

SPECTRAL REPRESENTATIONS, AND APPROXIMATIONS, OF DIVERGENCE-FREE VECTOR FIELDS

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Abstract. Special solutions of the equation for a solenoidal vector field subject to prescribed flux boundary conditions are described. A unique gradient solution is found and proved to be the least energy solution of the problem. This solution has a representation in terms of certain $\Sigma - \text{Steklov}$ -eigenvalues and eigenfunctions. Error estimates for finite approximations of these solutions are obtained. Some results of computational simulations for two-dimensional and axisymmetrical problems are presented.

1. Introduction. This paper describes some results about the solutions of the equation $\operatorname{div} \mathbf{v}(x) \equiv 0$ when the boundary flux $\mathbf{v} \cdot \nu$ is prescribed. This boundary-flux problem is an underdetermined problem that has an infinite dimensional affine subspace of solutions. Here a unique gradient solution of the problem is found and shown to be the *least energy solution* of this system. This solution may be represented using an eigenfunction expansion, and partial sums involving just a few terms are shown to provide good approximations that are qualitatively similar to observed results.

This equation arises in many classical field theories. For expository purposes, the field \mathbf{v} will generally be treated as a velocity field in this paper, though the analysis applies equally to magnetic flux densities and other solenoidal fields.

Specifically we treat the problem of solving

$$\operatorname{div} \mathbf{v}(x) \equiv 0 \quad \text{on } \Omega \quad \text{subject to} \quad \mathbf{v}(x) \cdot \nu(x) = \eta(x) \quad \text{on } \partial\Omega. \quad (1.1)$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded region with a Lipschitz boundary $\partial\Omega$ and unit outward normal $\nu(x)$, and our interest is in describing the dependence on solutions on the given boundary flux η .

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For many fluid flow problems some parts of the boundary are impermeable, so an additional requirement is that the flux $\eta(x) \equiv 0$ on a subset Σ_0 of $\partial\Omega$. The case Σ_0 is the empty set will be allowable and is usually simpler.

A subspace of solutions of this equation that are gradient fields will be described, and an L^2 -orthogonal basis of this space is constructed. These solutions are then shown to be the least energy solutions of (1.1). The basis consists of certain harmonic Steklov eigenfunctions so that eigenfunction (i.e. spectral) expansions of the solutions are found. Steklov eigenproblems have the eigenparameter in the boundary condition rather than in the equation. Finite sums of these expansions are shown to converge strongly to the solution, and error bounds for approximations of the solutions are described in Theorem 5.1.

For two- and three-dimensional regions, the Steklov eigenvalues and eigenfunctions may be found either exactly in some simple cases or computationally using finite element methods and standard software packages. Then approximations of the gradient solutions of (1.1) for prescribed fluxes can be computed. Some computational simulations of these solutions are described in sections 6 and 7. For many of these problems quite good approximations of the special solutions are obtained using relatively few terms, so these approximations should be very useful for further applications.

There is a large literature on the representation of classes of solenoidal vector fields. Chapter 3 of Galdi [1] provides a detailed treatment of the problem of representations of fields with given divergence. In particular there are integral representations of the solutions subject to various boundary value conditions. It has an extensive bibliography and historical comments, but the particular solutions to be described here do not appear to have been studied previously. A description of results that are of importance in electromagnetic field theory may be found in chapter IX, volume 3 of Dautray and Lions [2] written by M. Cessenat.

The solutions described here do not involve finding, or using, the integral operators that are used in Boundary Integral Methods. Rather, direct formulae for the solutions are obtained in terms of the boundary data and the natural Steklov eigenfunctions of the problem. Moreover the associated series converge very well, and just a few terms often provide good approximations of a solution.

2. Notation and definitions. This boundary value problem will be studied using the framework of Sobolev spaces and weak solutions with the following definitions and conventions. The problem is posed on a region $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ and boundary $\partial\Omega$. A region is a non-empty, bounded, connected, open subset and Cartesian coordinates will be used in general.

Let $L^2(\Omega; \mathbb{R}^n)$ be the usual real Lebesgue space of all functions $\varphi : \Omega \rightarrow [-\infty, \infty]$ which are square integrable with respect to Lebesgue measure on Ω . Similarly $L^2(\Omega; \mathbb{R}^n)$ is the class of all vector fields \mathbf{v} on Ω whose components v_j all are in $L^2(\Omega)$. These are real Hilbert spaces under the usual inner products

$$\langle \varphi, \psi \rangle := \int_{\Omega} \varphi \psi \, d^n x \quad \text{and} \quad \langle \mathbf{u}, \mathbf{v} \rangle := \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, d^n x. \quad (2.1)$$

The 2-norm of a vector field $\mathbf{v} \in L^2(\Omega; \mathbb{R}^n)$ will be called the *energy* of the field.

All derivatives are to be taken in a weak sense so $D_j \varphi$ is the weak j -th derivative of a function φ . The spaces $W^{1,p}(\Omega), H^1(\Omega)$ are the standard Sobolev spaces. When $\varphi \in W^{1,1}(\Omega)$ then $\nabla \varphi := (D_1 \varphi, \dots, D_n \varphi)$ is the gradient of φ .

