar proved an exponential decay of solutions of a problem in a bounded domain for singular kernel of the form

$$g(t) = \frac{t^{-a}e^{-bt}}{\Gamma(1-a)}, \quad b > 0, \quad 0 < a < 1, \quad t > 0.$$

For more results, we refer to Hursa and Renardy [26]-[27].

Also, in [28], Aassila et al. considered a nonlinear model of the wave equation subject to an abstract nonlinear damping and memory source term acting in the boundary and showed the existence of solutions by means of the Faedo-Galerkin method and obtained the uniform decay by using the multiplier technique. In [29], Cavalcanti and Guesmia obtained that general decay rates of solutions to a nonlinear wave equation with boundary condition of memory type depend on the relaxation function. In particular, if the relaxation function decays exponentially (or polynomially), then the solution also decays exponentially (or polynomially) and with the same decay rate. In [30], Aassila and Cavalcanti proved the existence, uniqueness and uniform decay of strong and weak solutions of the nonlinear model of the wave equation

$$u_{tt} - \Delta u + f(u) + h(\nabla u) = 0$$

in bounded domains with nonlinear dissipative boundary conditions given by

$$\frac{\partial u}{\partial v} + g(ut) = 0.$$

The existence is proved by means of the Faedo-Galerkin method, and the asymptotic behavior is obtained making use of the multiplier technique due to Komornik and Zuazua.

All these works and others have been established for bounded domains. The literature is, however, very poor in studying unbounded domains.

As we introduced earlier, our aim in this paper is to determine a type of $h(t)$ and to study global existence and nonexistence of a nondissipative Cauchy viscoelastic problem (1). Then we aim to establish a polynomial decay rate of the energy of solutions. So our domain is the whole space or $\Omega = \mathbb{R}^n$ and a nonlinear source with time-dependent coefficient. Unlike the latter work, Poincaré's inequality, as well as some embedding inequalities, is no longer valid in the present case. We will exploit the nature of the wave propagation to overcome part of the encountered difficulties. Other difficulties caused by the domain and the nonlinearity are overcome by selecting new Lyapunov type functionals.

This paper is organized as follows. In Section 2, we present some material needed for the proof of our result. In Section 3, we show the global existence and nonexistence results. Section 4 is devoted to the statement and proof of the main result.

2. Preliminaries.

In this section we present some material needed for the proof of our result.

LEMMA 2.1 (Sobolev, Gagliardo, Nirenberg). Suppose that $1 \le p < n$. If $u \in W^{1,p}(\mathbb{R}^n)$, then $u \in L^{p*}(\mathbb{R}^n)$, with

$$\frac{1}{p*} = \frac{1}{p} - \frac{1}{n}.$$

Moreover there exists a constant $c = c(n,p)$ such that

$$||u||_{p^*} \le c||\nabla u||_p, \quad \forall u \in W^{1,p}(\mathbb{R}^n).$$

LEMMA 2.2 ([31]). Let $2 \le p \le \frac{2n}{n-2}$ if $n \ge 3$ and $2 \le p \le +\infty$ if $n = 1, 2$. Then there exists a positive constant C_* depending on n and p only such that

$$\|u\|_p \le C_* \|u\|_{H^1(\mathbb{R}^n)}, \quad \forall u \in H^1(\mathbb{R}^n).$$

LEMMA 2.3. If u is the solution of (1), then

$$||u(t)||_2 \le C(L+t)||\nabla u(t)||_2, \quad t \ge 0, \tag{8}$$

for some $C > 0$ and $L > 0$ where

$$\text{supp}\{u_0(x), u_1(x)\} \subset B(L) = \{x \in \mathbb{R}^n \mid |x| < L\}.$$

Proof. Using Lemma 2.1, with $p = 2$, we find

$$||u||_{p^*} \le C||\nabla u||_2, \qquad p^* = \frac{2n}{n-2} \quad \text{if } n \ge 3.$$

An application of the finite-speed propagation property and Hölder inequality yields

$$\begin{aligned}
\int_{\mathbb{R}^n} |u|^2 \, dx &= \int_{B(L+t)} |u|^2 \, dx \\
&\le \left(\int_{B(L+t)} 1 dx \right)^{1-\frac{2}{p*}} \left(\int_{B(L+t)} (|u|^2)^{\frac{p^*}{2}} dx \right)^{\frac{2}{p*}} \\
&\le C(L+t)^2 ||u(t)||_{p^*}^2.
\end{aligned}$$

Hence,

$$||u(t)||_2 \leq C(L+t)||u(t)||_{p^*} \leq C(L+t)||\nabla u(t)||_2. \qquad \square$$

REMARK. Without loss of generality, we will take $L = 1$ in the forthcoming estimations.

The following assumptions are to be imposed on g and h.

(H1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies, for some $a > 0$,

$$1 - \int_0^\infty g(s)ds = l > 0, \qquad g'(t) \leq -ag(t), \quad t \geq 0. \tag{9}$$

(H2) $h(t)$ is a positive nonincreasing differentiable function which satisfies, for some $\lambda > p$,

$$h(t) \leq (1+t)^{-\lambda} \quad \text{and} \quad |h'(s)| \leq \lambda (1+t)^{-1} h(t), \quad t \geq 0.$$

The following proposition is the local existence result which can be found in earlier works such as [7–10].

