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ON THE EVOLUTION OF TRAVELLING WAVE SOLUTIONS OF THE BURGERS-FISHER EQUATION

By

J. A. LEACH (*School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, United Kingdom*)

AND

E. HANAÇ (*School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, United Kingdom*)

Abstract. In this paper, we consider an initial-value problem for the Burgers-Fisher equation

$$u_t + kuu_x = u_{xx} + u(1-u), \quad -\infty < x < \infty, \quad t > 0,$$

where x and t represent dimensionless distance and time respectively and $k\,(\neq 0)$ is a parameter. In particular, we consider the case when the initial data has a discontinuous compressive step, where $u(x,0) = 1$ for $x \leq 0$ and $u(x,0) = 0$ for $x > 0$. The method of matched asymptotic coordinate expansions is used to obtain the large-t asymptotic structure of the solution to this problem, which exhibits the formation of a permanent form travelling wave propagating in the $+x$ direction with the minimum possible speed $c = c^*(k)$, where

$$c^*(k) = \begin{cases} 2, & -\infty < k \leq 2, \\ \frac{2}{k} + \frac{k}{2}, & 2 < k < \infty. \end{cases}$$

The rate of convergence of the solution of the initial-value problem to the permanent form travelling wave is found to be algebraic in t, as $t \to \infty$, when $k \in (-\infty, 2]$ and exponential in t, as $t \to \infty$, when $k \in (2, \infty)$.

1. Introduction. In this paper we consider the following initial-value problem for the Burgers-Fisher equation, namely,

$$u_t + kuu_x = u_{xx} + u(1-u), \quad -\infty < x < \infty, \quad t > 0, \tag{1.1}$$

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E-mail address: j.a.leach@bham.ac.uk
E-mail address: hana.esen@gmail.com

$$u(x,0) = \begin{cases} 1, & x \le 0, \\ 0, & x > 0, \end{cases} \tag{1.2}$$

$$u(x,t) \to \begin{cases} 1, & x \to -\infty, \\ 0, & x \to \infty, \end{cases} \qquad t \ge 0, \tag{1.3}$$

where $k \neq 0$ is a parameter and the initial distribution (1.2) is a discontinuous compressive step. In what follows we label initial-value problem (1.1)-(1.3) as **IVP**. Equation (1.1) is a canonical equation combining reaction, diffusion and convection and as such arises in the modelling of many physical phenomena involving reaction-diffusion-convection processes.

When $k = 0$ equation (1.1) reduces to the Fisher-Kolmogorov equation, which has been studied extensively (see for example [2], [3], [4], [5], [8] and [9]). In particular, Bramson [2] considered (1.1) (with $k = 0$) when the initial data has a step function profile (1.2) and determined that the solution exhibits the formation of a permanent form travelling wave solution with propagation speed given by $v(t) \sim 2 - \frac{3}{2}\frac{1}{t}$ as $t \to \infty$.

In the present paper we develop, via the method of matched asymptotic coordinate expansions, the complete large-t asymptotic structure of the solution to **IVP**. We employ the methodology developed by J. A. Leach and D. J. Needham in the context of reaction-diffusion equations (see for example the monograph [6]) which enables the complete large-t asymptotic structure of **IVP** to be obtained by careful consideration of the asymptotic structures as $t \to 0$ $(-\infty < x < \infty)$ and as $|x| \to \infty$ $(t \ge O(1))$. In particular, we establish that the solution of **IVP** exhibits the formation of a permanent form travelling wave propagating in the $+x$ direction with the minimum possible speed $c = c^*(k)$, where

$$c^*(k) = \begin{cases} 2, & -\infty < k \le 2, \\ \frac{2}{k} + \frac{k}{2}, & 2 < k < \infty. \end{cases}$$

Specifically, we establish that:

(i) When $k \in (2, \infty)$ the solution of $u(x,t)$ of **IVP** satisfies

$$u(z + s(t), t) = u_T\left(z; c^*(k)\right) + O\left(t^{-\frac{3}{2}} e^{-\left(\frac{[c^*(k)]^2}{4} - 1\right)t}\right)$$

as $t \to \infty$, uniformly in z, where $u_T(z; c^*)$ is the permanent form travelling wave solution with propagation speed $c^*(k) = \frac{2}{k} + \frac{k}{2}$ given by

$$u_T(z; c^*) = \frac{e^{-\frac{k}{2}z}}{1 + e^{-\frac{k}{2}z}},$$

$z = x - s(t)$ ($s(t)$ is a measure of the location of the wave front at time t) and

$$s(t) = \left(\frac{2}{k} + \frac{k}{2}\right)t + \phi_c + O\left(t^{-\frac{3}{2}} e^{-\left(\frac{[c^*(k)]^2}{4} - 1\right)t}\right)$$

as $t \to \infty$, where ϕ_c is a constant. We note that the correction to the propagation speed $\dot{s}(t)$ is exponential in t, as $t \to \infty$, being of $O\left(t^{-\frac{3}{2}} \exp\left\{-\left(\frac{[c^*(k)]^2}{4} - 1\right)t\right\}\right)$.

We further note that the rate of convergence of the solution of **IVP** to the permanent form travelling wave is exponential in t, as $t \to \infty$, being of

$$O\left(t^{-\frac{3}{2}} \exp\left\{-\left(\frac{[c^*(k)]^2}{4} - 1\right) t\right\}\right).$$

(ii) When $k \in (-\infty, 2]$ the solution of $u(x,t)$ of **IVP** satisfies

$$u(z + s(t), t) = u_T(z; 2) + O\left(t^{-1}\right)$$

as $t \to \infty$, uniformly in z, where $u_T(z; 2)$ is the permanent form travelling wave solution with propagation speed 2, $z = x - s(t)$ ($s(t)$ is a measure of the location of the wave front at time t) and

$$s(t) = 2t - \frac{3}{2} \ln t + O(1)$$

as $t \to \infty$. We note that the correction to the propagation speed $\dot{s}(t)$ is algebraic in t, as $t \to \infty$, being of $O\left(t^{-1}\right)$. We further note that the rate of convergence of the solution of **IVP** to the permanent form travelling wave is algebraic in t, as $t \to \infty$, being of $O\left(t^{-1}\right)$.

We conclude by noting that the methodology presented in this paper for initial-value problem **IVP** is applicable to a large class of nonlinear evolution equations, in particular, where a coherent structure forms the large-time attractor for the solution to an initial-value (or initial-boundary value) problem for a nonlinear evolution equation (or system of equations).

2. Permanent form travelling waves. In this section we review the main results concerning the existence and structure of permanent form travelling waves (PTWs) which may occur in the solution to **IVP** as $t \to \infty$. On introducing the travelling wave coordinate $z = x - ct$ (with $c > 0$ being constant wave speed), a PTW is a solution to the following nonlinear boundary value problem:

$$\begin{aligned} &u_{zz} - kuu_z + cu_z + u(1-u) = 0, \quad -\infty < z < \infty, \\ &u(z) \geq 0, \quad -\infty < z < \infty, \\ &u(z) \to 0 \quad \text{as} \quad z \to \infty, \\ &u(z) \to 1 \quad \text{as} \quad z \to -\infty. \end{aligned} \tag{2.1}$$

The nonlinear boundary value problem (2.1) can be regarded as an eigenvalue problem for the travelling wave propagation speed c (> 0). Any solution to (2.1) with $c > 0$ provides a permanent form travelling wave solution which could develop as the primary large-time structure in the solution of the initial-value problem **IVP**. The nonlinear eigenvalue problem (2.1) has received considerable attention, and it is convenient to summarize the main results in the following theorem.

THEOREM 1. Boundary value problem (2.1) has a unique PTW solution (say $u = u_T(z; c)$ with translational invariance fixed so that $u_T(0; c) = \frac{1}{2}$) for each $c \geq c^*(k)$, where

$$c^*(k) = \begin{cases} 2, & -\infty < k \leq 2, \\ \frac{2}{k} + \frac{k}{2}, & 2 < k < \infty. \end{cases}$$

Moreover,

(i) When $k \in (-\infty, 2]$

$$u_T(z;c) \sim \begin{cases} (A^*z + D^*)e^{-z} & \text{as} \quad z \to \infty, \quad c = 2, \\ B^* e^{\lambda_+ z} & \text{as} \quad z \to \infty, \quad c > 2. \end{cases} \tag{2.2}$$

(ii) When $k \in (2, \infty)$

$$u_T(z;c) \sim \begin{cases} e^{-\frac{k}{2}z} & \text{as} \quad z \to \infty, \quad c = \frac{2}{k} + \frac{k}{2}, \\ B^* e^{\lambda_+ z} & \text{as} \quad z \to \infty, \quad c > \frac{2}{k} + \frac{k}{2}, \end{cases} \tag{2.3}$$

where

$$\lambda_+ = -\frac{c}{2} + \frac{1}{2}\sqrt{c^2 - 4}.$$

Further, in each of the above cases,

$$u_T(z;c) = 1 - O\left(e^{\widehat{\lambda}_+ z}\right) \quad \text{as} \quad z \to -\infty,$$

where

$$\widehat{\lambda}_+ = -\frac{(c-k)}{2} + \frac{1}{2}\sqrt{(c-k)^2 + 4}.$$

Proof. See for example [10] and [11]. □

In the above, A^*, B^* and D^* are constants which can in principle be determined. We note that when $k \in (2, \infty)$ the exact solution of the minimum speed travelling wave can be obtained as

$$u_T\left(z; \frac{2}{k} + \frac{k}{2}\right) = \frac{e^{-\frac{k}{2}z}}{1 + e^{-\frac{k}{2}z}} \qquad \begin{cases} 1 - e^{\frac{k}{2}z} & \text{as} \quad z \to -\infty, \\ e^{-\frac{k}{2}z} & \text{as} \quad z \to \infty, \end{cases}$$

where the translational invariance has been fixed so that $u_T\left(0; \frac{2}{k} + \frac{k}{2}\right) = \frac{1}{2}$.