For our analysis we only require some mild regularity conditions on Ω and $\partial\Omega$. Our basic criterion is that the boundary $\partial\Omega$ is Lipschitz in the sense of Evans and Gariepy, [3], section 4.2. Let $\sigma, d\sigma$ represent Hausdorff $(n-1)$ -dimensional measure and integration with respect to this measure respectively. This measure will be called surface area measure. When $\partial\Omega$ is Lipschitz, the unit outer normal $\nu(x)$ is defined σ a.e. on $\partial\Omega$, and the boundary traces of functions in $W^{1,p}(\Omega)$ are well-defined. The trace operator will be denoted γ and usually will be omitted in boundary integrals. See section 4.3 of [3]. In particular, the divergence theorem holds in the form

$$\int_{\Omega} D_j \varphi \, d^n x = \int_{\partial\Omega} \varphi \nu_j \, d\sigma \quad \text{for all } 1 \leq j \leq n, \varphi \in W^{1,1}(\Omega). \quad (2.2)$$

This follows from Theorem 1, section 4.3 of [3] with $p = 1$.

Our requirement is that Ω satisfies

CONDITION A1. Ω is a bounded open region in \mathbb{R}^n , and $\partial\Omega$ is a finite union of Lipschitz surfaces with finite surface area.

When (A1) holds and $\psi \in H^1(\Omega)$, then trace theorems imply that $\gamma(\psi) \in L^p(\partial\Omega; d\sigma)$ for $p \in [1, p_T]$ where $p_T = 2(n-1)/(n-2)$ when $n \geq 3$ and for all $p \in [1, \infty)$ when $n = 2$. In particular they are in $L^2(\partial\Omega, d\sigma)$ for every n .

The condition that $\eta \equiv 0$ on a subset Σ_0 of $\partial\Omega$ is interpreted in the sense that

$$\int_{\Sigma_0} \eta \varphi \, d\sigma = 0 \quad \text{for all } \varphi \in C_c(\Sigma_0)$$

where $C_c(\Sigma_0)$ is the set of all real-valued continuous functions on Σ_0 with compact support. This is equivalent to saying that $\eta(x) = 0$ σ a.e. on Σ_0 .

Define Σ to be the complement of the closure $\overline{\Sigma_0}$ of Σ_0 in the boundary $\partial\Omega$. This is an open subset of $\partial\Omega$ and must be non-empty. From the divergence theorem, a necessary condition for solvability of (1.1) is that $\int_{\partial\Omega} \eta \, d\sigma = 0$. When η has zero trace on Σ_0 , this becomes

$$\int_{\Sigma} \eta \, d\sigma = 0. \quad (2.3)$$

Let $L^2(\Sigma, d\sigma), L^2(\partial\Omega, d\sigma)$ be the usual space of L^2 -functions on $\Sigma, \partial\Omega$ respectively. The inner product and norm on $L^2(\Sigma, d\sigma)$ are

$$\langle \phi, \chi \rangle_{\Sigma} := |\Sigma|^{-1} \int_{\Sigma} \phi \chi \, d\sigma \quad \text{and} \quad \|\phi\|_{\Sigma}^2 := \langle \phi, \phi \rangle_{\Sigma}. \quad (2.4)$$

Here $|\Sigma| = \sigma(\Sigma)$ is the surface area of Σ .

Our assumption about the boundary flux η is that it is zero on Σ_0 and satisfies the compatibility condition for solvability of (1.1). Specifically

CONDITION A2. Σ is a non-empty open subset of $\partial\Omega$, η is in $L^2(\partial\Omega, d\sigma)$, has zero trace on Σ_0 and satisfies (2.3).

The case $\Sigma = \partial\Omega$ (or $\Sigma_0 = \emptyset$) is permitted here.

3. Hilbert subspaces of $H(\operatorname{div}, \Omega)$. In this section an appropriate real Hilbert space for investigating this problem will be introduced and weak forms of (1.1) will be posed.

Let $H(\operatorname{div}, \Omega)$ be the usual subspace of fields $\mathbf{v} \in L^2(\Omega; \mathbb{R}^n)$ with $\operatorname{div} \mathbf{v}$ in $L^2(\Omega)$. It is a real Hilbert space under the inner product

$$[\mathbf{v}, \mathbf{w}]_{\operatorname{div}} := \int_{\Omega} [\mathbf{v}(x) \cdot \mathbf{w}(x) + \operatorname{div} \mathbf{v}(x) \operatorname{div} \mathbf{w}(x)] \, d^n x. \tag{3.1}$$

The associated norm will be denoted $\|\cdot\|_{\operatorname{div}}$. Unfortunately the boundary flux of vector fields in $H(\operatorname{div}, \Omega)$ need not be L^2 - functions on the boundary. See Cessenat’s chapter 9 in [2] for an introduction to these spaces and results about this.

Here the space $H(\operatorname{div}, \partial) \subset H(\operatorname{div}, \Omega)$ of vector fields that have boundary fluxes $\mathbf{v} \cdot \nu$ in $L^2(\partial\Omega, d\sigma)$ will be used. Define an inner product on $H(\operatorname{div}, \partial)$ by

$$[\mathbf{v}, \mathbf{w}]_{\operatorname{div}, \partial} := [\mathbf{v}, \mathbf{w}]_{\operatorname{div}} + \int_{\partial\Omega} (\mathbf{v} \cdot \nu)(\mathbf{w} \cdot \nu) \, d\sigma. \tag{3.2}$$

The divergence theorem (2.2) applied to $\varphi \mathbf{v}$ yields that

$$\int_{\Omega} \varphi \operatorname{div} \mathbf{v} \, dx = \int_{\partial\Omega} \varphi (\mathbf{v} \cdot \nu) \, d\sigma - \int_{\Omega} \mathbf{v} \cdot \nabla \varphi \, dx \tag{3.3}$$

when $\varphi \in H^1(\Omega)$ and \mathbf{v} is continuous on $\overline{\Omega}$ with each $D_j v_j \in L^2(\Omega)$. When (A1) holds this identity extends to fields $\mathbf{v} \in H(\operatorname{div}, \partial)$.