PROPOSITION 2.4. Let $2 < p \leq \frac{2(n-1)}{n-2}$ if $n \geq 3$ and $2 < p \leq +\infty$ if $n = 1, 2$. Assume that (H1) and (H2) hold and $u_0, u_1 \in H^1(\mathbb{R}^n)$ are of compact support. Then problem (1) possesses a unique local weak solution such that

$$u, u_t \in C(0, T_m; H^1(\mathbb{R}^n)),$$

for some $T_m > 0$ small enough.

LEMMA 2.5 (Gronwall-Bellman [32]). Let a, b, K, ψ be nonnegative continuous functions on $I = (0, T)(0 < T \leq \infty)$; let $\omega : (0, \infty) \to \mathbf{R}$ be a positive, nondecreasing continuous function with $\omega(0) = 0$ and $\omega(u) > 0$ for $u > 0$; and let $A(t) := \max_{0 \leq s \leq t} a(s)$, $B(t) := \max_{0 \leq s \leq t} b(s)$. Assume that

$$\psi(t) \leq a(t) + b(t) \int_0^t K(s)\omega(\psi(s))ds, \quad t \in I.$$

Then

$$\psi(t) \leq H^{-1} \left[H(A(t)) + B(t) \int_0^t K(s)ds \right], \quad t \in (0, T_1),$$

where $H(v) := \int_{v_0}^v \frac{d\tau}{\omega(\tau)} (v \geq v_0 > 0)$, H^{-1} is the inverse of H and $T_1 > 0$ is such that $H(A(t)) + B(t) \int_0^t K(s)ds \in D(H^{-1})$ for all $t \in (0, T_1)$.

At the end of this section, we introduce the functionals

$$I(t) = \left(1 - \int_0^t g(s)ds \right) \|\nabla u\|_2^2 - h(t) \|u\|_p^p,$$

$$J(t) = \frac{1}{2} \left(1 - \int_0^t g(s)ds \right) \|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u) - \frac{1}{p} h(t) \|u\|_p^p,$$

and the energy functional

$$E(t) = J(t) + \frac{1}{2} \|u_t\|_2^2,$$

where

$$(g \circ v)(t) = \int_0^t g(t-s) \int_{\mathbb{R}^n} |v(s) - v(t)|^2 \, dxds, \quad \forall v \in L^2(\mathbb{R}^n).$$

3. Global existence of solutions. In this section, we show that the solution of (1) is uniformly bounded and global in time.

LEMMA 3.1. If u is a solution of (1), then the energy satisfies

$$E'(t) = \frac{1}{2}(g' \circ \nabla u) - \frac{1}{2} g(t) \|\nabla u\|_2^2 - h'(t) \|u\|_p^p, \quad t \geq 0. \tag{10}$$

Proof. This follows by a straightforward computation. □

Note here $h'(t) \leq 0$ and hence $E'(t)$ is not necessarily nonpositive. This means that the energy is not necessarily decreasing in t at this stage. In the following lemma, we show that the energy is uniformly bounded.

Because of the time-dependent coefficient, some difficulties arose here making the existing arguments in the literature impossible to carry out verbatim. In particular, we have appealed to the Gronwall-Bellman inequality in addition to some estimates and extra conditions to overcome these difficulties.

LEMMA 3.2. Suppose that (H1) and (H2) hold and let $2 < p \leq \frac{2(n-1)}{n-2}$ if $n \geq 3$ and $2 < p \leq +\infty$ if $n = 1, 2$. Assume further that $u_0, u_1 \in H^1(\mathbb{R}^n)$, with compact support, and they satisfy

$$\begin{cases} \beta = \left(\frac{2p}{p-2}\right)^{\frac{p-2}{2}} \left[\dfrac{2^p C_*^p}{l^{p/2}\left((E(0))^{1-p/2} - \frac{2^p C_*^p \lambda}{\lambda - p}\left(\frac{2p}{l(p-2)}\right)^{p/2}\right)}\right] < 1, \\ I(0) > 0, \end{cases}$$

for then $I(t) > 0$, for each $t \in [0, T_m)$.