3. The evolution of permanent form travelling waves in the solution of IVP. In this section we develop the asymptotic structure of the solution to **IVP** as $t \to \infty$ with particular attention to PTW formation. We must begin by examining the asymptotic structure of the solution to **IVP** as $t \to 0$.

3.1. *Asymptotic solution to* **IVP** *as* $t \to 0$. Consideration of the initial data (1.2) indicates that the structure of the asymptotic solution to **IVP** as $t \to 0$ has three asymptotic regions for $x \in (-\infty, \infty)$, namely,

$$\left.\begin{array}{lll} \text{region } \mathbf{I}: & x = o(1), & u(x,t) = O(1) \\ \text{region } \mathbf{II}^+: & x = O(1)\ (>0), & u(x,t) = o(1) \\ \text{region } \mathbf{II}^-: & x = O(1)\ (<0), & u(x,t) = 1 - o(1) \end{array}\right\} \quad \text{as} \quad t \to 0. \tag{3.1}$$

For brevity we summarize the structure of the solution of **IVP** in each of the above regions (the details following, after some modification, those given for a similar problem [7]):

Region I

$$u(\eta, t) = \frac{1}{2}\text{erfc}\left(\frac{\eta}{2}\right) + o(1) \tag{3.2}$$

as $t \to 0$ with $\eta = xt^{-\frac{1}{2}} = O(1)$, and where erfc(.) is the standard complementary error function (see [1]).

Region II$^+$

$$u(x,t) = \exp\left(-\frac{x^2}{4t} + \frac{1}{2}\ln t - \ln x - \ln\sqrt{\pi} + o(1)\right) \tag{3.3}$$

as $t \to 0$ with $x = O(1)(> 0)$.

Region II$^-$

$$u(x,t) = 1 - \exp\left(-\frac{x^2}{4t} + \frac{1}{2}\ln t + \frac{k}{2}x - \ln(-x) - \ln\sqrt{\pi} + o(1)\right) \tag{3.4}$$

as $t \to 0$ with $x = O(1)(< 0)$.

This completes the asymptotic structure as $t \to 0$, with expansions (3.2), (3.3) and (3.4) providing a uniform approximation in $x \in (-\infty, \infty)$ to the solution of **IVP** as $t \to 0$. We next consider the asymptotic structure of **IVP** as $|x| \to \infty$ with $t = O(1)$.

3.2. *Asymptotic solution to* **IVP** *as* $|x| \to \infty$. Again for brevity we summarize the asymptotic structure of the solution to **IVP** as $x \to \infty$ and $x \to -\infty$ (the details following, after some modification, those given in [7]):

Region III$^+$

$$u(x,t) = \exp\left(-\frac{x^2}{4t} - \ln x + t + \frac{1}{2}\ln t - \ln\sqrt{\pi} + o(1)\right) \tag{3.5}$$

as $x \to \infty$ with $t = O(1)$. Expansion (3.5) remains uniform for $t \gg 1$ provided that $x \gg t$, but becomes nonuniform when $x = O(t)$ as $t \to \infty$.

Region III$^-$

$$u(x,t) = 1 - \exp\left(-\frac{x^2}{4t} + \frac{k}{2}x - \ln(-x) - \left(1 + \frac{k^2}{4}\right)t + \frac{1}{2}\ln t - \ln\sqrt{\pi} + o(1)\right) \tag{3.6}$$

as $x \to -\infty$ with $t = O(1)$. Expansion (3.6) remains uniform for $t \gg 1$ provided that $(-x) \gg t$, but becomes nonuniform when $(-x) = O(t)$ as $t \to \infty$.

3.3. *Asymptotic solution to* **IVP** *as* $t \to \infty$. As $t \to \infty$, the asymptotic expansions (3.5) and (3.6) of regions III$^+$ ($x \to \infty, t = O(1)$) and III$^-$ ($x \to -\infty, t = O(1)$) respectively continue to remain uniform provided $|x| \gg t$. However, as already noted, a nonuniformity develops when $|x| = O(t)$.

We begin by considering the asymptotic structure as $t \to \infty$ for $x > 0$. To proceed we introduce a new region, region IV$^+$, when $x = O(t)$ as $t \to \infty$. To examine region IV$^+$ we introduce the scaled coordinate $y = \frac{x}{t}$, where $y = O(1)$ as $t \to \infty$, and write (as suggested by (3.5))

$$u(y,t) = e^{-f(y,t)t}, \tag{3.7}$$

where

$$f(y,t) = f_0(y) + f_1(y)\frac{\ln t}{t} + f_2(y)\frac{1}{t} + o\left(\frac{1}{t}\right) \tag{3.8}$$

as $t \to \infty$, with $y = O(1)$. On substituting (3.7) and (3.8) into equation (1.1) (when written in terms of y and t) we obtain the leading order problem as

$$(f_0')^2 - yf_0' + f_0 + 1 = 0, \quad y > 0, \tag{3.9}$$

$$f_0(y) > 0, \quad y > 0, \tag{3.10}$$

$$f_0(y) = \frac{y^2}{4} - 1 \quad \text{as} \quad y \to \infty. \tag{3.11}$$

The final condition, (3.11), arises from matching expansion (3.7) ($y \gg 1$) with expansion (3.5) ($x = O(t)$). Equation (3.9) admits the constant solution $f_0 = -1$, the one-parameter family of linear solutions

$$f_0(y) = A\left(y - \left[A + \frac{1}{A}\right]\right), \quad y \in (-\infty, \infty), \tag{3.12}$$

for each $A \in \mathbb{R}$, together with the associated envelope (singular) solution,

$$f_0(y) = \frac{y^2}{4} - 1, \quad y \in (-\infty, \infty). \tag{3.13}$$

Combinations of (3.12) and (3.13) which remain continuous and differentiable also provide solutions to (3.9) (envelope-touching solutions).

Therefore, the solution of (3.9)-(3.11) is given by either:

(i) the envelope solution

$$f_0(y) = \frac{y^2}{4} - 1, \quad 2 < y < \infty, \text{ or} \tag{3.14}$$

(ii) the envelope-touching solution

$$f_0(y) = \begin{cases} \frac{y^2}{4} - 1, & 2A < y < \infty, \\ A\left(y - \left[A + \frac{1}{A}\right]\right) & A + \frac{1}{A} < y \le 2A, \end{cases} \tag{3.15}$$

where $A > 1$.

We conclude that a nonuniformity occurs in expansion (3.7), (3.8) as $y \to y_c^+$ (≥ 2) where

$$y_c = \begin{cases} 2, & A = 1, \\ A + \frac{1}{A}\,(> 2), & A > 1, \end{cases}$$

for some $A \ge 1$ (when $A = 1$, $f_0(y)$ is given by (3.14), whilst when $A > 1$, $f_0(y)$ is given by (3.15)). A consideration of further terms in (3.8) demonstrates that this nonuniformity occurs when

$$y = y_c + O(t^{-1}) \quad \text{with} \quad u = O(1)$$

as $t \to \infty$. We must introduce a further region which we denote as region TW. In this region we write

$$y = y_c + \frac{z}{t}, \tag{3.16}$$

with $z = O(1)$ as $t \to \infty$, and expand as

$$u(z,t) = u_c(z) + o(1) \tag{3.17}$$

as $t \to \infty$ with $z = O(1)$. On substituting expansion (3.17) into (1.1) (when written in terms of z and t) we obtain the leading order problem as

$$\begin{aligned} &u_{zz} - kuu_z + y_c u_z + u(1-u) = 0, \quad -\infty < z < \infty, \\ &u(z) \geq 0, \quad -\infty < z < \infty, \\ &u(z) \to 0 \quad \text{as} \quad z \to \infty, \\ &u(z) \quad \text{bounded as} \quad z \to -\infty. \end{aligned} \tag{3.18}$$

Condition $(3.18)_3$ arises from matching expansion (3.17) (as $z \to \infty$) with expansion (3.7), (3.8) (as $y \to y_c^+$). Moreover, a phase plane analysis of equation $(3.18)_1$ with conditions $(3.18)_2$ and $(3.18)_3$ allows boundary condition $(3.18)_4$ to be replaced by

$$u_c(z) \to 1 \quad \text{as} \quad z \to -\infty. \tag{3.19}$$

We now recognize the nonlinear boundary value problem as being precisely (2.1), its solutions representing permanent form travelling wave structures. We can now appeal to Theorem 1: boundary value problem (3.18), (3.19) has a unique solution $u_c(z) = u_T(z; y_c)$ for each

$$y_c \in \begin{cases} [2, \infty), & -\infty < k \leq 2, \\ \left[\frac{2}{k} + \frac{k}{2}, \infty\right), & k > 2, \end{cases}$$

where y_c is the wave speed.

