QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXXIV, NUMBER 3
SEPTEMBER 2016, PAGES 539–570
http://dx.doi.org/10.1090/qam/1432
Article electronically published on June 16, 2016

GLOBAL SMOOTH SOLUTION TO A HYPERBOLIC SYSTEM ON AN INTERVAL WITH DYNAMIC BOUNDARY CONDITIONS

By

GILBERT PERALTA (*Department of Mathematics and Computer Science, University of the Philippines Baguio, Governor Pack Road, Baguio City, 2600, Philippines – and – Institut für Mathematik und Wissenschaftliches Rechnen, NAWI Graz, Karl-Franzens-Universität Graz, Heinrichstrasse 36, A-8010 Graz, Austria*)

AND

GEORG PROPST (*Institut für Mathematik und Wissenschaftliches Rechnen, NAWI Graz, Karl-Franzens-Universität Graz, Heinrichstrasse 36, A-8010 Graz, Austria*)

Abstract. We consider a hyperbolic two component system of partial differential equations in one space dimension with ODE boundary conditions describing the flow of an incompressible fluid in an elastic tube that is connected to a tank at each end. Using the local-existence theory together with entropy methods, the existence and uniqueness of a global-in-time smooth solution is established for smooth initial data sufficiently close to the equilibrium state. Energy estimates are derived using the relative entropy method for zero order estimates while constructing entropy-entropy flux pairs for the corresponding diagonal system of the shifted Riemann invariants to deal with higher order estimates. Finally, using the linear theory and interpolation estimates, we show that the solution converges exponentially to the equilibrium state.

1. Introduction. Consider a horizontal elastic tube of length ℓ filled with an incompressible liquid. Each end of the tube is connected to a vertical tank, each of which has horizontal cross-section A_T. The velocity $u(t,x)$ of the fluid inside the tube, the cross-section $A(t,x)$ of the tube and level heights $h_0(t)$ and $h_\ell(t)$ of the fluid in the tanks are

Received July 6, 2015.
2010 *Mathematics Subject Classification.* Primary 35L40, 35F61.
The first author was supported by the grant Technologiestipendien Südostasien in the frame of ASEA-Uninet granted by the Austrian Agency for International Cooperation in Education and Research (OeAD-GmbH) and financed by the Austrian Federal Ministry for Science and Research (BMWF).
E-mail address: grperalta@upb.edu.ph
E-mail address: georg.propst@uni-graz.at

©2016 Brown University

modeled by a hyperbolic PDE on $(t,x) \in (0,\infty) \times (0,\ell)$ with ODE boundary conditions

$$\begin{cases} A_t + uA_x + Au_x = 0, & t > 0,\ 0 < x < \ell, \\ u_t + \kappa^2 A^{-\frac{1}{2}} A_x + uu_x = -\beta u, & t > 0,\ 0 < x < \ell, \\ A_T h_0'(t) = -A(t,0)u(t,0), & t > 0, \\ A_T h_\ell'(t) = A(t,\ell)u(t,\ell), & t > 0, \\ A(t,0) = (a_0 + bh_0(t))^2, & t > 0, \\ A(t,\ell) = (a_\ell + bh_\ell(t))^2, & t > 0, \end{cases} \tag{1.1}$$

and initial conditions

$$A(0,x) = A^0(x), \qquad u(0,x) = u^0(x), \qquad h_0(0) = h_0^0, \qquad h_\ell(0) = h_\ell^0. \tag{1.2}$$

A prime $'$ denotes a derivative with respect to time t.

Physically, the coefficients in (1.1) are given by

$$\kappa^2 = \frac{sE}{2\rho r_0 \sqrt{A_0}}, \qquad \beta = \frac{8\pi\mu}{\rho A_0}, \qquad b = \frac{r_0 \rho g \sqrt{A_0}}{sE},$$
$$a_0 = \sqrt{A_0}\left(1 + \frac{r_0 p_{f0}}{sE}\right), \qquad a_\ell = \sqrt{A_0}\left(1 + \frac{r_0 p_{f\ell}}{sE}\right),$$

where r_0 represents the inner rest radius of the circular tube, A_0 is the corresponding rest cross-section, E and s are Young's modulus and the thickness of the tube material, ρ and μ are the constant density and viscosity of the fluid, p_{f0} and $p_{f\ell}$ are constant pressures above the fluid in the left and right tank respectively, and g is the gravitational constant. All parameters appearing in the model are positive except for the viscosity μ, which is only nonnegative. However, for global existence the assumption $\mu > 0$, or equivalently $\beta > 0$, will be reinforced. For the derivation of this model we refer to [15, 19].

The first two equations in (1.1) have the same form as isentropic flow in Eulerian coordinates of a thermoelastic polytropic fluid in a duct, e.g., [5, p. 198]. Models similar to (1.1) have been considered in the literature both for bounded and unbounded intervals, for instance, [2, 3, 7, 15, 20]. In a recent work [18], the linearized model has been analyzed with respect to stability and controllability. We will use the stability result to prove the exponential convergence of the state to the equilibrium for the nonlinear system (1.1).

The goal of the present paper is to use the local-existence theory together with entropy and energy methods to prove a global existence result and describe the asymptotic behavior of the solution, at least for sufficiently smooth data close to the equilibrium state, for the nonlinear system (1.1).

It is well-known that in general, solutions of first order quasilinear hyperbolic partial differential equations even with smooth initial data may not exist globally in time and singularities may develop in finite time, such as shocks, mass explosion, etc. However, it is observed that the presence of a linear damping term can prevent shock formation at least for small and smooth initial data. A simple example illustrating these phenomena is given by Burgers' equation; see for instance [5, Section 4.2]. Necessary and sufficient conditions for the existence of global solutions both for general and physical systems have

been developed in the past years; see [4, 8, 10, 11, 20]. However, there are only a few works dealing with bounded domains. In one-space dimension, Ruan et al. [20] investigated the global existence of smooth solutions of a network of 2×2 systems of balance laws in bounded intervals under a *dissipative condition* on the boundary conditions. This condition is similar to what has been considered in [10, Chapter 5]. However, the dissipative condition is not satisfied for instance by the isentropic Euler system, by systems with relaxation, for boundary conditions arising in blood flow models, nor by system (1.1).

Two main tools are used to prove the global existence of solutions, namely, the entropy and energy methods. The energy method was used by Nishida [14] and Kawashima [9] for hyperbolic and hyperbolic-parabolic equations. This was then used by several authors for isothermal Euler equations [4], partially dissipative systems with convex entropies [1, 8, 23], relaxation models with nonconvex flux [13], systems arising in blood flow models [20] and others. The main idea is to define an energy functional and to derive an estimate for this functional. Lower order estimates can be obtained using the relative entropy method [8]. The relative entropy associated with a strictly convex entropy, loosely speaking, can serve as a *distance* between solutions, e.g., classical, strong, weak, of conservation laws or balance laws; cf. [5]. For higher order estimates involving terms that do not have a dissipative term, one useful criterion, at least for Cauchy problems, is the Shizuta-Kawashima condition, which was formulated in [21]. However on a bounded interval, a different method was used in [20], namely the construction of entropy-entropy flux pairs for the Riemann invariants in deriving higher order estimates. In the case of bounded domains, boundary terms arise, and this causes some difficulty in obtaining the necessary estimates. The dissipative condition plays a crucial role in the proof of the estimates. Most of the existence results use the smallness assumptions on the initial data. Even with this restriction the proofs are not trivial.

Here, we will also use the relative entropy method to obtain lower order estimates for the energy functionals and use appropriate entropy-entropy flux pairs for higher order estimates. The main idea is to construct entropy-entropy flux pairs (η, q) such that

$$\eta_t + q_x = M$$

for some source term M which is, roughly speaking, dominated by the damping term, which is the velocity u in our case, or its derivatives. We will not assume the dissipative condition as in [20], but we use the special structure of the boundary conditions in (1.1).

2. Equilibrium and statement of the main result. The volume of the fluid inside the tube and the tanks at time $t \geq 0$ is given by

$$V(t) = \int_0^\ell A(t,x)\,\mathrm{d}x + A_T h_0(t) + A_T h_\ell(t). \tag{2.1}$$

If (A, u, h_0, h_ℓ) is a smooth solution of (1.1) on $[0, T]$, then $V(t)$ is conserved on $[0, T]$, i.e., $V(t) = V(0)$ for all $t \in [0, T]$. This can be seen immediately by taking the derivative of V and using the first, third and fourth equations in (1.1). In this paper, by a smooth solution we mean that each state component is at least continuously differentiable. The

equilibrium state of (1.1) is given by $(A_e, 0, h_{0e}, h_{\ell e})$ where

$$A_e = (a_0 + bh_{0e})^2 = (a_\ell + bh_{\ell e})^2. \tag{2.2}$$

For a given fixed volume and with the assumption that the pressures p_{f0} or $p_{f\ell}$ are given (not too large), the equilibrium is uniquely determined. Indeed, if V_0 denotes the fixed volume, then we have $V_0 = A_e\ell + A_T h_{0e} + A_T h_{\ell e}$. The latter equality together with (2.2) provides explicit expressions for A_e, h_{0e} and $h_{\ell e}$ in terms of V_0.

In [17], the mth order compatibility condition of the initial data is defined and the following local-in-time existence result and blow-up criterion are shown.

THEOREM 2.1 (Local existence and blow-up criterion). Let $(A^0, u^0, h_0^0, h_\ell^0) \in H^m(0,\ell) \times H^m(0,\ell) \times \mathbb{R}^2$ be compatible up to order $m-1$ for some integer $m \geq 3$. Suppose that the range of (A^0, u^0) lies on a compact and convex subset of $\mathcal{U} := \{(A,u) \in (0,\infty) \times \mathbb{R} : |u| < \kappa A^{1/4}\}$. Then there exists $T > 0$ such that (1.1)–(1.2) has a unique solution (A, u, h_0, h_ℓ) such that $A, u \in \bigcap_{k=0}^m C^{m-k}([0,T]; H^m(0,\ell))$ and $h_0, h_\ell \in H^{m+1}(0,T)$. Furthemore, if the maximal time T^* of existence is finite, then either (A, u, h_0, h_ℓ) leaves every compact set of $\mathcal{U} \times \mathbb{R}^2$ or

$$\lim_{t \uparrow T^*} (\|A_x(t)\|_{L^\infty[0,\ell]} + \|u_x(t)\|_{L^\infty[0,\ell]}) = +\infty.$$

If the maximal time is finite, the first scenario is typical for ODEs, while the second one is called shock formation. For the first one, the state approaches the boundary of $\mathcal{U}$, and as a result the flux matrix will become singular. In the region $\mathcal{U}$, there is one negative eigenvalue and one positive eigenvalue for the flux matrix, and the flow in this case is subsonic. On the other hand, the shock formation is a typical behavior for first order quasilinear PDEs where waves are compressed within finite time, and therefore wave profiles can have arbitrarily large slope. However, for data close enough to an equilibrium state and with dissipative terms these will not happen. This assertion with regard to (1.1) is the main result of this paper.

THEOREM 2.2 (Global existence). In the framework of Theorem 2.1 and $\beta > 0$, there exists $\delta_0 > 0$ such that if $E_0 := \|A^0 - A_e\|_{H^2}^2 + \|u^0\|_{H^2}^2 + |h_0^0 - h_{0e}|^2 + |h_\ell^0 - h_{\ell e}|^2 \leq \delta_0$, then there is a unique global solution (A, u, h_0, h_ℓ) of (1.1)–(1.2) such that

$$A, u \in C([0,\infty); H^2(0,\ell)) \cap C^1([0,\infty); H^1(0,\ell)), \quad h_0, h_\ell \in C^2[0,\infty),$$

and

$$\sup_{t \geq 0} (\|A(t) - A_e\|_{H^2}^2 + \|u(t)\|_{H^2}^2 + |h_0(t) - h_{0e}|^2 + |h_\ell(t) - h_{\ell e}|^2) + \int_0^\infty \|A_x(t)\|_{H^1}^2 + \|u(t)\|_{H^1}^2 \, \mathrm{d}t \;\leq\; CE_0$$

for some $C > 0$.