THEOREM 3.1. Suppose $H(\operatorname{div}, \partial)$, $[\cdot, \cdot]_{\operatorname{div}, \partial}$ are defined as above and (A1) holds. Then $[\cdot, \cdot]_{\operatorname{div}, \partial}$ is an inner product and $H(\operatorname{div}, \partial)$ is a real Hilbert space.

Proof. It is straightforward to check that the symmetric bilinear form in (3.2) satisfies the conditions to be an inner product. Suppose that $\{\mathbf{v}_m : m \geq 1\}$ is a Cauchy sequence in $H(\operatorname{div}, \partial)$. Then it is a Cauchy sequence in $L^2(\Omega; \mathbb{R}^n)$, $\operatorname{div} \mathbf{v}_m$ is a Cauchy sequence in $L^2(\Omega)$ and $\mathbf{v}_m \cdot \nu$ is a Cauchy sequence in $L^2(\partial\Omega, d\sigma)$. Since each of these spaces is complete there are limits $\tilde{\mathbf{v}}, \psi, \eta$ such that the sequences converge to these limits respectively. From the divergence theorem (3.3) one has

$$\int_{\Omega} \varphi \operatorname{div} \mathbf{v}_m \, dx = \int_{\partial\Omega} \varphi (\mathbf{v}_m \cdot \nu) \, d\sigma - \int_{\Omega} \mathbf{v}_m \cdot \nabla \varphi \, dx$$

for all $\varphi \in H^1(\Omega)$. Take limits as $m \rightarrow \infty$ to find that

$$\int_{\Omega} \varphi \psi \, dx = \int_{\partial\Omega} \varphi \eta \, d\sigma - \int_{\Omega} \tilde{\mathbf{v}} \cdot \nabla \varphi \, dx.$$

This implies that $\operatorname{div} \tilde{\mathbf{v}} = \psi$ on Ω as this holds for all $\varphi \in H_0^1(\Omega)$. Then the divergence theorem implies that $\tilde{\mathbf{v}} \cdot \nu = \eta$, σ *a.e.* on $\partial\Omega$. Thus the sequence converges to $\tilde{\mathbf{v}}$ and $H(\operatorname{div}, \partial)$ is complete. □

A field $\mathbf{v} \in L^2(\Omega; \mathbb{R}^n)$ is said to be *solenoidal*, or divergence-free, provided

$$\int_{\Omega} \mathbf{v} \cdot \nabla \varphi \, d^n x = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{3.4}$$

That is, the (weak) divergence of \mathbf{v} is zero *a.e.* on Ω , so any such field is in $H(\operatorname{div}, \Omega)$. The class of all such solenoidal vector fields is denoted $N(\operatorname{div})$ and is a closed subspace of $H(\operatorname{div}, \Omega)$. Define $N(\operatorname{div}, \partial)$ to be the class of all solenoidal vector fields \mathbf{v} in $H(\operatorname{div}, \partial)$.

The problem to be addressed here is to find vector fields $\mathbf{v} \in H(\text{div}, \partial)$ that satisfy

$$\int_{\Omega} \mathbf{v} \cdot \nabla \varphi \, dx = \int_{\Sigma} \eta \varphi \, d\sigma \quad \text{for all } \varphi \in H^1(\Omega). \tag{3.5}$$

Solutions of this are in $N(\text{div}, \partial)$ since (3.4) holds. This is a weak (i.e. integral) version of (1.1) as may be seen by using (3.3) and the assumptions (A2) about η .

It is easy to check that this is an underdetermined problem with a large affine space of solutions. So first the problem of finding solutions of (3.5) that are gradient fields will be considered. That is, we want to describe the $\psi \in H^1(\Omega)$ that satisfy

$$\int_{\Omega} \nabla \psi \cdot \nabla \varphi \, dx = \int_{\Sigma} \eta \varphi \, d\sigma \quad \text{for all } \varphi \in H^1(\Omega). \tag{3.6}$$

This is the weak form of a Neumann problem for the Laplacian subject to $D_{\nu} \psi \equiv \eta$ on the boundary.

It is straightforward to prove an existence result for this equation using variational methods. Our interest here is finding formulae for, and approximations of, this solution. To do this, an eigenfunction expansion for the solution of (3.6) in terms of an appropriate class of Steklov eigenfunctions will be derived.

4. Harmonic Σ -Steklov eigenproblems. This section develops some results about a class of Steklov eigenproblems that provide a basis for the space of all solutions of (3.6). The Steklov eigenfunctions are harmonic functions on Ω whose gradients constitute an L^2 -orthonormal set of solenoidal vector fields on Ω that are appropriate for satisfying the boundary conditions in (1.1). An early paper on the use of Steklov expansions to obtain representation results for solutions of elliptic boundary value problems may be found in Auchmuty [4]. The following results are similar to the first author’s description of trace spaces in Auchmuty [5] with modifications arising from the fact that Σ may be a proper subset of $\partial\Omega$.