We next match expansion (3.7), (3.8) of region IV$^+$ (as $y \to y_c^+$) to expansion (3.17) (as $z \to \infty$) to leading order in each of the distinct cases $k \in (2, \infty)$ and $k \in (-\infty, 2]$. When $k \in (2, \infty)$ we have, via (3.17) and (2.3), that

$$u \sim \begin{cases} e^{-\frac{k}{2}z} & \text{as} \quad z \to \infty, \quad c = \frac{2}{k} + \frac{k}{2}, \\ B^* e^{\lambda_+ z} & \text{as} \quad z \to \infty, \quad c > \frac{2}{k} + \frac{k}{2}. \end{cases}$$

On expanding expansion (3.7), (3.8) to $O(1)$ in region TW we obtain that

$$u = O\left(e^{-Az}\right) \quad \text{for} \quad z \gg 1, \ A \geq 1.$$

Therefore, matching following the matching principle of Van Dyke [13] requires that

$$A = \frac{k}{2} \ (> 1),$$

giving that $y_c = \frac{2}{k} + \frac{k}{2}$ and the travelling wave solution of minimum propagation speed $\frac{2}{k} + \frac{k}{2}$ is selected in region TW. When $k \in (-\infty, 2]$ we have, via (3.17) and (2.2), that

$$u \sim \begin{cases} (A^* z + D^*) e^{-z} & \text{as} \quad z \to \infty, \quad c = 2, \\ B^* e^{\lambda_+ z} & \text{as} \quad z \to \infty, \quad c > 2. \end{cases}$$

In this case matching requires that

$$A = 1,$$

giving that $y_c = 2$ and the travelling wave solution of minimum propagation speed 2 is selected in region TW.

In conclusion, we have established that the travelling wave solution of minimum speed, $c = c^*(k)$, is selected in region TW where

$$c^*(k) = \begin{cases} 2, & -\infty < k \le 2, \\ \frac{2}{k} + \frac{k}{2}, & 2 < k < \infty. \end{cases}$$

Further, we have established that when $k \in (-\infty, 2]$, $f_0(y)$ is given by (3.14), while when $k \in (2, \infty)$, $f_0(y)$ is given by (3.15).

In what follows we will consider the cases $k \in (-\infty, 2]$ and $k \in (2, \infty)$ separately.

3.3.1. $k \in (2, \infty)$. In this case $f_0(y)$ is given by (3.15) with $A = \frac{k}{2}$, that is,

$$f_0(y) = \begin{cases} \frac{y^2}{4} - 1, & k < y < \infty, \\ \frac{k}{2}\left(y - \left[\frac{2}{k} + \frac{k}{2}\right]\right) & c^*(k) < y \le k, \end{cases} \tag{3.20}$$

and $c^*(k) = \frac{2}{k} + \frac{k}{2}$. Figure 1 gives a sketch of the solution (3.20). We observe that although $f_0(y)$ and $f_0'(y)$ are continuous at $y = k$, the second derivative $f_0''(y)$ is discontinuous at the point $y = k$. This indicates that a thin transition region, region TR$^+$, exists in the neighbourhood of the point $y = k$, in which second derivatives are retained at leading order to smooth out the discontinuity in curvature. Hence, region IV$^+$ is replaced by three regions, region IV$^+$(a) ($k < y < \infty$), region TR$^+$ (transition region) and region IV$^+$(b) ($c^*(k) < y < k$). We first consider region IV$^+$(a). On substituting (3.7), (3.8) (with $(3.20)_1$) into equation (1.1) (when written in terms of y and t) gives, on solving at each order in turn and matching to expansion (3.5) as $y \to \infty$, that

$$u(y,t) = \exp\left(-\left(\frac{y^2}{4} - 1\right)t - \frac{1}{2}\ln t - H(y) + o(1)\right) \tag{3.21}$$

as $t \to \infty$ with $y = O(1)$ ($\in (k, \infty)$), where the function $H(y)$ remains undetermined, but having

$$H(y) \sim \ln y + \ln\sqrt{\pi} \quad \text{as} \quad y \to \infty.$$

Further, we make the assumption (which we will verify as consistent) that

$$H(y) \sim \ln(y - k) + \gamma_1 \quad \text{as} \quad y \to k^+,$$

where γ_1 is a constant to be determined. It is instructive to note that the expansion in region IV$^+$(b) is given by

$$u(y,t) = \exp\left(-\frac{k}{2}\left(y - \left[\frac{2}{k} + \frac{k}{2}\right]\right)t - c_1\ln t - c_1\ln(k - y) - c_2 + o(1)\right) \tag{3.22}$$

as $t \to \infty$ with $y = O(1)$ ($\in (c^*(k), k)$), and where c_1 and c_2 are constants to be determined.

Before considering region TR$^+$ it is instructive to examine the asymptotic structure for $y < c^*(k)$. To proceed we introduce a new region, region IV$^-$, and write (as suggested by (3.6))

$$u(y,t) = 1 - e^{-g(y,t)t}, \tag{3.23}$$

where

$$g(y,t) = g_0(y) + g_1(y)\frac{\ln t}{t} + g_2(y)\frac{1}{t} + o\left(\frac{1}{t}\right) \tag{3.24}$$

as $t \to \infty$, with $y = O(1)$ ($\in (-\infty, c^*(k))$) and where $g_0(y) > 0$. On substituting (3.23) and (3.24) into equation (1.1) (when written in terms of y and t) we obtain the leading order problem as

$$(g_0')^2 + (k - y)g_0' + g_0 - 1 = 0, \quad y < c^*(k), \tag{3.25}$$

$$g_0(y) > 0, \quad y < c^*(k), \tag{3.26}$$

$$g_0(y) = \frac{(y-k)^2}{4} + 1 \quad \text{as} \quad y \to -\infty, \tag{3.27}$$

$$g_0(y) = -\frac{k}{2}\big(y - c^*(k)\big) \quad \text{as} \quad y \to c^*(k)^-. \tag{3.28}$$

Condition (3.27) arises from matching expansion (3.23) ($-y \gg 1$) with expansion (3.6) ($-x = O(t)$), while the final condition (3.28) is the matching condition to allow matching with expansion (3.17) (as $z \to -\infty$) of region TW. Equation (3.25) admits the constant solution $g_0 = 1$, the one-parameter family of linear solutions

$$g_0(y) = \widehat{A}\left(y - \left[\widehat{A} + k - \frac{1}{\widehat{A}}\right]\right), \quad y \in (-\infty, \infty), \tag{3.29}$$

for each $\widehat{A} \in \mathbb{R}$, together with the associated envelope (singular) solution

$$g_0(y) = \frac{(y-k)^2}{4} + 1, \quad y \in (-\infty, \infty). \tag{3.30}$$

Combinations of (3.29) and (3.30) which remain continuous and differentiable also provide solutions to (3.9) (envelope-touching solutions). It is straightforward to establish that the required solution of (3.25)-(3.28) is given by the envelope-touching solution

$$g_0(y) = \begin{cases} \frac{(y-k)^2}{4} + 1, & -\infty < y < 0, \\ -\frac{k}{2}\left(y - c^*(k)\right) & 0 \le y < c^*(k), \end{cases} \tag{3.31}$$

where $\widehat{A} = -\frac{k}{2}$. Figure 1 gives a sketch of the solution (3.31). Therefore, region IV^- is replaced by three regions, region IV^-(a) ($-\infty < y < 0$), region TR^- (transition region) and region IV^-(b) ($0 < y < c^*(k)$). On continuing expansion (3.23), (3.24) in regions IV^-(a) (where $g_0(y)$ is given by $(3.31)_1$) and IV^-(b) (where $g_0(y)$ is given by $(3.31)_2$) we obtain

Region IV^-(a)

$$u(y,t) = 1 - \exp\left(-\left(\frac{(y-k)^2}{4} + 1\right)t - \frac{1}{2}\ln t - \widehat{H}(y) + o(1)\right) \tag{3.32}$$

as $t \to \infty$ with $y = O(1)$ ($\in (-\infty, 0)$), where the function $\widehat{H}(y)$ remains undetermined, but having

$$\widehat{H}(y) \sim \ln(-y) + \ln\sqrt{\pi} \quad \text{as} \quad y \to -\infty.$$

Further, we make the assumption (which we will verify as consistent) that

$$\widehat{H}(y) \sim \ln(-y) + \hat{\gamma}_1 \quad \text{as} \quad y \to k^+,$$

where $\hat{\gamma}_1$ is a constant to be determined.