3. Entropy-entropy flux pairs. Entropies of the system (1.1) can be obtained by solving a wave equation as shown in the following. For a more general result of a similar model and in the case of $\beta = 0$ we refer to the paper of Lions, Perthame and Tadmor [12].

PROPOSITION 3.1. Let $\eta \in C^2((0,\infty)\times\mathbb{R}) \cap C^1([0,\infty)\times\mathbb{R})$ satisfy the wave equation

$$\frac{\partial^2 \eta}{\partial A^2}(A,u) = \kappa^2 A^{-\frac{3}{2}} \frac{\partial^2 \eta}{\partial u^2}(A,u), \qquad \text{in } (0,\infty)\times\mathbb{R}. \tag{3.1}$$

Then any smooth functions A and u satisfying the first two equations in (1.1) also satisfy the *entropy dissipation identity*

$$\frac{\partial}{\partial t}\eta(A,u) + \frac{\partial}{\partial x} q(A,u) = -\beta u \frac{\partial}{\partial u}\eta(A,u), \qquad \text{in } (0,\infty)\times\mathbb{R}, \tag{3.2}$$

where $q \in C^2((0,\infty)\times\mathbb{R})$ is given by

$$q(A,u) = \int_0^u v\eta_u(A,v) + A\eta_A(A,v)\,\mathrm{d}v + \int_0^A \kappa^2 a^{-\frac{1}{2}}\eta_u(a,0)\,\mathrm{d}a. \tag{3.3}$$

Proof. The regularity of q stated above follows immediately from the regularity of η. Since u and A satisfy the first two equations in (1.1), the PDE (3.2) is equivalent to

$$u_x(q_u - u\eta_u - A\eta_A) + A_x(q_A - \kappa^2 A^{-\frac{1}{2}}\eta_u - u\eta_A) = 0. \tag{3.4}$$

The first term vanishes due to the construction of q since $q_u = u\eta_u + A\eta_A$. We show that the second term also vanishes. Differentiating the latter equality with respect to A and using (3.1) we have

$$q_{Au} = q_{uA} = u\eta_{uA} + \eta_A + A\eta_{AA} = (u\eta_A + \kappa^2 A^{-\frac{1}{2}}\eta_u)_u. \tag{3.5}$$

Integrating (3.5) twice, first with respect to u and then with respect to A, we have

$$q(A,u) = \int_0^A u\eta_A(a,u) + \kappa^2 a^{-\frac{1}{2}}\eta_u(a,u)\,\mathrm{d}a + F(A) \tag{3.6}$$

for some function F. Taking $u = 0$ in (3.3) and (3.6) shows that $F \equiv 0$. Thus, differentiating (3.6) with respect to A shows that the second term in (3.4) is identically zero. Hence (3.4) is satisfied and so is (3.2). □

The function η is called an *entropy* and q is the corresponding *entropy flux*. The entropy dissipation identity (3.2) is commonly called a *companion law* to the first two equations in (1.1). Let $\eta_p = a_1 u + a_2 A + a_3 uA + a_4$ where the a_i's are constants. Notice that the wave equation is invariant under perturbations of the form η_p; i.e., if η satisfies (3.1), then so does $\eta + \eta_p$.

A common entropy of the above system is

$$\eta(A,u) = \frac{1}{2}Au^2 + \frac{4}{3}\kappa^2 A^{\frac{3}{2}},$$

called the *mechanical energy*, and it is strictly convex in the variables $(A, Au) \in (0,\infty)\times\mathbb{R}$. This particular entropy satisfies the boundary conditions $\eta(0,u) = 0$ and $\eta_A(0,u) = \frac{1}{2}u^2$. Such entropies are called weak entropies [12]. However, for our purpose we will modify this entropy. We want an entropy η_0 such that $\eta_0(A_e,0) = 0$ and $D\eta_0(A_e,0) = (0,0)$. This can be done by choosing

$$\begin{aligned}\eta_0(A,u) &= \eta(A,u) - \eta(A_e,0) - (D\eta(A_e,0),(A-A_e,u)) \\ &= \frac{1}{2}Au^2 + \frac{4}{3}\kappa^2(A^{\frac{3}{2}} - A_e^{\frac{3}{2}}) - 2\kappa^2 A_e^{\frac{1}{2}}(A - A_e).\end{aligned} \tag{3.7}$$

In the literature, η_0 is referred to as the relative entropy with respect to the state $(A_e, 0)$. Notice that the difference of the mechanical energy η and its modified version η_0 is a function of the form η_p stated above. By invariance, η_0 also satisfies the wave equation (3.1), and therefore if (A, u) satisfies the first two equations in (1.1), η_0 also satisfies the entropy dissipation identity (3.2) with the corresponding entropy flux

$$q_0(A, u) = \frac{1}{2}Au^3 + 2\kappa^2(A^{\frac{1}{2}} - A_e^{\frac{1}{2}})uA \tag{3.8}$$

obtained from (3.3). Moreover, η_0 is also strictly convex in the variables (A, uA). This entropy-entropy flux pair will be used in the next section to obtain zero order estimates. By a second order Taylor expansion we can see that there exist constants $c_K, C_K > 0$ such that

$$c_K(|uA|^2 + |A - A_e|^2) \leq \eta_0(A, u) \leq C_K(|uA|^2 + |A - A_e|^2) \tag{3.9}$$

for every $(A, u) \in K$ where $K \subset (0, \infty) \times \mathbb{R}$ is a compact set containing $(A_e, 0)$. Thus the relative entropy serves as a distance between the smooth solutions of the system and the constant equilibrium state.

The next step is to develop entropy-entropy flux pairs to deal with first order and second order estimates as done by Ruan et al. [20]. This will be done using an appropriate diagonal form of the system. The eigenvalues of the associated flux matrix of (1.1) are $\tilde{\lambda} = u - \kappa A^{\frac{1}{4}}$ and $\tilde{\mu} = u + \kappa A^{\frac{1}{4}}$. Multiplying the first two equations in (1.1) by $(\kappa A^{-\frac{3}{4}}, 1)$ and by $(\kappa A^{-\frac{3}{4}}, -1)$ we obtain a diagonal system

$$\begin{aligned} \tilde{w}_t + \tilde{\lambda}(\tilde{w}, \tilde{z})\tilde{w}_x &= \frac{\beta}{2}(\tilde{z} - \tilde{w}) \\ \tilde{z}_t + \tilde{\mu}(\tilde{w}, \tilde{z})\tilde{z}_x &= -\frac{\beta}{2}(\tilde{z} - \tilde{w}) \end{aligned}$$

where $\tilde{w} = -u + 4\kappa A^{\frac{1}{4}}$, $\tilde{z} = u + 4\kappa A^{\frac{1}{4}}$, $\tilde{\lambda} = -\frac{5}{8}\tilde{w} + \frac{3}{8}\tilde{z}$ and $\tilde{\mu} = -\frac{3}{8}\tilde{w} + \frac{5}{8}\tilde{z}$. If (A, u) is close to the equilibrium state $(A_e, 0)$, then (w, z) is close to $(4\kappa A_e^{\frac{1}{4}}, 4\kappa A_e^{\frac{1}{4}})$. With this in mind, we shall consider the shifted Riemann invariants $w = \tilde{w} - 4\kappa A_e^{\frac{1}{4}}$ and $z = \tilde{z} - 4\kappa A_e^{\frac{1}{4}}$ so that

$$w = -u + 4\kappa(A^{\frac{1}{4}} - A_e^{\frac{1}{4}}), \qquad z = u + 4\kappa(A^{\frac{1}{4}} - A_e^{\frac{1}{4}}). \tag{3.10}$$

Therefore the state variables (A, u) and the shifted Riemann invariants (w, z) are related according to

$$u = \frac{1}{2}(z - w), \qquad A^{\frac{1}{4}} - A_e^{\frac{1}{4}} = \frac{1}{8\kappa}(z + w). \tag{3.11}$$

Using the Riemann invariants, the system (1.1) can be written in diagonal form:

$$\begin{cases} w_t + \lambda(w,z)w_x = \dfrac{\beta}{2}(z-w), & t>0,\ 0<x<\ell, \\ z_t + \mu(w,z)z_x = -\dfrac{\beta}{2}(z-w), & t>0,\ 0<x<\ell, \\ h_0'(t) = -\theta(w(t,0),z(t,0))(z(t,0)-w(t,0)), & t>0, \\ h_\ell'(t) = \theta(w(t,\ell),z(t,\ell))(z(t,\ell)-w(t,\ell)), & t>0, \\ z(t,0)+w(t,0) = \zeta_0(h_0(t))(h_0(t)-h_{0e}), & t>0, \\ z(t,\ell)+w(t,\ell) = \zeta_\ell(h_\ell(t))(h_\ell(t)-h_{\ell e}), & t>0, \end{cases} \tag{3.12}$$

where the coefficient functions are given by

$$\lambda(w,z) = -\frac{5}{8}w + \frac{3}{8}z - \frac{1}{4}C_e, \qquad C_e = 4\kappa A_e^{\frac{1}{4}}, \tag{3.13}$$

$$\mu(w,z) = -\frac{3}{8}w + \frac{5}{8}z + \frac{1}{4}C_e, \tag{3.14}$$

$$\theta(w,z) = \frac{1}{2^{13}\kappa^4 A_T}(w+z+2C_e)^4, \tag{3.15}$$

$$\zeta_k(h) = b(\sqrt{a_k+bh}+\sqrt{a_k+bh_{ke}})^{-1}, \qquad k=0,\ell. \tag{3.16}$$

Differentiating the first two equations in (3.12) with respect to x once and twice we have

$$(\partial_x^k w)_t + \lambda(w,z)(\partial_x^k w)_x = F_k \tag{3.17}$$

$$(\partial_x^k z)_t + \mu(w,z)(\partial_x^k z)_x = G_k \tag{3.18}$$

for $k=1,2$ where

$$F_1 = -\lambda_x w_x + \frac{\beta}{2}(z_x - w_x) \tag{3.19}$$

$$G_1 = -\mu_x z_x - \frac{\beta}{2}(z_x - w_x) \tag{3.20}$$

$$F_2 = -2\lambda_x w_{xx} - \lambda_{xx} w_x + \frac{\beta}{2}(z_{xx} - w_{xx}) \tag{3.21}$$

$$G_2 = -2\mu_x z_{xx} - \mu_{xx} z_x - \frac{\beta}{2}(z_{xx} - w_{xx}). \tag{3.22}$$