Let Ω, Σ be sets that satisfy conditions (A1) and (A2). δ is said to be a *harmonic Σ – Steklov eigenvalue* provided there is a non-zero solution $s \in H^1(\Omega)$ of

$$\int_{\Omega} \nabla s \cdot \nabla \chi \, dx = \delta \langle s, \chi \rangle_{\Sigma} = \delta |\Sigma|^{-1} \int_{\Sigma} s \chi \, d\sigma \quad \text{for all } \chi \in H^1(\Omega). \tag{4.1}$$

Any such solution s is said to be a *harmonic Σ -Steklov eigenfunction* corresponding to δ .

First note that any solution of this system is a weakly harmonic function on Ω as (4.1) holds for all $\chi \in C_c^1(\Omega)$. (4.1) also implies that the eigenfunctions satisfy a weak form of the boundary conditions

$$D_{\nu} s = 0 \quad \text{on } \Sigma_0 \quad \text{and} \quad D_{\nu} s = \delta |\Sigma|^{-1} s \quad \text{on } \Sigma. \tag{4.2}$$

This eigenproblem has the following properties since it may be treated as a particular example of the abstract results described in [6]. There is an infinite sequence $\Lambda := \{\delta_j : j \geq 0\}$ of real eigenvalues obeying $\delta_{j+1} \geq \delta_j$. $\delta_0 = 0$ is the least eigenvalue of this system corresponding to the eigenfunction $s_0(x) \equiv 1$ on Ω . This eigenvalue is simple as Ω is connected and the other eigenvalues of (4.1) are strictly positive.

The first non-zero Σ – Steklov eigenvalue δ_1 may be found by maximizing the weakly continuous functional $\mathcal{M}(s) := \int_{\Sigma} s^2 d\sigma$ on the closed bounded convex subset of $H^1(\Omega)$ of functions satisfying

$$\int_{\Omega} |\nabla s|^2 dx + |\Sigma|^{-1} \int_{\Sigma} s^2 d\sigma \leq 1 \quad \text{and} \quad \int_{\Sigma} s d\sigma = 0.$$

The maximizers of this problem are eigenfunctions of (4.1) corresponding to δ_1 .

To prove this use the notation of section 3 of [6] with $V = H^1(\Omega)$, $\lambda = (\delta - 1)/|\Sigma|$,

$$m(u, v) := |\Sigma|^{-1} \int_{\Sigma} uv d\sigma \quad \text{and} \quad a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v dx + m(u, v).$$

Then equation (2.1) of [6] is the same as (4.1) above and conditions (A1)-(A4) there are satisfied from standard results about functions in $H^1(\Omega)$.

Then section 4 of that paper describes an algorithm for obtaining successive eigenvalues and eigenfunctions by maximizing the functional \mathcal{M} subject to the same inequality constraint and m -orthogonality to all previous eigenfunctions.

Assume that these Σ – Steklov eigenfunctions are normalized so that

$$\int_{\Sigma} s_j s_k d\sigma = 0 \quad \text{when } j \neq k \quad \text{and} \quad \int_{\Sigma} s_j^2 d\sigma = |\Sigma|. \tag{4.3}$$

Let $\Lambda := \{\delta_j : j \geq 0\}$ be the set of Σ – Steklov eigenvalues repeated according to multiplicity and in increasing order and let $\mathcal{S} := \{s_j : j \geq 0\}$ be a sequence of harmonic Σ -Steklov eigenfunctions associated with the δ_j that satisfy (4.3) and are constructed by the sequence of problems described in section 4 of [6]. Then the following theorem holds.

THEOREM 4.1. Assume (A1) holds and Σ is a non-empty open subset of $\partial\Omega$. If Λ and \mathcal{S} are sequences of harmonic Σ -Steklov eigenfunctions that satisfy (4.3), then

- (i) Λ is a sequence that increases to ∞ with no finite accumulation point,
- (ii) \mathcal{S} is a maximal orthonormal set in $L^2(\Sigma, d\sigma)$, and
- (iii) the sequence $\{\nabla s_j : j \geq 1\}$ is a maximal a -orthogonal set in $H^1(\Omega)$.

Proof. This holds as an application of results from section 4 of [6]. First since (A1)-(A4) hold, Theorem 4.3 implies that Λ is a sequence that increases to infinity and that the associated set of eigenfunctions is a maximal a -orthogonal set. Hence (i) and (iii) hold. Also the trace theorem for $H^1(\Omega)$ implies that condition (A5) there holds, so the eigenfunctions also are a maximal L^2 -orthogonal set on Σ . The normalization in equation (4.3) implies that these functions are orthonormal under the inner product of (2.4). \square

For each $j \geq 1$, define $\tilde{\mathbf{w}}_j := \nabla s_j$. Then (4.1) and (4.3) yield that each $\tilde{\mathbf{w}}_j$ is solenoidal and

$$\int_{\Omega} \tilde{\mathbf{w}}_j \cdot \tilde{\mathbf{w}}_k dx = \begin{cases} 0 & \text{when } j \neq k \\ \delta_j & \text{when } j = k. \end{cases} \tag{4.4}$$

That is, the gradients of these Steklov eigenfunctions are L^2 -orthogonal on Ω . Moreover each $\tilde{\mathbf{w}}_j$ is a C^∞ field on Ω as each s_j is harmonic.