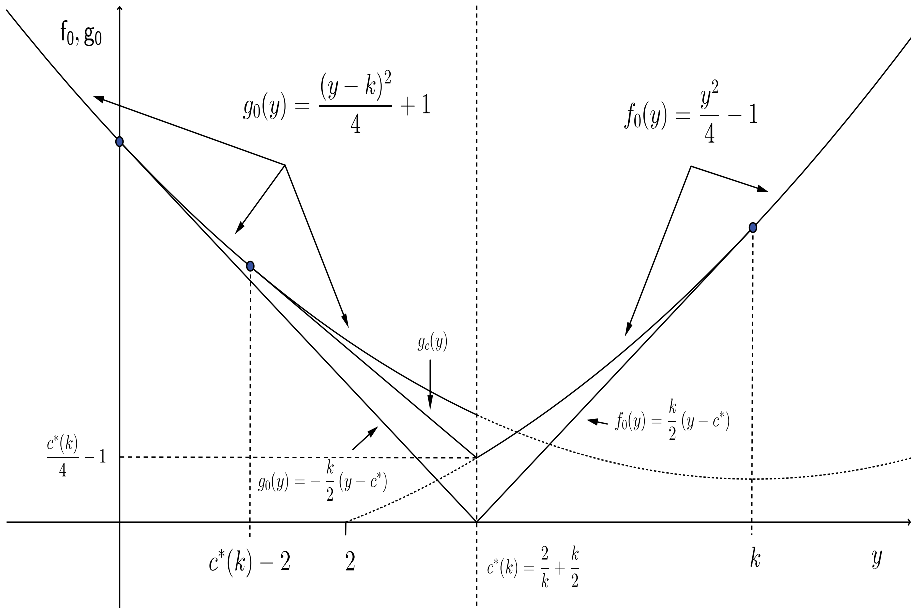

FIG. 1. A sketch of the functions $f_0(y)$ and $g_0(y)$ when $k \in (2, \infty)$. We note that $g_c(y) = \widehat{A}_c \left(y - \left(\widehat{A}_c + k - \frac{1}{\widehat{A}_c} \right) \right)$ for $y \in (c^*(k) - 2, c^*(k))$ and where $\widehat{A}_c = \frac{1}{2}\left(\frac{2}{k} - \frac{k}{2}\right) - 1$.

Region IV$^-$(b)

$$u(y,t) = 1 - \exp\left(\frac{k}{2}\left(y - c^*(k)\right)t - \hat{c}_1 \ln t - \hat{c}_1 \ln(y) - \hat{c}_2 + o(1) \right) \tag{3.33}$$

as $t \to \infty$ with $y = O(1)$ $(\in (0, c^*(k)))$, and where $\hat{c}_1$ and $\hat{c}_2$ are constants to be determined.

We will return and complete the transition region, region TR$^-$, once the constant $\hat{c}_1$ has been determined.

We now return to region TW and recall, via (3.17), that

$$u(z,t) = u_T(z; c^*(k)) + o(1) \tag{3.34}$$

as $t \to \infty$ with $z = O(1)$, where $z = x - s(t)$ and $s(t) = c^*(k)t + \theta(t) + \phi_c + \chi(t)$ as $t \to \infty$. Here $1 \ll \theta(t) \ll t$, ϕ_c is a constant, and $\chi(t) = o(1)$ as $t \to \infty$ and are as yet undetermined gauge functions (to be fixed on matching with regions IV$^\pm$(b)), whilst $u_T(z; c^*(k))$ represents the minimum speed permanent form travelling wave solution. We recall from Section 2 that in this case

$$u_T(z; c^*(k)) = \frac{e^{-\frac{k}{2}z}}{1 + e^{-\frac{k}{2}z}}, \tag{3.35}$$

with the following asymptotic properties:

$$u_T(z; c^*(k)) \sim \begin{cases} e^{-\frac{k}{2}z}, & \text{as} \quad z \to \infty, \\ 1 - e^{\frac{k}{2}z}, & \text{as} \quad z \to -\infty. \end{cases} \tag{3.36}$$

Now on matching expansion (3.22) of region IV$^+$(b) (as $y \to c^*(k)^+$) to expansion (3.34) (with $(3.36)_1$) of region TW we obtain that

$$\theta(t) = -\frac{2c_1}{k}\ln t, \quad \frac{k}{2}\phi_c + c_2 + c_1 \ln(k - c^*(k)) = 0,$$
$$\chi(t) = \begin{cases} -\frac{4c_1^2}{k^2}\frac{1}{(k-c^*(k))}\frac{\ln t}{t}, & c_1 \neq 0, \\ O(\text{EXP}), & c_1 = 0, \end{cases} \tag{3.37}$$

where $O(\text{EXP})$ is exponentially small in t as $t \to \infty$. On matching expansion (3.33) of region IV$^-$(b) (as $y \to c^*(k)^-$) to expansion (3.34) (with $(3.36)_2$) of region TW we obtain that

$$\theta(t) = \frac{2\hat{c}_1}{k}\ln t, \quad \frac{k}{2}\phi_c - \hat{c}_2 - \hat{c}_1 \ln c^*(k) = 0,$$
$$\chi(t) = \begin{cases} -\frac{4\hat{c}_1^2}{k^2}\frac{1}{c^*(k)}\frac{\ln t}{t}, & c_1 \neq 0, \\ O(\text{EXP}), & c_1 = 0. \end{cases} \tag{3.38}$$

Consideration of (3.37) and (3.38) requires for consistency that

$$\hat{c}_1 = c_1 = 0, \quad \hat{c}_2 = -c_2 = \frac{k}{2}\phi_c, \tag{3.39}$$

and

$$\chi(t) = O(\text{EXP}) \quad \text{as} \quad t \to \infty, \quad \theta(t) = 0. \tag{3.40}$$

Now that we have determined that $c_1 = 0$ and $c_2 = -\frac{k}{2}\phi_c$ we can return to region TR$^+$. An examination of expansion (3.21) (as $y \to k^+$) and expansion (3.22) (as $y \to k^-$) indicates that in this region $y = k + O(t^{-\frac{1}{2}})$ as $t \to \infty$. Therefore, to examine region TR$^+$ we introduce the scaled coordinate $\eta = (y-k)t^{\frac{1}{2}}$, where $\eta = O(1)$ as $t \to \infty$, and look for an expansion of the form

$$u(\eta, t) = \Big(F(\eta) + o(1)\Big)\exp\left\{-\left(\frac{k^2}{4} - 1\right)t - \frac{k}{2}\eta t^{\frac{1}{2}}\right\} \tag{3.41}$$

as $t \to \infty$ with $\eta = O(1)$. On substitution of (3.41) into equation (1.1) (when written in terms of η and t) we obtain at leading order

$$F_{\eta\eta} + \frac{\eta}{2}F_\eta = 0, \quad -\infty < \eta < \infty. \tag{3.42}$$

Equation (3.42) is to be solved subject to the matching conditions

$$F(\eta) \sim \begin{cases} e^{\frac{k}{2}\phi_c} & \text{as} \quad \eta \to -\infty, \\ \frac{e^{-\gamma_1}}{\eta}e^{-\frac{\eta^2}{4}} & \text{as} \quad \eta \to \infty. \end{cases} \tag{3.43}$$

The solution to (3.42) subject to conditions (3.43) is readily obtained as

$$u(\eta, t) = \left(\frac{1}{2}e^{\frac{k}{2}\phi_c}\text{erfc}\left(\frac{\eta}{2}\right) + o(1)\right)\exp\left\{-\left(\frac{k^2}{4} - 1\right)t - \frac{k}{2}\eta t^{\frac{1}{2}}\right\} \tag{3.44}$$

as $t \to \infty$ with $\eta = O(1)$, and where the constant $\gamma_1 = -\frac{k}{2}\phi_c + \ln\sqrt{\pi}$. The details of the transition region, region TR$^+$, are now complete.

The details of region TR$^-$ follow, after some minor modification, those given above for region TR$^+$ and are summarized here for brevity.

Region TR^-

$$u(\zeta,t) = \left(e^{-\frac{k}{2}\phi_c}\left[1-\frac{1}{2}\mathrm{erfc}\left(\frac{\zeta}{2}\right)\right]+o(1)\right)\exp\left\{-\left(\frac{k^2}{4}+1\right)t+\frac{k}{2}\zeta t^{\frac{1}{2}}\right\} \tag{3.45}$$

as $t\to\infty$ with $\zeta = yt^{\frac{1}{2}} = O(1)$, and where the constant $\hat{\gamma}_1 = \frac{k}{2}\phi_c + \ln\sqrt{\pi}$.

Now that we have been able to complete the transition regions, regions $\mathrm{TR}^{\pm}$, we can obtain the correction terms to expansions (3.22) (of region IV^+(b)) and (3.33) (of region IV^-(b)). In particular, the correction term to expansion (3.22) is required in order to be able to determine the rate of convergence of the solution of **IVP** to the PTW. We begin by developing expansion (3.22).