Consider differentiable functions $\phi_k = \phi_k(t,x,W)$ and $\psi_k = \psi_k(t,x,Z)$ for $k=1,2$. Using the equation (3.17) we have, for a smooth solution (w,z) of the system (3.12),

$$\begin{aligned} &\partial_t\phi_k(t,x,\partial_x^k w(t,x)) + \partial_x(\lambda(t,x)\phi_k(t,x,\partial_x^k w(t,x))) \\ &= \phi_{kt}(t,x,\partial_x^k w(t,x)) + \phi_{kW}(t,x,\partial_x^k w(t,x))\partial_t(\partial_x^k w(t,x)) \\ &\quad + \lambda_x(t,x)\phi_k(t,x,\partial_x^k w(t,x)) + \lambda(t,x)\phi_{kx}(t,x,\partial_x^k w(t,x)) \\ &\quad + \lambda(t,x)\phi_{kW}(t,x,\partial_x^k w(t,x))\partial_x(\partial_x^k w(t,x)) \\ &= \phi_{kt}(t,x,\partial_x^k w(t,x)) + \lambda_x(t,x)\phi_k(t,x,\partial_x^k w(t,x)) + \lambda(t,x)\phi_{kx}(t,x,\partial_x^k w(t,x)) \\ &\quad + \phi_{kW}(t,x,\partial_x^k w(t,x))F_k(t,x) \end{aligned} \tag{3.23}$$

for $k = 1, 2$. Similarly, using (3.18) we get

$$\begin{aligned}
&\partial_t \psi_k(t, x, \partial_x^k z(t, x)) + \partial_x(\mu(t, x)\psi_k(t, x, \partial_x^k z(t, x))) \\
&= \psi_{kt}(t, x, \partial_x^k z(t, x)) + \mu_x(t, x)\psi_k(t, x, \partial_x^k z(t, x)) + \mu(t, x)\psi_{kx}(t, x, \partial_x^k z(t, x)) \\
&\quad + \psi_{kZ}(t, x, \partial_x^k z(t, x))G_k(t, x) \qquad (3.24)
\end{aligned}$$

for $k = 1, 2$. Subtracting (3.23) from (3.24) we obtain the partial differential equation

$$\partial_t(\psi_k - \phi_k) + \partial_x(\mu\psi_k - \lambda\phi_k) = M_k(\psi_k, \phi_k) \qquad (3.25)$$

where

$$\begin{aligned}
M_k(\psi_k, \phi_k) \;=\; &(\psi_{kt} - \phi_{kt}) + (\mu_x\psi_k - \lambda_x\phi_k) + (\mu\psi_{kx} - \lambda\phi_{kx}) \\
&+ (\psi_{kZ}G_k - \phi_{kW}F_k). \qquad (3.26)
\end{aligned}$$

Integrating (3.25) over $[0, t] \times [0, \ell]$ and using Fubini's theorem we have

$$\begin{aligned}
&\int_0^\ell \eta_k(t, x) - \eta_k(0, x)\,\mathrm{d}x + \int_0^t q_k(\tau, \ell) - q_k(\tau, 0)\,\mathrm{d}\tau \\
&= \int_0^t \int_0^\ell M_k(\psi_k, \phi_k)\,\mathrm{d}x\,\mathrm{d}\tau \qquad (3.27)
\end{aligned}$$

where

$$\begin{aligned}
\eta_k(t, x) \;&=\; \psi_k(t, x, \partial_x^k w(t, x)) - \phi_k(t, x, \partial_x^k w(t, x)) \\
q_k(t, x) \;&=\; \mu(t, x)\psi_k(t, x, \partial_x^k w(t, x)) - \lambda(t, x)\phi_k(t, x, \partial_x^k w(t, x)).
\end{aligned}$$

The point is that solutions (w, z) of (3.12) that are sufficiently smooth satisfy (3.27) for $k = 1, 2$. Equation (3.27) will be of great importance in deriving the energy estimates. This is done by choosing appropriate functions ψ_k and ϕ_k such that the term M_k will be, in some sense, dominated by the velocity u or its derivatives.

4. Energy estimates. For $T > 0$ define the solution space

$$X_T = (C([0, T]; H^2(0, \ell)^2) \cap C^1([0, T]; H^1(0, \ell)^2) \cap C^2([0, T]; L^2(0, \ell)^2)) \times C^2[0, T]^2.$$

By using classical embedding results we can see that X_T is continuously embedded in $C^1([0, T] \times [0, \ell])^2 \times C^2[0, T]^2$. All throughout this section (A, u, h_0, h_ℓ) will be a smooth solution to the system on the time interval $[0, T]$, provided that such solution exists on such interval. Define the energy functionals $N_k : [0, \infty) \to [0, \infty)$ for $k = 0, 1, 2$ by

$$\begin{aligned}
N_k^2(t) \;=\; &\sup_{\tau \in [0,t]} (\|u(\tau)\|_{H^k}^2 + \|A^{\frac{1}{4}}(\tau) - A_e^{\frac{1}{4}}\|_{H^k}^2 + |h_0(\tau) - h_{0e}|^2 + |h_\ell(\tau) - h_{\ell e}|^2) \\
&+ \int_0^t \|u(\tau)\|_{H^k}^2 + k\|(A^{\frac{1}{4}})_x(\tau)\|_{H^{k-1}}^2\,\mathrm{d}\tau.
\end{aligned}$$

In the following estimates, C_δ and $C_{i\delta}$ will denote generic positive constants that depend on the system parameters and may depend on $\delta > 0$, and

$$C_\delta \text{ and } C_{i\delta} \text{ remain bounded as long as } \delta \text{ stays on a bounded set in } (0, \infty). \qquad (4.1)$$

Before we proceed we state the following equivalence of norms of the state variables u, A and the Riemann invariants

$$2\|\partial_x^k u(t)\|_{L^2}^2 + 32\kappa^2\|\partial_x^k(A^{\frac{1}{4}}(t) - A_e^{\frac{1}{4}})\|_{L^2}^2 = \|\partial_x^k w(t)\|_{L^2}^2 + \|\partial_x^k z(t)\|_{L^2}^2 \tag{4.2}$$

for $k = 0, 1, 2$ and for $t \in [0, T]$. This follows immediately from the identity $2w^2 + 2z^2 = (z - w)^2 + (z + w)^2$ in $\mathbb{R}$ and the transformations given in (3.11). This norm equivalence will be used in converting an estimate involving the Riemann invariants into an estimate involving the state variables and vice versa. Furthermore, if $0 < \delta < A_e$, then $|A - A_e| \le \delta$ implies that

$$C_{1\delta}|A - A_e| \le |A^{\frac{1}{4}} - A_e^{\frac{1}{4}}| \le C_{2\delta}|A - A_e|. \tag{4.3}$$

This can be seen from the elementary identity $A - A_e = (A^{\frac{1}{4}} - A_e^{\frac{1}{4}})(A^{\frac{1}{4}} + A_e^{\frac{1}{4}})(A^{\frac{1}{2}} + A_e^{\frac{1}{2}})$ whenever $A, A_e > 0$.

Lemma 4.1 (Zero order estimate). There exist $\delta > 0$ and $C_\delta > 0$ such that for any solution $(A, u, h_0, h_\ell) \in X_T$ satisfying $N_2^2(T) \le \delta$, it also satisfies the energy estimate

$$N_0^2(t) \le C_\delta \left(N_0^2(0) + \sup_{\tau\in[0,t]} \|u(\tau)\|_{H^1} \int_0^t \|u(\tau)\|_{H^1}^2 \,\mathrm{d}\tau \right) \tag{4.4}$$

for all $t \in [0, T]$.

Proof. Recall that η_0 and q_0 given in (3.7) and (3.8), respectively, satisfy the entropy dissipation identity (3.2). Integrating (3.2) over $[0, t] \times [0, \ell]$ and using Fubini's Theorem yield

$$\begin{aligned} &\int_0^\ell \eta_0(A(t,x), u(t,x)) - \eta_0(A(0,x), u(0,x)) \,\mathrm{d}x \\ &+ \int_0^t q_0(A(\tau,\ell), u(\tau,\ell)) - q_0(A(\tau,0), u(\tau,0)) \,\mathrm{d}\tau = -\beta \int_0^t \int_0^\ell (Au^2)(\tau,x) \,\mathrm{d}x \,\mathrm{d}\tau. \end{aligned} \tag{4.5}$$

Let us estimate the left hand side of (4.5) from below and its right hand side from above. According to (3.9) and (4.3) it holds that, choosing $\delta > 0$ sufficiently small,

$$\begin{aligned} &\int_0^\ell \eta_0(A(t,x), u(t,x)) - \eta_0(A(0,x), u(0,x)) \,\mathrm{d}x \\ &\ge C_\delta(\|(uA)(t)\|_{L^2}^2 + \|A^{\frac{1}{4}}(t) - A_e^{\frac{1}{4}}\|_{L^2}^2 - \|(uA)(0)\|_{L^2}^2 - \|A^{\frac{1}{4}}(0) - A_e^{\frac{1}{4}}\|_{L^2}^2). \end{aligned} \tag{4.6}$$

Using (2.2) and the last four equations of (1.1) in (3.8) we have

$$\begin{aligned} q_0(A(\tau,\ell), u(\tau,\ell)) &= \frac{1}{2}(Au^3)(\tau,\ell) + 2A_T\kappa^2 b(h_\ell(\tau) - h_{\ell e})h_\ell'(\tau) \\ q_0(A(\tau,0), u(\tau,0)) &= \frac{1}{2}(Au^3)(\tau,0) - 2A_T\kappa^2 b(h_0(\tau) - h_{0e})h_0'(\tau). \end{aligned}$$

Plugging these into the second integral in (4.5) and using the Sobolev embedding theorem we have

$$\begin{aligned}
&\int_0^t q_0(A(\tau,\ell),u(\tau,\ell)) - q_0(A(\tau,0),u(\tau,0))\,\mathrm{d}\tau \\
&\geq C(|h_0(t)-h_{0e}|^2 + |h_\ell(t)-h_{\ell e}|^2 - |h_0^0-h_{0e}|^2 - |h_\ell^0-h_{\ell e}|^2) \\
&\quad - C_\delta \sup_{\tau\in[0,t]} \|u(\tau)\|_{H^1} \int_0^t \|u(\tau)\|_{H^1}^2\,\mathrm{d}\tau.
\end{aligned} \tag{4.7}$$

Moreover, the Sobolev embedding theorem again implies that

$$-\beta \int_0^t \int_0^\ell (Au^2)(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau \leq -\beta C_\delta \int_0^t \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau. \tag{4.8}$$

Now it can be seen that (4.4) follows from (4.5)–(4.8) and the fact that the L^2-norm of $(uA)(t)$ and $u(t)$ are equivalent for each t provided that $\delta > 0$ is small enough. □

The next step is to derive estimates involving the spatial derivatives of the state components u and $A^{\frac{1}{4}}$. To this end we recall two classical inequalities frequently used in deriving estimates. The first one is Young's inequality: For each real number a, b and $\epsilon > 0$ we have $ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$. The second one is the following modified Sobolev embedding.

PROPOSITION 4.2. Let $a < b$. For every $\vartheta > 0$ there exists $C(a,b,\vartheta) > 0$ such that

$$\|u\|_{L^\infty(a,b)}^2 \leq \vartheta \|u_x\|_{L^2(a,b)}^2 + C(a,b,\vartheta)\|u\|_{L^2(a,b)}^2 \tag{4.9}$$

for all $u \in H^1(a,b)$.

Proof. Let $a \leq x_0 \leq \frac{a+b}{2}$. Consider the linear multiplier $m(x) = \frac{2}{b-x_0}(x-x_0) - 1$ satisfying $\|m\|_{L^\infty[x_0,b]} = 1$. Thus

$$\begin{aligned}
|u(x_0)|^2 + |u(b)|^2 &= \int_{x_0}^b (mu^2)_x\,\mathrm{d}x = \frac{2}{b-x_0}\int_{x_0}^b u^2\,\mathrm{d}x + 2\int_{x_0}^b muu_x\,\mathrm{d}x \\
&\leq \vartheta\|u_x\|_{L^2(x_0,b)}^2 + \left(\frac{4}{b-a} + \frac{1}{\vartheta}\right)\|u\|_{L^2(x_0,b)}^2
\end{aligned}$$

where we used Young's inequality in the last step. A similar process can be done for the case $\frac{a+b}{2} \leq x_0 \leq b$, now using the multiplier $n(x) = \frac{2}{x_0-a}(x-x_0) + 1$ and integration over $[a, x_0]$. These estimates imply (4.9). □

The proposition is useful when dealing with higher order estimates. For example, in obtaining estimates for z_x and w_x, we will put a small factor, if necessary, to these terms, but the drawback is the occurrence of a large factor to lower order terms. However, this will not cause problems when we have already derived estimates for the lower order terms, specifically, the one given in Lemma 4.1.