Define $\mathbf{w}_j := \tilde{\mathbf{w}}_j / \sqrt{\delta_j}$; then the set $\mathcal{W} := \{\mathbf{w}_j : j \geq 1\}$ is an L^2 -orthonormal subset of both $L^2(\Omega; \mathbb{R}^n)$ and $H(\text{div}, \Omega)$. From the divergence theorem one also has that, for

each $\chi \in H^1(\Omega)$,

$$\int_{\Omega} \mathbf{w}_j \cdot \nabla \chi \, dx = \int_{\partial\Omega} (\mathbf{w}_j \cdot \nu) \chi \, d\sigma.$$

Thus the fields \mathbf{w}_j have boundary fluxes that satisfy

$$\mathbf{w}_j \cdot \nu = \mathbf{0} \quad \sigma \text{ a.e. on } \Sigma_0 \quad \text{and} \quad \mathbf{w}_j \cdot \nu = \frac{\sqrt{\delta_j} \mathbf{s}_j}{|\Sigma|} \quad \sigma \text{ a.e. on } \Sigma, \tag{4.5}$$

using the fact that (4.1) holds for all functions χ that are continuous on $\bar{\Omega}$ and in $H^1(\Omega)$. In particular each \mathbf{w}_j is in $N(\text{div}, \partial)$.

Thus a finite linear combination of fields $\mathbf{v} := \sum_{j=1}^M c_j \mathbf{w}_j$ in \mathcal{W} will be solenoidal and obeys

$$\mathbf{v} \cdot \nu = 0 \quad \text{on } \Sigma_0 \quad \text{and} \quad \mathbf{v} \cdot \nu = \frac{1}{|\Sigma|} \sum_{j=1}^M \sqrt{\delta_j} c_j s_j \quad \text{on } \Sigma. \tag{4.6}$$

Moreover the orthogonality properties (4.3) and (4.4) of these functions and fields yield that

$$\|\mathbf{v}\|_{\text{div}}^2 = \sum_{j=1}^M |c_j|^2 \quad \text{and} \quad \int_{\Sigma} |\mathbf{v} \cdot \nu|^2 \, d\sigma = |\Sigma|^{-1} \sum_{j=1}^M \delta_j |c_j|^2. \tag{4.7}$$

Define W to be the subspace of $H(\text{div}, \Omega)$ spanned by the orthonormal set \mathcal{W} , and assume that $\mathbf{w} := \sum_{j=1}^{\infty} c_j \mathbf{w}_j$ is in W . Then the following result follows from the theorem.

COROLLARY 4.2. Under the assumptions of the theorem, a vector field $\mathbf{w} := \sum_{j=1}^{\infty} c_j \mathbf{w}_j$ is in $H(\text{div}, \partial)$ if and only if $\sum_{j=1}^{\infty} (1 + \frac{\delta_j}{|\Sigma|}) |c_j|^2 < \infty$.

Proof. Take limits as $M \rightarrow \infty$ in equation (4.7) to see that

$$\|\mathbf{w}\|_{\text{div}}^2 = \sum_{j=1}^{\infty} |c_j|^2 \quad \text{and} \quad \int_{\Sigma} |\mathbf{w} \cdot \nu|^2 \, d\sigma = |\Sigma|^{-1} \sum_{j=1}^{\infty} \delta_j |c_j|^2,$$

so the result follows from the definition (3.2). □

Define $W_S := W \cap N(\text{div}, \partial)$. Then W_S is a subspace of solenoidal fields that have L^2 -flux through the boundary and no flux through Σ_0 , so it is an appropriate class of fields for seeking solutions of the boundary value problem (3.6).

5. Representation and approximation of solutions. We now are in a position to find approximations of, and explicit formulae for, the solution of (3.6). These come from the expansion of the solution in terms of the basis of harmonic Σ -Steklov eigenfunctions. Similar results for elliptic boundary value problems have been described by the first author in [4] and [7].

Suppose that $\hat{\mathbf{v}}$ is a solution of (3.5) and $\hat{\mathbf{v}}(x) := \nabla \hat{\psi}$ where $\hat{\psi}$ is a solution of (3.6). From part (iii) of Theorem 4.1, there are c_j such that

$$\nabla \hat{\psi}(x) := \sum_{j=1}^{\infty} c_j \nabla s_j(x). \tag{5.1}$$

Substitute s_j for φ in (3.6); then orthogonality yields, for each $j \geq 1$,

$$c_j = |\Sigma| \frac{\eta_j}{\delta_j} \quad \text{with} \quad \eta_j := \langle \eta, s_j \rangle_\Sigma. \tag{5.2}$$

Note that $\eta_0 = 0$ from the assumption (A2). Then $\hat{\mathbf{v}} := S\eta$ will be a solution of (3.6) where $S : L^2(\Sigma, d\sigma) \rightarrow N(\text{div}, \partial)$ is the linear transformation defined by

$$S\eta(x) := |\Sigma| \sum_{j=1}^{\infty} \frac{\eta_j}{\sqrt{\delta_j}} \mathbf{w}_j(x). \tag{5.3}$$

Let $\hat{\psi}_m(x) := \sum_{j=1}^m c_j s_j(x)$ be the m -th approximation of $\hat{\psi}$ on Ω and $S_m : L^2(\Sigma, d\sigma) \rightarrow N(\text{div}, \partial)$ be defined by

$$S_m\eta(x) := \nabla \hat{\psi}_m(x) := |\Sigma| \sum_{j=1}^m \frac{\eta_j}{\sqrt{\delta_j}} \mathbf{w}_j(x). \tag{5.4}$$

This will be called the m -th Steklov approximation of $S\eta$.

