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GLOBAL EXISTENCE AND BLOW-UP FOR THE FAST DIFFUSION EQUATION WITH A MEMORY BOUNDARY CONDITION

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Abstract. In this paper, we study the long-time behavior of solutions to the fast diffusion equation with a memory boundary condition. The problem corresponds to a model introduced in previous studies of tumor-induced angiogenesis. We establish global existence and finite time blow-up results for the problem.

1. Introduction. In this paper, we study the long-time behavior of solutions to the fast diffusion equation with a memory boundary condition:

$$\begin{aligned} &u_t = \Delta(u^m) && x \in \Omega, \quad t > 0, \\ &\nabla(u^m) \cdot \boldsymbol{n} = u^q(x,t) \int_0^t u^p(x,s)ds && x \in \partial\Omega, \quad t > 0, \\ &u(x,0) = u_0(x) && x \in \overline{\Omega}, \end{aligned} \tag{1.1}$$

where $0 < m < 1$, $p > 0$, $q \geq 0$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and $\boldsymbol{n}$ is the outward normal. The initial condition u_0 is a nonnegative, continuous function on $\overline{\Omega}$.

Our first motivation for the study of (1.1) comes from a previously introduced model of capillary growth in solid tumors as initiated by angiogenesis growth factor [5], wherein Levine et al. developed a model for the transmission of growth factor across a capillary wall that takes the following form:

$$\begin{aligned} &u_t = \nabla \cdot (\nabla\phi(x,t,u) + \boldsymbol{b}(x,t,u)) + h(x,t,u) && x \in \Omega, \quad t > 0, \\ &(\nabla\phi(x,t,u) + \boldsymbol{b}(x,t,u)) \cdot \boldsymbol{n} = g(x,t,u,v) && x \in \partial\Omega, \quad t > 0, \\ &u = u_0 && x \in \overline{\Omega} \end{aligned} \tag{A}$$

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with $g_v \geq 0$ on $\partial\Omega \times \{t > 0\}$; $g_v \equiv 0$ on $(\partial\Omega \setminus \Sigma) \times \{t > 0\}$; and

$$\begin{aligned} &v_t = f(x,t,u,v) + G(u)_t && x \in \Sigma, \quad t > 0, \\ &v = v_0 && x \in \overline{\Sigma}. \end{aligned} \tag{B}$$

Here, Σ is a relatively open subset of $\partial\Omega$, which represents the capillary wall.

In (B), if we let

$$v(x,t) \equiv \int_0^t u^p(x,s)ds$$

and $\Sigma = \partial\Omega$, then the problem (1.1) may be seen to be of the type (A)-(B) with $\phi = u^m$, $g = u^q v$, $f = u^p$, $G = 0$, and $v_0 = 0$.

Our second motivation comes from a corresponding localized model which has been extensively studied in the literature:

$$\begin{aligned} &u_t = \Delta(u^m) && x \in \Omega, \quad t > 0, \\ &\nabla(u^m) \cdot \boldsymbol{n} = u^{q+p} && x \in \partial\Omega, \quad t > 0, \\ &u(x,0) = u_0(x) && x \in \overline{\Omega}. \end{aligned} \tag{1.2}$$

For the one-dimensional case, in [4] by introducing auxiliary functions and the comparison principle, Filo proved that if $p+q < (m+1)/2$, every solution of (1.2) is global, whereas if $p + q > (m + 1)/2$, all solutions blow up in finite time. Wu [8] then showed that the critical exponent $p + q = (m + 1)/2$ belongs to the global existence case. Later, Wolanski [7] generalized these results to an N-dimensional ball via integral estimates and the maximum principle. For a general bounded domain Ω in $\mathbb{R}^N$, Wang [6] successfully established the same results by constructing suitable supersolutions and subsolutions.

Because of the presence of the memory term in (1.1), however, all the arguments used for the localized model (1.2) seem not applicable to (1.1). Therefore, for (1.1) in one-dimensional space, we developed various integral estimates to prove the global existence result and constructed appropriate subsolutions to show finite time blow-up in [3]. It turns out that our results for (1.1) in one-dimensional space are the same as those in [4,8] for (1.2). Nevertheless, certain techniques used in [3] cannot be extended to (1.1) in N-dimensional space. Hence, our main objective here is to establish the global existence and blow-up results for (1.1) on $\Omega \in \mathbb{R}^N$. Specifically, in the sequel we will establish the following results.

Theorem 1.1. If $p + q < (m + 1)/2$, every solution of (1.1) exists globally, whereas if $p + q > (m + 1)/2$, all solutions of (1.1) blow up in finite time.