The structure of (3.44) for $(-\eta)\gg 1$ (as we move into region IV^+(b)) indicates that the correction term to expansion (3.22) is $O\left(t^{-\frac{1}{2}}\exp\left\{-\left(\frac{y^2}{4}-1\right)t\right\}\right)$ as $t\to\infty$. The expansion in region $\mathrm{IV}^+(b)$ is now given (after some calculation) as

$$u(y,t) = \exp\left(-\frac{k}{2}\left(y-c^*(k)\right)t+\frac{k}{2}\phi_c\right)-\exp\left(-\left(\frac{y^2}{4}-1\right)t-\frac{1}{2}\ln t - H_1(y)\right) + o\left(t^{-\frac{1}{2}}\exp\left(-\left(\frac{y^2}{4}-1\right)t\right)\right) \tag{3.46}$$

as $t\to\infty$ with $y=O(1)$ $(\in (c^*(k),k))$. The function $H_1(y)$ remains undetermined at this order, but matching with (3.44) as $y\to k^-$ requires that

$$H_1(y)\sim \ln(k-y)-\frac{k}{2}\phi_c+\ln\sqrt{\pi}\quad \text{as}\quad y\to k^-.$$

Further, we make the assumption (which we will verify as consistent) that

$$H_1(y)\sim \beta_0\ln(y-c^*(k))+\beta_1\quad\text{as}\quad y\to c^*(k)^+,$$

where β_0 and β_1 are constants to be determined. As $y\to c^*(k)^+$ we move from region IV^+(b) into region TW. On writing expansion (3.46) in terms of the travelling wave variable z we obtain that

$$u(z,t)\sim e^{-\frac{k}{2}z}\left(1-\frac{k}{2}\chi(t)+\dots\right)-\frac{t^{\beta_0-\frac{1}{2}}}{z^{\beta_0}e^{\beta_1}}e^{-\frac{1}{2}c^*(k)[z+\phi_c]}e^{-\left(\frac{[c^*(k)]^2}{4}-1\right)t}+\dots \tag{3.47}$$

as $t\to\infty$ with $z\gg 1$. We conclude from (3.47) that in region TW we must have

$$u(z,t) = u_T(z;c^*(k)) + O(\chi(t)) \tag{3.48}$$

as $t\to\infty$ with $z=O(1)$, and where $u_T(z;c^*(k))$ is given by (3.35). On substituting (3.48) into equation (1.1) (when written in terms of z and t) we require that

$$\dot{\chi}(t) = O(\chi(t))\quad\text{as}\quad t\to\infty. \tag{3.49}$$

We conclude, via (3.49), that $\chi(t)$ must be exponentially small in t as $t\to\infty$. Thus we set

$$\chi(t) = \bar{A}t^{\epsilon}e^{-\sigma t}\left[1+o(1)\right] \tag{3.50}$$

as $t \to \infty$, with the constants $\bar{A}$, ϵ and $\sigma(>0)$ to be determined. We now continue the expansion in region TW as

$$u(z,t) = u_T(z; c^*(k)) + u_1(z)\chi(t) + o(\chi(t)) \tag{3.51}$$

as $t \to \infty$ with $z = O(1)$. On substituting (3.51) into equation (1.1) (when written in terms of z and t) we obtain at $O(\chi(t))$ that

$$u_1'' + \alpha_0(z)\, u_1' + \alpha_1(z)\, u_1 = \sigma u_T'(z; c^*(k)) \tag{3.52}$$

where $\alpha_0(z) = c^*(k) - k\, u_T(z; c^*(k))$ and $\alpha_1 = \sigma - k\, u_T'(z; c^*(k)) - 2\, u_T(z; c^*(k)) + 1$. We now determine the asymptotic properties of $u_1(z)$ as $|z| \to \infty$. For $z \gg 1$, $\alpha_0(z) \sim c^*(k)$ and $\alpha_1(z) \sim (\sigma + 1)$, giving that

$$u_1(z) \sim \begin{cases} A_1 e^{s_+ z} + B_1 e^{s_- z} - \frac{k}{2} e^{-\frac{k}{2} z} & \text{if} \quad \sigma < \frac{[c^*(k)]^2}{4} - 1, \\ (A_1 z + B_1) e^{-\frac{c^*(k)}{2} z} - \frac{k}{2} e^{-\frac{k}{2} z} & \text{if} \quad \sigma = \frac{[c^*(k)]^2}{4} - 1, \end{cases} \tag{3.53}$$

as $z \to \infty$, where A_1 and B_1 are constants and

$$s_{\pm} = -\frac{c^*(k)}{2} \pm \frac{1}{2}\sqrt{[c^*(k)]^2 - 4\sigma - 4}.$$

We note that the case when $\sigma > \frac{[c^*(k)]^2}{4} - 1$ can be excluded, as this would lead to oscillatory solutions and matching with (3.47) would not be possible. For $(-z) \gg 1$, $\alpha_0(z) \sim c^*(k) - k$ and $\alpha_1(z) \sim (\sigma - 1)$, giving that

$$u_1(z) = O\left(e^{m_+ z}\right) \tag{3.54}$$

as $z \to -\infty$, where

$$m_+ = -\frac{1}{2}\left(\frac{2}{k} - \frac{k}{2}\right) + \frac{1}{2}\sqrt{[c^*(k)]^2 - 4\sigma}.$$

We note that $m_+ > 0$ for $k \in (2, \infty)$.

On matching expansion (3.51) (with (3.50) and (3.53)) as $z \to \infty$ with expansion (3.47) up to exponentially small terms of $O\left(t^{\epsilon} e^{-\sigma t}\right)$ requires immediately that

$$\sigma = \frac{[c^*(k)]^2}{4} - 1 \tag{3.55}$$

and

$$\epsilon = -\frac{3}{2}, \quad \beta_0 = -1, \quad B_1 = 0, \quad \bar{A} = -e^{-\frac{c^*(k)}{2}\phi_c - \beta_1}.$$

Further, for this selected value of σ we have that

$$m_+ = -\frac{1}{2}\left(\frac{2}{k} - \frac{k}{2}\right) + 1.$$

Therefore, we have established that when $k \in (2, \infty)$ the large-time structure in region TW is dominated by the evolution of the PTW with speed $c = c^*(k)(= \frac{2}{k} + \frac{k}{2})$ (this being the minimum speed available). In summary, we have in region TW that

$$u(z,t) = u_T(z; c^*(k)) + O\left(t^{-\frac{3}{2}} e^{-\left(\frac{[c^*(k)]^2}{4} - 1\right)t}\right) \tag{3.56}$$

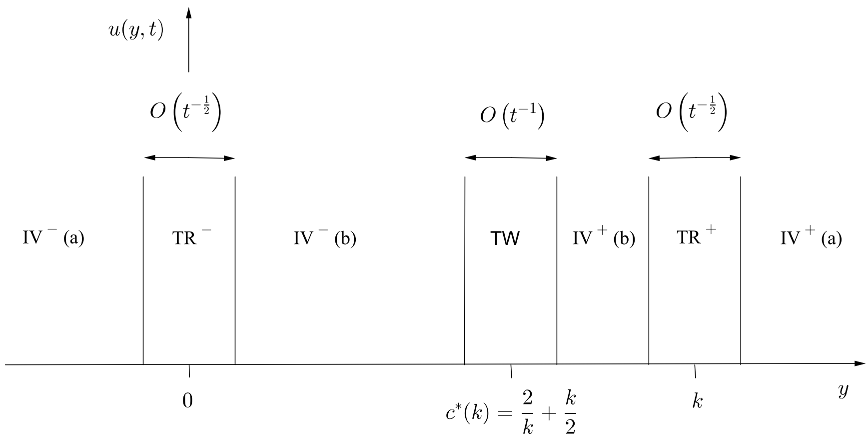

FIG. 2. Schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ when $k \in (2, \infty)$.

as $t \to \infty$ with $z = O(1)$, and where $u_T(z; c^*(k))$ is given by (3.35). We observe from (3.56) that the rate of convergence of the solution to **IVP** to the PTW is exponential in t as $t \to \infty$.

Although we will not pursue the full details of the correction to expansion (3.33) in region IV$^-$(b), it is instructive to consider the leading order structure of the correction term. On recalling that $\hat{c}_1 = 0$ and $\hat{c}_2 = \frac{k}{2}\phi_c$, we have in region IV$^-$(b) that

$$u(y,t) = 1 - \exp\left(\frac{k}{2}\left(y - c^*(k)\right)t - \frac{k}{2}\phi_c\right) + e^{-G(y,t)t}, \tag{3.57}$$

where

$$G(y,t) = g_c(y) + o(1) \tag{3.58}$$

as $t \to \infty$ with $y = O(1)$ $(\in (0, c^*(k)))$. It is straightforward to establish that $g_c(y)$ is given by the envelope-touching solution

$$g_c(y) = \begin{cases} \frac{(y-k)^2}{4} + 1, & 0 < y < c^*(k) - 2, \\ \widehat{A}_c\left(y - \left(\widehat{A}_c + k - \frac{1}{\widehat{A}_c}\right)\right), & c^*(k) - 2 \leq y < c^*(k), \end{cases} \tag{3.59}$$

where $\widehat{A}_c = \frac{1}{2}\left(\frac{2}{k} - \frac{k}{2}\right) - 1$. The envelope-touching solution (3.59) is sketched in Figure 1. This change of structure in $g_c(y)$ is required to facilitate matching with region TW (as $y \to c*(k)^-$) and region TR$^-$ (as $y \to 0^+$). Therefore, to accommodate this change in structure of $g_c(y)$, region IV$^-$(b) would need to be divided into three further regions, but we do not pursue this here.

This then completes the asymptotic structure of the solution of **IVP** as $t \to \infty$ in the case when $k \in (2, \infty)$. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Figure 2.