Proof. Since $I(0) > 0$, by continuity, there exists $T_* \leq T_m$ such that $I(t) \geq 0$ for all $t \in [0, T_*)$. This implies that, for all $t \in [0, T_*)$,

$$\begin{aligned} J(t) &= \frac{1}{2}\left(1 - \int_0^t g(s)ds\right) \|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u) - \frac{h(t)}{p} \|u\|_p^p \\ &\geq \frac{p-2}{2p}\left[\left(1 - \int_0^t g(s)ds\right) \|\nabla u\|_2^2 + (g \circ \nabla u)\right] + \frac{1}{p} I(t) \\ &\geq \frac{p-2}{2p}\left[\left(1 - \int_0^t g(s)ds\right) \|\nabla u\|_2^2 + (g \circ \nabla u)\right]. \end{aligned}$$

Hence

$$l \|\nabla u\|_2^2 \leq \left(1 - \int_0^t g(s)ds\right) \|\nabla u\|_2^2 \leq \frac{2p}{p-2} J(t) \leq \frac{2p}{p-2} E(t), \quad \forall t \in [0, T_*).$$

Using Lemma 2.2, Lemma 2.3 and (H2), we estimate

$$\begin{aligned} h(t)\|u\|_p^p &\le C_*^p h(t)\|u\|_{H^1(\mathbb{R}^n)}^p \le 2^{p-1}C_*^p h(t)\left(\|u\|_2^p+\|\nabla u\|_2^p\right)\\ &\le 2^{p-1}C_*^p h(t)\left((1+t)^p\|\nabla u\|_2^p+\|\nabla u\|_2^p\right)\le 2^p C_*^p h(t)(1+t)^p\|\nabla u\|_2^p\\ &\le 2^p C_*^p(1+t)^{-\lambda+p}\|\nabla u\|_2^p. \end{aligned} \tag{11}$$

Therefore,

$$\begin{aligned} E(t) &\le E(0)+\int_0^t |h'(s)|\,\|u(s)\|_p^p\,ds \le E(0)+\int_0^t \lambda(1+s)^{-1}h(s)\|u(s)\|_p^p\,ds\\ &\le E(0)+2^pC_*^p\lambda\int_0^t (1+s)^{-\lambda+p-1}\|\nabla u\|_2^p\,ds\\ &\le E(0)+2^pC_*^p\lambda\left(\frac{2p}{l(p-2)}\right)^{p/2}\int_0^t(1+s)^{-\lambda+p-1}E^{p/2}(s)ds. \end{aligned}$$

By the Gronwall-Bellman inequality with $\omega(s)=s^{p/2}$, $p/2>1$, using (H2) and small enough $E(0)$, we get

$$\begin{aligned} E(t) &\le \frac{1}{\left[(E(0))^{1-p/2}-2^pC_*^p\lambda\left(\frac{2p}{l(p-2)}\right)^{p/2}\int_0^t(1+s)^{-\lambda+p-1}\,ds\right]^{2/(p-2)}}\\ &\le \frac{1}{\left[(E(0))^{1-p/2}-\frac{2^pC_*^p\lambda}{\lambda-p}\left(\frac{2p}{l(p-2)}\right)^{p/2}\right]^{2/(p-2)}}. \end{aligned}$$

Now, using (11), we arrive at

$$\begin{aligned} h(t)\|u\|_p^p &\le \frac{2^pC_*^p}{l}\|\nabla u\|_2^{p-2}\,l\,\|\nabla u\|_2^2 \le \frac{2^pC_*^p}{l^{p/2}}\left(\frac{2p}{p-2}E(t)\right)^{\frac{p-2}{2}} l\,\|\nabla u\|_2^2\\ &\le \frac{2^pC_*^p}{l^{p/2}}\left(\frac{2p}{p-2}\mu\right)^{\frac{p-2}{2}} l\,\|\nabla u\|_2^2\\ &\le \beta l\|\nabla u\|_2^2 \le \beta\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2\\ &< \left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2,\quad \forall t\in[0,T_*). \end{aligned} \tag{12}$$

Therefore,

$$I(t)=\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2-h(t)\|u\|_p^p>0,\quad \forall t\in[0,T_*).$$

By repeating this procedure and using the fact that

$$\lim_{t\to T_*} \frac{2^p C_*^p}{l^{p/2}} \left(\frac{2p}{p-2}E(t)\right)^{\frac{p-2}{2}} \leq \beta < 1,$$

T_* is extended to T_m. □

THEOREM 3.3. Suppose that (H1) and (H2) hold and $2 < p \leq \frac{2(n-1)}{n-2}$ if $n \geq 3$ and $2 < p \leq +\infty$ if $n = 1, 2$. If $u_0, u_1 \in H^1(\mathbb{R}^n)$ and satisfy the conditions of Lemma 3.2, then the solution of (1) is uniformly bounded and global in time.

Proof. It suffices to show that $\|\nabla u\|_2^2$ is bounded independently of t. From the proof of Lemma 3.2, since $I(t) > 0,\ t \in [0, T_m)$, we see that $E(t)$ is uniformly bounded. Clearly,

$$\begin{aligned} \mu &\geq E(t) \\ &\geq \frac{p-2}{2p}\left[\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + (g \circ \nabla u)\right] + \frac{1}{p}I(t) \\ &\geq \frac{l\,(p-2)}{2p}\|\nabla u\|_2^2, \end{aligned}$$

since $I(t)$ and $(g \circ \nabla u)$ are nonnegative. Therefore,

$$\|\nabla u\|_2^2 \leq C_\mu,$$

where C_μ is a positive constant depending only on $E(0)$, p and l but independent of t. □

4. Decay of solutions.

The main result is

THEOREM 4.1. Let (H1), for large a, and (H2) hold. Assume further that $2 < p \leq \frac{2(n-1)}{n-2}$ if $n \geq 3$ and $2 < p \leq +\infty$ if $n = 1, 2$. If $u_0, u_1 \in H^1(\mathbb{R}^n)$ and they satisfy the conditions of Lemma 3.2, then there exist positive constants K, k and t_0 such that the solution of (1) satisfies

$$E(t) \leq K\,(1+t)^{-k}, \quad \forall t \geq t_0.$$

In order to prove this result we need the following lemmas.