The critical case $p + q = (m + 1)/2$ cannot be discussed by the techniques employed in the current work, and it will be left for a future study.

2. Proof of global existence for $p + q < (m + 1)/2$. In order to carry out our program, we first present an important property possessed by solutions of problem (1.1), which will be used in the sequel.

LEMMA 2.1. Let $u(x,t)$ be the solution of the following problem:

$$\begin{aligned} &u_t = \Delta(u^m) && x \in \Omega, \quad t > 0, \\ &\nabla(u^m)\cdot \boldsymbol{n} = u^q(x,t)\int_0^t u^p(x,s)ds && x \in \partial\Omega, \quad t > 0, \\ &u(x,0) \equiv c && x \in \overline{\Omega}, \end{aligned} \tag{2.1}$$

where $0 < m < 1$, $p > 0$, $q \geq 0$, and c is a positive constant. Then $u(x,t)$ is nondecreasing in time.

Proof. By the subsolution comparison principle established in [2], it is easy to see that the solution of (2.1) satisfies $u(x,t) \geq c$. Let $v(x,t) = u(x,t+\kappa)$ $(\kappa > 0)$; then v satisfies

$$v_t = \Delta(v^m) \quad \text{and} \quad v(x,0) = u(x,\kappa) \geq c.$$

Moreover, for $x \in \partial\Omega$,

$$\nabla(v^m)\cdot \boldsymbol{n} = u^q(x,t+\kappa)\int_0^{t+\kappa} u^p(x,s)ds \geq v^q(x,t)\int_0^t u^p(x,s)ds.$$

By the comparison principle for the fast diffusion equation with a local boundary condition (cf. [1]), $v(x,t) \geq u(x,t)$ for $x \in \overline{\Omega}$, $t \geq 0$, which implies that u is nondecreasing in t. □

Clearly, the solution of (1.1) is a subsolution of the following problem:

$$\begin{aligned} &u_t = \Delta(u^m) && x \in \Omega, \quad t > 0, \\ &\nabla(u^m)\cdot \boldsymbol{n} = u^q(x,t)\int_0^t u^p(x,s)ds && x \in \partial\Omega, \quad t > 0, \\ &u(x,0) = \|u_0\|_\infty && x \in \overline{\Omega}. \end{aligned} \tag{2.2}$$

Furthermore, by Lemma 2.1, the solution of (2.2) satisfies

$$\int_0^t u^p(x,s)ds \leq \int_0^t u^p(x,t)ds = tu^p(x,t).$$

Thus, by the comparison principle for the fast diffusion equation with a local boundary condition, to establish the global existence result for (1.1), it suffices to study the following problem:

$$\begin{cases} u_t = \Delta(u^m) & x \in \Omega, \quad t > 0, \\ \nabla u \cdot \boldsymbol{n} = \dfrac{t}{m} u^{1-m+p+q}(x,t) & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = \|u_0\|_\infty & x \in \overline{\Omega}. \end{cases} \tag{2.3}$$

Since (2.3) is a localized problem, motivated by [6], we seek a global supersolution of (2.3). Consider two cases.

CASE 1 ($p + q \leq m$). Let $\alpha = 1 - m + p + q$; then $0 < \alpha \leq 1$. We define φ to satisfy the following:

$$\begin{cases} \varphi'(\zeta) = \dfrac{1}{m}\varphi^\alpha(\zeta) & \zeta > 0, \\ \varphi(0) = \|u_0\|_\infty. \end{cases} \tag{2.4}$$

Then $\varphi(\zeta)$ exists globally, and $\lim_{\zeta\to\infty}\varphi(\zeta) = \infty$.

Let $h(x)$ be the solution of the problem

$$\begin{cases} \Delta h = k & \text{in } \Omega, \\ \nabla h \cdot \boldsymbol{n} = 1 & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

where $k = |\partial\Omega|/|\Omega|$. Since $h(x) + c$ is also a solution of (2.5) for any positive constant c, we may assume that $h(x) > 0$ on $\overline{\Omega}$.