3.3.2. $k \in (-\infty, 2]$. In this case $f_0(y)$ is given by (3.14) with $A = 1$, that is,

$$f_0(y) = \frac{y^2}{4} - 1, \quad y \in (2, \infty), \tag{3.60}$$

and $c^*(k) = 2$. Figure 3 gives a sketch of the solution (3.60) for $y \geq 2$. On continuing expansion (3.7), (3.8) we have in region IV$^+$ that

$$u(y,t) = \exp\left(-\left(\frac{y^2}{4} - 1\right)t - \frac{1}{2}\ln t - H(y) + o(1)\right) \tag{3.61}$$

as $t \to \infty$ with $y = O(1)$ ($\in (2,\infty)$), where the function $H(y)$ remains undetermined but having

$$H(y) \sim \ln y + \ln\sqrt{\pi} \quad \text{as} \quad y \to \infty.$$

As $y \to 2^+$ we move out of region IV$^+$ into region TW (the PTW region), where $y = 2 \pm O(t^{-1})$ as $t \to \infty$. In region TW we have that

$$u(z,t) = u_T(z,2) + O\left(\dot{s}(t) - 2\right) \tag{3.62}$$

as $t \to \infty$ with $z = O(1)$, where $u_T(z;2)$ is the permanent for travelling wave solution with propagation speed 2, $z = x - s(t)$ ($s(t)$ is a measure of the location of the wave front at time t) and

$$s(t) = 2t + \theta(t) + \phi_c$$

as $t \to \infty$, where ϕ_c is a constant and $1 \ll \theta(t) \ll t$ as $t \to \infty$. We further recall from Section 2 that

$$u_T(z;2) \sim \left(A^*z + D^*\right)e^{-z} \quad \text{as} \quad z \to \infty. \tag{3.63}$$

Matching expansion (3.62) with (3.63) as $z \to \infty$ to expansion (3.61) (as $y \to 2^+$) requires that

$$H(y) \sim \ln(y-2) \quad \text{as} \quad y \to 2^+, \quad \theta(t) = -\frac{3}{2}\ln t, \quad \phi_c = -\ln A^*.$$

Therefore, we have established that when $k \in (-\infty, 2]$ the large-time structure in region TW is dominated by the evolution of the PTW with speed $c = c^*(k)(= 2)$ (this being the minimum speed available). In summary, we have in region TW that

$$u(z,t) = u_T(z;2) + O\left(t^{-1}\right) \tag{3.64}$$

as $t \to \infty$ with $z = O(1)$, where $u_T(z;c^*(k))$ is the permanent for travelling wave solution with propagation speed 2, and

$$s(t) = 2t - \frac{3}{2}\ln t - \ln A^*. \tag{3.65}$$

We observe from (3.65) that the rate of convergence of the solution to **IVP** to the PTW is algebraic in t as $t \to \infty$.

We conclude this section by summarizing the asymptotic structure of the solution of **IVP** as $t \to \infty$ in $y < 2$. The asymptotic structure of the solution of **IVP** in $y < 0$ in this case follows, after some minor modifications, that given in Section 3.3.1. In particular, we note that

$$g_0(y) = \begin{cases} \frac{(y-k)^2}{4} + 1, & -\infty < y < k - 2\widehat{\lambda}_+, \\ -\widehat{\lambda}_+\,(y-2) & k - 2\widehat{\lambda}_+ \leq y < 2, \end{cases} \tag{3.66}$$

where $\widehat{A} = -\widehat{\lambda}_+$. Function $g_0(y)$ is sketched in Figure 3 for $y \in (-\infty, 2]$. Therefore, region IV$^-$ is as in Section 3.3.1 subdivided into three regions IV$^-$(a), TR$^-$ and IV$^-$(b)

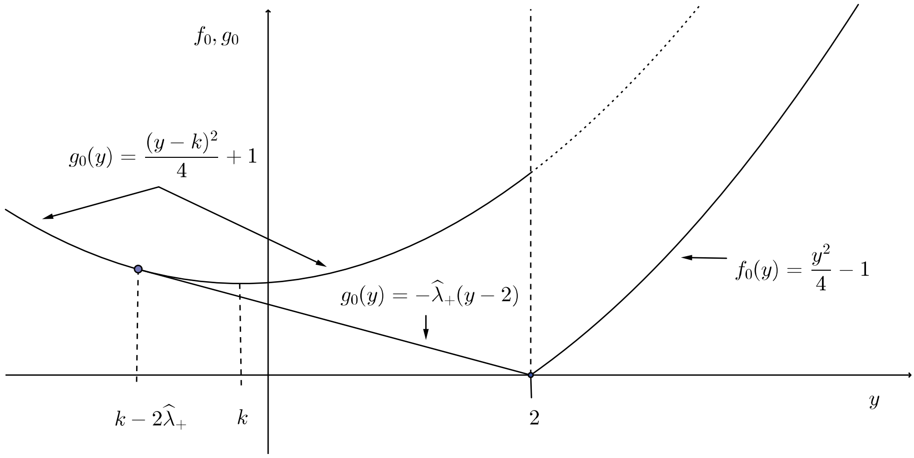

FIG. 3. A sketch of the functions $f_0(y)$ and $g_0(y)$ when $-\infty < k \leq 2$.

to accommodate the change in structure of $g_0(y)$. The details of these regions are summarized below:

Region IV$^-$(a)

$$u(y,t) = 1 - \exp\left(-\left(\frac{(y-k)^2}{4} + 1\right)t - \frac{1}{2}\ln t - \widehat{H}(y) + o(1)\right) \tag{3.67}$$

as $t \to \infty$ with $y = O(1)\ \big(\in (-\infty, k - 2\widehat{\lambda}_+)\big)$, where the function $\widehat{H}(y)$ remains undetermined but having

$$\widehat{H}(y) \sim \begin{cases} \ln(-y) + \ln\sqrt{\pi} & \text{as} \quad y \to -\infty, \\ \ln((k - 2\widehat{\lambda}_+) - y) + \frac{k}{2}\phi_c + \ln\sqrt{\pi} & \text{as} \quad y \to (k - 2\widehat{\lambda}_+)^-. \end{cases} \tag{3.68}$$

Region TR$^-$

$$u(\zeta,t) = \left(e^{-\frac{k}{2}\phi_c}\left[1 - \frac{1}{2}\text{erfc}\left(\frac{\zeta}{2}\right)\right] + o(1)\right)\exp\left\{-\left(\widehat{\lambda}_+^2 + 1\right)t + \widehat{\lambda}_+\zeta t^{\frac{1}{2}}\right\} \tag{3.69}$$

as $t \to \infty$ with $\zeta = (y - (k - 2\widehat{\lambda}_+))t^{\frac{1}{2}} = O(1)$.

Region IV$^-$(b)

$$u(y,t) = 1 - \exp\left(\widehat{\lambda}_+(y-2)t - \frac{k}{2}\phi_c + o(1)\right) \tag{3.70}$$

as $t \to \infty$ with $y = O(1)\ \big(\in (k - 2\widehat{\lambda}_+, 2)\big)$.

This then completes the asymptotic structure of the solution of **IVP** as $t \to \infty$ in the case when $k \in (-\infty, 2]$. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Figure 4.

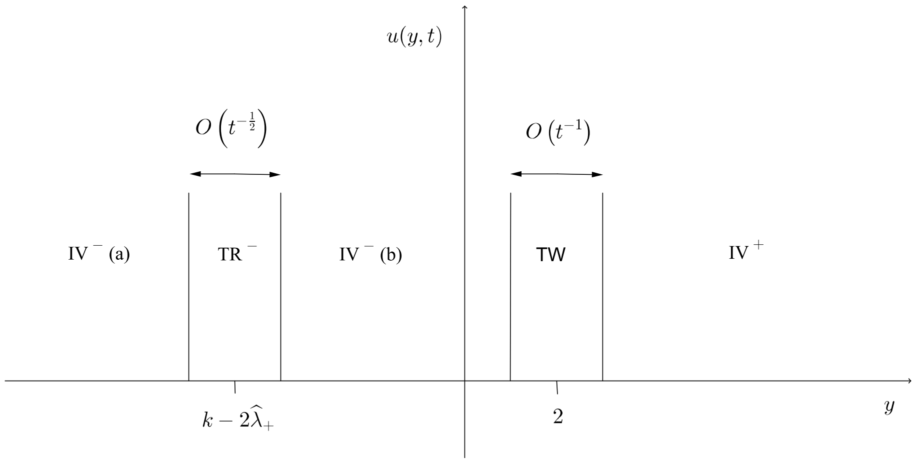

FIG. 4. Schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ when $k \in (-\infty, 2]$.

4. Numerical solution of the initial-value problem. Finally, we present numerical solutions of **IVP** which both support and illustrate the detailed asymptotic analysis given in the preceding sections. The numerical simulations were performed using a numerical algorithm based on the method of finite differences, solving over the spatial interval $(-L, L)$ using M equal width intervals, and over the time interval $(0, T)$ using N equal time steps. We define $\delta x = 2L/M$ and $\delta t = T/N$; typically $\delta x = 0.1$ and $\delta t = 0.005$ were found to give sufficient accuracy, although the x axes in the plots have been shortened in order to better display the features of interest. The interested reader is referred to the excellent text [12] for details of the numerical method employed here. We present numerical solutions of **IVP** for $k = 4$ and $k = -4$ (representative values of the problem parameter k in each of the ranges of interest, that is, $k = -4 \in (-\infty, 2]$ and $k = 4 \in (2, \infty)$). We consider each case in turn:

(i) When $k = 4$ the numerical results are given in Figures 5-7. In Figure 5 the PTW of wave speed $c = c^*(4) = 2.5$ (the minimum wave speed available in this case) is seen to develop rapidly, and the correction to $\dot{s}(t)$ as $t \to \infty$ appears to be exponentially small in t as $t \to \infty$, in line with the theory of Section 3.3.1. Figure 6 shows the numerically obtained curve of $\dot{s}(t)$ ($s(t)$ is the x location where $u = 0.5$) against t for $t \in (0, 25]$. As predicted by the theory $\dot{s}(t)$ rapidly approaches the minimum wave speed $c^*(4) = 2.5$ as $t \to \infty$. Figure 7 shows the numerically obtained curve of $s(t) - c^*(4)t$ against t for $t \in (0, 25]$. As predicted by the theory $s(t) - c^*(4)t$ rapidly approaches the constant $\phi_c \approx -0.1632$.