LEMMA 4.2. The functional

$$\Phi_1(t) := (1+t)^{-1}\int_{\mathbb{R}^n} u u_t dx$$

satisfies, for some $t_0 > 0$,

$$\Phi_1'(t) \leq (1+t)^{-1}\left(1 + \frac{C^2}{l}\right)\|u_t\|_2^2 - (1+t)^{-1}\frac{l}{4}\|\nabla u\|_2^2 + (1+t)^{-1}\frac{1}{l}(g \circ \nabla u), \quad t \geq t_0. \tag{13}$$

Proof. By differentiating $\Phi_1(t)$, we have

$$\Phi_1'(t) = -(1+t)^{-2}\int_{\mathbb{R}^n} u u_t dx + (1+t)^{-1}\left(\|u_t\|_2^2 + \int_{\mathbb{R}^n} u u_{tt} dx\right). \tag{14}$$

Along solutions of (1), we find

$$\int_{\mathbb{R}^n} u u_{tt} dx = -\|\nabla u\|_2^2 + \int_{\mathbb{R}^n} \nabla u. \int_0^t g(t-s)\nabla u(s) ds dx + h(t)\|u\|_p^p. \tag{15}$$

Using Lemma 2.3, we get the estimate

$$-\int_{\mathbb{R}^n} u u_t dx \le C(1+t)\left(\frac{l}{4C}\|\nabla u\|_2^2 + \frac{C}{l}\|u_t\|_2^2\right) \tag{16}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^n} \nabla u(t). \int_0^t g(t-s)\nabla u(s) ds dx \qquad (17)\\ = & \int_{\mathbb{R}^n} \nabla u(t). \int_0^t g(t-s)\left[(\nabla u(s) - \nabla u(t)) + \nabla u(t)\right] ds dx \\ \le & \frac{l}{4}\|\nabla u\|_2^2 + \frac{1}{l}\left(\int_0^t g(s)ds\right)(g\circ\nabla u) + \left(\int_0^t g(s)ds\right)\|\nabla u\|_2^2. \end{aligned}$$

Also, using (11) and the uniform boundedness of $\|\nabla u\|_2$,

$$h(t)\|u\|_p^p \le C(1+t)^{-\lambda+p}\|\nabla u\|_2^p \le C(1+t)^{-\lambda+p}\|\nabla u\|_2^2.$$

As $\lambda > p$, if t_0 is chosen such that

$$C(1+t_0)^{-\lambda+p} < \frac{l}{4},$$

then, for all $t \ge t_0$, we get

$$h(t)\|u\|_p^p \le \frac{l}{4}\|\nabla u\|_2^2.$$

Now combining the above estimations, we arrive at

$$\begin{aligned} \Phi_1'(t) \le\ & (1+t)^{-1}\left(\frac{l}{4}\|\nabla u\|_2^2 + \frac{C^2}{l}\|u_t\|_2^2\right) \\ & + (1+t)^{-1}\left[\|u_t\|_2^2 - \|\nabla u\|_2^2 + \frac{l}{2}\|\nabla u\|_2^2\right. \\ & \left. + \frac{1}{l}\left(\int_0^t g(s)ds\right)(g\circ\nabla u) + \left(\int_0^t g(s)ds\right)\|\nabla u\|_2^2\right], \quad t \ge 0. \end{aligned}$$

Recalling (H1) and arranging terms in the right hand side of the last relation we obtain (13). □

Lemma 4.3. The functional

$$\Phi_2(t) := -(1+t)^{-1}\int_{\mathbb{R}^n} u_t \int_0^t g(t-s)(u(t)-u(s)) ds dx, \quad t \ge 0,$$

satisfies, along the solution of (1) and for any $\delta_1 > 0$,

$$\begin{aligned}
\Phi_2'(t) \leq & -(1+t)^{-1}\left[\int_0^t g(s)ds - \delta_1 (C+1)\right]\|u_t\|_2^2 \\
& + (1+t)^{-1}\left(2\delta_1 + \frac{\delta_1^p}{p}\right)\|\nabla u\|_2^2 \\
& + C(1+t)^{-1}\left[1 + \frac{1}{\delta_1} + \delta_1^{-p/(p-1)}\right](g \circ \nabla u) \\
& - (1+t)^{-1}\frac{g(0)}{4\delta_1}(g' \circ u)(t), \qquad t \geq 0.
\end{aligned} \tag{18}$$