We then let $g(t)$ be the solution of the following problem:

$$\begin{cases} g'(t) = (k + L^2)\left[m\varphi^{m-1}(g(t)) + (m+\alpha-1)\varphi^{m+\alpha-2}(g(t))\right] & t > 0, \\ g(0) = 0, \end{cases}$$

where $L = \max_{\overline{\Omega}}(h(x) + |\nabla h(x)|)$. In view of (2.4), since $m + \alpha - 1 = p + q < 1$, $m + 2\alpha - 2 = 2(p+q) - m < 1$, $g(t)$ exists globally. We now construct a supersolution of (2.3) as follows:

$$\bar{u}(x,t) = \varphi\left(g(t^3 + t^2) + th(x)\right).$$

For simplicity, let $s = t^3 + t^2$ and $\tau = g(s) + th(x)$. Then by (2.4) and (2.5), we have that

$$\begin{aligned} \bar{u}_t &= \varphi'(\tau)\left[g'(s)(3t^2 + 2t) + h(x)\right], \\ \nabla\bar{u} &= t\varphi'(\tau)\nabla h(x), \\ \Delta\bar{u} &= \frac{\alpha}{m}t^2\varphi^{\alpha-1}(\tau)\varphi'(\tau)|\nabla h|^2 + kt\varphi'(\tau). \end{aligned}$$

Thus we further have

$$\begin{aligned} \Delta(\bar{u}^m) &= m\bar{u}^{m-1}\Delta\bar{u} + m(m-1)\bar{u}^{m-2}|\nabla\bar{u}|^2 \\ &= \varphi'(\tau)\left[mkt\varphi^{m-1}(\tau) + (m+\alpha-1)t^2\varphi^{m+\alpha-2}(\tau)|\nabla h|^2\right]. \end{aligned}$$

Since $m < 1$, $m + \alpha - 2 = p + q - 1 < 0$, $h(x) > 0$, and $\varphi'(\tau) > 0$, we then find that on $\Omega \times \{t > 0\}$,

$$\begin{aligned} \bar{u}_t &\geq \varphi'(\tau)(3t^2 + 2t)g'(s) \\ &= \varphi'(\tau)(3t^2 + 2t)(k + L^2)\left[m\varphi^{m-1}(g(s)) + (m+\alpha-1)\varphi^{m+\alpha-2}(g(s))\right] \\ &\geq \varphi'(\tau)(3t^2 + 2t)(k + L^2)\left[m\varphi^{m-1}(\tau) + (m+\alpha-1)\varphi^{m+\alpha-2}(\tau)\right] \\ &\geq \Delta(\bar{u}^m). \end{aligned}$$

Furthermore, on the parabolic boundary, we have

$$\nabla\bar{u} \cdot \boldsymbol{n} = t\varphi'(\tau)\nabla h \cdot \boldsymbol{n} = \frac{t}{m}\varphi^{\alpha}(\tau) = \frac{t}{m}\bar{u}^{1-m+p+q}(x,t) \quad x \in \partial\Omega,\ t > 0$$

and

$$\bar{u}(x,0) = \varphi(0) = \|u_0\|_\infty \quad x \in \overline{\Omega}.$$

Hence, $\bar{u}(x,t)$ is a desired supersolution of problem (2.3).

CASE 2 ($m < p + q < (m+1)/2$). Again let $\alpha = 1 - m + p + q$; then $\alpha > 1$. We define $\varphi(\zeta)$ to be

$$\varphi(\zeta) = \left[\varepsilon + \frac{1}{m}(1-\alpha)\zeta\right]^{\frac{1}{1-\alpha}}, \qquad \zeta \geq 0,$$

where $\varepsilon > 0$ is sufficiently small. Let $\zeta_0(\varepsilon) = m\varepsilon/(\alpha - 1)$. Then $\varphi(\zeta) \to \infty$ as $\zeta \to \zeta_0(\varepsilon)$, and

$$\varphi'(\zeta) = \frac{1}{m}\varphi^{\alpha}(\zeta), \quad \zeta > 0. \tag{2.6}$$

Let λ_0 be the first eigenvalue of the Laplace equation with homogeneous Dirichlet boundary condition and h_0 be the corresponding eigenfunction, that is,

$$\begin{cases} \Delta h_0 = -\lambda_0 h_0 & x \in \Omega, \\ h_0 = 0 & x \in \partial\Omega. \end{cases}$$

Then $\lambda_0 > 0$, $h_0 > 0$ in Ω, and $\nabla h_0 \cdot \boldsymbol{n} < 0$ on $\partial\Omega$. Without loss of generality, we may assume $h_0(x) \leq 1/2$. We then introduce the notation $c_1 = \max_{\overline{\Omega}} |\nabla h_0(x)|$, $c_2 = \min_{\overline{\partial\Omega}}(-\nabla h_0 \cdot \boldsymbol{n}) > 0$, and $l = 1/c_2$.

We now let $g(t)$ be the solution of the problem

$$\begin{cases} g'(t) = M\left[\varepsilon + \dfrac{1}{m}(1-\alpha)g(t)\right] = M\varphi^{1-\alpha}(g(t)) & t > 0, \\ g(0) = 0, \end{cases}$$

where M is a positive constant to be determined. Specifically, g takes the form

$$g(t) = \frac{m\varepsilon\left[1 - e^{-\frac{M}{m}(\alpha-1)t}\right]}{\alpha - 1} < \frac{m\varepsilon}{\alpha - 1} = \zeta_0(\varepsilon),$$

and $\lim_{t\to\infty} g(t) = \zeta_0(\varepsilon)$.

