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SINGULARITIES OCCURRING IN MULTIMATERIALS WITH TRANSPARENT BOUNDARY CONDITIONS

By

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Abstract. Computation of waves at the interface between two materials with different wave velocities is an important engineering problem. Transparent boundary conditions on the input and output boundaries are known for a single velocity wave. Adapting them in case of two velocities, singularities appear in the computation of the waves. We precisely exhibit these singularities both with theoretical and numerical points of view.

Introduction. The theory of transparent boundary conditions is a large and deep domain of mathematical studies. To our knowledge, the first main theoretical paper on the subject concerns simulations of solutions of a linear wave equation in the exterior of a bounded domain and is due to B. Engquist and A. Majda ([5]): the authors suggest a method, based on a Fourier transform in time and the transverse direction, which leads to exact and non-local transparent boundary conditions. They also perform several series developments (at different orders) and they obtain approximate but local transparent boundary conditions. Let us also notice the works of L. Halpern ([10] [11]): she (and co-authors) studied well-posedness of different boundary conditions and their numerical schemes.

When one studies solutions of linear wave equations with a unique velocity, the boundary conditions on the input and output boundary are transparent boundary conditions: they ensure that waves should go out of the domain (there is no reflection) and they avoid singularity (see Figure 2). In case of a transmission problem, the difference between wave velocities involves singularities which are localized at the intersections between the boundary of the domain and the interface of transmission (see Figure 3 and Figure 4). We give a precise description of them with a mathematical analysis of this phenomenon:

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we prove that the singularities are involved by the gap across the interface on the lateral boundary of the antisymmetric part of the Neumann derivative of the solution. Let us notice that there is no singularity for homogeneous Neumann boundary conditions on the whole boundary of Ω.

Let us illustrate our claim with the following different situations: Figures 1, 2, 3 and 4 below concern solutions of a transmission problem (see system (1)) in the square $\Omega =]0,1[\times]0,1[$ and during the time $T = 1.2$. The interface is localized at $y = 0.2$, and the velocities are $c = c_-$ if $y < 0.2$ and $c = c_+$ if $y > 0.2$ with $0 < c_- \leq c_+$. Initial data are null and the right hand side f is localized in the region with velocity c_-. Figure 1 shows the support of the right hand side f. Numerical computations are performed in Matlab. A finite element approximation using first degree polynomials has been used. The time step integration is performed with a central difference scheme. The damping term is estimated by a backward difference scheme. But an upgrade is introduced using a characteristic method for the boundary terms implied. Concerning the intersection point we used the average velocity. The time step of order $0.001T$ and space steps (both in the x and y directions) of $\frac{1}{120} = 0.0083$. Figures 2, 3 and 4 are snapshots at time $t = 0.41294$ or $t = 0.41367$ in the three following situations: on Figure 2, one has $c_- = c_+ = 1$ and the right hand side is a high frequency time excitation. One can see that there is no singularity. In Figure 3, one has $c_+ = 2$ and $c_- = 1$, and the right hand side f is a low frequency time excitation. In Figure 4, one has $c_+ = 2$ and $c_- = 0.5$, and the right hand side is a high frequency time excitation. In both Figures 3 and 4, one can see singularities appearing at the intersection between the interface $y = 0.2$ and the the boundary $x = 0$ and $x = 1$. This is what we explain in the following. Other numerical computations are given in order to exhibit more precisely these singularities.

In a forthcoming paper, we suggest a way to avoid these singularities introducing efficient new transparent boundary conditions. Let us notice that our method leads to a non-local computation which is classical in the derivation of exact transparent boundary conditions at higher order but new for first order.

Our plan is the following one : in the first section, we present the mathematical problem and we focus on the singularity. In the second section, we study existence and regularity results of its solutions. In the third section, we focus on singularities on the boundary of the interface. In this section, we state our main theorem. All along, the paper is completed by numerical simulations which have been performed with Matlab and which illustrate our theoretical results.

1. The wave model used for the discussion. Let us consider a two dimensional open set $\Omega =]0, L[\times] - a, a[$ ($L > 0$ and $a > 0$) of $\mathbb{R}^2$ as shown on Figure 5. A point x of Ω has coordinates $x = (x_1, x_2)$. The wave velocity is denoted by c and is piecewise constant. It is c_+ in $\Omega_+ = \Omega \cap (x_2 > 0)$ and c_- in $\Omega_- = \Omega \cap (x_2 < 0)$, and we assume that $0 < c_- \leq c_+$.