LEMMA 4.3 (First order estimate). There exist $\delta > 0$ and $C_\delta > 0$ such that for any solution $(A, u, h_0, h_\ell) \in X_T$ satisfying $N_2^2(T) \le \delta$ we have

$$\|u_x(t)\|_{L^2}^2 + \|(A^{\frac{1}{4}})_x(t)\|_{L^2}^2 + \int_0^t \|u_x(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \;\le\; C_\delta N_1^2(0) \tag{4.10}$$
$$+ C_\delta \sup_{\tau\in[0,t]} (\|u(\tau)\|_{H^2} + \|A^{\frac{1}{4}}(\tau) - A_e^{\frac{1}{4}}\|_{H^2}) \int_0^t \|u(\tau)\|_{H^1}^2 + \|(A^{\frac{1}{4}})_x(\tau)\|_{L^2}^2 \,\mathrm{d}\tau$$

for all $t \in [0, T]$.

Proof. To prove the lemma we will utilize the system satisfied by the (shifted) Riemann invariants (3.12). Let us consider the entropy $\eta_1 = \psi_1 - \phi_1$ where

$$\begin{aligned} \psi_1(t,x,Z) &= \theta(w(t,x), z(t,x))\mu(t,x)Z^2 \\ \phi_1(t,x,W) &= \theta(w(t,x), z(t,x))\lambda(t,x)W^2. \end{aligned}$$

We will estimate each integral in (3.27) with these particular functions.

Suppose that $N_2^2(T) \le \delta$. If $\delta > 0$ is sufficiently small, then there exist positive constants $C_{i\delta}$ such that $C_{1\delta} \le \zeta_k(h_k(t)) \le C_{2\delta}$ for $k = 0, \ell$, $-C_{3\delta} \le \lambda(t,x) \le -C_{4\delta}$, $C_{5\delta} \le \mu(t,x) \le C_{6\delta}$ and $C_{7\delta} \le \theta(w(t,x), z(t,x)) \le C_{8\delta}$ for all $(t,x) \in [0,T] \times [0,\ell]$. We shall use these properties throughout without mentioning them anymore.

We estimate each of the integrals on the left hand side of (3.27) from below and estimate the integral on the right hand side from above. For ease of reading, we divide the process into three steps. To make the terms more compact we also introduce the variable $V = (w, z)$.

STEP 1. ESTIMATE FROM BELOW. The preceding remarks about θ, λ and μ show that

$$C_{1\delta}(w_x^2(t,x) + z_x^2(t,x)) \le \eta_1(t,x) \le C_{2\delta}(w_x^2(t,x) + z_x^2(t,x)) \tag{4.11}$$

for all $(t,x) \in [0,T] \times [0,\ell]$. Thus

$$\int_0^\ell \eta_1(t,x) - \eta_1(0,x)\,\mathrm{d}x \ge C_\delta(\|V_x(\tau)\|_{L^2}^2 - \|V_x(0)\|_{L^2}^2). \tag{4.12}$$

Next, we deal with boundary terms. Let us note the identity

$$\begin{aligned} q_1 &= \theta(w,z)((\mu z_x)^2 - (\lambda w_x)^2) \\ &= \theta(w,z)\left(\left(-z_t - \frac{\beta}{2}(z-w)\right)^2 - \left(-w_t + \frac{\beta}{2}(z-w)\right)^2\right) \\ &= \theta(w,z)(z_t^2 - w_t^2 + \beta(z_t + w_t)(z-w)) \end{aligned}$$

obtained from the first two equations in (3.12). Each term of the above equality is evaluated at either $(t,0)$ or (t,ℓ). Consider the case where it is evaluated at $(t,0)$. Differentiating the fifth equation in (3.12) and using the third equation we arrive at

$$z_t(t,0) + w_t(t,0) = [\zeta_0'(h_0(t))(h_0(t) - h_{0e}) + \zeta_0(h_0(t))]h_0'(t) \tag{4.13}$$
$$= -S_1(t)(z(t,0) - w(t,0)) \tag{4.14}$$

where $S_1(t) = \theta(w(t,0), z(t,0))[\zeta_0'(h_0(t))(h_0(t) - h_{0e}) + \zeta_0(h_0(t))]$. Thus we have

$$\begin{aligned} -q_1(t,0) &= -\theta(w(t,0), z(t,0))(z_t^2(t,0) - w_t^2(t,0)) \\ &\quad - \beta\theta(w(t,0), z(t,0))S_1(t)(z(t,0) - w(t,0))^2 =: \Psi_1(t) + \Psi_2(t). \end{aligned} \tag{4.15}$$

Using the estimate in Propostion 4.2, the Sobolev embedding theorem and the equality $2u = z - w$ we have

$$\int_0^t \Psi_2(\tau)\,\mathrm{d}\tau \geq -C_\delta \vartheta \int_0^t \|u_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau - C_{\delta,\vartheta}\int_0^t \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau. \tag{4.16}$$

Differentiating the third equation in (3.12) gives

$$\begin{aligned} h_0''(t) \;=\; &-\theta_1(w(t,0), z(t,0))(z_t(t,0) + w_t(t,0))(z(t,0) - w(t,0)) \\ &- \theta(w(t,0), z(t,0))(z_t(t,0) - w_t(t,0)) \end{aligned} \tag{4.17}$$

where $\theta_1(w,z) = \frac{1}{2^{11}\kappa A_T}(w + z + 2C_e)^3$. Multiplying the left hand side of (4.13) by the right hand side of (4.17), rearranging the terms and then using (4.14) we obtain

$$\Psi_1(t) = S_2(t)(z(t,0) - w(t,0))^3 + \frac{1}{2}S_3(t)\frac{\mathrm{d}}{\mathrm{d}t}|h'(t)|^2 \tag{4.18}$$

where $S_2(t) = \theta_1(w(t,0), z(t,0))S_1^2(t)$ and $S_3(t) = \zeta_0'(h_0(t))(h_0(t) - h_{0e}) + \zeta_0(h_0(t))$. Let us integrate (4.18) from 0 to t. The first term of the integral can be estimated as

$$\int_0^t S_2(\tau)(z(\tau,0) - w(\tau,0))^3\,\mathrm{d}\tau \geq -C_\delta \sup_{\tau\in[0,t]}\|u(\tau)\|_{H^1}\int_0^t \|u(\tau)\|_{H^1}^2\,\mathrm{d}\tau. \tag{4.19}$$

For the remaining term we integrate by parts, use the the third equation in (1.1), and apply the Sobolev embedding and Proposition 4.2 to obtain

$$\begin{aligned} \frac{1}{2}\int_0^t S_3(\tau)\frac{\mathrm{d}}{\mathrm{d}t}|h'(\tau)|^2\,\mathrm{d}\tau &= \frac{1}{2}S_3(t)|h_0'(t)|^2 - \frac{1}{2}S_3(0)|h_0'(0)|^2 \\ &\quad - \frac{1}{2}\int_0^t [\zeta_0''(h_0(\tau))(h_0(\tau) - h_{0e}) + 2\zeta_0'(h_0(\tau))]h_0'(\tau)^3\,\mathrm{d}\tau \\ &\geq -C_\delta\Big(\vartheta\|u_x(t)\|_{L^2}^2 + C_\vartheta\|u(t)\|_{L^2}^2 + \|u(0)\|_{H^1}^2 \\ &\qquad + \sup_{\tau\in[0,t]}\|u(\tau)\|_{H^1}\int_0^t \|u(\tau)\|_{H^1}^2\,\mathrm{d}\tau\Big). \end{aligned} \tag{4.20}$$

Therefore, (4.15) and the inequalities (4.16), (4.19) and (4.20) give us the estimate

$$\begin{aligned} -\int_0^t q_1(\tau,0)\,\mathrm{d}\tau \;&=\; \int_0^t \Psi_1(\tau)\,\mathrm{d}\tau + \int_0^t \Psi_2(\tau)\,\mathrm{d}\tau \\ \geq &-C_\delta\Big(\vartheta\|u_x(t)\|_{L^2}^2 + \vartheta\int_0^t \|u_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau + C_\vartheta\|u(t)\|_{L^2}^2 + C_\vartheta\int_0^t \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau \\ &\quad + \|u(0)\|_{H^1}^2 + \sup_{\tau\in[0,t]}\|u(\tau)\|_{H^1}\int_0^t \|u(\tau)\|_{H^1}^2\,\mathrm{d}\tau\Big). \end{aligned}$$

In an analogous manner we can obtain the same form of estimate from below for the integral $\int_0^t q_1(\tau,\ell)\,\mathrm{d}\tau$. Combining the estimates that we have obtained so far, we have

the following estimate from below for the left hand side of (3.27):

$$\begin{aligned}
&\int_0^\ell \eta_1(t,x) - \eta_1(0,x)\,\mathrm{d}x + \int_0^t q_1(\tau,\ell) - q_1(\tau,0)\,\mathrm{d}\tau \\
&\geq C_\delta\Bigg((1-\vartheta)\|V_x(t)\|_{L^2}^2 - \vartheta\int_0^t \|u_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau - C_\vartheta\|V(t)\|_{L^2}^2 \\
&\qquad - C_\vartheta\int_0^t \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau - \|V(0)\|_{H^1}^2 - \sup_{\tau\in[0,t]}\|u(\tau)\|_{H^1}\int_0^t \|u(\tau)\|_{H^1}^2\,\mathrm{d}\tau\Bigg).
\end{aligned} \tag{4.21}$$

Step 2. Estimate from above. First we will express the derivative of the eigenvalues λ and μ with respect to t in terms of the Riemann invariants w and z. A straightforward calculation and application of the two PDEs in (3.12) give us

$$\begin{aligned}
\mu_t &= -\frac{3C_e}{32}w_x - \frac{5C_e}{32}z_x - \frac{\beta}{2}(z-w) + R_1 \\
\lambda_t &= -\frac{5C_e}{32}w_x - \frac{3C_e}{32}z_x - \frac{\beta}{2}(z-w) + R_2
\end{aligned}$$

where $R_k = c_{k1}ww_x + c_{k2}zw_x + c_{k3}wz_x + c_{k4}zz_x$, $k = 1,2$, for some constants c_{kj}. Therefore, each term of μ_t and λ_t contains at least one factor among $z-w, w_x, z_x$. Consequently, the same is true for w_t and z_t according to the PDE and in turn for $\theta_t(w,z) = \theta_1(w,z)(w_t+z_t)$. This observation is important because we want to avoid the term $\int_0^t \|A^{\frac{1}{4}}(\tau) - A_e^{\frac{1}{4}}\|_{L^2}\,\mathrm{d}\tau$, which is not present in the energy functional N_2.