We then construct a supersolution $\overline{u}(x,t)$ of (2.3) as follows:

$$\begin{aligned} \overline{u}(x,t) &= \varphi\left(g(t^3+t^2) + \delta t(1-h_0(x))^{l/\delta}\right) \\ &= \left[\varepsilon e^{-\frac{M}{m}(\alpha-1)(t^3+t^2)} - \frac{\delta t(\alpha-1)}{m}(1-h_0(x))^{l/\delta}\right]^{\frac{1}{1-\alpha}}. \end{aligned}$$

Here $\delta = \varepsilon^{\gamma}$ with $\gamma = (2-m-\alpha)/(\alpha-1) = [1-(p+q)]/[(p+q)-m] > 1$. To ensure the positivity of $\bar{u}$, it suffices to require that

$$\varepsilon e^{-\frac{M}{m}(\alpha-1)(t^3+t^2)} - \frac{\delta t(\alpha-1)}{m}(1-h_0)^{l/\delta} > \varepsilon e^{-\frac{M}{m}(\alpha-1)(t^3+t^2)} - \frac{\delta t(\alpha-1)}{m} > 0,$$

or equivalently,

$$\frac{M}{m}(\alpha-1)(t^3+t^2) + \ln t < -\ln\left(\frac{\alpha-1}{m}\varepsilon^{\sigma-1}\right). \tag{2.7}$$

Since $\ln t < t$, if M is large enough such that $M(\alpha-1)/m \geq 1$, then (2.7) is valid if we require that

$$\frac{M}{m}(\alpha-1)(t^3+t^2+t) < \frac{M}{m}(\alpha-1)(t+1)^3 < -\ln\left(\frac{\alpha-1}{m}\varepsilon^{\gamma-1}\right).$$

Therefore, for sufficiently small ε we set

$$T(\varepsilon) = \left[-\frac{m}{M(\alpha-1)}\ln\left(\frac{\alpha-1}{m}\varepsilon^{\gamma-1}\right)\right]^{1/3} - 1.$$

Then, $\bar{u}(x,t)$ exists on $\overline{\Omega}\times[0,T(\varepsilon))$ and is positive there. For simplicity, let $s=t^3+t^2$ and $\tau=g(s)+\delta t(1-h_0(x))^{l/\delta}$. Making use of (2.6) and the equation $\Delta h_0=-\lambda_0 h_0$, we have

$$\begin{aligned}
\bar{u}_t &= \varphi'(\tau)\left[g'(t)(3t^2+2t)+\delta(1-h_0)^{l/\delta}\right],\\
\nabla\bar{u} &= -\,l\varphi'(\tau)t(1-h_0)^{(l-\delta)/\delta}\nabla h_0,\\
\Delta\bar{u} &= \frac{\alpha}{m}l^2\varphi^{\alpha-1}(\tau)\varphi'(\tau)t^2(1-h_0)^{2(l-\delta)/\delta}|\nabla h_0|^2\\
&\quad + l\left(\frac{l-\delta}{\delta}\right)\varphi'(\tau)t(1-h_0)^{(l-2\delta)/\delta}|\nabla h_0|^2\\
&\quad + l\lambda_0\varphi'(\tau)t(1-h_0)^{(l-\delta)/\delta}h_0.
\end{aligned}$$

Thus, we further have

$$\begin{aligned}
\Delta\left(\bar{u}^m\right) &= m\bar{u}^{m-1}\Delta\bar{u}+m(m-1)\bar{u}^{m-2}|\nabla\bar{u}|^2\\
&= \varphi'(\tau)\Big[l^2(m+\alpha-1)\varphi^{m+\alpha-2}(\tau)t^2(1-h_0)^{2(l-\delta)/\delta}|\nabla h_0|^2\\
&\quad + ml\left(\frac{l-\delta}{\delta}\right)\varphi^{m-1}(\tau)t(1-h_0)^{(l-2\delta)/\delta}|\nabla h_0|^2\\
&\quad + ml\lambda_0\varphi^{m-1}(\tau)t(1-h_0)^{(l-\delta)/\delta}h_0\Big]\\
&\le l\varphi'(\tau)\left[lc_1^2(m+\alpha-1)\varphi^{m+\alpha-2}(\tau)+m\frac{l}{\delta}c_1^2\varphi^{m-1}(\tau)+m\lambda_0\varphi^{m-1}(\tau)\right](t^2+t).
\end{aligned}$$