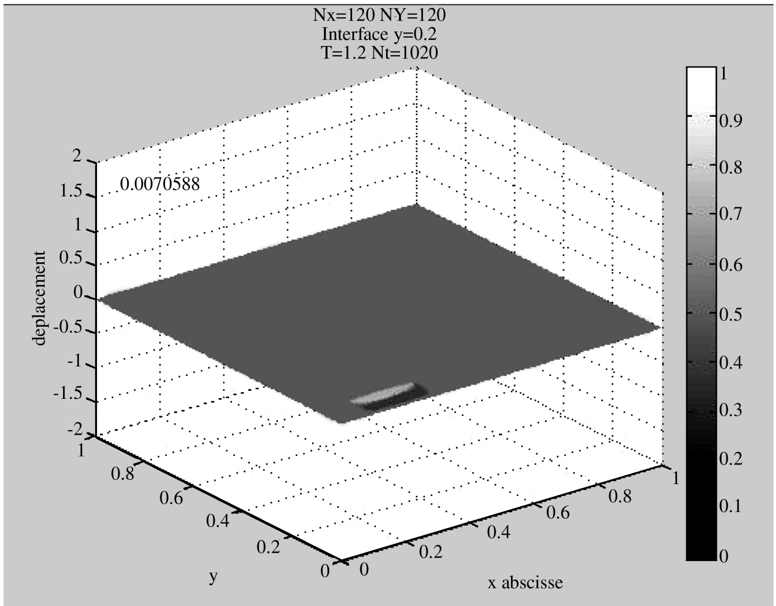


FIG. 1. The right hand side's support

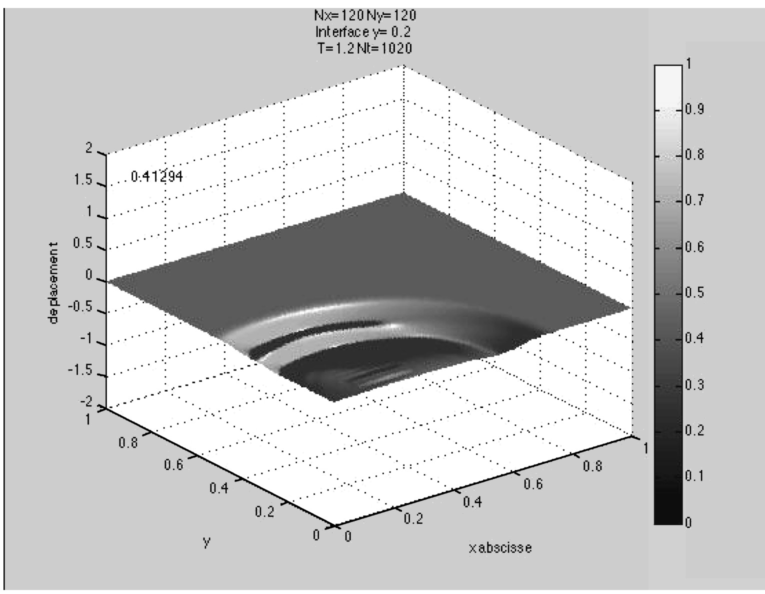


FIG. 2. One material

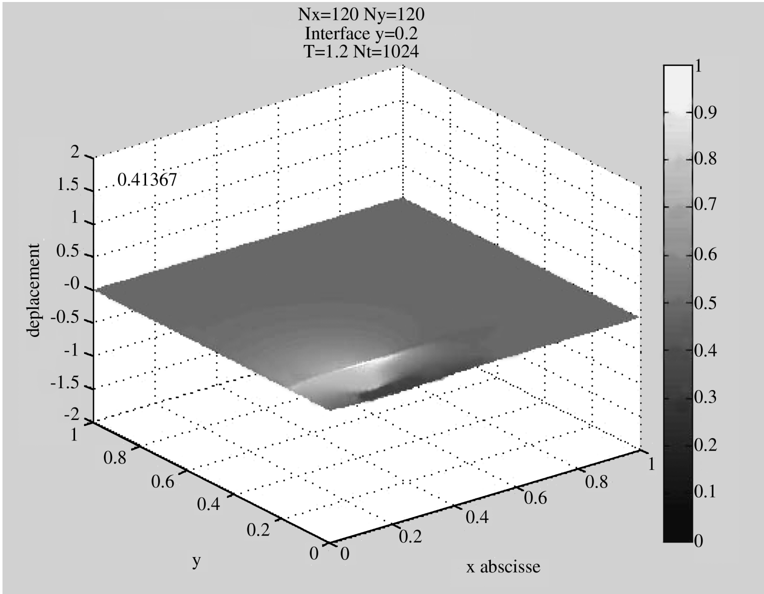


FIG. 3. Two materials and low frequency time excitation

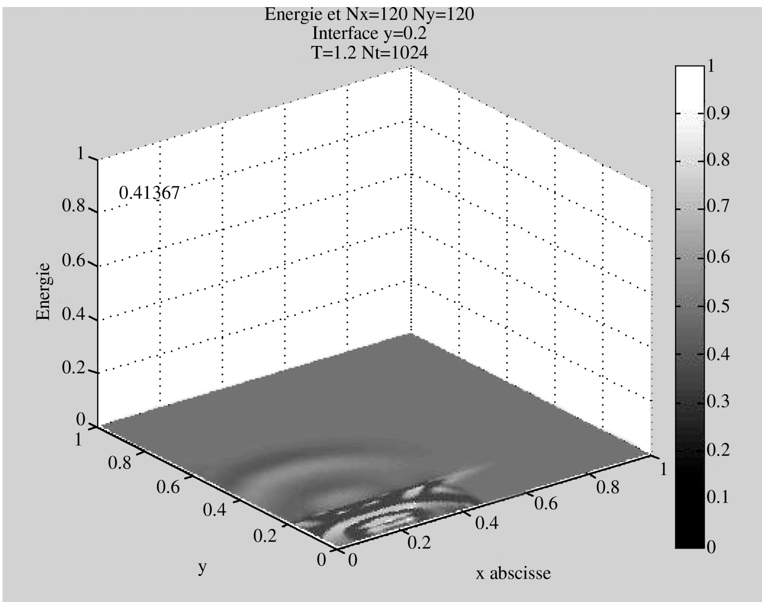


FIG. 4. Two materials and high frequency time excitation

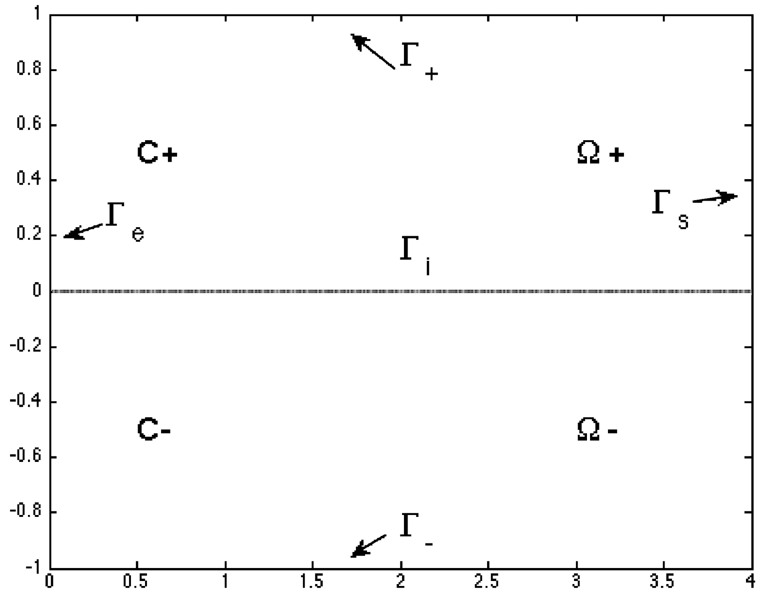


FIG. 5. The open set Ω and the notation for the theoretical discussion

For any given functions $f = f(x,t)$, $u_0 = u_0(x)$ and $u_1 = u_1(x)$, let us consider the $u = u(x,t)$ solution of the following mathematical model:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \operatorname{div}(c^2 \nabla u) = f \ \text{ in } \ Q = \Omega \times]0,T[, \\[2ex] \dfrac{\partial u}{\partial \nu} = 0 \ \text{ on } (\Gamma_+ \cup \Gamma_-) \times]0,T[, \\[2ex] \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial \nu} = 0 \ \text{ on } \ (\Gamma_e \cup \Gamma_s) \times]0,T[, \\[2ex] u(x,0) = u_0(x) \ \ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \ \text{ in } \ \Omega. \end{cases} \tag{1}$$

We write $\Gamma_{e_+} = \Gamma_e \times (x_2 > 0)$, $\Gamma_{e_-} = \Gamma_e \times (x_2 < 0)$, $\Sigma_+ = \Gamma_+ \times]0,T[$, $\Sigma_- = \Gamma_- \times]0,T[$, $\Sigma_e = \Gamma_e \times]0,T[$, and $\Sigma_s = \Gamma_s \times]0,T[$. The interface is $\Gamma_i = \Omega \cap [x_2 = 0]$ and $\Sigma_i = \Gamma_i \times]0,T[$. The letter ν denotes the unit outward normal vector on the boundary Γ of Ω. For $x \in \mathbb{R}^2$, we denote by V_x a small enough open and non-empty neighborhood of x in $\mathbb{R}^2$. We write 1_O for the characteristic function of a set O of $\mathbb{R}^2$ and $\overset{\circ}{\Gamma}_i$ for the interior of Γ_i.

Existence and uniqueness of solutions of system (1) are classical results, even if the transparent boundary conditions on $\Gamma_e \cup \Gamma_s$ (which imply first order time derivative) require a slightly different strategy from the usual one. Let us summarize them in the following statement.

Proposition 1. Let us assume that $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $f \in L^2(Q)$. Then, there exists a unique solution u to the system (1) with:

$$u \in L^\infty(]0,T[; H^1(\Omega)) \cap W^{1,\infty}(]0,T[; L^2(\Omega)).$$

We don't give the full proof of Proposition 1.1; one can refer to [4] in case of interest: it is based on a Galerkin method, a priori estimate and weak convergence of a subsequence solution of a finite dimensional approximation. □

Let us recall the variational formulation of (1).

- The function u is the solution of

$$\begin{aligned} \forall v \in H^1(\Omega), \quad & \int_\Omega \frac{\partial^2 u}{\partial t^2}(x,t)v(x)dx + \int_\Omega c^2 \nabla u(x,t).\nabla v(x)dx \\ &+ \int_{\Gamma_e \cup \Gamma_s} c\frac{\partial u}{\partial t}(x,t)v(x) = \int_\Omega f(x,t)v(x)dx. \end{aligned} \tag{2}$$

- The energy is defined by

$$E(t) = \frac{1}{2}\int_\Omega \frac{\partial u}{\partial t}(x,t)^2 dx + \frac{1}{2}\int_\Omega c^2 |\nabla u(x,t)|^2 dx, \tag{3}$$

and one has for $f = 0$:

$$E(t) + \int_{\Sigma_e \cup \Sigma_s} c\frac{\partial u}{\partial t}(x,t)^2 dx = E(0).$$

Therefore it decreases with respect to the time variable.

In the general case, there exists a constant $d > 0$ such that for every data $(u_0, u_1, f) \in H^1(\Omega) \times L^2(\Omega) \times L^2(Q)$:

$$E(t) + \int_{\Sigma_e \cup \Sigma_s} c|\frac{\partial u}{\partial t}(x,t)|^2 dx \leq d[E(0) + ||f||_Q]^2,$$

where $||\ ||_Q$ denotes the $L^2(Q)-$ norm. These results prove that

$$\frac{\partial u}{\partial t} \in L^2(\Sigma_e \cup \Sigma_s).$$

Since $\frac{\partial u}{\partial \nu} = -\frac{1}{c}\frac{\partial u}{\partial t}$, one has $\frac{\partial u}{\partial \nu} \in L^2(\Sigma_e \cup \Sigma_s)$.

Furthermore, even if $\frac{\partial u}{\partial t} \in H^{-1}(0,T; H^1(\Omega))$, the discontinuity of c implies that in general $\frac{\partial u}{\partial \nu} \notin \mathcal{D}'(0,T; H^{1/2}(\Gamma_e))$ and $\frac{\partial u}{\partial \nu} \notin \mathcal{D}'(0,T; H^{1/2}(\Gamma_s))$.

The function u, the unique solution of (1), satisfies locally:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c_+^2 \Delta u = f \ \text{ in } \ Q_+ = \Omega_+ \times]0,T[, \\[2ex] \dfrac{\partial^2 u}{\partial t^2} - c_-^2 \Delta u = f \ \text{ in } \ Q_- = \Omega_- \times]0,T[, \\[2ex] c_+^2 \dfrac{\partial u}{\partial \nu_i} - c_-^2 \dfrac{\partial u}{\partial \nu_i} = 0 \ \text{ on } \ \Sigma_i, \end{cases} \tag{4}$$

where ν_i here is one of the two unit normal vectors of Γ_i and $\frac{\partial u}{\partial \nu_i} = \nabla u.\nu_i$.

The third equation of (4) implies that whatever the smoothness of the data are there is in general no hope to have $u \in D'(0,T;H^2_{loc}(\Omega))$ if c is not constant across the separation line Γ_i between the two materials.

We now turn to the regularity study of solutions of (1).

2. On the smoothness of u in time and space. Let us study the regularity of u first with respect to the time variable and then with respect to space variables. Singularities will be discussed in the next section.

Based on derivative methods with respect to the time t, one can upgrade the results stated in Proposition 1 by assuming that the initial condition u_0 doesn't cross the interface Γ_i. Let us define by $\mathcal{O}+$, respectively $\mathcal{O}_-$, two open subsets whose closures are subsets of Ω_+ and Ω_-.