(ii) When $k = -4$ the numerical results are given in Figures 8-9. In Figure 8 the PTW of wave speed $c = c^*(-4) = 2$ (the minimum wave speed available in this case) is seen to develop, and the correction to $\dot{s}(t)$ as $t \to \infty$ appears to be algebraically small in t as $t \to \infty$, in line with the theory of Section 3.3.2. This is in contrast to the case when $k = 4$.

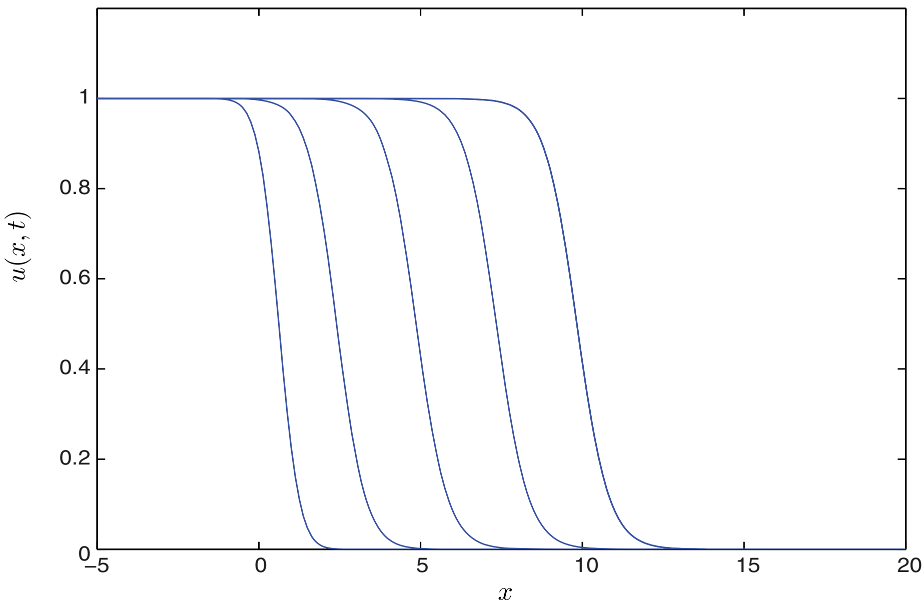

FIG. 5. Graphs of the solution to **IVP** when $k = 4$ for $t = 0.5, 1, 2, 3, 4$.

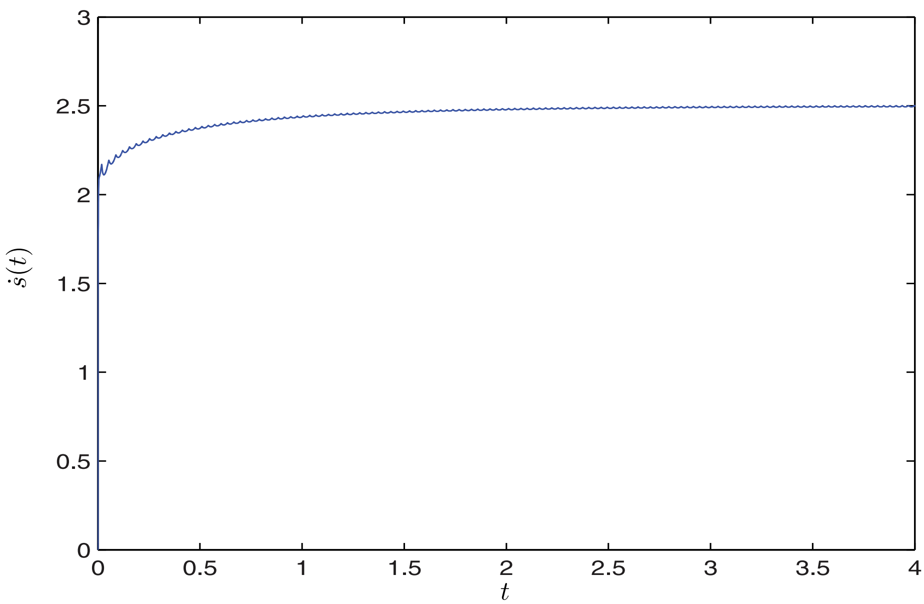

FIG. 6. The graph of $\dot{s}(t)$ against t when $k = 4$.

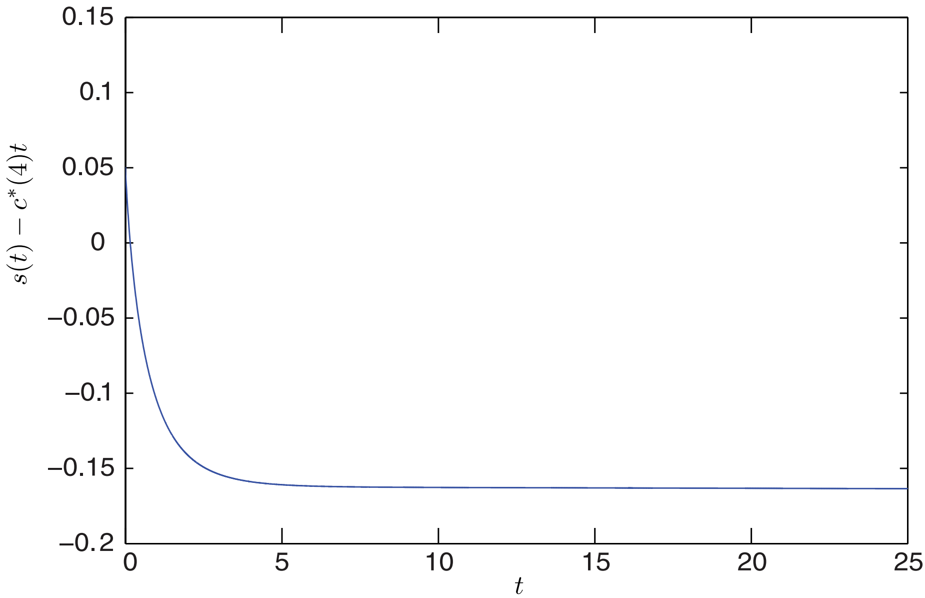

FIG. 7. The graph of $s(t) - c^*(4)t$ against t when $k = 4$. $\phi_c \approx -0.1634$.

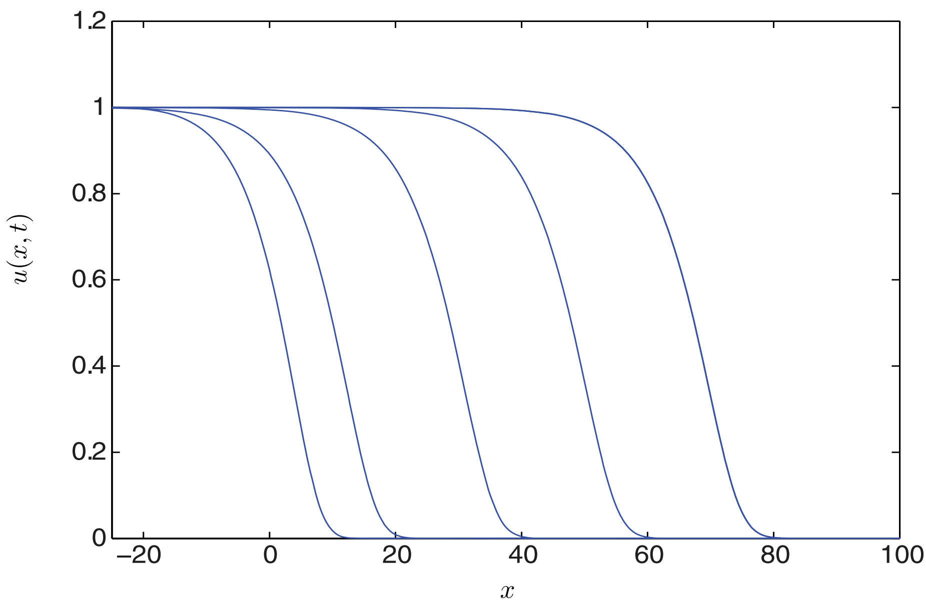

FIG. 8. Graphs of the solution to **IVP** when $k = -4$ for $t = 5, 10, 20, 30, 40$.