Now the first three pairs appearing in (3.26) for $k=1$ are given by

$$\begin{aligned}
\psi_{1t} - \phi_{1t} &= (\theta_t\mu + \theta\mu_t)z_x^2 - (\theta_t\lambda + \theta\lambda_t)w_x^2 \\
\mu_x\psi_1 - \lambda_x\phi_1 &= \theta\mu\mu_x z_x^2 - \theta\lambda\lambda_x w_x^2 \\
\mu\psi_{1x} - \lambda\phi_{1x} &= \mu(\theta_x\mu + \theta\mu_x)z_x^2 - \lambda(\theta_x\lambda + \theta\lambda_x)w_x^2.
\end{aligned}$$

From the previous remarks we notice that the factors of z_x^2 and w_x^2 appearing on the right hand sides of the last three equations are polynomials of degree at least 1 in z, w, z_x, w_x. Applying the Sobolev embedding theorem for these factors and then taking the supremum over $[0,t]$ we have

$$\begin{aligned}
&\int_0^t\int_0^\ell (\psi_{1t} - \phi_{1t}) + (\mu_x\psi_1 - \lambda_x\phi_1) + (\mu\psi_{1x} - \lambda\phi_{1x})\,\mathrm{d}\tau\,\mathrm{d}x \\
&\qquad \leq C_\delta \sup_{\tau\in[0,t]}\|V(\tau)\|_{H^2}\int_0^t \|V_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau.
\end{aligned} \tag{4.22}$$

The last term in M_1 is more delicate since it contains second order terms. Indeed, we have

$$\begin{aligned}
\psi_{1Z}G_1 - \phi_{1W}F_1 &= 2\theta\mu z_x G_1 - 2\theta\lambda w_x F_1 \\
&= 2\theta\mu z_x\left(-\mu_x z_x - \frac{\beta}{2}(z_x - w_x)\right) - 2\theta\lambda w_x\left(-\lambda_x w_x + \frac{\beta}{2}(z_x - w_x)\right) \\
&= -\frac{\theta_c C_e\beta}{4}(z_x - w_x)^2 + R_3
\end{aligned} \tag{4.23}$$

where $\theta_c > 0$ is the constant term of θ. Here R_3 are terms of degree at least 3 that contain either z_x^2, w_x^2, or $w_x z_x$. Hence

$$\int_0^t \int_0^\ell \psi_{1Z} G_1 - \phi_{1W} F_1 \,\mathrm{d}\tau \,\mathrm{d}x \;\leq\; -\,\tilde{C} \int_0^t \|u_x(\tau)\|_{L^2}^2 \,\mathrm{d}\tau + C_\delta \sup_{\tau\in[0,t]} \|V(\tau)\|_{H^2} \int_0^t \|V_x(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \tag{4.24}$$

where $\tilde{C} = \frac{\theta_c C_e \beta}{4} > 0$, if $\beta > 0$, independent of δ. Adding (4.22) and (4.24) we arrive at

$$\int_0^t \int_0^\ell M_1(\psi_1, \phi_1) \,\mathrm{d}\tau \,\mathrm{d}x \;\leq\; -\,\tilde{C} \int_0^t \|u_x(\tau)\|_{L^2}^2 \,\mathrm{d}\tau + C_\delta \sup_{\tau\in[0,t]} \|V(\tau)\|_{H^2} \int_0^t \|V_x(\tau)\|_{L^2}^2 \,\mathrm{d}\tau. \tag{4.25}$$

Step 3. Let us combine the estimates obtained from Step 1 and Step 2. Choosing $\vartheta > 0$ small enough so that $\tilde{C} - C_\delta \vartheta > 0$ we have

$$\|V_x(t)\|_{L^2}^2 + \int_0^t \|u_x(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \;\leq\; C_\delta \|V(t)\|_{L^2}^2 + C_\delta \|V(0)\|_{H^1}^2 + C_\delta \int_0^t \|u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau + C_\delta \sup_{\tau\in[0,t]} \|V(\tau)\|_{H^2} \int_0^t \|V_x(\tau)\|_{L^2}^2 + \|u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau. \tag{4.26}$$

We can use Lemma 4.1 to bound the first and third terms on the right hand side of (4.26) from above. Consequently, (4.10) follows from (4.26), (4.4) and (4.2). □

To complete the estimate for the energy functional N_1 we need the following additional estimate.

Lemma 4.4. There exist $\delta > 0$ and $C_\delta > 0$ such that for any solution $(A, u, h_0, h_\ell) \in X_T$ satisfying $N_2^2(T) \leq \delta$ we have

$$\int_0^t \|(A^{\frac{1}{4}})_x(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \;\leq\; C_\delta N_1^2(0) + C_\delta \sup_{\tau\in[0,t]} (\|u(\tau)\|_{H^2} + \|A^{\frac{1}{4}}(\tau) - A_e^{\frac{1}{4}}\|_{H^2}) \int_0^t \|u(\tau)\|_{H^1}^2 + \|(A^{\frac{1}{4}})_x(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \tag{4.27}$$

for all $t \in [0, T]$.

Proof. The proof of the lemma is basically the same as the proof of Lemma 4.3; the main difference is the particular choice of the entropy appearing in (3.27). In the current situation, we consider the entropy $\tilde{\eta}_1 = \tilde{\psi}_1 - \tilde{\phi}_1$ with corresponding entropy flux $\tilde{q}_1 = \mu\tilde{\psi}_1 - \lambda\tilde{\phi}_1$ where

$$\tilde{\phi}_1(t, x, W) \;=\; \frac{\theta(w(t,x), z(t,x))}{\lambda(t,x)} \left(\lambda(t,x)W - \frac{\beta}{2}(z(t,x) - w(t,x))\right)^2$$

$$\tilde{\psi}_1(t, x, Z) \;=\; \frac{\theta(w(t,x), z(t,x))}{\mu(t,x)} \left(\mu(t,x)Z + \frac{\beta}{2}(z(t,x) - w(t,x))\right)^2.$$

Let $F_0 = \frac{\beta}{2}(z-w)$. Using Young's inequality

$$\begin{aligned}\tilde{\eta}_1 &= \theta\mu^{-1}\left(\mu^2 z_x^2 + 2\mu F_0 z_x + F_0^2\right) - \theta\lambda^{-1}(\lambda^2 w_x^2 - 2\lambda F_0 w_x + F_0^2)\\ &= \theta\left(\mu z_x^2 - \lambda w_x^2 + 2F_0 z_x + 2F_0 w_x + (\mu^{-1}-\lambda^{-1})F_0^2\right)\\ &\geq C_\delta(w_x^2 + z_x^2) - C_\delta(\epsilon z_x^2 + 2\epsilon^{-1}F_0^2 + \epsilon w_x^2) + C_\delta F_0^2\\ &\geq C_\delta(w_x^2 + z_x^2) - C_\delta(w^2 + z^2)\end{aligned}$$

for some $\epsilon \in (0,1)$ small enough. Similarly, $\tilde{\eta}_1 \leq C_\delta(w_x^2 + z_x^2 + w^2 + z^2)$. Thus

$$\int_0^\ell \tilde{\eta}_1(t,x) - \tilde{\eta}_1(0,x)\,\mathrm{d}x \geq C_\delta(\|V_x(t)\|_{L^2}^2 - \|V(t)\|_{L^2}^2 - \|V(0)\|_{H^1}^2). \tag{4.28}$$

From (3.12), (4.18), (4.19) and (4.20), and according to the statement following (4.20) we immediately get

$$\begin{aligned}&\int_0^t \tilde{q}_1(\tau,\ell) - \tilde{q}_1(\tau,0)\,\mathrm{d}\tau = -\int_0^t \theta(w(\tau,0), z(\tau,0))(z_\tau^2(\tau,0) - w_\tau^2(\tau,0))\,\mathrm{d}\tau\\ &\geq -C_\delta\Bigg(\vartheta\|u_x(t)\|_{L^2}^2 + C_\vartheta\|u(t)\|_{L^2}^2 + \|u(0)\|_{H^1}^2\\ &\qquad\qquad + \sup_{\tau\in[0,t]}\|u(\tau)\|_{H^1}\int_0^t \|u(\tau)\|_{H^1}^2\,\mathrm{d}\tau\Bigg).\end{aligned} \tag{4.29}$$

The remaining task is to obtain estimates from above. As in the previous lemma, we need to look carefully at each pair appearing in $\tilde{M}_1$ since some of them contain terms of degree only 2. For the rest of the proof R_i will denote terms that are degree at least 3 and contain at least two factors among $z-w, w_x, z_x$. Note that using (3.12) we have

$$z_t - w_t = -\frac{C_e}{4}(z_x + w_x) - \beta(z-w) + \hat{R}_0 \tag{4.30}$$

where $\hat{R}_0 = c_1 w w_x + c_2 z w_x + c_3 w z_x + c_4 z z_x$ for some constants c_i. Thus we have

$$\begin{aligned}\tilde{\psi}_{1t} - \tilde{\phi}_{1t} &= (\mu z_x + F_0)^2\frac{\mu\theta_t - \theta\mu_t}{\mu^2} + \frac{2\theta}{\mu}(\mu z_x + F_0)(\mu_t z_x + F_{0t})\\ &\quad - (\lambda w_x - F_0)^2\frac{\lambda\theta_t - \theta\lambda_t}{\lambda^2} - \frac{2\theta}{\lambda}(\lambda w_x - F_0)(\lambda_t z_x - F_{0t})\\ &= 2\theta\left(z_x + w_x + \left(\frac{1}{\mu} - \frac{1}{\lambda}\right)F_0\right)F_{0t} + \frac{1}{\lambda^2\mu^2}R_4\\ &= -\frac{C_e\beta\theta}{4}(z_x + w_x)^2 - \beta^2\theta(z_x + w_x)(z-w)\\ &\quad - \frac{C_e\beta^2\theta}{8}\left(\frac{1}{\mu} - \frac{1}{\lambda}\right)(z_x + w_x)(z-w) - \frac{\beta^3\theta}{2}\left(\frac{1}{\mu} - \frac{1}{\lambda}\right)(z-w)^2\\ &\quad + \frac{1}{\lambda^2\mu^2}R_5.\end{aligned}$$

By Young's inequality and the Sobolev embedding theorem we have

$$\tilde{\psi}_{1t} - \tilde{\phi}_{1t} \leq \left(-\frac{\theta_c C_e\beta}{4} + C_\delta\epsilon\right)(z_x + w_x)^2 + C_{\delta,\epsilon}(z-w)^2 + \frac{1}{\lambda^2\mu^2}R_5. \tag{4.31}$$

For the second pair we can see that

$$\mu_x\tilde{\psi}_1 - \lambda_x\tilde{\phi}_1 = \frac{\theta}{\mu}\mu_x(\mu z_x + F_0)^2 - \frac{\theta}{\lambda}\lambda_x(\lambda w_x - F_0)^2 = \frac{1}{\lambda\mu}R_6. \tag{4.32}$$

The third pair can be computed as in the first pair and we get

$$\begin{aligned}
\mu\tilde{\psi}_{1x} - \lambda\tilde{\phi}_{1x} &= (\mu z_x + F_0)^2\frac{\mu\theta_x - \theta\mu_x}{\mu} + 2\theta(\mu z_x + F_0)(\mu_x z_x + F_{0x}) \\
&\quad - (\lambda w_x - F_0)^2\frac{\lambda\theta_x - \theta\lambda_x}{\lambda} - 2\theta(\lambda w_x - F_0)(\lambda_x w_x - F_{0x}) \\
&= 2\theta\left(\left(\mu z_x + \frac{\beta}{2}(z-w)\right) + \left(\lambda w_x - \frac{\beta}{2}(z-w)\right)\right)\frac{\beta}{2}(z_x - w_x) \\
&\quad + \frac{1}{\lambda\mu}R_7 \\
&= \frac{\theta_c C_e\beta}{4}(z_x - w_x)^2 + \frac{1}{\lambda\mu}R_8.
\end{aligned} \tag{4.33}$$

Finally, for the last pair we use (3.19) and (3.20) to obtain

$$\begin{aligned}
\tilde{\psi}_{1Z}G_1 - \tilde{\phi}_{1W}F_1 &= \frac{2\theta}{\mu}(\mu z_x + F_0)\mu G_1 - \frac{2\theta}{\lambda}(\lambda w_x - F_0)\lambda F_1 \\
&= 2\theta\left(\frac{C_e}{4}z_x + \frac{\beta}{2}(z-w) + \hat{R}_1\right)\left(-\mu_x z_x - \frac{\beta}{2}(z_x - w_x)\right) \\
&\quad - 2\theta\left(-\frac{C_e}{4}w_x - \frac{\beta}{2}(z-w) + \hat{R}_2\right)\left(-\lambda_x w_x + \frac{\beta}{2}(z_x - w_x)\right) \\
&= -\frac{\theta_c C_e\beta}{4}(z_x - w_x)^2 + R_9
\end{aligned} \tag{4.34}$$

where $\hat{R}_1$, $\hat{R}_2$ are of degree 2 and have the same form as $\hat{R}_0$.