THEOREM 5.1. Assume (A1) and (A2) hold with $\mathcal{W}, N(\text{div}, \partial), W_S, S$ and S_m as above. Then $\hat{\mathbf{v}} = S\eta$ is the unique solution of (3.5) in W_S . S is a continuous linear transformation of $L^2(\Sigma, d\sigma)$ into $N(\text{div}, \partial)$ with

$$\|S\|_{\text{div}, \partial}^2 = |\Sigma| \left(1 + \frac{|\Sigma|}{\delta_1}\right). \tag{5.5}$$

The sequence $S_m\eta$ converges strongly to $S\eta$ as $m \rightarrow \infty$ with

$$\|(S - S_m)\eta\|_{\text{div}, \partial}^2 \leq |\Sigma| \left(1 + \frac{|\Sigma|}{\delta_{m+1}}\right) \|\eta - \eta_m\|_\Sigma^2. \tag{5.6}$$

Proof. We have seen that $\hat{\psi}$ satisfies (3.6) for all $\varphi = s_j$. From the last part of Theorem 4.1, \mathcal{S} is a maximal orthogonal set in $H^1(\Omega)$, so $\hat{\psi} \in H^1(\Omega)$ is a solution of (3.6). If ψ is another solution of (3.6), then $\varphi := \psi - \hat{\psi}$ satisfies

$$\int_\Omega \nabla \varphi \cdot \nabla \chi \, dx = 0 \quad \text{for all} \quad \chi \in H^1(\Omega),$$

so φ is constant on Ω , and thus $\hat{\mathbf{v}} := S\eta$ is the only solution of (3.5) in W_S .

The fields \mathbf{w}_j are L^2 -orthogonal and solenoidal, so

$$\|S\eta\|_{\text{div}, \partial}^2 = \sum_{j=1}^{\infty} |\Sigma| \left(1 + \frac{|\Sigma|}{\delta_j}\right) \eta_j^2 \leq |\Sigma| \left(1 + \frac{|\Sigma|}{\delta_1}\right) \|\eta\|_\Sigma^2.$$

Hence S is continuous and has the specified norm. Similarly

$$\|(S - S_m)\eta\|_{\text{div}, \partial}^2 = |\Sigma| \sum_{j=m+1}^{\infty} \left(1 + \frac{|\Sigma|}{\delta_j}\right) \eta_j^2 \leq |\Sigma| \left(1 + \frac{|\Sigma|}{\delta_{m+1}}\right) \|\eta - \eta_m\|_\Sigma^2,$$

so the last statement of the theorem holds. □

When $\mathbf{v} \in H(\text{div}, \partial)$ is a solution of (3.5) and $\hat{\mathbf{v}} = S\eta$ is the solution from (5.3), then $\mathbf{w} := \mathbf{v} - \hat{\mathbf{v}}$ is a solution of the system

$$\int_\Omega \mathbf{w} \cdot \nabla \varphi \, dx = 0 \quad \text{for all} \quad \varphi \in H^1(\Omega). \tag{5.7}$$

That is, \mathbf{w} is a weak solution of $\operatorname{div} \mathbf{w} = 0$ on Ω subject to $\mathbf{w} \cdot \nu = 0$ on $\partial\Omega$.

This system has a large subspace of solutions. Any field of the form $\mathbf{w} := \operatorname{curl} A$ where \mathbf{A} is a C^1 -field with compact support in Ω is a solution of (5.7). Let V_η be the affine subspace of all solutions of (3.5) in $N(\operatorname{div}, \partial)$ and consider the problem of minimizing the energy $\|\mathbf{v}\|$ on this subspace. We were surprised to observe the following simple result.

THEOREM 5.2. Assume (A1) and (A2) hold. Then $\hat{\mathbf{v}}$ defined by (5.3) is the least energy solution in V_η of (3.5).

Proof. Let $\mathbf{v} := \hat{\mathbf{v}} + \mathbf{w}$ be a solution of (3.5). Then $\mathbf{w} \in N(\operatorname{div}, \partial)$ and $\gamma(\mathbf{w}) = 0$. Also

$$\int_{\Omega} |\mathbf{v}|^2 d^n x = \int_{\Omega} [|\nabla \hat{\phi}|^2 + 2\mathbf{w} \cdot \nabla \hat{\phi} + |\mathbf{w}|^2] dx.$$

The second integral on this right hand side is zero from the properties of \mathbf{w} and the divergence theorem (3.3), so the result follows. \square

In view of this result we call $\hat{\mathbf{v}} := S\eta(x)$ the least energy solution of (1.1), while the representation (5.3) is called a Steklov (spectral) representation of this solution.

6. Approximation of least energy solenoidal fields. These last two sections will describe some results about the computational approximation of the solutions described above. A natural question is whether these spectral formulae for the solutions provide useful ways of approximating these solutions.

A typical application is to model inviscid fluid flows in a region Ω with specific inflows and outflows on parts of the boundary and no flux boundary conditions elsewhere. This could range from a simple pipe flow where there is an inlet and an outlet to systems such as sprinklers with a single inlet and multiple possible outlets.

The analysis of section 5 implies that the least energy solutions of (1.1) should be well-approximated by gradient fields of the form (5.4). To determine the least energy flows with prescribed boundary flux η through Σ one:

- (i) computes sufficiently many Σ -Steklov eigenvalues and eigenfunctions for the region Ω ,
- (ii) evaluates the relevant Steklov coefficients η_j , and
- (iii) uses the formulae for $\hat{\psi}_m$ for contour or level surface plots and (5.4) for the flow field.