Proof. Direct differentiation of $\Phi_2(t)$ yields

$$\begin{aligned}
\Phi_2'(t) = & (1+t)^{-2}\int_{\mathbb{R}^n} u_t \int_0^t g(t-s)(u(t)-u(s))\,dsdx \\
& -(1+t)^{-1}\left[\int_{\mathbb{R}^n} u_{tt}\int_0^t g(t-s)(u(t)-u(s))\,dsdx\right. \\
& + \int_{\mathbb{R}^n} u_t \int_0^t g'(t-s)(u(t)-u(s))\,dsdx \\
& \left. + \left(\int_0^t g(s)ds\right)\|u_t\|_2^2\right].
\end{aligned} \tag{19}$$

Along (1), we find

$$\begin{aligned}
& -\int_{\mathbb{R}^n} u_{tt}\int_0^t g(t-s)(u(t)-u(s))\,dsdx \\
= & \int_{\mathbb{R}^n} \nabla u . \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\,dsdx \\
& -\int_{\mathbb{R}^n}\left(\int_0^t g(t-s)\nabla u(s)ds . \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)dx \\
& -h(t)\int_{\mathbb{R}^n}|u|^{p-2}u\int_0^t g(t-s)(u(t)-u(s))\,dsdx.
\end{aligned} \tag{20}$$

On the other hand, (H1) and (8) allow us to write

$$\begin{aligned} &\int_{\mathbb{R}^n} \left| \int_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right) ds \right|^2 dx \\ \leq\ & \left(\int_0^t g(s)ds \right) (g \circ \nabla u)(t) \leq (1-l)\,(g \circ \nabla u)(t) \end{aligned} \tag{21}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^n} \left| \int_0^t g(t-s)\left(u(t) - u(s)\right) ds \right|^2 dx \\ \leq\ & \left(\int_0^t g(s)ds \right) (g \circ u)(t) \leq C\,(1-l)\,(1+t)(g \circ \nabla u)(t). \end{aligned}$$

Therefore, for $\delta_1 > 0$,

$$\begin{aligned} &\int_{\mathbb{R}^n} u_t \int_0^t g(t-s)\left(u(t) - u(s)\right) ds dx \\ \leq\ & C(1+t)\left(\delta_1 \left\| u_t \right\|_2^2 + \frac{(1-l)}{4\delta_1} (g \circ \nabla u)(t) \right), \end{aligned} \tag{22}$$

$$\begin{aligned} &\int_{\mathbb{R}^n} \nabla u. \int_0^t g(t-s)\left(\nabla u(t) - \nabla u(s)\right) ds dx \\ \leq\ & \delta_1 \left\| \nabla u \right\|_2^2 + \frac{1}{4\delta_1}\,(1-l)\,(g \circ \nabla u)(t) \end{aligned} \tag{23}$$

and

$$-\int_{\mathbb{R}^n} u_t \int_0^t g'\,(t-s)\left(u(t) - u(s)\right) ds dx \leq \delta_1 \left\| u_t \right\|_2^2 - \frac{g(0)}{4\delta_1}\,(g' \circ u)\,(t)\,. \tag{24}$$

Moreover, the estimate (21) implies that

$$\begin{aligned}
& -\int_{\mathbb{R}^n}\left(\int_0^t g(t-s)\nabla u(s)ds\right).\left(\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)ds\right)dx \\
= & -\int_{\mathbb{R}^n}\left(\int_0^t g(t-s)\nabla u(t)ds\right).\left(\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)ds\right)dx \qquad (25)\\
& -\int_{\mathbb{R}^n}\left(\int_0^t g(t-s)\left(\nabla u(s)-\nabla u(t)\right)ds\right).\left(\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)ds\right)dx \\
\leq & \left(\delta_1\left\|\nabla u\right\|_2^2+\frac{C}{4\delta_1}(g\circ\nabla u)\right)+(1-l)\left(g\circ\nabla u\right) \\
\leq & \ \delta_1\left\|\nabla u\right\|_2^2+C\left(1+\frac{1}{\delta_1}\right)(g\circ\nabla u), \quad t\geq 0.
\end{aligned}$$

Finally, the last term in (20) can be estimated using Young's inequality as follows:

$$\begin{aligned}
& -\int_{\mathbb{R}^n} u\left|u\right|^{p-2}\int_0^t g(t-s)\left(u(t)-u(s)\right)dsdx \\
\leq & \ \frac{\delta_1^p}{p}\left\|u\right\|_p^p+\frac{p-1}{p}\delta_1^{-p/(p-1)}\int_{\mathbb{R}^n}\left|\int_0^t g(t-s)\left(u(t)-u(s)\right)ds\right|^p dx \\
\leq & \ \frac{\delta_1^p}{p}\left\|u\right\|_p^p+\frac{p-1}{p}\delta_1^{-p/(p-1)}\left[\left(\int_0^t g(s)ds\right)^{p-1}\int_0^t g(t-s)\left\|(u(t)-u(s))\right\|_p^p ds\right] \\
\leq & \ \frac{\delta_1^p}{p}\left\|u\right\|_p^p+\frac{p-1}{p}\delta_1^{-p/(p-1)}\left[\int_0^t g(t-s)\left\|(u(t)-u(s))\right\|_p^p ds\right].
\end{aligned}$$