THEOREM 1. If the initial conditions are such that $u_0 \in H^2(\Omega)$ with $supp(u_0) \subset O_+ \cup O_-$, $u_1 \in H^1(\Omega)$ and $f \in H^1(]0,T[;L^2(\Omega))$, then the solution u of (1) is such that

$$u \in W^{1,\infty}(]0,T[;H^1(\Omega)) \cap W^{2,\infty}(]0,T[;L^2(\Omega)).$$

From classical inclusions (see [3]), this implies for instance that

$$u \in \mathcal{C}^0([0,T];H^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega)).$$

Proof. Let us set $\dot{u} = \dfrac{\partial u}{\partial t}$. The function $\dot{u}$ is the solution of (1) with data: $\dot{f} = \dfrac{\partial f}{\partial t} \in L^2(Q)$, $\dot{u}(x,0) = u_1(x) \in H^1(\Omega)$ and $\dfrac{\partial \dot{u}}{\partial t}(x,0) = f(x,0) + \operatorname{div}(c^2\nabla u_0)$.

The assumption on the initial data u_0 ensures that ∇u_0 is null on a neighborhood of the interface Γ_i in Ω and thus $div(c^2\nabla u_0) \in L^2(\Omega)$. Moreover, $f \in C^0([0,T];L^2(\Omega))$; therefore $f(x,0)$ is in $L^2(\Omega)$. Applying Proposition 1, we easily deduce that $u \in W^{1,\infty}(]0,T[;H^1(\Omega)) \cap W^{2,\infty}(]0,T[;L^2(\Omega))$. □

By iterating Theorem 1, one easily gets:

THEOREM 2. With the same notation as in Theorem 1, and if the initial conditions are such that $(u_0,u_1) \in H^3(\Omega) \times H^2(\Omega)$ both with compact supports in $O_+ \cup O_-$, and if $f \in H^2(0,T;L^2(\Omega))$, then the solution u of (1) is such that

$$u \in W^{2,\infty}(]0,T[;H^1(\Omega)) \cap W^{3,\infty}(]0,T[;L^2(\Omega)),$$

and thus $u \in C^1([0,T],H^1(\Omega)) \cap C^2([0,T],L^2(\Omega))$.

REMARK. Let us first recall that if γ is a non-empty and open part of the whole boundary Γ of Ω, the restriction operator to γ maps $H^{1/2}(\Gamma)$ onto $H^{1/2}(\gamma)$, whereas the extension by zero of a function of $H^{1/2}(\gamma)$ is not in general a function of $H^{1/2}(\Gamma)$. In both cases of Theorems 1 and 2, one has

$$u_{|\Gamma_e \cup \Gamma_s} \in \mathcal{C}^1([0,T];H^{1/2}(\Gamma_e \cup \Gamma_s))$$

and therefore

$$c\frac{\partial u}{\partial \nu}_{|\Gamma_e \cup \Gamma_s} \in \mathcal{C}^0([0,T];H^{1/2}(\Gamma_e \cup \Gamma_s)).$$

□

The function c is discontinuous across Γ_i and therefore in general (even a piecewise continuous function is not globally $H^{1/2}$ of the whole open set)

$$\frac{\partial u}{\partial \nu}_{|\Gamma_e} \notin D'(0,T;H^{1/2}(\Gamma_e)) \text{ and } \frac{\partial u}{\partial \nu}_{|\Gamma_s} \notin D'(0,T;H^{1/2}(\Gamma_s)). \tag{5}$$

We get at least for $0 \leq s < 1/2$,

$$\frac{\partial u}{\partial \nu}_{|\Gamma_e} \in C^0([0,T];H^s(\Gamma_e)) \text{ and } \frac{\partial u}{\partial \nu}_{|\Gamma_s} \in C^0([0,T];H^s(\Gamma_s)).$$

Let us now turn to the smoothness with respect to the space variables; there are several cases.

THEOREM 3. Let $f \in H^1(Q)$ and let us assume the initial data as in Theorem 1. Let u be the solution of system (1). One has:

(1) $u1_{\Omega_+} \in L^\infty(]0,T[;H^2_{loc}(\Omega_+))$ and $u1_{\Omega_-} \in L^\infty(]0,T[;H^2_{loc}(\Omega_-))$.
(2) If $x \in \Gamma_+ \cup \Gamma_-$, then $u \in L^\infty(]0,T[;H^2(V_x \cap \bar{\Omega}))$.
(3) If $x \in \Gamma_{e_+} \cup \Gamma_{e_-} \cup \Gamma_{s_+} \cup \Gamma_{s_-}$, then $u \in L^\infty(]0,T[;H^2(V_x \cap \bar{\Omega}))$.
(4) If $x \in \overset{\circ}{\Gamma}_i$, then $\dfrac{\partial u}{\partial x_1} \in L^\infty(]0,T[;H^1(V_x))$.

Proof. (1) On $\Omega_+ \times]0,T[$, one has

$$\operatorname{div}(c^2 \nabla u) = c_+^2 \Delta(u) = \frac{\partial^2 u}{\partial t^2} \in L^\infty(]0,T[;L^2(\Omega_+)).$$

A classical localization argument leads to $u|_{\Omega_+} \in L^\infty(]0,T[;H^2_{loc}(\Omega_+))$. Of course, the same argument can be applied in Ω_-.

(2) Let $x \in \Gamma_+$ for example. The boundary conditions $\dfrac{\partial u}{\partial \nu} = 0$ on Σ_+ allow us to apply a symmetry argument and to consider an extension $\bar{u}$ of u across Γ_+, which is still the solution of (1) in a neighborhood V_x of x. The first point of this theorem leads to $\bar{u} \in L^\infty(]0,T[;H^2(V_x))$ and thus $u \in L^\infty(]0,T[;H^2(V_x \cap \Omega))$.

(3) Let $x \in \Gamma_{e_+}$. Since $\dfrac{\partial u}{\partial t} \in L^\infty(]0,T[;H^1(\Omega))$ and $\dfrac{\partial u}{\partial \nu} = -\dfrac{1}{c}\dfrac{\partial u}{\partial t}$, we obtain $\dfrac{\partial u}{\partial \nu} \in L^\infty(]0,T[;H^{1/2}(\Gamma_{e_+}))$. After localization, we get

$$\Delta u \in L^2((V_x \cap \Omega) \times (0,T)) \text{ and } \frac{\partial u}{\partial \nu} \in L^\infty(0,T;H^{1/2}(\partial(V_x \cap \Omega)).$$

Classically, this leads to $u \in L^\infty(0,T;H^2(V_x \cap \bar{\Omega}))$.

(4) Let $x \in \overset{\circ}{\Gamma}_i$. In this case, let us notice that we can assume that $V_x \subset \bar{\Omega}$. There is no hope (in general) that $u \in D'(0,T;H^2_{loc}(V_x))$ except for $c_+ = c_-$. Let us consider $\rho_x \in \mathcal{D}(\Omega)$ with $\rho_x = 1$ on V_x and let us write $w_1 = \rho_x \dfrac{\partial u}{\partial x_1}$. Since Γ_i is parallel to the axis and boundary x_1, the function w_1 is still the solution of a transmission problem similar to (1) with

$$\begin{cases} \dfrac{\partial^2 w_1}{\partial t^2} - \operatorname{div}(c^2 \nabla w_1) \in L^2(Q), \\ \dfrac{\partial w_1}{\partial \nu} \in L^\infty(]0,T[;H^{1/2}(\partial V_x)), \end{cases}$$

and thus $w_1 \in L^\infty(]0,T[;H^1(V_x))$. □

REMARK. One can easily prove that $\dfrac{\partial u}{\partial x_1}$ is $L^\infty(]0,T[, H^1(]b_1,b_2[\times]-a,a[))$ in any rectangle $]b_1,b_2[\times]-a,a[$ with $0 < b_1 < b_2 < L$. □

We now turn to the main point: the description of the singularities focusing at the point $(x_1, x_2) = (0,0)$ (see Figure 5).

3. Polylog-2 singularities at the junction of the interface Γ_i and the boundary Γ_e (or Γ_s). We focus our study at the point with the coordinates $(0,0)$, but, of course, an analogous result is valid at the point with coordinates $(L,0)$. We introduce an even function ρ with $\rho = \rho(x_2) \in C^\infty(\mathbb{R})$:

$$\begin{cases} \rho(x_2) = 1 \quad \text{for} \quad -\dfrac{a}{2} < x_2 < \dfrac{a}{2} \\ \rho(x_2) = 0 \quad \text{for} \quad |x_2| > \dfrac{3a}{4} \end{cases} . \tag{6}$$

Let us denote $W =]0,b[\times]-\frac{a}{2},\frac{a}{2}[$ where $0 < b < L$. We consider a function $\rho_W \in C^\infty(\Omega)$ such that

$$\begin{cases} \rho_W = 1 \ \text{ in } \ W, \\ \rho_W = 0 \ \text{ in a neighborhood of } \ \Gamma_+ \cup \Gamma_- \cup \Gamma_s. \end{cases} \tag{7}$$

Let us set

$$\begin{aligned} \mathcal{V} = \{w \in H^1(\Omega), \quad & \rho w 1_{\Omega_+} \in H^2(\Omega_+), \rho w 1_{\Omega_-} \in H^2(\Omega_-), \\ & \text{and } \frac{\partial \rho w}{\partial x_1} \in H^1(W)\}. \end{aligned} \tag{8}$$

For a given function h, we denote by T_s and T_a the following symmetric and antisymmetric part of h defined by

$$T_s(h)(x_1,x_2) = \frac{1}{2}[h(x_1,x_2) + h(x_1,-x_2)] \tag{9}$$

and

$$T_a(h)(x_1,x_2) = \frac{1}{2}[h(x_1,x_2) - h(x_1,-x_2)]. \tag{10}$$

When no mistake can be made, we write h_s and h_a instead of $T_s(h)$ and $T_a(h)$. Of course, $h = h_s + h_a$.