We first considered the problem of finding solutions when Ω is a rectangular region of finite length with given fluxes on the left and right ends and no flux through the upper and lower sides. This models 2d fluid flow between two parallel plates with prescribed end-fluxes.

Specifically take $\Omega := (-L, L) \times (-d, d)$ to be the planar rectangular region of length $2L$ and height $2d$ with Σ_0 being the top and bottom sides with $y = \pm d$ and Σ being the left and right sides with $x = \pm L$. The fluxes through the left and right ends are assumed to be a given function η that satisfies the conservation law (2.3). For convenience, the left hand side will be called the inlet and the right hand side the outlet.

For this problem the Σ - Steklov eigenvalues and eigenfunctions may be found explicitly. The first eigenfunction satisfying (4.1) and (4.3) is $s_0(x, y) \equiv 1$ and the associated eigenvalue is $\delta_0 = 0$. Using separation of variables, the successive harmonic

Σ -eigenvalues and eigenfunctions may be determined. There are two families given by

$$\delta_{k,1} = 2k\pi \tanh \nu_k L \quad \text{and} \quad \delta_{k,2} = 2k\pi \coth \nu_k L \quad \text{with} \quad \nu_k = \frac{k\pi}{2d} \quad (6.1)$$

and k an integer. When k is odd the associated (unnormalized) eigenfunctions are

$$s_{k,1}(x, y) = \cosh \nu_k x \sin \nu_k y \quad \text{and} \quad s_{k,2}(x, y) = \sinh \nu_k x \sin \nu_k y. \quad (6.2)$$

When k is even they are

$$s_{k,1}(x, y) = \cosh \nu_k x \cos \nu_k y \quad \text{and} \quad s_{k,2}(x, y) = \sinh \nu_k x \cos \nu_k y. \quad (6.3)$$

As k increases, $(\delta_{k,2} - \delta_{k,1})$ converges to zero rapidly, so $2k\pi$ is close to being a double eigenvalue of this problem.

These eigenfunctions and eigenvalues were also found computationally using the software package FreeFEM++ [8]. When this was done the interesting observation was that as the eigenvalue δ_j increased, the associated Σ -Steklov-eigenfunctions became very “flat” near the center and oscillated only in small regions near the inlet and outlet. This is consistent with the explicit formulae, as when k increases some sinh and cosh terms become very large near the inlet and outlet at $x = \pm L$.

For illustration, a contour plot of the third and fourth Steklov eigenfunctions is given in Figure 1 with $L = 2, d = 1$.

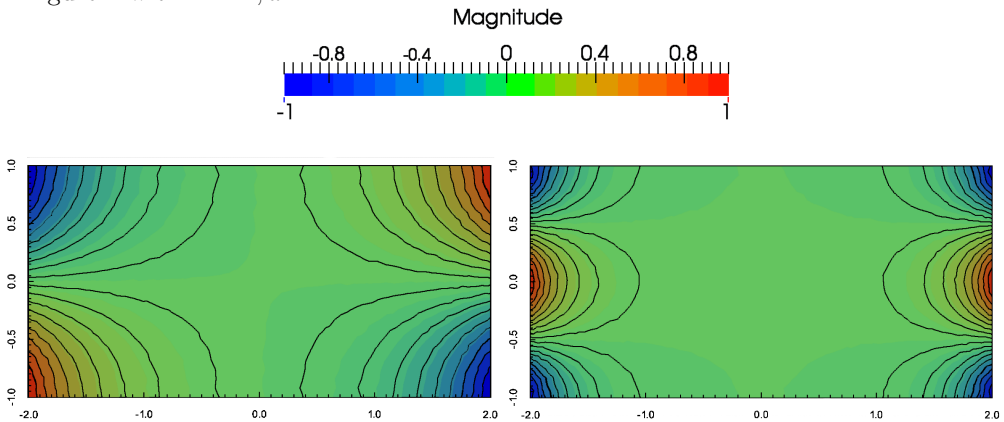


FIG. 1. Contours of the 3rd (left) and 4th (right) Steklov eigenfunctions.

(5.2) with the explicit formulae for the eigenfunctions was used to obtain analytical approximate expressions for solenoidal fields subject to given data η . Solutions of these problems were also obtained using direct finite element codes and also finite element computations of the Σ -Steklov eigenfunctions.

Note that the fluxes at $x = \pm L$ are described by standard Fourier series, so there is an extensive literature about the approximation of this data by finite approximations such as η_m . Then the last part of (5.1) provides error bounds on the computational solutions.

Two typical contour plots are shown in Figure 2 with parameters $L = 2, d = 1$ and $j = 4$. In the first plot the boundary fluxes are $\eta(-2, y) = (1 - y^2)(y + 2/3)$ and $\eta(2, y) = (4/9 - y)$ and $\eta(x, 1) \equiv \eta(x, -1) \equiv 0$ on the top and bottom. In the second

plot the boundary fluxes are $\eta(-2, y) = 3y(1 - y^2)$ and $\eta(2, y) = y$.