This estimation, together with (12), Lemma 2.2 and Lemma 2.3, implies that

$$
\begin{aligned}
& h(t) \int_{\mathbb{R}^n} u\,|u|^{p-2} \int_0^t g(t-s)\,(u(t)-u(s))\,dsdx \\
\leq\ & \frac{\delta_1^p}{p}\,\|\nabla u\|_2^2 + \frac{p-1}{p}\delta_1^{-p/(p-1)} h(t) \int_0^t g(t-s)\,\|u(t)-u(s)\|_p^p\,ds \\
\leq\ & \frac{\delta_1^p}{p}\,\|\nabla u\|_2^2 + C\delta_1^{-p/(p-1)} h(t) \int_0^t g(t-s)\,[(1+t)\,\|\nabla u(t)-\nabla u(s)\|_2 \\
& \qquad + \|\nabla u(t)-\nabla u(s)\|_2]^p\,ds \\
\leq\ & \frac{\delta_1^p}{p}\,\|\nabla u\|_2^2 + C\delta_1^{-p/(p-1)} h(t) \int_0^t g(t-s)\,((1+t)^p\,\|\nabla u(t)-\nabla u(s)\|_2^p \\
& \qquad + \|\nabla u(t)-\nabla u(s)\|_2^p)\,ds.
\end{aligned}
$$

Then the uniform boundedness of $\|\nabla u\|_2^2$ and the facts that $h(t)\,(1+t)^p \leq 1$ and $h(t) \leq 1$ give

$$
\begin{aligned}
& h(t) \int_{\mathbb{R}^n} u\,|u|^{p-2} \int_0^t g(t-s)\,(u(t)-u(s))\,dsdx \\
\leq\ & \frac{\delta_1^p}{p}\,\|\nabla u\|_2^2 + C\delta_1^{-p/(p-1)} h(t) \\
& \cdot \int_0^t g(t-s)\left((1+t)^p\,\|\nabla u(t)-\nabla u(s)\|_2^2 + \|\nabla u(t)-\nabla u(s)\|_2^2\right) ds \\
\leq\ & \frac{\delta_1^p}{p}\,\|\nabla u\|_2^2 + C\delta_1^{-p/(p-1)} (g \circ \nabla u). \qquad (26)
\end{aligned}
$$

It suffices to combine (19)-(26) to conclude the result. □

LEMMA 4.4. The functional $\Phi_3(t)$ defined by

$$\Phi_3(t) := (1+t)^{-1} \int_{\mathbb{R}^n} \int_0^t G(t-s)\,|u(t)-u(s)|^2\,dsdx,$$

with

$$G(t) := e^{-\alpha t} \int_t^\infty e^{\alpha s}\,(-g'(s))\,ds,$$

satisfies, for any $\delta_2 > 0$ and $\alpha < a$,

$$\frac{d}{dt}\left[\Phi_3(t)\right] \leq -\,(1+t)^{-1}\left[\left(1+(1+t)\left(\alpha - \frac{1}{\delta_2}\overline{G}\right)\right)\Phi_3(t) - (g' \circ u) - \delta_2\,\|u_t\|_2^2\right], \quad (27)$$

where $\overline{G} = \int_0^\infty G(t)dt$.

Proof. Using (H1) and integration by parts, we can easily find that

$$e^{\alpha s} g(s) \le c e^{(\alpha - a)s} \to 0 \quad \text{as } s \to \infty$$

and

$$\int_t^\infty e^{\alpha s} \left(-g'(s)\right) ds = \left[-e^{\alpha s} g(s)\right]_{s=t}^{s=\infty} + \alpha \int_t^\infty e^{\alpha s} g(s) ds \le e^{\alpha t} g(t) + \frac{\alpha}{a} \int_t^\infty e^{\alpha s} \left(-g'(s)\right) ds;$$

therefore,

$$\int_t^\infty e^{\alpha s} \left(-g'(s)\right) ds \le \left(\frac{a}{a-\alpha}\right) e^{\alpha t} g(t).$$

Hence,

$$G(t) \le \left(\frac{a}{a-\alpha}\right) g(t)$$

and

$$0 \le \overline{G} = \int_0^\infty G(t) dt \le \left(\frac{a}{a-\alpha}\right) \int_0^\infty g(t) dt < \infty,$$

which means that $\overline{G}$ is finite.