We denote by $Im(z)$ the imaginary part of the complex number z and we introduce the two following functions:

Li_2 is the *polylogarithm function of order* 2 defined by

$$Li_2(z) = \sum_{n\geq 1} \frac{z^n}{n^2} \quad \text{for} \quad |z| \leq 1.$$

With $z = \exp[-\frac{\pi}{a}(x_1 - ix_2)]$, we consider the function S defined by

$$S(x_1,x_2) = \frac{2a}{\pi^2(c_+^2 + c_-^2)} Im[Li_2(z) - Li_2(-z)].$$

Let us notice that $S \in H^1(\Omega)$ and that S is null on the interface Γ_i. The graphs of the function S and its partial derivatives are given in Figure 6.

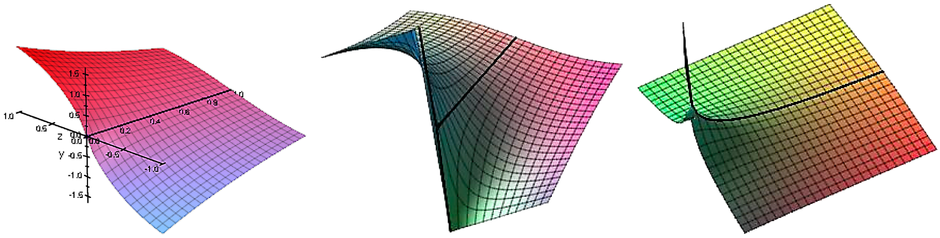


FIG. 6. Graphs of the function S and its partial derivatives : $\dfrac{\partial S}{\partial x_1}$ and $\dfrac{\partial S}{\partial x_2}$

Our main result in the paper is the following one: it gives a splitting of the solution u in a regular part (u_r) and a singular one (u_{sg}).

THEOREM 4. (i) Let $f \in H^1(\Omega\times]0,T[)$, $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega)$. Then, there exist two functions u_r and u_{sg} in $L^2(]0,T[;H^1(\Omega))$ with $u = u_r + u_{sg}$ in Q. The function u_r satisfies $\dfrac{\partial u_r}{\partial x_1} \in H^{-2}(]0,T[;\mathcal{V})$ (where $\mathcal{V}$ is defined in (8)).

Furthermore, there exist $\alpha_0 = \alpha_0 \in H^{-2}(0,T)$ and $\varepsilon > 0$, such that in $]0,\varepsilon[\times]-\varepsilon,\varepsilon[$, we have

$$u_{sg}(x_1,x_2,t) = (c_+ - c_-)\alpha_0(t)S(x_1,x_2)\,[\frac{c_-}{c_+}1_{\Omega_+} + \frac{c_+}{c_-}1_{\Omega_-}]$$

and

$$\frac{\partial u_{sg}}{\partial x_2} = \frac{\alpha_0(c_+ - c_-)}{\pi(c_+^2 + c_-^2)}\mathrm{Ln}|\frac{1 + 2e^{-\dfrac{\pi x_1}{a}}\cos(\dfrac{\pi x_2}{a}) + e^{-2\dfrac{\pi x_1}{a}}}{1 - 2e^{-\dfrac{\pi x_1}{a}}\cos(\dfrac{\pi x_2}{a}) + e^{-2\dfrac{\pi x_1}{a}}}|[\frac{c_-}{c_+}1_{\Omega_+} + \frac{c_+}{c_-}1_{\Omega_-}]. \quad (11)$$

(ii) If $f \in H^1(\Omega\times]0,T[)$, $u_0 \in H^2(\Omega)$ and $u_1 \in H^1(\Omega)$ both with compact supports in $O_+ \cup O_-$, then $\alpha_0 \in H^{-1}(0,T)$.

(iii) If $f \in H^2(\Omega\times]0,T[)$, $u_0 \in H^3(\Omega)$ and $u_1 \in H^2(\Omega)$ both with compact supports in $O_+ \cup O_-$, then $\alpha_0 \in L^2(0,T)$.

REMARK. Let us notice that u_{sg} is null if $c_+ = c_-$; thus the existence of the singularity is due to the presence of two materials. We have (with $r = \sqrt{x_1^2 + x_2^2}$) for $c_+ \neq c_-$ and near $(0,0)$:

$$\frac{\partial u_{sg}}{\partial x_2} \sim \frac{2\alpha_0(c_+ - c_-)}{\pi(c_+^2 + c_-^2)}|\ \mathrm{Ln}(r)|.$$

We then get (still with $c_+ \neq c_-$ and $\alpha_0 \neq 0$)

$$\lim_{x_2\to 0^+}\frac{\partial u_{sg}}{\partial x_2}(0,x_2) = \lim_{x_2\to 0^-}\frac{\partial u_{sg}}{\partial x_2}(0,x_2) = \pm\infty.$$

□

Numerical illustrations. In Figures 7 and 8, the velocities satisfy $c_+ = 2$ and $c_- = 0.5$. The graph of $\dfrac{\partial u}{\partial x_2}$ is plotted on Γ_e in Figure 7(a), on an interior line $x_1 = 0.5$ in Figure

7(b), whereas Figures 8(a) and 8(b) point out the singularity on the outgoing side at the interface. One can see that the function $\dfrac{\partial u}{\partial x_2}$ presents singularities on the boundaries Γ_e, Γ_s. On the line $x_1 = 0.5$ of the domain Ω (see Figure 7(b)), there is a gap due to the transmission boundary conditions on the interface Γ_i. One can point out that the energy of the solution is mainly in the softest media Ω_-. This is in agreement with the fact that Love waves which are localized in this part of Ω act as an energy trap.

In Figures 9 (a), (b) and 10 (a) and (b), the graph of $\dfrac{\partial u}{\partial x_2}$ is given for the same data and time as in Figures 7 and 8 but in the case where $c_+ = c_-$. One can obviously see that there is no singularity in this homogeneous case.

The end of the paper is devoted to the proof of Theorem 4.

Proof of Theorem 4. The idea is the following: we prove that the singular part of u on Γ_e comes from the antisymmetric part of the singular function $-\dfrac{1}{c}\dfrac{\partial u}{\partial t}$ on the boundary Γ_e. More precisely, the singularity is strictly connected to the gap at the origin $(0,0)$ of this function, a gap that we have to define. In order to point out this fact, we split the solution of (1) into several parts and write

$$g = -\frac{1}{c}\frac{\partial u}{\partial t} 1_{\Sigma_e \cup \Sigma_s}. \tag{12}$$

Let us recall that for $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$ and $f \in L^2(Q)$, we have $g \in L^2(\Sigma)$ but $g \notin H^{-1}(]0,T[; H^{1/2}(\Gamma))$. In the same way, if $(u_0, u_1) \in H^2(\Omega) \times H^1(\Omega)$ with $supp(u_0) \subset K$ where K is a compact set in $\Omega_+ \cup \Omega_-$, even if $f \in H^1(Q)$, then $g \notin L^2(]0,T[; H^{1/2}(\Gamma))$. Since our interest is not far from the interface Γ_i, we first localize the function u introducing $\tilde{u} = \rho u$ where ρ is defined at (6). The function $\tilde{u}$ is the solution of the following system:

$$\begin{cases} \dfrac{\partial^2 \tilde{u}}{\partial t^2} - \operatorname{div}(c^2 \nabla \tilde{u}) = \tilde{f} \ \text{ in } \ Q = \Omega \times]0,T[, \\[2ex] \tilde{u} = 0 \ \text{ on } (\Gamma_+ \cup \Gamma_-) \times]0,T[, \\[2ex] \dfrac{\partial \tilde{u}}{\partial t} + c\dfrac{\partial \tilde{u}}{\partial \nu} = 0 \ \text{ on } \ (\Gamma_e \cup \Gamma_s) \times]0,T[, \\[2ex] \tilde{u}(x,0) = \rho u_0(x) \ \ \dfrac{\partial \tilde{u}}{\partial t}(x,0) = \rho u_1(x) \ \text{ in } \ \Omega, \end{cases} \tag{13}$$

with $\tilde{f} = \rho f + c^2 \nabla\rho . \nabla u + \operatorname{div}(c^2 u \nabla \rho)$. Since $\rho = 1$ in a neighborhood of Γ_i, we have $\operatorname{div}(c^2 u \nabla\rho) \in L^2(Q)$. We denote by $\tilde{g}$ the following function:

$$\tilde{g} = -\frac{1}{c}\frac{\partial \tilde{u}}{\partial t} 1_{\Sigma_e \cup \Sigma_s}. \tag{14}$$

We get $\tilde{g} \in L^2(\Sigma)$ and $\tilde{g} = g = -\dfrac{1}{c}\dfrac{\partial u}{\partial t}$ in a neighborhood in Γ_e of the point $(0,0)$.