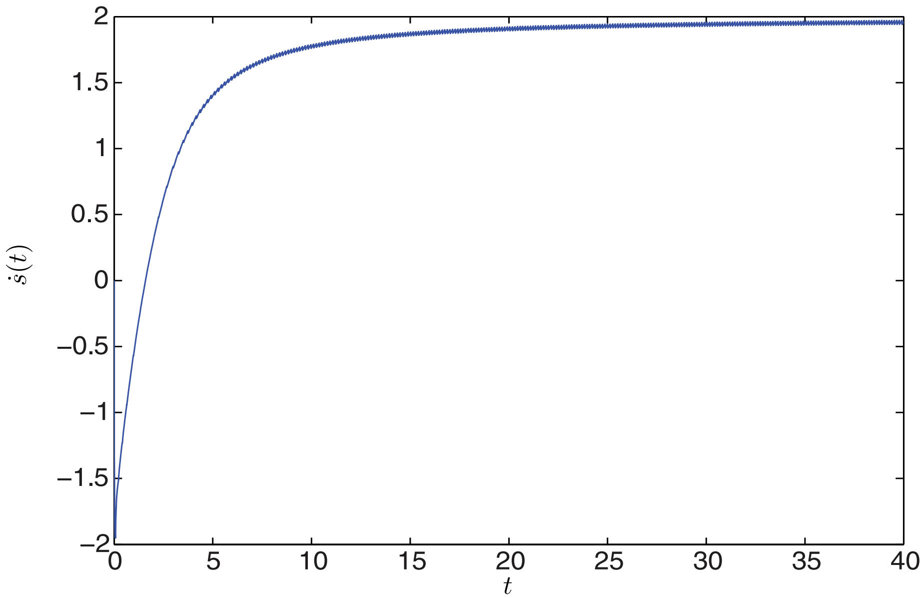

FIG. 9. The graph of $\dot{s}(t)$ against t when $k = -4$.

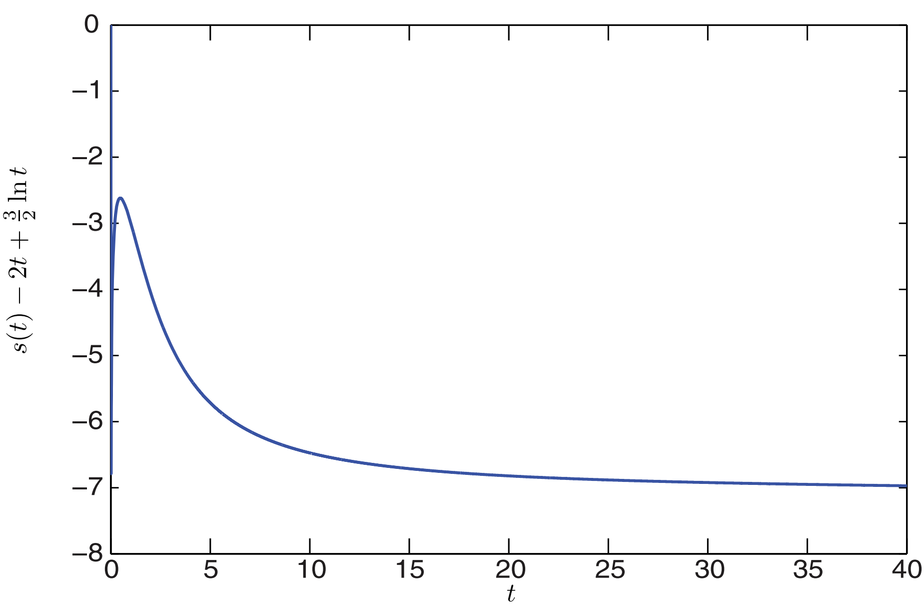

FIG. 10. The graph of $s(t) - 2t + \frac{3}{2}\ln t$ against t when $k = -4$.

Figure 9 shows the numerically obtained curve of $\dot{s}(t)$ ($s(t)$ is the x location where $u = 0.5$) against t for $t \in (0, 40]$. As predicted by the theory $\dot{s}(t)$ approaches the minimum wave speed 2 as $t \to \infty$. However, the rate of convergence is considerably slower than the case $k = 4$ above and appears to be algebraically small in t as $t \to \infty$. Finally, Figure 10 shows the numerically obtained curve of $s(t)-2t+\frac{3}{2}\ln t$ against t for $t \in (0, 40]$. As predicted by the theory $s(t)-2t+\frac{3}{2}\ln t$ approaches a constant as $t \to \infty$. This then confirms (3.65) and supports the theory that the rate of convergence of **IVP** as $t \to \infty$ in this case is algebraically small in t as $t \to \infty$, being of $O(t^{-1})$.

5. Conclusions. In this paper we have used the method of matched asymptotic coordinate expansions to develop the complete large-t solution of **IVP** for all values of the parameter k (excluding the case $k = 0$ when equation (1.1) reduces to the Fisher-Kolmogorov equation, the initial-value problem in this case having been considered by a number of authors; see for example [2]). In particular, we have established that the solution of **IVP** exhibits the formation of a permanent form travelling wave propagating in the $+x$ direction with the minimum possible speed $c = c^*(k)$, where

$$c^*(k) = \begin{cases} 2, & -\infty < k \leq 2, \\ \frac{2}{k} + \frac{k}{2}, & 2 < k < \infty. \end{cases}$$

Specifically, we have established that:

(i) When $k \in (2, \infty)$ the solution of $u(x, t)$ of **IVP** satisfies

$$u(z + s(t), t) = u_T\left(z; c^*(k)\right) + O\left(t^{-\frac{3}{2}} e^{-\left(\frac{[c^*(k)]^2}{4} - 1\right)t}\right)$$

as $t \to \infty$, uniformly in z, where $u_T(z; c^*)$ is the permanent for travelling wave solution with propagation speed $c^*(k) = \frac{2}{k} + \frac{k}{2}$ given by

$$u_T(z; c^*) = \frac{e^{-\frac{k}{2}z}}{1 + e^{-\frac{k}{2}z}},$$

$z = x - s(t)$ ($s(t)$ is a measure of the location of the wave front at time t) and

$$s(t) = \left(\frac{2}{k} + \frac{k}{2}\right)t + \phi_c + O\left(t^{-\frac{3}{2}} e^{-\left(\frac{[c^*(k)]^2}{4} - 1\right)t}\right)$$

as $t \to \infty$, where ϕ_c is a constant. We note that the correction to the propagation speed $\dot{s}(t)$ is exponential in t, as $t \to \infty$, being of $O\Big(t^{-\frac{3}{2}} \exp\Big\{-\Big(\frac{[c^*(k)]^2}{4} - 1\Big)t\Big\}\Big)$. We further note that the rate of convergence of the solution of **IVP** to the permanent form travelling wave is exponential in t, as $t \to \infty$, being of

$$O\left(t^{-\frac{3}{2}} \exp\left\{-\left(\frac{[c^*(k)]^2}{4} - 1\right)t\right\}\right). \tag{5.1}$$

(ii) When $k \in (-\infty, 2]$ the solution of $u(x, t)$ of **IVP** satisfies

$$u(z + s(t), t) = u_T(z; 2) + O\left(t^{-1}\right)$$

as $t \to \infty$, uniformly in z, where $u_T(z;2)$ is the permanent for travelling wave solution with propagation speed 2, $z = x - s(t)$ ($s(t)$ is a measure of the location of the wave front at time t) and

$$\dot{s}(t) = 2 - \frac{3}{2}\frac{1}{t} + o\left(\frac{1}{t}\right) \tag{5.2}$$

as $t \to \infty$. We note that the correction to the propagation speed $\dot{s}(t)$ is algebraic in t, as $t \to \infty$, being of $O\left(t^{-1}\right)$. We further note that the rate of convergence of the solution of **IVP** to the permanent form travelling wave is algebraic in t, as $t \to \infty$, being of $O\left(t^{-1}\right)$. It is worthwhile to point out that the above result (5.2) is in agreement with the result obtained by Bramson [2], who considered (1.1) (with $k = 0$) when the initial data has a step function profile (1.2).

These results are supported by the numerical simulations of Section 4.

We conclude by noting that in Section 3.3 the large-t solution of **IVP** was obtained by the careful consideration of the functions $f_0(y)$ and $g_0(y)$, the solutions of the Clairaut equations (3.9) and (3.25) respectively. Clairaut equations admit constant solutions, one-parameter families and linear solutions and associated envelope (singular) solutions. Further, as already mentioned in Section 3.3 combinations of these solutions which remain continuous and differentiable also provide solutions to the Clairaut equation in question. Figures 1 and 3 give sketches of the required forms for $f_0(y)$ and $g_0(y)$ in the cases $k \in (2,\infty)$ and $k \in (-\infty, 2]$, respectively. It is instructive to note that in the case $k \in (2,\infty)$ (Figure 1) the point $\left(c^*(k), \frac{[c^*(k)]^2}{4} - 1\right)$ lying on the envelope solution $f_0(y) = \frac{y^2}{4} - 1$ approaches the point $(2, 0)$ as $k \to 2^+$, the y ordinate of this point being associated with the argument of the exponential correction term to the PTW in this case. $k = 2$ can then be considered a bifurcation point marking a change in structure of the large-t solution of **IVP**. When $k \in (2,\infty)$ the correction to the PTW is exponential in t as $t \to \infty$, being given by (5.1), while when $k \in (-\infty, 2]$ the correction to the PTW is algebraic in t, being of $O(t^{-1})$ as $t \to \infty$. Finally, we note that equations of Clairaut type appear to play a central role in the analysis of many problems arising in the areas of mathematical chemistry, biology and physics and warrant further investigation.

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