Next, by differentiating $\Phi_3(t)$ and using Young's inequality, we have

$$\begin{aligned} \frac{d}{dt}\left[\Phi_3(t)\right] &= -\left(1+t\right)^{-1} \Phi_3(t) - \alpha \Phi_3(t) \\ &\quad + \left(1+t\right)^{-1} \left[(g' \circ u) + 2 \int_{\mathbb{R}^n} u_t \int_0^t G(t-s) \left(u(t) - u(s)\right) ds dx \right] \\ &\le -\left(1+t\right)^{-1} \Phi_3(t) - \alpha \Phi_3(t) + \left(1+t\right)^{-1} \left[(g' \circ u) + \delta_2 \left\| u_t \right\|_2^2 \right] + \frac{\overline{G}}{\delta_2} \Phi_3(t) \\ &\le -\left(1+t\right)^{-1} \left[\left(1 + (1+t) \left(\alpha - \frac{1}{\delta_2} \overline{G} \right) \right) \Phi_2(t) - (g' \circ u) - \delta_2 \left\| u_t \right\|_2^2 \right]. \end{aligned}$$

Thus (27) follows. □

Lemma 4.5. The functional defined by

$$\Phi_4(t) := -\left(1+t\right)^{-\lambda} \int_t^\infty \left(1+s\right)^\lambda h'(s) \left\| u(s) \right\|_p^p ds, \qquad t \ge 0,$$

satisfies

$$\Phi_4'(t) = -\lambda \left(1+t\right)^{-1} \Phi_4(t) + h'(t) \left\| u(t) \right\|_p^p, \qquad t \ge 0. \tag{28}$$

Proof. By differentiating $\Phi_4(t)$, we have

$$\begin{aligned} \Phi_4'(t) &= \lambda \left(1+t\right)^{-\lambda-1} \int_t^\infty \left(1+s\right)^\lambda h'(s) \left\| u(s) \right\|_p^p ds \\ &\quad - \left(1+t\right)^{-\lambda} \left(-\left(1+t\right)^\lambda h'(t) \left\| u(t) \right\|_p^p \right) \\ &= -\lambda \left(1+t\right)^{-1} \Phi_4(t) + h'(t) \left\| u(t) \right\|_p^p. \end{aligned}$$

□

LEMMA 4.6. For some γ_1, γ_2 small enough and γ_3, γ_4 positive constants, the functional

$$F(t) := E(t) + \sum_{i=1}^{4} \gamma_i \Phi_i(t), \qquad t \geq 0, \tag{29}$$

satisfies

$$\xi_1 E(t) \leq F(t) \leq \xi_2 [E(t) + \Phi_3(t) + \Phi_4(t)], \quad t \geq 0, \tag{30}$$

where ξ_1 and ξ_2 are positive constants.

Proof. We estimate some terms in the above functionals using Cauchy-Schwarz inequality, Young inequality, (8) and (9) as follows:

$$\begin{aligned} \Phi_1(t) &= (1+t)^{-1} \int_{\mathbb{R}^n} u u_t dx \\ &\leq (1+t)^{-1} \left(\int_{\mathbb{R}^n} |u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |u_t|^2 dx \right)^{\frac{1}{2}} \\ &\leq C \|\nabla u\|_2 \|u_t\|_2 \leq C \left(\|u_t\|_2^2 + \frac{1}{4} \|\nabla u\|_2^2 \right) \end{aligned} \tag{31}$$

and

$$\begin{aligned} &(1+t)^{-1} \int_{\mathbb{R}^n} \left| \int_0^t g(t-s) \left(u(t) - u(s) \right) ds \right|^2 dx \\ \leq\ & (1+t)^{-1} \left(\int_0^t g(s) ds \right) \int_{\mathbb{R}^n} \int_0^t g(t-s) |u(t) - u(s)|^2 ds dx \leq \Phi_3(t). \end{aligned} \tag{32}$$

So, from the last inequality, we deduce that

$$\Phi_2(t) \leq \frac{1}{2} \left[\|u_t\|_2^2 + \Phi_3(t) \right]. \tag{33}$$

By inserting (31) and (33) into (29), we obtain, for some $\xi_2 > 0$,

$$\begin{aligned} F(t) &\leq \left[\frac{1}{2} + \left(C\gamma_1 + \frac{\gamma_2}{2} \right) \right] \|u_t\|_2^2 + \left[\frac{1}{2} + \gamma_1 \frac{C}{4} \right] \|\nabla u\|_2^2 \\ &\quad + \frac{1}{2} (g \circ \nabla u) - \frac{h(t)}{p} \|u\|_p^p \\ &\quad + \left(\gamma_3 + \frac{\gamma_2}{2} \right) \Phi_3(t) + \gamma_4 \Phi_4(t) \\ &\leq \xi_2 [E(t) + \Phi_3(t) + \Phi_4(t)], \quad t \geq 0. \end{aligned} \tag{34}$$

Again the same estimates, using (11), give

$$\begin{aligned} F(t) &\geq E(t) + \gamma_1 \Phi_1(t) + \gamma_2 \Phi_2(t) + \gamma_3 \Phi_3(t) \\ &\geq \left[\frac{1}{2} - \left(C\gamma_1 + \frac{\gamma_2}{2} \right) \right] \|u_t\|_2^2 + \left[\frac{l}{2} - \gamma_1 \frac{C}{4} \right] \|\nabla u\|_2^2 \\ &\quad + \frac{1}{2} (g \circ \nabla u) - \frac{h(t)}{p} \|u\|_p^p + \left(\gamma_3 - \frac{\gamma_2}{2} \right) \Phi_3(t). \end{aligned}$$

Taking γ_1, γ_2 small enough, we arrive at

$$F(t) \geq \xi_1 E(t), \quad \text{for some} \quad \xi_1 > 0. \tag{35}$$

Combining (34) and (35), the result follows. □

Now, we are in a position to prove the main result.