Waves in a bimaterial with interface at $x_2 = 0.2$

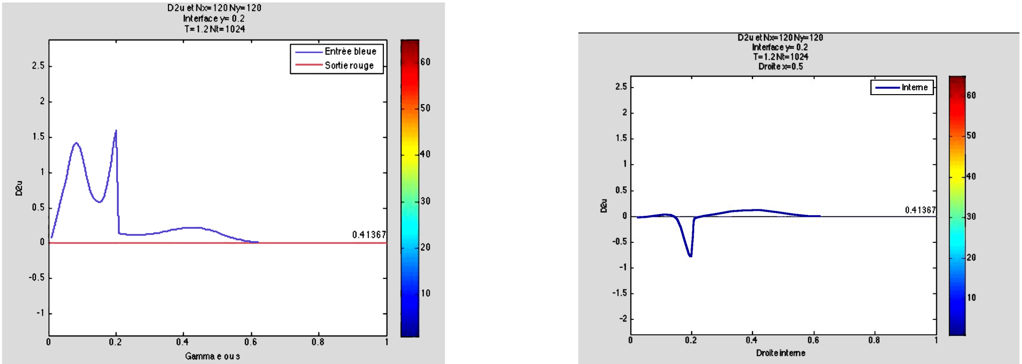


FIG. 7. (a) $\dfrac{\partial u}{\partial x_2}$ on Γ_e (b) $\dfrac{\partial u}{\partial x_2}$ on $x_1 = 0.5$

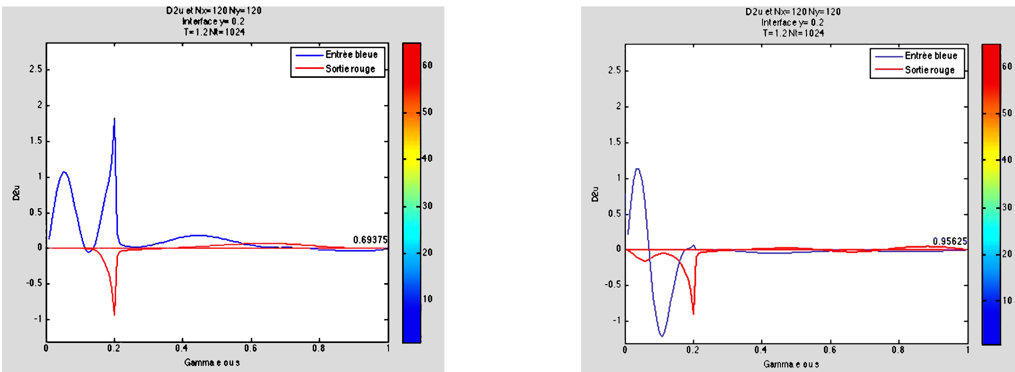


Fig. 8. (a) $\dfrac{\partial u}{\partial x_2}$ both on Γ_e and Γ_s (b) singularity of $\dfrac{\partial u}{\partial x_2}$ at the interface on Γ_s

We recall that g_s (respectively g_a) is the symmetric (respectively antisymmetric) part of g. We introduce $u^{(1)} \in L^2(0,T;H^1(\Omega))$ and $u^{(2)} \in L^2(0,T;H^1(\Omega))$ solutions of:

$$\left\{\begin{array}{l} \operatorname{div}(c^2\nabla u^{(2)}(t)) = 0 \ \text{ in } \ Q = \Omega\times]0,T[, \\[2ex] u^{(2)}(t) = 0 \ \text{ on } \ \{\Gamma_+ \cup \Gamma_-\}\times]0,T[, \\[2ex] \dfrac{\partial u^{(2)}}{\partial \nu}(t) = \tilde{g}_a(t) \ \text{ on } \ \{\Gamma_e \cup \Gamma_s\}\times]0,T[, \end{array}\right. \tag{15}$$

Waves in a material with $c_+ = c_- = 1$ and interface at $x_2 = 0.2$

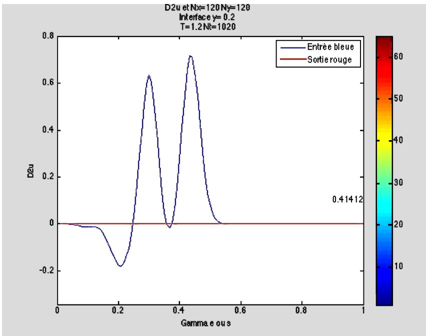
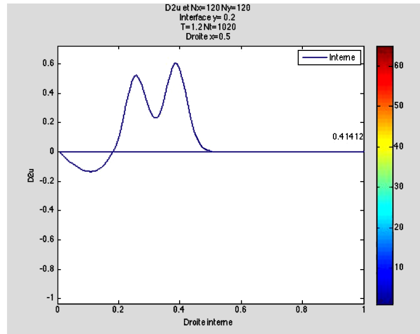


Fig. 9. (a) $\dfrac{\partial u}{\partial x_2}$ on Γ_e



(b) $\dfrac{\partial u}{\partial x_2}$ on $x_1 = 0.5$

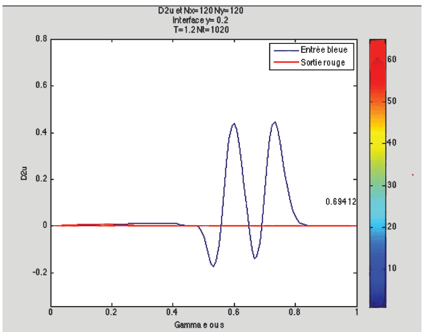
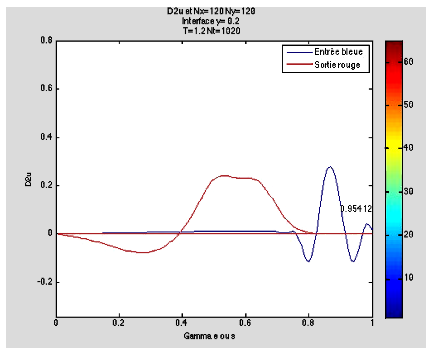


Fig. 10. (a) $\dfrac{\partial u}{\partial x_2}$ both on Γ_e and Γ_s



(b) $\dfrac{\partial u}{\partial x_2}$ at the interface on Γ_s

and $u^{(1)} = \tilde{u} - u^{(2)}$. We obtain that $u^{(1)} \in L^2(0,T; H^1(\Omega))$ is the solution of

$$\begin{cases} \operatorname{div}(c^2 \nabla u^{(1)}) = \operatorname{div}(c^2 \nabla \tilde{u}) = \ddot{\tilde{u}} - \tilde{f} \ \text{ in } \ Q, \\ u^{(1)} = 0 \ \text{ on } \{\Gamma_+ \cup \Gamma_-\} \times]0,T[, \\ \dfrac{\partial u^{(1)}}{\partial \nu} = \tilde{g}_s \ \text{ on } \ (\Gamma_e \cup \Gamma_s) \times]0,T[. \end{cases} \tag{16}$$

The following two lemmas will be useful for the study of the regularity of $u^{(1)}$.

Lemma 1. Let $w \in H^1(\Omega)$ be a solution of

$$\begin{cases} \operatorname{div}(c^2 \nabla w) = h \ \text{ in } \ \Omega, \\ w = 0 \ \text{ on } \Gamma_+ \cup \Gamma_-, \\ \dfrac{\partial w}{\partial \nu} = g_0 \ \text{ on } \ \Gamma_e \cup \Gamma_s, \end{cases} \tag{17}$$

with $h \in L^2(\Omega)$, $g_0 1_{\Gamma_e} \in H^{1/2}(\Gamma_e)$ and $g_0 1_{\Gamma_s} \in H^{1/2}(\Gamma_s)$. Then $w \in \mathcal{V}$ (defined in (8)).

Proof of Lemma 1. We refer to [12] for the main properties of the Sobolev spaces of order 1/2. The function $w_1 = \rho w$ is the solution of

$$\begin{cases} \operatorname{div}(c^2\nabla w_1) = h_1 \text{ in } \Omega, \\ w_1 = 0 \text{ on } \Gamma_+ \cup \Gamma_-, \\ \dfrac{\partial w_1}{\partial \nu} = g_1 \text{ on } \Gamma_e \cup \Gamma_s, \end{cases} \tag{18}$$

with

$$h_1 = \rho h + c^2\nabla\rho.\nabla w + \operatorname{div}(c^2 w\nabla\rho) \quad \text{and} \quad g_1 = \rho g_0 + w\nabla\rho.\nu.$$

Since $\nabla\rho = 0$ in a neighborhood of Γ_i, we deduce that $c^2\nabla\rho \in C^\infty(\mathbb{R})$ and $\operatorname{div}(c^2 w\nabla\rho) \in L^2(Q)$ and thus $h_1 \in L^2(Q)$.