Taking the sum of (4.31)–(4.34), choosing $\epsilon > 0$ small enough so that $\tilde{C}_1 = \frac{\theta_c C_e\beta}{4} - C_\delta\epsilon > 0$, using the Sobolev embedding theorem and the transformations (3.11) we obtain

$$\begin{aligned}
\int_0^t\int_0^\ell \tilde{M}_1(\tilde{\psi}_1, \tilde{\phi}_1)\,\mathrm{d}x\,\mathrm{d}\tau \;\le\; & -\tilde{C}_1\int_0^t \|(A^{\frac{1}{4}})_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau \\
& + C_\delta \sup_{\tau\in[0,t]}\|V(\tau)\|_{H^2}\int_0^t \|V_x(\tau)\|_{L^2}^2 + \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau.
\end{aligned} \tag{4.35}$$

Now it can be seen that (4.27) follows from (4.28), (4.29), (4.35), Lemma 4.1, and from the equivalence of norms in (4.2). □

REMARK 4.5. It is worth pointing out that by an appropriate modification of the entropy-entropy flux pair we saw in the proof of Lemma 4.4 that the term u_x^2, or equivalently $(z_x - w_x)^2$, which appears on the right hand side of (3.27) cancels when adding (4.33) and (4.34). Moreover it was replaced by a term involving $(A^{\frac{1}{4}})_x^2$, or equivalently $(z_x + w_x)^2$. The appearance of $(A^{\frac{1}{4}})_x^2$ is precisely what we want in order to prove Lemma 4.4. This observation will also be used in the following two lemmas.

Before we proceed in obtaining estimates for the second spatial derivatives of the state variables, we will derive some identities from the two PDEs in the diagonal system (3.12).

In the following, we concentrate on the linear terms and state only the properties of the higher degree terms. Differentiating the first equation in (3.12) with respect to t we get

$$\lambda w_{xt} = -w_{tt} - \lambda_t w_x + \frac{\beta}{2}(z_t - w_t). \tag{4.36}$$

However, we note from (3.17) for $k = 1$ that

$$\lambda w_{tx} = -\lambda^2 w_{xx} + \lambda F_1. \tag{4.37}$$

Thus, according to (4.36), (4.37) and (3.19) we have

$$w_{tt} = \lambda^2 w_{xx} + \frac{\beta}{2}(z_t - w_t) - \frac{\beta\lambda}{2}(z_x - w_x) + \lambda\lambda_x w_x - \lambda_t w_x. \tag{4.38}$$

In a similar way we have the following equation for z_{tt}:

$$z_{tt} = \mu^2 z_{xx} - \frac{\beta}{2}(z_t - w_t) + \frac{\beta\mu}{2}(z_x - w_x) + \mu\mu_x z_x - \mu_t z_x. \tag{4.39}$$

Taking the derivative of both sides of (4.30) with respect to x, we have

$$z_{tx} - w_{tx} = -\frac{C_e}{4}(z_{xx} + w_{xx}) - \beta(z_x - w_x) + \hat{R}_3 \tag{4.40}$$

where $\hat{R}_3 = \sum_{j+k=2} c_{jk}(\partial_x^j w)(\partial_x^k z)$ for some constants c_{jk}. Subtracting (4.38) from (4.39) and using (4.30) we have

$$z_{tt} - w_{tt} = \frac{C_e^2}{16}(z_{xx} - w_{xx}) + \frac{\beta C_e}{2} z_x + \beta^2(z - w) + \hat{R}_4 \tag{4.41}$$

where $\hat{R}_4$ are terms of degree at least 2 and contain at least one factor among $z - w, w_x, z_x, z_{xx}, w_{xx}$. However, each term has at most one factor among w_{xx}, z_{xx}.

Lemma 4.6 (Second order estimate). There exist $\delta > 0$ and $C_\delta > 0$ such that for any solution $(A, u, h_0, h_\ell) \in X_T$ satisfying $N_2^2(T) \le \delta$ it holds that

$$\begin{aligned} \|u_{xx}(t)\|_{L^2}^2 + \|(A^{\frac{1}{4}})_{xx}(t)\|_{L^2}^2 + \int_0^t \|u_{xx}(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \;&\le\; C_\delta N_2^2(0) \\ + C_\delta \sup_{\tau\in[0,t]} (\|u(\tau)\|_{H^2} + \|A^{\frac{1}{4}}(\tau) - A_e^{\frac{1}{4}}\|_{H^2}) \int_0^t \|u(\tau)\|_{H^2}^2 + \|(A^{\frac{1}{4}})_x(\tau)\|_{H^1}^2 \,\mathrm{d}\tau \end{aligned} \tag{4.42}$$

for all $t \in [0, T]$.

Proof. Again we will proceed in the same manner, now with the entropy $\eta_2 = \psi_2 - \phi_2$ where

$$\begin{aligned} \psi_2(t,x,Z) \;&=\; \frac{\theta(w,z)}{\mu}\left(\mu^2 Z - \frac{\beta}{2}(z_t - w_t) + \frac{\beta\mu}{2}(z_x - w_x) + \mu\mu_x z_x - \mu_t z_x\right)^2 \\ \phi_2(t,x,W) \;&=\; \frac{\theta(w,z)}{\lambda}\left(\lambda^2 W + \frac{\beta}{2}(z_t - w_t) - \frac{\beta\lambda}{2}(z_x - w_x) + \lambda\lambda_x w_x - \lambda_t w_x\right)^2. \end{aligned}$$

We estimate (3.27) with these particular functions and as before we divide the procedure into three steps, namely, the derivation of estimates of the left hand side of (3.27) from below, estimates of the right hand side of (3.27) from above and finally combining the two.

STEP 1. ESTIMATE FROM BELOW. For brevity let us set

$$\tilde{N} = -\frac{\beta}{2}(z_t - w_t) + \frac{\beta\mu}{2}(z_x - w_x) + \mu\mu_x z_x - \mu_t z_x \tag{4.43}$$

$$\tilde{P} = \frac{\beta}{2}(z_t - w_t) - \frac{\beta\lambda}{2}(z_x - w_x) + \lambda\lambda_x w_x - \lambda_t w_x. \tag{4.44}$$

Using Young's inequality we have, for $\delta > 0$ small enough,

$$\begin{aligned}\psi_2(t,x,z_{xx}(t,x)) &= \theta\mu^{-1}(\mu^4 z_{xx}^2 + 2\mu^2 z_{xx}\tilde{N} + \tilde{N}^2)\\ &\geq \theta\mu^3 z_{xx}^2 - \theta\mu(\epsilon z_{xx}^2 + C_\epsilon \tilde{N}^2) + \theta\mu^{-1}\tilde{N}^2\\ &= (\theta\mu^3 - \theta\mu\epsilon)z_{xx}^2 - (\theta\mu C_\epsilon - \theta\mu^{-1})\tilde{N}^2\end{aligned}$$

for every $\epsilon > 0$. We removed the arguments (t,x) on the right hand sides for simplicity. Using the definition of $\tilde{N}_2$ and replacing the term $z_t - w_t$ by the right hand side of (4.30), we can see that

$$\tilde{N}(t,x)^2 \leq C_\delta(w(t,x)^2 + z(t,x)^2 + w_x(t,x)^2 + z_x(t,x)^2).$$

This follows immediately from the fact that $\tilde{N}$ consists of terms that are at least degree 1 in w, z, w_x, z_x, and so $\tilde{N}^2$ will have at least degree 2 terms in these variables. Then we retain two factors and take the supremum of the rest, employing the Sobolev embedding theorem to estimate the supremum and finally use the assumption that $N_2^2(T) \leq \delta$, for $\delta > 0$ small enough.

Now, choosing $\epsilon > 0$ sufficiently small we have

$$\psi_2(t,x,z_{xx}(t,x)) \geq C_\delta z_{xx}^2(t,x) - C_\delta(|V(t,x)|^2 + |V_x(t,x)|^2) \tag{4.45}$$

for all $(t,x) \in [0,T] \times [0,\ell]$. Recall that $V = (w,z)$. Similarly, we have the upper bound

$$\psi_2(t,x,z_{xx}(t,x)) \leq C_\delta z_{xx}^2(t,x) + C_\delta(|V(t,x)|^2 + |V_x(t,x)|^2) \tag{4.46}$$

for all $(t,x) \in [0,T] \times [0,\ell]$. Doing the same process with ϕ_2 and recalling that λ is negative for small enough $\delta > 0$ we have

$$-C_\delta w_{xx}^2 - C_\delta(|V|^2 + |V_x|^2) \leq \phi_2 \leq -C_\delta w_{xx}^2 + C_\delta(|V|^2 + |V_x|^2). \tag{4.47}$$

From (4.45)–(4.47) we have

$$\int_0^\ell \eta_2(t,x) - \eta_2(0,x)\,\mathrm{d}x \geq C_\delta(\|V_{xx}(\tau)\|_{L^2}^2 - \|V(0)\|_{H^2}^2). \tag{4.48}$$

According to (4.38) and (4.39) we can see that

$$-\int_0^t q_2(\tau,0)\,\mathrm{d}\tau = -\int_0^t \theta(w(\tau,0),z(\tau,0))(z_{\tau\tau}^2(\tau,0) - w_{\tau\tau}^2(\tau,0))\,\mathrm{d}\tau. \tag{4.49}$$

Let us use the boundary conditions to rewrite the integrand in terms of w, z and their first derivatives with respect to x. For convenience, the functions in the following discussions are to be evaluated at $(t,0)$ or t, or with other variables representing time, where they make sense. First, we notice from (4.13) that

$$z_t + w_t = S(h_0)\theta(w,z)(z - w) \tag{4.50}$$

where $S(h_0) = -\zeta_0'(h_0)(h_0 - h_{0e}) - \zeta_0(h_0)$. Let

$$p_1(w, z, w_x, z_x) = -\frac{C_e}{4}(z_x + w_x) - \beta(z - w) + \hat{R}_0, \tag{4.51}$$

and from (4.30) we have $z_t - w_t = p_1(w, z, w_x, z_x)$. Using (4.50) in (4.17) yields

$$\begin{aligned} h_0'' &= -S(h_0)\theta_1(w, z)\theta(w, z)(z - w)^3 - \theta(w, z)p_1(w, z, w_x, z_x) \\ &=: p_2(w, z, w_x, z_x). \end{aligned} \tag{4.52}$$

Taking the derivative of both sides of (4.13) gives us

$$\begin{aligned} z_{tt} + w_{tt} &= [\zeta_0''(h_0)(h_0 - h_{0e}) + 2\zeta_0'(h_0)](h_0')^2 + [\zeta_0'(h_0)(h_0 - h_{0e}) + \zeta_0(h_0)]h_0'' \\ &=: S_1(h_0)(h_0')^2 + S_2(h_0)h_0''. \end{aligned} \tag{4.53}$$

Thus, (4.52) implies that

$$\begin{aligned} z_{tt} + w_{tt} &= S_1(h_0)\theta(w, z)^2(z - w)^2 + S_2(h_0)p_2(w, z, w_x, z_x) \\ &=: p_3(w, z, w_x, z_x). \end{aligned} \tag{4.54}$$

We also take the derivative of (4.17) and apply (4.50) and (4.54) to obtain

$$\begin{aligned} h_0^{(3)} &= -\theta_2(w, z)(z_t + w_t)^2(z - w) - \theta_1(w, z)(z_{tt} + w_{tt})(z - w) \\ &\quad -2\theta_1(w, z)(z_t + w_t)(z_t - w_t) - \theta(w, z)(z_{tt} - w_{tt}) \\ &=: p_4(w, z, w_x, z_x) - \theta(w, z)(z_{tt} - w_{tt}) \end{aligned} \tag{4.55}$$

where $\theta_2(w, z) = \frac{12}{A_T}(w + z + 2C_e)^2$ and

$$\begin{aligned} p_4(w, z, w_x, z_x) &= -S(h_0)^2\theta_2(w, z)\theta(w, z)^2(z - w)^3 \\ &\quad - \theta_1(w, z)(z - w)p_3(w, z, w_x, z_x) \\ &\quad - 2\theta_1(w, z)S(h_0)\theta(w, z)(z - w)p_1(w, z, w_x, z_x). \end{aligned} \tag{4.56}$$

Note that p_1, p_2 and p_3 contain terms that are degree at least 1 and have at least one factor among $z - w, w_x, z_x$, while p_4 has terms with degree at least 2 that contain at least two factors among $z - w, w_x, z_x$. Moreover, we note that each S_i is bounded as long as its arguments stay on a bounded subset of $(0, \infty)$, which is the case due to the assumption that $|h_0(t) - h_{0e}|^2 \le \delta$ for small enough $\delta > 0$.