Since $m+2\alpha-3=2(p+q)-m-1<0$ and $(m+\alpha-2)/(1-\alpha)=\gamma$, if $\varepsilon\le 1$, we find that

$$\begin{aligned}
\varphi^{m+\alpha-2}(\tau) &= \varphi^{1-\alpha}(\tau)\left[\varepsilon+\frac{1}{m}(1-\alpha)\tau\right]^{(m+2\alpha-3)/(1-\alpha)}\\
&\le \varphi^{1-\alpha}(\tau)\varepsilon^{(m+2\alpha-3)/(1-\alpha)}\\
&\le \varphi^{1-\alpha}(\tau)
\end{aligned}\tag{2.8}$$

and

$$\begin{aligned}
\varphi^{m-1}(\tau) &= \varphi^{1-\alpha}(\tau)\left[\varepsilon+\frac{1}{m}(1-\alpha)\tau\right]^{(m+\alpha-2)/(1-\alpha)}\\
&\le \varphi^{1-\alpha}(\tau)\varepsilon^{(m+\alpha-2)/(1-\alpha)}\\
&= \delta\varphi^{1-\alpha}(\tau).
\end{aligned}\tag{2.9}$$

In view of (2.8) and (2.9), we then find

$$\Delta(\bar{u}^m)\le l\left[lc_1^2(m+\alpha-1)+mlc_1^2+m\lambda_0\right]\varphi'(\tau)\varphi^{1-\alpha}(\tau)(t^2+t).$$

If we choose $M=\max\left\{m/(\alpha-1), l\left[lc_1^2(m+\alpha-1)+mlc_1^2+m\lambda_0\right]\right\}$, then M is independent of ε. Since $1-\alpha<0$, $\tau\ge g(s)$, and $\varphi'(\tau)>0$, we further find that on

$\Omega \times (0, T(\varepsilon))$,

$$\begin{aligned}\Delta(\bar{u}^m) &\le M\varphi'(\tau)\varphi^{1-\alpha}(\tau)(t^2+t)\\ &\le M\varphi'(\tau)\varphi^{1-\alpha}(g(s))(t^2+t)\\ &\le \varphi'(\tau)g'(s)(t^2+t)\\ &\le \bar{u}_t.\end{aligned}$$

On the other hand, by (2.6) and the fact that $h_0 = 0$ on $\partial\Omega$, we have that for $(x,t) \in \partial\Omega \times (0, T(\varepsilon))$,

$$\begin{aligned}\nabla\bar{u}\cdot\boldsymbol{n} &= l\varphi'(\tau)t(-\nabla h_0\cdot\boldsymbol{n})\\ &\ge \frac{t}{m}lc_2\varphi^{\alpha}(\tau)\\ &= \frac{t}{m}\varphi^{\alpha}(\tau)\\ &= \frac{t}{m}\bar{u}^{1-m+p+q}(x,t).\end{aligned}$$

Moreover, if ε is sufficiently small, we have

$$\bar{u}(x,0) = \varphi(0) = \varepsilon^{\frac{1}{1-\alpha}} \ge \|u_0\|_\infty.$$

Hence, $\bar{u}(x,t)$ is indeed a supersolution of problem (2.3), and it follows that the solution $u(x,t)$ of problem (1.1) exists on $\overline{\Omega}\times[0,T(\varepsilon))$. Because $T(\varepsilon)\to\infty$ as $\varepsilon\to 0$, the solution $u(x,t)$ of (1.1) exists globally.

3. Proof of blow-up in finite time for $p+q > (m+1)/2$. In order to show that the solution of problem (1.1) blows up in finite time, by the subsolution comparison principle established in [2] and Lemma 2.1, it suffices to consider the following problem:

$$\begin{cases} u_t = \Delta(u^m) & x\in\Omega, \quad t>0,\\ \nabla u\cdot\boldsymbol{n} = \dfrac{1}{m}\displaystyle\int_0^t u^{1-m+p+q}(x,s)ds & x\in\partial\Omega, \quad t>0,\\ u(x,0) = \min\limits_{\overline{\Omega}} u_0(x) & x\in\overline{\Omega}.\end{cases} \tag{3.1}$$

As in [3], we construct a subsolution of (3.1), which blows up in finite time. Consider two cases.