Furthermore, $g_0 1_{\Gamma_e} \in H^{1/2}(\Gamma_e)$ implies that $g_0\rho 1_{\Gamma_e} \in H_{00}^{1/2}(\Gamma_e)$ (and the same argument is valid on Γ_s). Let us denote by $\bar{g}_1$ the extension by zero on the whole boundary of the function g_1. We then have $\bar{g}_1 \in H^{1/2}(\Gamma)$.

Let us set $w_2 = \dfrac{\partial w_1}{\partial x_1} = \rho\dfrac{\partial w}{\partial x_1}$. We obtain

$$\begin{cases} \operatorname{div}(c^2\nabla w_2) = \dfrac{\partial h_1}{\partial x_1} \text{ in } \Omega, \\ w_2 = 0 \text{ on } \Gamma_+ \cup \Gamma_-, \\ w_2 = -g_1 \text{ on } \Gamma_e \text{ and } w_2 = g_1 \text{ on } \Gamma_s. \end{cases} \tag{19}$$

Since $g_1 1_{\Gamma_e} \in H_{00}^{1/2}(\Gamma_e)$ (and the same on Γ_s), we get that the function $1_{\Gamma_s} g_1 - 1_{\Gamma_e} g_1 \in H^{1/2}(\Gamma)$. Introducing now $G_1 \in H^1(\Omega)$ with $G_1 = 1_{\Gamma_s} g_1 - 1_{\Gamma_e} g_1$ on Γ, and $w_3 = w_2 - G_1$, we deduce that

$$\begin{cases} \operatorname{div}(c^2\nabla w_3) = \dfrac{\partial h_1}{\partial x_1} - \operatorname{div}(c^2\nabla G_1) \text{ in } \Omega, \\ w_3 = 0 \text{ on } \Gamma, \end{cases} \tag{20}$$

with $h_3 = \dfrac{\partial h_1}{\partial x_1} - \operatorname{div}(c^2\nabla G_1) \in H^{-1}(\Omega)$.

We easily deduce that $w_3 \in H^1(\Omega)$; thus $w_2 = \rho\dfrac{\partial w}{\partial x_1} \in H^1(\Omega)$.

Assertions $\rho w 1_{\Omega_+} \in H^2(\Omega_+)$ and $\rho w 1_{\Omega_-} \in H^2(\Omega_-)$ are easy consequences of $\Delta(\rho w 1_{\Omega_+}) \in L^2(\Omega_+)$ and $\dfrac{\partial \rho w}{\partial x_1} \in H^1(\Omega)$ (the same for Ω_-). Of course, there is no hope to have $w \in H^2(W)$. Lemma 1 is therefore proved. □

LEMMA 2. Let $g \in L^2(\Gamma_e)$ with $g1_{\Gamma_{e_+}} \in H^{1/2}(\Gamma_{e+})$ and $g1_{\Gamma_{e_-}} \in H^{1/2}(\Gamma_{e-})$. Then, we have $g_s \in H^{1/2}(\Gamma_e)$.

Let us suppose that Lemma 2 is proved. Since the function $\tilde{g}$ satisfies $\tilde{g}1_{\Gamma_{e+}} \in H^{-1}(0, T, H^{1/2}(\Gamma_{e_+}))$ (and the same on Γ_{e_-}), we obtain that $\tilde{g}_s \in H^{-1}(0, T; H^{1/2}(\Gamma_e))$.

Moreover, $\ddot{\tilde{u}} - \tilde{f} \in H^{-1}(0,T;L^2(\Omega))$. We can apply Lemma 1 to the function $u^{(1)}$, and we get that $u^{(1)} \in H^{-1}(0,T;\mathcal{V})$ if $(u_0,u_1) \in H^1(\Omega)\times L^2(\Omega)$. In the case where $(u_0,u_1) \in H^2(\Omega)\times H^1(\Omega)$ with $supp(u_0) \subset K$ where K is a compact set in $\Omega_+ \cup \Omega_-$, one can easily prove that $u^{(1)} \in L^2(0,T;\mathcal{V})$ since $\ddot{\tilde{u}} - \tilde{f} \in L^2(0,T;L^2(\Omega))$ $\tilde{g}_s \in L^2(0,T;H^{1/2}(\Gamma_e))$.

Proof of Lemma 2. Let us point out that the time variable is a parameter in Lemma 2 and thus by linearity, the regularity in time comes from that of the function g. There is nothing to prove for this.

Let us introduce

$$\mathcal{T}_s(g_1,g_2)(x_2) = \begin{cases} \dfrac{1}{2}[g_1(x_2)+g_2(-x_2)] \;\text{ if } x_2>0 \\ [6pt]\dfrac{1}{2}[g_2(x_2)+g_1(-x_2)] \;\text{ if } x_2<0 \end{cases}$$

for $g_1 \in L^2(\Gamma_{e_+})$ and $g_2 \in L^2(\Gamma_{e_-})$.

We easily get

$$\mathcal{T} \in \mathcal{L}(L^2(\Gamma_{e_+})\times L^2(\Gamma_{e_-});L^2(\Gamma_e)).$$

If $(g_1,g_2) \in H^1(\Gamma_{e_+})\times H^1(\Gamma_{e_-})$, then $\mathcal{T}_s(g_1,g_2) \in H^1(\Gamma_e)$ (there is no gap across $x_2=0$); hence

$$\mathcal{T} \in \mathcal{L}(H^1(\Gamma_{e_+})\times H^1(\Gamma_{e_-});H^1(\Gamma_e)).$$

By interpolation of order 1/2, we deduce Lemma 2. □

We now turn to the proof of Theorem 4 with the study of $u^{(2)}$, the solution of (15), and we first prove the following result concerning the solution u: it defines the gap of $u|_{\Gamma_e}$ at the origin.

PROPOSITION 2. Suppose $(u_0,u_1)\in H^1(\Omega)\times L^2(\Omega)$. We have $\tilde{u}1_{\Omega_+} \in L^2(0,T;H^{3/2}(\Omega_+))$, $\tilde{u}1_{\Omega_-} \in L^2(]0,T[;H^{3/2}(\Omega_-))$ and we can write

$$\tilde{g}_a = \mathcal{T}_a(\tilde{g}) = \alpha(t)\,\mathrm{sign}(x_2) + g_{ar},$$

where $\alpha \in H^{-2}(0,T)$ and $g_{ar} \in H^{-2}(]0,T[;H^{1/2}(\Gamma_e))$.

Furthermore, in the case (ii) (respectively (iii)) of Theorem 4, we have $\alpha(t) \in H^{-1}(0,T)$ (respectively $\alpha(t) \in L^2(0,T)$).

Proof of Proposition 2. Let us prove that $\tilde{u}1_{\Omega_+} \in L^2(]0,T[;H^{3/2}(\Omega_+))$. We introduce $\tilde{u}^S$ and $\tilde{u}^A$ defined by $\tilde{u}^A = \tilde{u} - \tilde{u}^S$ and

$$\tilde{u}^S(x_1,x_2,t) = \begin{cases} \dfrac{c_+^2\tilde{u}(x_1,x_2,t)+c_-^2\tilde{u}(x_1,-x_2,t)}{c_+^2+c_-^2} \quad \text{if} \quad x_2>0 \\[10pt] \dfrac{c_+^2\tilde{u}(x_1,-x_2,t)+c_-^2\tilde{u}(x_1,x_2,t)}{c_+^2+c_-^2} \quad \text{if} \quad x_2<0. \end{cases}$$

We have $\tilde{u}^S \in L^2(0,T;H^1(\Omega))$ and $\tilde{u}^S = \tilde{u}$ on Γ_i. Let us prove that $\tilde{u}^S \in H^{-1}(0,T;H^2(\Omega))$. We obtain in Ω_+ :

$$\frac{\partial\tilde{u}^S}{\partial x_2} = \frac{1}{c_+^2+c_-^2}[c_+^2\frac{\partial\tilde{u}}{\partial x_2}(x_1,x_2,t) - c_-^2\frac{\partial\tilde{u}}{\partial x_2}(x_1,-x_2,t)];$$

thus (recall that $\tilde{u}$ satisfies the transmission condition), we get $\dfrac{\partial \tilde{u}^S}{\partial \nu} = 0$ on Γ_i. An analogous calculus on Ω_- proves the normal derivative of $\tilde{u}^S$ through the interface Γ_i is also null. Since $\tilde{u}^S \in L^2(0,T;H^1(\Omega))$, $\Delta(\tilde{u}^S 1_{\Omega_+}) \in H^{-1}(0,T;L^2(\Omega_+))$, $\Delta(\tilde{u}^S 1_{\Omega_-}) \in H^{-1}(0,T;L^2(\Omega_-))$, with no gap of the normal derivative through Γ_i, we deduce that $\Delta \tilde{u}^S \in H^{-1}(0,T;L^2(\Omega))$.