From (4.53), (4.54) and (4.55) we can now rewrite (4.49) as

$$\begin{aligned} -\int_0^t q_2(\tau, 0)\,\mathrm{d}\tau &= \int_0^t (h_0^{(3)} - p_4(w, z, w_x, z_x))(S_1(h_0)(h_0')^2 + S_2(h_0)h_0'')\,\mathrm{d}\tau \\ &= \int_0^t S_1(h_0)(h_0')^2 h_0^{(3)}\,\mathrm{d}\tau + \frac{1}{2}\int_0^t S_2(h_0)\frac{\mathrm{d}}{\mathrm{d}t}|h_0''|^2\,\mathrm{d}\tau \\ &\quad - \int_0^t p_4(w, z, w_x, z_x)p_3(w, z, w_x, z_x)\,\mathrm{d}\tau \\ &=: J_1 + J_2 + J_3. \end{aligned}$$

Integrating by parts and using (4.52) we get

$$\begin{aligned} J_1 &= S_1(h_0(\tau))h_0'(\tau)^2 h_0''(\tau)\Big|_{\tau=0}^{\tau=t} - \int_0^t S_1'(h_0)(h_0')^3 h_0'' + 2S_1(h_0)h_0'(h_0'')^2 \,\mathrm{d}\tau \\ &= S_1(h_0(\tau))\theta(w,z)^2(z-w)^2 p_2(w,z,w_x,z_x)\Big|_{\tau=0}^{\tau=t} \\ &\quad + \int_0^t S_1'(h_0)\theta(w,z)^3(z-w)^3 p_2 + 2S_1(h_0)\theta(w,z)(z-w)p_2^2 \,\mathrm{d}\tau. \end{aligned}$$

Applying Proposition 4.2 to the terms having either $z_x(\tau,0)$ or $w_x(\tau,0)$ appearing in the first term of the above last expression and using the Sobolev embedding theorem for the rest, we obtain the inequality

$$\begin{aligned} J_1 \geq &- C_\delta \vartheta \|V_{xx}(t)\|_{L^2}^2 - C_{\delta,\vartheta}\|V(t)\|_{H^1}^2 - C_\delta \|V(0)\|_{H^2}^2 \\ &- C_\delta \sup_{\tau\in[0,t]} \|V(\tau)\|_{H^2} \int_0^t \|V_x(\tau)\|_{H^1}^2 + \|u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau. \end{aligned}$$

In the above computations it is important to note the properties of p_2.

In a similar way we can integrate by parts and use the same techniques to obtain

$$\begin{aligned} J_2 \geq &- C_\delta \vartheta \|V_{xx}(t)\|_{L^2}^2 - C_{\delta,\vartheta}\|V(t)\|_{H^1}^2 - C_\delta \|V(0)\|_{H^2}^2 \\ &- C_\delta \sup_{\tau\in[0,t]} \|V(\tau)\|_{H^2} \int_0^t \|V_x(\tau)\|_{H^1}^2 + \|u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau. \end{aligned}$$

Furthermore, invoking the properties of p_3 and p_4 we have

$$J_3 \geq -C_\delta \sup_{\tau\in[0,t]} \|V(\tau)\|_{H^2} \int_0^t \|V_x(\tau)\|_{H^1}^2 + \|u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau.$$

Adding the lower bounds for J_1, J_2 and J_3 gives us a lower bound of $-\int_0^t q_2(\tau,0)\,\mathrm{d}\tau$, which has essentially the form of the lower bound for J_1. We can repeat the same process for $\int_0^t q_2(\tau,\ell)\,\mathrm{d}\tau$ and obtain a lower bound having the same form as stated above. With these, we finally obtain

$$\begin{aligned} \int_0^t q_2(\tau,\ell) - q_2(\tau,0)\,\mathrm{d}\tau \geq &- C_\delta \vartheta \|V_{xx}(t)\|_{L^2}^2 - C_{\delta,\vartheta}\|V(t)\|_{H^1}^2 - C_\delta \|V(0)\|_{H^2}^2 \\ &- C_\delta \sup_{\tau\in[0,t]} \|V(\tau)\|_{H^2} \int_0^t \|V_x(\tau)\|_{H^1}^2 + \|u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau. \end{aligned} \tag{4.57}$$

Inequalities (4.48) and (4.57) give us the desired estimate from below.

STEP 2. ESTIMATE FROM ABOVE. In this step R_i will denote terms of degree at least 3 containing at least two factors among $z-w, w_x, z_x, z_{xx}, w_{xx}$ and containing at most

two among z_{xx}, w_{xx}. First, we have

$$\begin{aligned}
\psi_{2t} - \phi_{2t} &= (\mu^2 z_{xx} + \tilde{N})^2 \frac{\mu\theta_t - \theta\mu_t}{\mu^2} + \frac{2\theta}{\mu}\Big(\mu^2 z_{xx} - \frac{\beta}{2}(z_t - w_t) + \frac{\beta\mu}{2}(z_x - w_x) \\
&+ \mu\mu_x z_x - \mu_t z_x\Big)\Big(2\mu\mu_t z_{xx} - \frac{\beta}{2}(z_{tt} - w_{tt}) + \frac{\beta}{2}\mu_t(z_x - w_x) + \frac{\beta\mu}{2}(z_{tx} - w_{tx}) \\
&+ (\mu\mu_x z_x - \mu_t z_x)_t\Big) - (\lambda^2 w_{xx} + \tilde{P})^2 \frac{\lambda\theta_t - \theta\lambda_t}{\lambda^2} - \frac{2\theta}{\lambda}\Big(\lambda^2 w_{xx} + \frac{\beta}{2}(z_t - w_t) \\
&- \frac{\beta\lambda}{2}(z_x - w_x) + \lambda\lambda_x w_x - \lambda_t w_x\Big)\Big(2\lambda\lambda_t w_{xx} + \frac{\beta}{2}(z_{tt} - w_{tt}) - \frac{\beta}{2}\lambda_t(z_x - w_x) \\
&- \frac{\beta\lambda}{2}(z_{tx} - w_{tx}) + (\lambda\lambda_x w_x - \lambda_t w_x)_t\Big) \\
&= -\theta\beta(\mu z_{xx} + \lambda w_{xx})(z_{tt} - w_{tt}) + \frac{\theta\beta^2}{2}\left(\frac{1}{\mu} - \frac{1}{\lambda}\right)(z_t - w_t)(z_{tt} - w_{tt}) \\
&+ \theta\beta(\mu^2 z_{xx} + \lambda^2 w_{xx})(z_{tx} - w_{tx}) + \frac{\theta\beta^2}{2}(\mu - \lambda)(z_x - w_x)(z_{tx} - w_{tx}) + \frac{R_1}{\lambda^2\mu^2} \\
&=: I_1 + I_2 + I_3 + I_4 + \frac{R_1}{\lambda^2\mu^2}. \qquad (4.58)
\end{aligned}$$

Consider each I_i. According to (4.41) and Young's inequality we have

$$\begin{aligned}
I_1 &= -\frac{\theta_c C_e \beta}{4}(z_{xx} - w_{xx})\left(\frac{C_e^2}{16}(z_{xx} - w_{xx}) + \frac{\beta C_e}{2} z_x + \beta^2(z - w)\right) + R_2 \\
&\le \left(-\frac{\theta_c \beta C_e^3}{64} + C_\delta \epsilon\right)(z_{xx} - w_{xx})^2 + C_{\delta,\epsilon}(z_x^2 + (z - w)^2) + R_2. \qquad (4.59)
\end{aligned}$$

Also, from (4.30) and (4.41)

$$\begin{aligned}
I_2 &= \frac{\theta_c \beta^2}{2}\left(\frac{1}{\mu} - \frac{1}{\lambda}\right)\left(-\frac{C_e}{4}(z_x + w_x) - \beta(z - w)\right)\left(\frac{C_e^2}{16}(z_{xx} - w_{xx})\right. \\
&\quad \left. + \frac{\beta C_e}{2} z_x + \beta^2(z - w)\right) + R_3 \\
&\le C_\delta \epsilon (z_{xx} - w_{xx})^2 + C_{\delta,\epsilon}(z_x^2 + w_x^2 + (z - w)^2) + R_3. \qquad (4.60)
\end{aligned}$$

From (4.40) we see that

$$\begin{aligned}
I_3 &= \frac{\theta_c \beta C_e^2}{16}(z_{xx} + w_{xx})\left(-\frac{C_e}{4}(z_{xx} + w_{xx}) - \beta(z_x - w_x)\right) + R_4 \\
&= -\frac{\theta_c \beta C_e^3}{64}(z_{xx} + w_{xx})^2 - \frac{\theta_c \beta^2 C_e^2}{16}(z_{xx} + w_{xx})(z_x - w_x) + R_4 \qquad (4.61)
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= \frac{\theta_c \beta^2 C_e}{4}(z_x - w_x)\left(-\frac{C_e}{4}(z_{xx} + w_{xx}) - \beta(z_x - w_x)\right) + R_5 \\
&= -\frac{\theta_c \beta^2 C_e^2}{16}(z_{xx} + w_{xx})(z_x - w_x) - \frac{\theta_c \beta^3 C_e}{4}(z_x - w_x)^2 + R_5. \qquad (4.62)
\end{aligned}$$

Adding (4.59)–(4.62) we have

$$\begin{aligned}\psi_{2t} - \phi_{2t} \;\le\; & \left(-\frac{\theta_c\beta C_e^3}{64} + C_\delta\epsilon\right)(z_{xx} - w_{xx})^2 - \frac{\theta_c\beta C_e^3}{64}(z_{xx} + w_{xx})^2 \\ & - \frac{\theta_c\beta^2 C_e^2}{8}(z_{xx} + w_{xx})(z_x - w_x) + C_{\delta,\epsilon}(z_x^2 + w_x^2 + (z-w)^2) \\ & + C_\delta(z_x - w_x)^2 + \frac{R_7}{\lambda^2\mu^2}.\end{aligned} \tag{4.63}$$