CASE 1 ($(m+1)/2 < p+q < 2-m$). We define φ to satisfy the following:

$$\begin{cases}\varphi(\zeta)\equiv\sigma & 0\le\zeta\le 1,\\ \varphi'(\zeta) = \dfrac{1}{m}\displaystyle\int_1^\zeta \varphi^{\alpha}(s)ds & \zeta>1,\end{cases}$$

where $\sigma = \min_{\overline{\Omega}} u_0(x) > 0$ and $\alpha = 2(1+p+q-2m)/3+1$. Since $0<m<1$ and $p+q > (m+1)/2 > m$, $\alpha > 1$. Clearly,

$$\varphi''(\zeta)\equiv 0 \ \text{ for } 0\le\zeta<1 \quad\text{and}\quad \varphi''(\zeta) = \frac{1}{m}\varphi^{\alpha}(\zeta) \ \text{ for } \zeta>1. \tag{3.2}$$

Multiplying the second equation in (3.2) by $\varphi'(\zeta)$ and integrating over $(1,\zeta)$, we obtain

$$\varphi'(\zeta) = \left[\frac{2}{m(\alpha+1)}\left(\varphi^{\alpha+1}(\zeta) - \sigma^{\alpha+1}\right)\right]^{\frac{1}{2}} \quad \text{for } \zeta > 1. \tag{3.3}$$

We then have that for $\zeta > 1$,

$$\varphi'(\zeta) \le \sqrt{\frac{2}{m(\alpha+1)}}\varphi^{\frac{\alpha+1}{2}}(\zeta). \tag{3.4}$$

On the other hand, since $\varphi'(\zeta) > 0$ for $\zeta > 1$ and $\varphi(1) = \sigma$, there exists $\zeta_1 > 1$ such that $\varphi(\zeta_1) = 2\sigma$. In view of (3.3), we then have that for $\zeta \ge \zeta_1$,

$$\begin{aligned}\varphi'(\zeta) &\ge \left\{\frac{2}{m(\alpha+1)}\left[\varphi^{\alpha+1}(\zeta) - \left(\frac{\varphi(\zeta)}{2}\right)^{\alpha+1}\right]\right\}^{\frac{1}{2}} \\ &= \left[\frac{2}{m(\alpha+1)}\left(1 - \frac{1}{2^{\alpha+1}}\right)\varphi^{\alpha+1}(\zeta)\right]^{\frac{1}{2}} \\ &\ge \sqrt{\frac{1}{m(\alpha+1)}}\varphi^{\frac{\alpha+1}{2}}(\zeta).\end{aligned} \tag{3.5}$$

Let $c_3 = \sqrt{\frac{1}{m(\alpha+1)}}$ and $c_4 = \sqrt{\frac{2}{m(\alpha+1)}}$. By (3.4) and (3.5), we obtain

$$c_3\varphi^{\frac{\alpha+1}{2}}(\zeta) \le \varphi'(\zeta) \le c_4\varphi^{\frac{\alpha+1}{2}}(\zeta) \quad \text{for } \zeta \ge \zeta_1. \tag{3.6}$$

We then integrate the above inequality from ζ_1 to ζ to find that for $\zeta \ge \zeta_1$,

$$\left\{\frac{c_3(\alpha-1)}{2}\left[\frac{2(2\sigma)^{\frac{1-\alpha}{2}}}{c_3(\alpha-1)} + \zeta_1 - \zeta\right]\right\}^{\frac{2}{1-\alpha}} \le \varphi(\zeta) \le \left\{\frac{c_4(\alpha-1)}{2}\left[\frac{2(2\sigma)^{\frac{1-\alpha}{2}}}{c_4(\alpha-1)} + \zeta_1 - \zeta\right]\right\}^{\frac{2}{1-\alpha}}. \tag{3.7}$$

Since $\alpha > 1$, the lower bound on $\varphi(\zeta)$ in (3.7) shows that $\varphi(\zeta)$ blows up in finite time. Thus, there exists $\hat{\zeta} > \zeta_1$ such that if $\zeta \to \hat{\zeta}$, $\varphi(\zeta) \to \infty$. Let $g(t)$ be the solution of the following problem:

$$\begin{cases} g'(t) = M\varphi^{m-1+\frac{\alpha-1}{2}}(g(t)+1) & t > 0, \\ g(0) = 0, \end{cases} \tag{3.8}$$

where M (> 0) is to be determined, and $m - 1 + (\alpha-1)/2 = (p+q+m-2)/3 < 0$. In view of the upper bound on φ in (3.7), there exists $t_1 > 0$ such that $g(t_1) + 1 = \zeta_1$ and $g(t) + 1 \ge \zeta_1$ for $t \ge t_1$. Applying the lower bound on φ in (3.7), we have

$$g'(t) \le M\left\{\frac{c_3(\alpha-1)}{2}\left[\frac{2(2\sigma)^{\frac{1-\alpha}{2}}}{c_3(\alpha-1)} + \zeta_1 - g(t) - 1\right]\right\}^{\frac{2m+\alpha-3}{1-\alpha}}.$$

Since $p+q > (m+1)/2$, $(2m+\alpha-3)/(1-\alpha) = (2-p-q-m)/(1+p+q-2m) < 1$. Thus, $g(t)$ exists for all $t \ge 0$, which, in conjunction with (3.7), implies that there exists $T < \infty$ such that $g(T) + 1 = \hat{\zeta}$.