On the other hand, we have on Ω_+:

$$\frac{\partial \tilde{u}^S}{\partial x_1} = \frac{1}{c_+^2 + c_-^2}[c_+^2 \frac{\partial \tilde{u}}{\partial x_1}(x_1,x_2,t) + c_-^2 \frac{\partial \tilde{u}}{\partial x_1}(x_1,-x_2,t)]$$

and on Ω_-:

$$\frac{\partial \tilde{u}^S}{\partial x_1} = \frac{1}{c_+^2 + c_-^2}[c_-^2 \frac{\partial \tilde{u}}{\partial x_1}(x_1,x_2,t) + c_+^2 \frac{\partial \tilde{u}}{\partial x_1}(x_1,-x_2,t)];$$

thus $\dfrac{\partial \tilde{u}^S}{\partial \nu}$ is a symmetric function on Γ_e and on Γ_s. We can apply Lemma 2 and get that $\dfrac{\partial \tilde{u}^S}{\partial \nu} \in H^{-1}(0,T;H^{1/2}(\Gamma_e \cup \Gamma_s))$. We have proved that $\tilde{u}^S$ is the solution of

$$\begin{cases} \Delta \tilde{u}^S \in H^{-1}(0,T;L^2(\Omega)) \\ \tilde{u}^S = 0 \quad \text{on} \quad \Gamma_+ \cup \Gamma_- \\ \dfrac{\partial \tilde{u}^S}{\partial \nu} \in H^{-1}(0,T;H^{1/2}(\Gamma_e \cup \Gamma_s)). \end{cases}$$

Since $\tilde{u}^S$ is null in a neighborhood of Γ_+ and Γ_-, we deduce that $\tilde{u}^S \in H^{-1}(0,T;H^2(\Omega))$.

Let us now prove that $\tilde{u}^A 1_{\Omega_+} = (\tilde{u} - \tilde{u}^S)1_{\Omega_+} \in H^{-1}(0,T;H^{3/2}(\Omega_+))$. We have $\tilde{u}^A \in L^2(0,T;H^1(\Omega))$. Since $\tilde{u} = \tilde{u}^S$ on Γ_i, we get $\tilde{u}^A|_{\Gamma_i} = 0$. Moreover, using that the normal direction on Γ_e (respectively Γ_s) is the $-x_1$'s (respectively x_1's) direction, one can easily prove that u_A satisfies in $\Omega_+ \times (0,T)$:

$$\begin{cases} \Delta \tilde{u}^A \in H^{-1}(0,T;L^2(\Omega_+)) \\ \\ \dfrac{\partial \tilde{u}^A}{\partial \nu} = \dfrac{\partial \tilde{u}}{\partial \nu} - \dfrac{\partial \tilde{u}^S}{\partial \nu} \in H^{-1}(0,T;H^{1/2}(\Gamma_{e_+} \cup \Gamma_{s+})) \\ \\ u^A = 0 \ \text{ on } \ \Gamma_i \cup \Gamma_+ \end{cases}$$

and therefore $\tilde{u}^A \in H^{-1}(]0,T[;H^{3/2}(\Omega_+))$ if $(u_0,u_1) \in H^1(\Omega) \times L^2(\Omega)$. We proved that $\tilde{u} \in H^{-1}(]0,T[;H^{3/2}(\Omega_+)) + H^{-1}(]0,T[;H^2(\Omega_+))$ and thus $\tilde{u} \in H^{-1}(]0,T[;H^{3/2}(\Omega_+))$ for initial data (u_0,u_1) in $H^1(\Omega) \times L^2(\Omega)$. Of course, the same is valid in Ω_-.

We deduce that $\dfrac{\partial \tilde{u}}{\partial t}|_{\Gamma_{e_+}}$ and $\dfrac{\partial u}{\partial t}|_{\Gamma_{e_-}}$ make sense in $H^{-2}(]0,T[;H^1(\Gamma_{e_+}))$ and $H^{-2}(]0,T[;H^1(\Gamma_{e_-}))$ and thus their values at point $(0,0)$ exist. Furthermore, $\dfrac{\partial \tilde{u}}{\partial t} \in H^{-1}(]0,T[;H^{1/2}(\Gamma_e))$; thus these values are equal. Let $\alpha = \tilde{g}_a|_{\Gamma_{e_+}}$ be the trace of the function $\tilde{g}_a$ on Γ_{e_+}. We have

$$\begin{aligned}\alpha(t) &= \tilde{g}_a(0+,t) = -\frac{1}{2}[\frac{1}{c^+}\frac{\partial \tilde{u}}{\partial t}(0,0+,t) - \frac{1}{c^-}\frac{\partial u}{\partial t}(0,0-,t)] \\ &= \frac{1}{2c_+c_-}(c_+ - c_-)\frac{\partial \tilde{u}}{\partial t}(0,0,t).\end{aligned}$$

We write

$$\tilde{g}_a(x_2,t) = \alpha(t)\ \ \text{sign}(x_2) + g_{ar}(x_2,t),$$

where sign denotes the sign function defined by $\text{sign}(x) = 1$ if $x > 0$ and $\text{sign}(x) = -1$ if $x < 0$. The function g_{ar} is an odd function with respect to x_2, $g_{ar}1_{\Gamma_{e_+}} \in H^{-2}(0,T;H^1(\Gamma_{e+}))$ and $g_{ar}1_{\Gamma_{e_-}} \in H^{-2}(0,T;H^1(\Gamma_{e-}))$, and their values at the point $(0,0)$ are null. Therefore $g_{ar} \in H^{-2}(]0,T[;H^{1/2}(\Gamma_e))$, and Proposition 2 is proved.

Finally, let us study the regularity of α in cases (ii) and (iii). In case (ii) of Theorem 4, we obtained that $\Delta\tilde{u}^S \in L^2(Q)$, $\Delta\tilde{u}^A \in L^2(Q)$, and $\dfrac{\partial \tilde{u}^S}{\partial \nu} \in L^2(0,T;H^{1/2}(\Gamma_e \cup \Gamma_s))$ and $\dfrac{\partial \tilde{u}^A}{\partial \nu} = \dfrac{\partial \tilde{u}}{\partial \nu} - \dfrac{\partial \tilde{u}^S}{\partial \nu} \in L^2(0,T;H^{1/2}(\Gamma_{e_+} \cup \Gamma_{s+}))$. Following the proof, it is not difficult to deduce that $u \in L^2(]0,T[;H^{3/2}(\Omega_+))$ and therefore $\alpha(t) \in H^{-1}(0,T)$. In case (iii) of Theorem 4, there is one more regularity with respect to the time variable which leads to $\alpha(t) \in L^2(0,T)$. □

Let us return to the proof of our main theorem. We write $u^{(2)} = u^{(3)} + u^{(4)}$ where $u^{(3)}$ (respectively $u^{(4)}$) is the solution of (15) with $\dfrac{\partial u^{(3)}}{\partial \nu} = g_{ar}$ (respectively $\dfrac{\partial u^{(4)}}{\partial \nu} = \alpha\ \text{sign}(x_2)$) on Γ_e. Lemma 1 can be applied to $u^{(3)}$, which leads to $u^{(3)} \in H^{-2}(0,T;\mathcal{V})$. We deduce that the singular part of the function u is involved by the lack of continuity at the origin of the odd part of the function $\dfrac{1}{c}\dfrac{\partial u}{\partial t}$.

Let us study $w = u^{(4)}$. We use the same splitting as in Proposition 2 and we write $w = w^S + w^A$. We get $w^S \in H^{-1}(0,T;H^2(\Omega))$, and thus it is sufficient to study $u^{(5)} = w^A$. We know that $\Delta u^{(5)} = 0$ separately in Ω_+ and Ω_- and that $u^{(5)} = 0$ on $\Gamma_i \cup \Gamma_+ \cup \Gamma_-$.

Let us compute $\dfrac{\partial u^{(5)}}{\partial \nu} = -\dfrac{\partial u^{(5)}}{\partial x_1}$ on Γ_{e_+} and Γ_{e_-}. We have on Γ_{e_+} :

$$\begin{aligned}\frac{\partial u^{(5)}}{\partial x_1}(x_1,x_2) &= \frac{\partial u^{(4)}}{\partial x_1}(x_1,x_2) - \frac{\partial w^S}{\partial x_1}(x_1,x_2) \\ &= -\alpha - \frac{c_+^2\dfrac{\partial u^{(4)}}{\partial x_1}(x_1,x_2) + c_-^2\dfrac{\partial u^{(4)}}{\partial x_1}(x_1,-x_2)}{c_+^2 + c_-^2} \\ &= -\alpha - \frac{c_+^2(-\alpha) + c_-^2\alpha}{c_+^2 + c_-^2} = -\alpha\frac{2c_-^2}{c_+^2 + c_-^2}\end{aligned}$$

and thus

$$\beta_+(t) = \frac{\partial u^{(5)}}{\partial \nu} = \frac{c_-}{c_+(c_+^2 + c_-^2)}(c_+ - c_-)\frac{\partial \tilde{u}}{\partial t}(0,0,t) \quad \text{on} \quad \Gamma_{e+}. \tag{21}$$