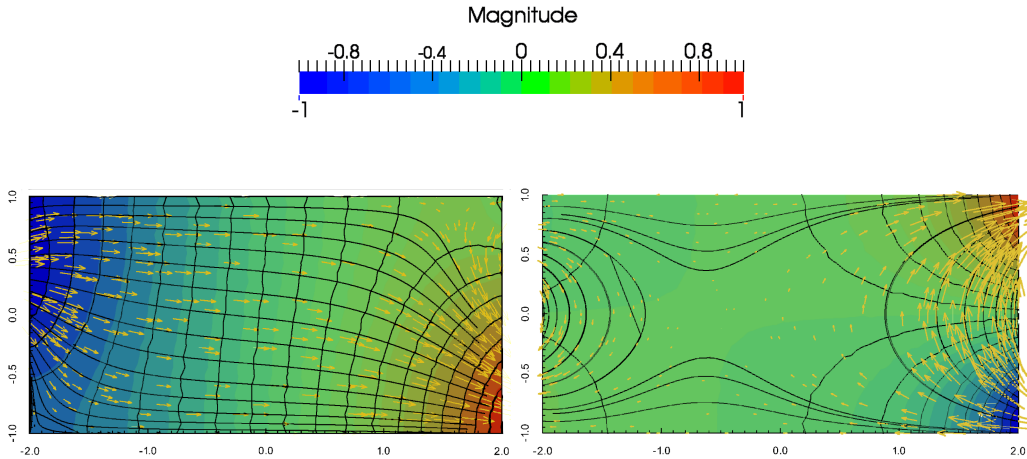


FIG. 2. Flow field and contour set of solutions using 8 Steklov eigenfunctions. Left picture: Data at $x = -2$ is $(1 - y^2)(y + 2/3)$, and data at $x = 2$ is $4/9 - y$. Right picture: Data at $x = -2$ is $(1 - y^2/4)(y - 1)$, and data at $x = 2$ is $y - 8/15$.

7. Computation of least energy solenoidal fields. The preceding section described some computational results with simple geometry where explicit formulae for the Σ - Steklov eigenvalues and eigenfunctions could be determined.

The second author also computed a number of 2d and axisymmetric flows in situations where the eigenproblem did not have explicit solutions. In this case, the first m Steklov eigenvalues and eigenfunctions were found computationally using FreeFEM++. Then the coefficients in the approximation (5.5) were found and both contour lines and flow fields of this approximation were computed. An example of a result for the solution of this problem in an axisymmetric funnel of length 4 and radius 2 at the left end and 1 at the right is given in Figure 3. When r is the radial variable in a cross-section, the boundary data are $\eta(-2, r) = (1 - r^2/4)(r - 1)$ for $0 \leq r \leq 2$ and $\eta(2, r) = r - 8/15$ for $0 \leq r \leq 1$. This was obtained using 8 Σ - Steklov eigenfunctions computed numerically using FreeFEM++.

More numerical results may be found in the project report Simpkins [10]. The primary observation from these experiments is that quite accurate simulations of flows may be found with relatively few (usually less than 12) Steklov eigenfunctions. That is, the solutions were quite comparable to finite element simulations of the solutions of the boundary value problem when the same grids and parameters were used as for determining the Steklov eigenfunctions.

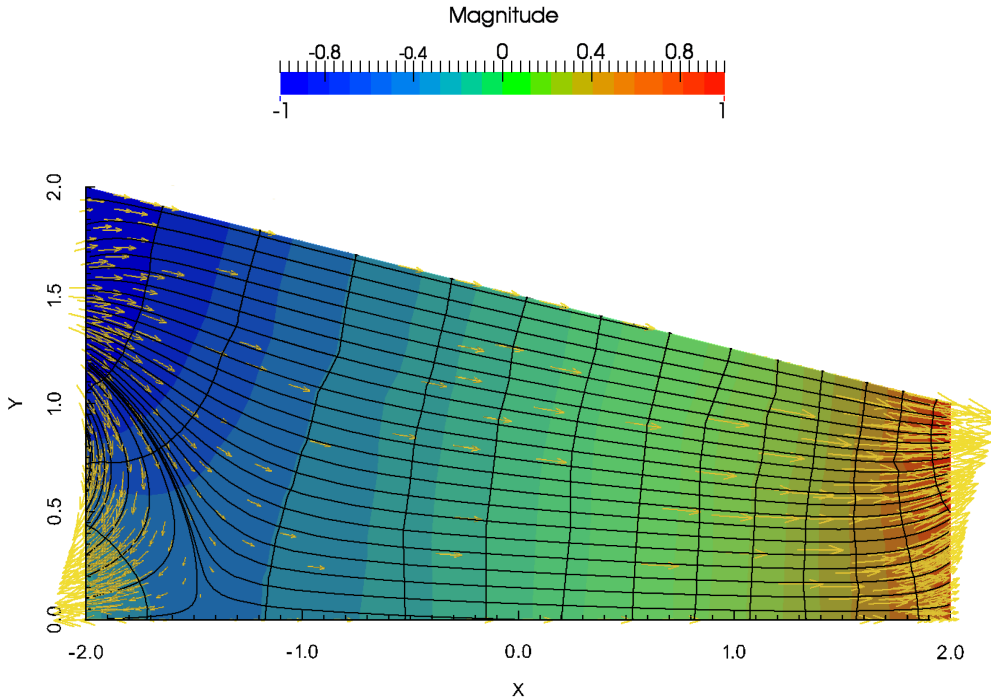


FIG. 3. Flow-field and contour sets of axisymmetric solution with 8 Steklov eigenfunctions. Data at $z = -2$ is $(1 - r^2/4)(r - 1)$, and data at $z = 2$ is $r - 8/15$.

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