We then introduce a function $h_1(x)$ defined by

$$h_1(x) = x_1 + x_2 + \cdots + x_N \quad \text{for } x = (x_1, x_2, \ldots, x_N) \in \overline{\Omega}$$

and we let $\lambda = \min\{1/N, 1/2Nd\}$, where $d > |x_i|$ $(1 \le i \le N)$ for $x \in \overline{\Omega}$. Then $\lambda|h_1(x)| \le \lambda N d \le 1/2$, and it follows that there exists $\eta \ge 1/2$ such that $0 \le \eta + \lambda h_1(x) \le 1$ for $x \in \overline{\Omega}$ and $\eta + \max_{\overline{\Omega}}\{\lambda h_1(x)\} = 1$. We now construct a subsolution of (3.1) as follows:

$$\underline{u}(x,t) = \varphi(g(t) + \eta + \lambda h_1(x)) \quad x \in \overline{\Omega},\ 0 \le t < T.$$

For simplicity, let $\tau = g(t) + \eta + \lambda h_1(x)$. Then we find that

$$\underline{u}_t = \varphi'(\tau)g'(t), \quad \nabla\underline{u} = \lambda\varphi'(\tau)\nabla h_1, \quad \text{and} \quad \Delta\underline{u} = \lambda^2\varphi''(\tau)|\nabla h_1|^2.$$

If $0 < \tau \le 1$, since $\varphi'(\tau) = \varphi''(\tau) = 0$, one can see that $\underline{u}_t = \Delta\left(\underline{u}^m\right)$. If $\tau > 1$, in view of (3.2), (3.3), and (3.6), we have that

$$\begin{aligned}\Delta\left(\underline{u}^m\right) &= m\underline{u}^{m-1}\Delta\underline{u} + m(m-1)\underline{u}^{m-2}|\nabla\underline{u}|^2\\ &= m\lambda^2\varphi^{m-1}(\tau)\varphi''(\tau)|\nabla h_1|^2 + m(m-1)\lambda^2\varphi^{m-2}(\tau)\left(\varphi'(\tau)\right)^2|\nabla h_1|^2\\ &= \lambda^2\varphi^{m+\alpha-1}(\tau)|\nabla h_1|^2 + \frac{2(m-1)}{\alpha+1}\lambda^2\varphi^{m-2}(\tau)\left[\varphi^{\alpha+1}(\tau) - \sigma^{\alpha+1}\right]|\nabla h_1|^2\\ &= \frac{2m+\alpha-1}{\alpha+1}\lambda^2\varphi^{m+\alpha-1}(\tau)|\nabla h_1|^2 + \frac{2(1-m)}{\alpha+1}\lambda^2\sigma^{\alpha+1}\varphi^{m-2}(\tau)|\nabla h_1|^2\\ &\ge \frac{N\lambda^2(2m+\alpha-1)}{\alpha+1}\varphi^{m+\alpha-1}(\tau)\\ &\ge M\varphi^{m-1+\frac{\alpha-1}{2}}(\tau)\varphi'(\tau),\end{aligned} \tag{3.9}$$

where $M = \min\{1,\ N\lambda^2(2m+\alpha-1)/c_4(\alpha+1)\}$, which depends only on m, p, q, σ, N, and d. Since $m - 1 + (\alpha-1)/2 = (p+q+m-2)/3 < 0$ and $\varphi' > 0$, one can see that

$$\underline{u}_t = M\varphi^{m-1+\frac{\alpha-1}{2}}(g(t)+1)\varphi'(\tau) \le \Delta(\underline{u}^m).$$

Thus, we find that $\underline{u}_t \le \Delta(\underline{u}^m)$ a.e. in $\Omega \times (0,T)$.