Proof of Theorem 4.1. Direct differentiation of (29), using (10) and (H1), yields

$$\begin{aligned} F'(t) &= E'(t) + \sum_{i=1}^{4} \gamma_i \Phi_i'(t) \leq \frac{1}{2}(g' \circ \nabla u) - h'(t) \|u\|_p^p + \sum_{i=1}^{4} \gamma_i \Phi_i'(t) \\ &\leq \frac{-a}{2}(g \circ \nabla u) - h'(t) \|u\|_p^p + \sum_{i=1}^{4} \gamma_i \Phi_i'(t). \end{aligned} \tag{36}$$

For all $t \geq t_0 > 0$, we have

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0 > 0.$$

By inserting (13), (18), (27) and (28) into (36), we get

$$\begin{aligned} F'(t) \leq &- (1+t)^{-1} \left(1 + (1+t)\left(\alpha - \frac{1}{\delta_2}\overline{G}\right)\right) \gamma_3 \Phi_3(t) - \lambda (1+t)^{-1} \gamma_4 \Phi_4(t) \\ &+ (\gamma_4 - 1) h'(t) \|u\|_p^p + (1+t)^{-1} \left[\gamma_3 - \gamma_2 \frac{g(0)}{4\delta_1}\right] (g' \circ u)(t) \\ &- (1+t)^{-1} \left[\frac{a}{2} - \gamma_2 C \left(1 + \frac{1}{\delta_1} + \delta_1^{-p/(p-1)}\right) - \frac{\gamma_1}{l}\right] (g \circ \nabla u) \\ &- (1+t)^{-1} \left[\gamma_2 (g_0 - \delta_1 (C+1)) - \gamma_1 \left(1 + \frac{C^2}{l}\right) - \gamma_3 \delta_2\right] \|u_t\|_2^2 \\ &- (1+t)^{-1} \left[\frac{\gamma_1 l}{4} - \gamma_2 \left(2\delta_1 + \frac{\delta_1^p}{p}\right)\right] \|\nabla u\|_2^2. \end{aligned} \tag{37}$$

At this point, we choose δ_1 small enough so that

$$\delta_1 < \frac{g_0}{C+2}$$

and satisfies

$$\frac{2\delta_1 + \frac{\delta_1^p}{p}}{\frac{l}{4}} < \frac{g_0 - \delta_1 (C+1)}{1 + \frac{C^2}{l}}.$$

Once δ_1 is fixed, any choice of γ_1, γ_2 satisfying

$$\frac{2\delta_1 + \frac{\delta_1^p}{p}}{\frac{l}{4}} \gamma_2 < \gamma_1 < \gamma_2 \frac{g_0 - \delta_1 (C+1)}{1 + \frac{C^2}{l}} \tag{38}$$

will yield

$$\begin{aligned} \gamma_2 (g_0 - \delta_1 (C+1)) - \gamma_1 \left(1 + \frac{C^2}{l}\right) &= k > 0, \\ \frac{\gamma_1 l}{4} - \gamma_2 \left(2\delta_1 + \frac{\delta_1^p}{p}\right) &> 0. \end{aligned}$$

So we choose γ_1 and γ_2 so small that (30) and (38) remain valid and further

$$\frac{a}{2} - \gamma_2 C \left(1 + \frac{1}{\delta_1} + \delta_1^{-p/(p-1)}\right) - \frac{\gamma_1}{l} > 0.$$

After that, we pick γ_3 large enough so that

$$\gamma_3 - \gamma_2 \frac{g(0)}{4\delta_1} > 0$$

and δ_2 small enough so that

$$k - \gamma_3 \delta_2 > 0.$$

Finally, if $\gamma_4 = 1$ and a is large so that we can choose $\alpha > \frac{1}{\delta_2}\overline{G}$, then (37) becomes, for some $c_1, c_2 > 0$ and $t \geq t_0$,

$$\begin{aligned} F'(t) &\leq -c_1 (1+t)^{-1} \left[\|u_t\|_2^2 + \|\nabla u\|_2^2 + (g \circ \nabla u) + \Phi_3(t) + \Phi_4(t)\right] \\ &\leq -c_2 (1+t)^{-1} \left[E(t) + \Phi_3(t) + \Phi_4(t)\right]. \end{aligned}$$

Using the equivalence relation in (30), we deduce that

$$F'(t) \leq -\frac{c_2}{\xi_2} (1+t)^{-1} F(t), \quad t \geq t_0. \tag{39}$$

Integration of (39) over (t_0, t) yields

$$F(t) \leq \frac{F(t_0)\,(1+t_0)^{\frac{c_2}{\xi_2}}}{(1+t)^{\frac{c_2}{\xi_2}}}.$$

Again the relation in (30) gives the desired result. The proof is completed.

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