On Γ_-, we get

$$\frac{\partial u^{(5)}}{\partial \nu} = -\alpha - \frac{c_-^2 \dfrac{\partial u^{(4)}}{\partial \nu}(x_1, x_2) + c_+^2 \dfrac{\partial u^{(4)}}{\partial \nu}(x_1, -x_2)}{c_+^2 + c_-^2}$$

$$= -\alpha - \frac{c_-^2(-\alpha) + c_+^2 \alpha}{c_+^2 + c_-^2} = -\alpha \frac{2c_+^2}{c_+^2 + c_-^2}$$

and thus

$$\beta_-(t) = \frac{\partial u^{(5)}}{\partial \nu} = -\frac{c_+}{c_-(c_+^2 + c_-^2)}(c_+ - c_-)\frac{\partial \tilde{u}}{\partial t}(0, 0, t) \quad \text{on} \quad \Gamma_{e_-}. \tag{22}$$

We deduce that the function $u^{(5)}$ is the solution of

$$\begin{cases} \Delta u^{(5)} = 0 \ \text{ in } \ \Omega_+, \\ u^{(5)} = 0 \ \text{ on } \Gamma_+ \cup \Gamma_i, \\ \dfrac{\partial u^{(5)}}{\partial \nu} = \beta_+ \ \text{ on } \ \Gamma_{e_+} \cup \Gamma_{s_+}, \end{cases} \quad \text{and} \quad \begin{cases} \Delta u^{(5)} = 0 \ \text{ in } \ \Omega_-, \\ u^{(5)} = 0 \ \text{ on } \Gamma_- \cup \Gamma_i, \\ \dfrac{\partial u^{(5)}}{\partial \nu} = \beta_- \ \text{ on } \ \Gamma_{e_-} \cup \Gamma_{s_-}. \end{cases} \tag{23}$$

Proposition 3. Let $w = w(x) \in H^1(\Omega_+)$ be the solution of

$$\begin{cases} \Delta w = 0 \ \text{ in } \ \Omega_+, \\ w = 0 \ \text{ on } \Gamma_+ \cup \Gamma_i \cup \Gamma_{s_+}, \\ \dfrac{\partial w}{\partial \nu} = 1 \ \text{ on } \ \Gamma_{e_+}. \end{cases} \tag{24}$$

Then there exists $\varepsilon > 0$ such that the behavior near the origin is given by

$$w(x) = \frac{2a}{\pi^2}\, Im[Li_2(e^{-\frac{\pi}{a}(x_1 - ix_2)}) - Li_2(-e^{-\frac{\pi}{a}(x_1 - ix_2)})] \ \text{ in }]0, \varepsilon[^2.$$

Suppose Proposition 3 is proved. With $\beta = \beta_+$ and then $\beta = \beta_-$ (and a symmetry argument) where β_+ and β_- are defined in (21) and (22), we then obtain

$$u^{(5)}(x_1, x_2) = \beta_+ w(x_1, x_2) 1_{\Omega_+} + \beta_- w(x_1, -x_2) 1_{\Omega_-}.$$

This expression is exactly u_{sg}. The gradient of u_{sg} is easily deduced from the gradient of the function w which is given below.

Proof. For the sake of simplicity we consider another system which leads to the same singularity (the problem is a local one). Let us write $\Omega_a = \mathbb{R}^{+*} \times]0, a[$ and consider the solution $v \in H^1(\Omega_a)$ of the system

$$\begin{cases} \Delta v = 0 \ \text{ in } \ \mathbb{R}^{+*}\times]0,a[, \\ v = 0 \ \text{ on } \mathbb{R}^{+*} \times \{0,a\}, \\ \dfrac{\partial v}{\partial \nu} = 1 \ \text{ on } \ \{0\}\times]0,a[. \end{cases} \tag{25}$$

We write for $n \in \mathbb{N}^*$, $w_n(x_2) = \sqrt{\dfrac{2}{a}}\sin(\dfrac{n\pi x_2}{a})$ and

$$v(x_1,x_2) = \sum_{n\geq 1} a_n w_n(x_2) e^{-\dfrac{n\pi x_1}{a}},$$

the series being convergent in the spaces $H^1(\Omega_a)$ and $H^2(\Omega_a \cap (x_2 > 0))$.

Let us recall that the functions w_n $(n > 0)$ represent an orthonormal basis of $L^2(]0,a[)$. We obtain for $p \in \mathbb{N}^*$,

$$\int_0^a v(x_1,x_2) w_p(x_2) dx_2 = a_p e^{-\dfrac{p\pi x_1}{a}}$$

and

$$\frac{\partial v}{\partial x_1}(x_1,x_2) = -\frac{\pi}{a}\sum_n a_n \, n w_n(x_2) e^{-\dfrac{n\pi x_1}{a}};$$

thus

$$\int_0^a \frac{\partial v}{\partial x_1}(0,x_2) w_p(x_2) dx_2 = -\frac{p\pi}{a} a_p,$$

which leads to

$$a_p = \frac{a}{p\pi}\int_0^a w_p(x_2) dx_2 = \frac{a^2}{p^2\pi^2}\sqrt{\frac{2}{a}}[1-(-1)^p].$$

We then get

$$a_{2p} = 0 \quad \text{and} \quad a_{2p+1} = \frac{2a\sqrt{2a}}{(2p+1)^2\pi^2}$$

and

$$v(x_1,x_2) = \frac{4a}{\pi^2}\sum_{p\geq 0}\frac{1}{(2p+1)^2}\sin(\frac{(2p+1)\pi}{a}x_2) e^{-\dfrac{(2p+1)\pi x_1}{a}}. \tag{26}$$

We set

$$z = e^{-\dfrac{\pi x_1}{a} + i\dfrac{\pi x_2}{a}}$$

and we get

$$v(x_1,x_2) = Im[\frac{4a}{\pi^2}\sum_{p\geq 0}\frac{z^{2p+1}}{(2p+1)^2}].$$

Let us recall that the *dilogarithm* function Li_2 is defined by $Li_2(z) = \sum_{n\geq 1} \dfrac{z^n}{n^2}$ ($|z| \leq 1$), and we refer to [15] for very surprising properties of this function. We proved that

$$v(x_1, x_2) = \frac{2a}{\pi^2} Im[Li_2(z) - Li_2(-z)].$$

Let us compute the gradient of v. One obtains with (26)

$$\frac{\partial v}{\partial x_1} = -\frac{4}{\pi} \sum_{p\geq 0} \frac{1}{(2p+1)} \sin(\frac{(2p+1)\pi}{a} x_2) e^{-\dfrac{(2p+1)\pi x_1}{a}}$$

and

$$\frac{\partial v}{\partial x_2} = \frac{4}{\pi} \sum_{p\geq 0} \frac{1}{(2p+1)} \cos(\frac{(2p+1)\pi}{a} x_2) e^{-\dfrac{(2p+1)\pi x_1}{a}};$$

thus (notice that $\dfrac{1+z}{1-z} \notin \mathbb{R}^-$)

$$\begin{aligned} \frac{\partial v}{\partial x_2} - i\frac{\partial v}{\partial x_1} &= \frac{4}{\pi} \sum_{p\geq 0} \frac{z^{2p+1}}{(2p+1)} = \frac{2}{\pi} \log(\frac{1+z}{1-z}) \\ &= \frac{2}{\pi} [\mathrm{Ln}|\frac{1+z}{1-z}| + iArg(\frac{1+z}{1-z})] \end{aligned}$$

where the function Arg takes its values in the open set $]-\pi, +\pi[$.

One deduces that

$$\frac{\partial v}{\partial x_2} = \frac{1}{\pi} \mathrm{Ln}| \frac{1 + 2e^{-\dfrac{\pi x_1}{a}} \cos(\dfrac{\pi x_2}{a}) + e^{-2\dfrac{\pi x_1}{a}}}{1 - 2e^{-\dfrac{\pi x_1}{a}} \cos(\dfrac{\pi x_2}{a}) + e^{-2\dfrac{\pi x_1}{a}}} |.$$

On Γ_i, one has

$$\frac{\partial v}{\partial x_2} = \frac{2}{\pi} \mathrm{Ln}| \frac{1 + e^{-\dfrac{\pi x_1}{a}}}{1 - e^{-\dfrac{\pi x_1}{a}}} | = \frac{4}{\pi} \mathrm{Ln}| \coth \frac{\pi x_1}{2a} |$$

and

$$\lim_{x_1 \to 0^+} \frac{\partial v}{\partial x_2} = +\infty.$$

In the same way, one has on Γ_{e_+},

$$\lim_{x_2 \to 0^+} \frac{\partial v}{\partial x_2}(0, x_2) = +\infty.$$

Theorem 3.1 is proved. □

Conclusion. In this paper, we have discussed the singularity which appears when one uses a transparent boundary condition for a bimaterial. This situation occurs for instance when one tries to simulate wave propagations in an infinite strip replaced by a finite one. The singularity which belongs to the Dilog family is an artifact which doesn't exist in the physical model. Therefore it seems necessary to eliminate it from the solution. The first step was obviously to make it explicit in order to be able to suggest a method which would improve the boundary condition. For instance, a numerical method could be to compute the contribution of this artificial singularity and to subtract it from the global solution. In fact, this is a way for defining an upgrade transparent boundary condition for this kind of problem. This will be discussed in a forthcoming paper. Finally, let us remark that these phenomena appear in the study of the detection of cracks at the interface between two materials, as noticed in [7].

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