On the parabolic boundary, we have that for $x \in \partial\Omega,\ 0 < t < T$,

$$\begin{aligned}\nabla\underline{u}\cdot\boldsymbol{n} = \lambda\varphi'(\tau)\nabla h_1\cdot\boldsymbol{n} &\le \frac{\lambda N}{m}\int_1^\tau \varphi^\alpha(\xi)d\xi\\ &\le \frac{1}{m}\int_0^t \varphi^\alpha(g(s)+\eta+\lambda h_1(x))g'(s)ds\\ &= \frac{M}{m}\int_0^t \varphi^\alpha(g(s)+\eta+\lambda h_1(x))\varphi^{m-1+\frac{\alpha-1}{2}}(g(s)+1)ds\\ &\le \frac{M}{m}\int_0^t \varphi^{m-1+\frac{3\alpha-1}{2}}(g(s)+\eta+\lambda h^*(x))ds\\ &\le \frac{1}{m}\int_0^t \underline{u}^{1-m+p+q}(x,s)ds,\end{aligned} \tag{3.10}$$

and for $x \in \overline{\Omega}$,

$$\underline{u}(x,0) = \varphi(\eta + \lambda h_1(x)) \le \varphi(1) = \sigma \le u_0(x).$$

Hence, $\underline{u}(x,t)$ is indeed a subsolution of (3.1). Since $\varphi(\zeta) \to \infty$ as $\zeta \to \hat{\zeta}$, and $g(T)+\eta+\max_{\overline{\Omega}}\{\lambda h_1(x)\} = g(T)+1 = \hat{\zeta}$, $\underline{u}(x,t)$ blows up in finite time, and so does the solution $u(x,t)$ of (1.1).

CASE 2 ($p+q \geq 2-m$). We define φ to satisfy the following:

$$\begin{cases} \varphi(\zeta) \equiv \sigma & 0 \leq \zeta \leq 1, \\ \varphi'(\zeta) = \dfrac{1}{m}\displaystyle\int_1^\zeta \varphi^\alpha(s)ds & \zeta > 1, \end{cases}$$

where $\sigma = \min_{\overline{\Omega}} u_0(x)$ and $\alpha = 1-m+p+q$. Since $p+q \geq 2-m > (m+1)/2 > m$, $\alpha > 1$. Analogously as in Case 1, there exist $\zeta_2 > 1$ and $\tilde{\zeta} > \zeta_2$ such that $\varphi(\zeta_2) = 2\sigma$ and $\varphi(\zeta) \to \infty$ as $\zeta \to \tilde{\zeta}$. In addition, we have

$$c_3\varphi^{\frac{\alpha+1}{2}}(\zeta) \leq \varphi'(\zeta) \leq c_4\varphi^{\frac{\alpha+1}{2}}(\zeta) \quad \text{for } \zeta \geq \zeta_2. \tag{3.11}$$

Let $h_1(x), \lambda$, and η be the same as in Case 1. We then construct a subsolution of (3.1) as follows:

$$\underline{u}(x,t) = \varphi(\mu t + \eta + \lambda h_1(x)),$$

where μ is a positive constant to be chosen. For simplicity, we let $\tau = \mu t + \eta + \lambda h_1(x)$. Repeating the same process as (3.9), we find

$$\Delta\left(\underline{u}^m\right) \geq \frac{N\lambda^2(2m+\alpha-1)}{(\alpha+1)}\varphi^{m+\alpha-1}(\tau).$$

Since $m-1+(\alpha-1)/2 = (p+q+m-2)/2 \geq 0$ and $\varphi' \geq 0$, taking (3.11) into consideration, we have

$$\underline{u}_t = \mu\varphi'(\tau) \leq \Delta\left(\underline{u}^m\right),$$

where the positive constant μ is chosen as

$$\mu = \min\left\{1,\ \frac{N\lambda^2(2m+\alpha-1)}{c_4(\alpha+1)}\sigma^{m-1+\frac{\alpha-1}{2}}\right\}.$$

On the other hand, we have that

$$\begin{aligned} \nabla\underline{u}\cdot\boldsymbol{n} = \lambda\varphi'(\tau)\nabla h_1\cdot\boldsymbol{n} &\leq \frac{\lambda N}{m}\int_1^\tau \varphi^\alpha(\xi)d\xi \\ &\leq \frac{\mu}{m}\int_0^t \varphi^\alpha(\mu s+\eta+\lambda h_1(x))ds \\ &\leq \frac{1}{m}\int_0^t \underline{u}^{1-m+p+q}(x,s)ds \end{aligned}$$

and

$$\underline{u}(x,0) = \varphi(\eta+\lambda h_1(x)) \leq \varphi(1) = \sigma.$$

Thus, $\underline{u}(x,t)$ is a subsolution of (3.1). Clearly, there exists $0 < T < \infty$ such that $\mu T + \eta + \max_{\overline{\Omega}}\{\lambda h_1(x)\} = \tilde{\zeta}$, which implies that $\underline{u}(x,t)$ blows up in finite time. The proof is completed.

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