It can be checked that

$$\mu_x\psi_2 - \lambda_x\phi_2 = \frac{1}{\lambda\mu}R_8. \tag{4.64}$$

Similarly for the third pair we have

$$\begin{aligned}&\mu\psi_{2x} - \lambda\phi_{2x} = (\mu^2 z_{xx} + \tilde{N})^2\frac{\mu\theta_x - \theta\mu_x}{\mu} + 2\theta\Big(\mu^2 z_{xx} - \frac{\beta}{2}(z_t - w_t) \\ &+ \frac{\beta\mu}{2}(z_x - w_x) + \mu\mu_x z_x - \mu_t z_x\Big)\Big(2\mu\mu_x z_{xx} - \frac{\beta}{2}(z_{tx} - w_{tx}) + \frac{\beta}{2}\mu_x(z_x - w_x) \\ &+ \frac{\beta\mu}{2}(z_{xx} - w_{xx}) + (\mu\mu_x z_x - \mu_t z_x)_x\Big) - (\lambda^2 w_{xx} + \tilde{P})^2\frac{\lambda\theta_x - \theta\lambda_x}{\lambda} \\ &- 2\theta\Big(\lambda^2 w_{xx} + \frac{\beta}{2}(z_t - w_t) - \frac{\beta\lambda}{2}(z_x - w_x) + \lambda\lambda_x w_x - \lambda_t w_x\Big)\Big(2\lambda\lambda_x w_{xx} \\ &+ \frac{\beta}{2}(z_{tx} - w_{tx}) - \frac{\beta}{2}\lambda_x(z_x - w_x) - \frac{\beta\lambda}{2}(z_{xx} - w_{xx}) + (\lambda\lambda_x w_x - \lambda_t w_x)_x\Big) \\ &= -\theta\beta(\mu^2 z_{xx} + \lambda^2 w_{xx})(z_{tx} - w_{tx}) - \frac{\theta\beta^2}{2}(\mu - \lambda)(z_x - w_x)(z_{tx} - w_{tx}) \\ &+ \theta\beta(\mu^3 z_{xx} + \lambda^3 w_{xx})(z_{xx} - w_{xx}) - \frac{\theta\beta^2}{2}(\mu - \lambda)(z_t - w_t)(z_{xx} - w_{xx}) \\ &+ \frac{\theta\beta^2}{2}(\mu^2 - \lambda^2)(z_x - w_x)(z_{xx} - w_{xx}) + \frac{R_9}{\lambda^2\mu^2} \\ &=: I_5 + I_6 + I_7 + I_8 + I_9 + \frac{R_9}{\lambda^2\mu^2}.\end{aligned} \tag{4.65}$$

From (4.30), (4.40) and Young's inequality we have

$$\begin{aligned}I_5 &= \frac{-\theta_c\beta C_e^2}{16}(z_{xx} + w_{xx})\left(-\frac{C_e}{4}(z_{xx} + w_{xx}) - \beta(z_x - w_x)\right) + R_{10} \\ &= \frac{\theta_c\beta C_e^3}{64}(z_{xx} + w_{xx})^2 + \frac{\theta_c\beta^2 C_e^2}{16}(z_{xx} + w_{xx})(z_x - w_x) + R_{10}\end{aligned} \tag{4.66}$$

$$\begin{aligned}I_6 &= -\frac{\theta_c\beta^2 C_e}{4}(z_x - w_x)\left(-\frac{C_e}{4}(z_{xx} + w_{xx}) - \beta(z_x - w_x)\right) + R_{11} \\ &= \frac{\theta_c\beta^2 C_e^2}{16}(z_{xx} + w_{xx})(z_x - w_x) + \frac{\theta_c\beta^3 C_e}{4}(z_x - w_x)^2 + R_{11}\end{aligned} \tag{4.67}$$

$$
\begin{aligned}
I_7 &= \frac{\theta_c \beta C_e^3}{64}(z_{xx} - w_{xx})^2 + R_{12} && (4.68)\\
I_8 &= -\frac{\theta_c \beta^2 C_e}{8}(z_t - w_t)(z_{xx} - w_{xx}) + R_{13}\\
&= -\frac{\theta_c \beta^2 C_e}{8}(z_{xx} - w_{xx})\left(-\frac{C_e}{4}(z_x + w_x) - \beta(z - w)\right) + R_{13}\\
&\le C_\delta \epsilon (z_{xx} - w_{xx})^2 + C_{\delta,\epsilon}((z_x + w_x)^2 + (z - w)^2) && (4.69)\\
I_9 &= R_{14}. && (4.70)
\end{aligned}
$$

The last equation is due to the fact that the terms in $\mu^2 - \lambda^2$ are of degree at least 1. Therefore from (4.66)–(4.70) we have

$$
\begin{aligned}
\psi_{2x} - \phi_{2x} \le{}& \left(\frac{\theta_c \beta C_e^3}{64} + C_\delta \epsilon\right)(z_{xx} - w_{xx})^2 + \frac{\theta_c \beta C_e^3}{64}(z_{xx} + w_{xx})^2 && (4.71)\\
&+ \frac{\theta_c \beta^2 C_e^2}{8}(z_{xx} + w_{xx})(z_x - w_x) + C_{\delta,\epsilon}((z_x + w_x)^2 + (z - w)^2)\\
&+ C_\delta (z_x - w_x)^2 + R_{15}.
\end{aligned}
$$

Finally for the last pair in M_2 we use (3.21) and (3.22) to obtain

$$
\begin{aligned}
\psi_{2Z} G_2 - \phi_{2W} F_2 &= \frac{2\theta}{\mu}(\mu^2 z_{xx} + \tilde{N})\mu^2 \left(-\frac{\beta}{2}(z_{xx} - w_{xx}) - 2\mu_x z_x - \mu_{xx} z_x\right)\\
&\quad - \frac{2\theta}{\lambda}(\lambda^2 w_{xx} + \tilde{P})\lambda^2 \left(\frac{\beta}{2}(z_{xx} - w_{xx}) - 2\lambda_x w_x - \lambda_{xx} w_x\right)\\
&= \theta_c \beta(-\mu^3 z_{xx}(z_{xx} - w_{xx}) - \lambda^3 w_{xx}(z_{xx} - w_{xx})) + R_{16}\\
&= -\frac{\theta_c \beta C_e^3}{64}(z_{xx} - w_{xx})^2 + R_{17} && (4.72)\\
&=: I_{10} + R_{17}.
\end{aligned}
$$

Adding (4.63), (4.64), (4.71), (4.72), choosing $\epsilon > 0$ small enough so that $\tilde{C}_2 = \frac{\theta_c \beta C_e^3}{64} - C_\delta \epsilon > 0$, where the first term is independent of δ and ϵ, using the Sobolev embedding for the terms R_i and finally invoking (3.11) yield

$$
\begin{aligned}
&\int_0^t \int_0^\ell M_2(\psi_2, \phi_2)\,\mathrm{d}x\,\mathrm{d}\tau\\
&\le -\tilde{C}_2 \int_0^t \|u_{xx}(\tau)\|_{L^2}^2\,\mathrm{d}\tau + C_\delta \Bigg(\int_0^t \|u(\tau)\|_{H^1}^2\,\mathrm{d}\tau + \int_0^t \|(A^{\frac{1}{4}})_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau\\
&\quad + \sup_{\tau \in [0,t]} \|V(\tau)\|_{H^2} \int_0^t \|V_x(\tau)\|_{H^1}^2 + \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau\Bigg). && (4.73)
\end{aligned}
$$

Step 3. The estimate (4.42) immediately follows from (4.48), (4.57), (4.73), Lemmas 4.1–4.4, (4.2) and by choosing $\vartheta > 0$ in Proposition 4.2 small enough. □

As in the case of first order estimates, we shall also need the following estimate in order to complete an estimate for the full energy functional N_2.

LEMMA 4.7. There exist $\delta > 0$ and $C_\delta > 0$ such that for any solution $(A, u, h_0, h_\ell) \in X_T$ satisfying $N_2^2(T) \leq \delta$ it holds that

$$\begin{aligned} \int_0^t \|(A^{\frac{1}{4}})_{xx}(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \;&\leq\; C_\delta N_2^2(0) \\ &+ C_\delta \sup_{\tau\in[0,t]} (\|u(\tau)\|_{H^2} + \|A^{\frac{1}{4}}(\tau) - A_e^{\frac{1}{4}}\|_{H^2}) \int_0^t \|u(\tau)\|_{H^2}^2 + \|(A^{\frac{1}{4}})_x(\tau)\|_{H^1}^2 \,\mathrm{d}\tau \end{aligned} \tag{4.74}$$

for all $t \in [0, T]$.

Proof. We modify the entropy of the previous lemma. We consider the entropy $\tilde{\eta}_2 = \tilde{\psi}_2 - \tilde{\phi}_2$ with corresponding entropy flux $\tilde{q}_2 = \mu\tilde{\psi}_2 - \lambda\tilde{\phi}_2$ where

$$\begin{aligned} \tilde{\psi}_2(t, x, Z) \;&=\; \frac{\theta}{\mu}\left(\mu^2 Z + \frac{\beta\mu}{2}(z_x - w_x) + \mu\mu_x z_x - \mu_t z_x\right)^2 \\ \tilde{\phi}_2(t, x, W) \;&=\; \frac{\theta}{\lambda}\left(\lambda^2 W - \frac{\beta\lambda}{2}(z_x - w_x) + \lambda\lambda_x w_x - \lambda_t w_x\right)^2. \end{aligned}$$

Doing the same process as in the first step of Lemma 4.6 we can show that

$$\int_0^\ell \tilde{\eta}_2(t, x) - \tilde{\eta}_2(0, x) \,\mathrm{d}x \geq C_\delta(\|V_{xx}(\tau)\|_{L^2}^2 - \|V(0)\|_{H^2}^2). \tag{4.75}$$

Using (4.39) and (4.38), a simple computation gives us

$$\begin{aligned} \tilde{q}_2 \;&=\; \theta(w, z)((z_{tt} + (\beta/2)(z_t - w_t))^2 - (w_{tt} - (\beta/2)(z_t - w_t))^2) \\ &=\; \theta(w, z)(z_{tt}^2 - w_{tt}^2) + \beta\theta(w, z)(z_{tt} + w_{tt})(z_t - w_t) \\ &=\; q_2 + \beta\theta(w, z)p_3(w, z, w_x, z_x)p_1(w, z, w_x, z_x) \end{aligned} \tag{4.76}$$

where q_2 is the entropy flux in the previous lemma and p_1 and p_3 are defined by (4.51) and (4.54), respectively. A straightforward calculation gives

$$\begin{aligned} p_3(w, z, w_x, z_x)p_1(w, z, w_x, z_x) \;&=\; -\theta(w, z)p_1^2(w, z, w_x, z_x) + \hat{R}_3 \\ &\geq\; -\, C_\delta(z_x + w_x)^2 - C_\delta(z - w)^2 + \hat{R}_4 \end{aligned}$$

where $\hat{R}_3$ and $\hat{R}_4$ are terms of degree at least 3 and contain at least two factors among $z - w, w_x, z_x$. By the estimate Proposition 4.2 and (4.2) we have

$$\begin{aligned} &\int_0^t \beta(\theta p_3 p_1)(\tau, \ell) - \beta(\theta p_3 p_1)(\tau, 0) \,\mathrm{d}\tau \\ &\geq -\, C_\delta\vartheta \int_0^t \|(A^{\frac{1}{4}})_{xx}(\tau)\|_{L^2}^2 \,\mathrm{d}\tau - C_{\delta,\vartheta} \int_0^t \|(A^{\frac{1}{4}})_x(\tau)\|_{L^2}^2 + \|u(\tau)\|_{H^1}^2 \,\mathrm{d}\tau \\ &-\, C_\delta \sup_{\tau\in[0,t]} \|V(\tau)\|_{H^2} \int_0^t \|V_x(\tau)\|_{H^1}^2 + \|u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau. \end{aligned} \tag{4.77}$$

Integrating (4.76) from 0 to t and using (4.57) and (4.77) we have

$$\begin{aligned}\int_0^t \tilde{q}_2(\tau,\ell) - \tilde{q}_2(\tau,0)\,\mathrm{d}\tau \geq &- C_\delta\vartheta\|V_{xx}(t)\|_{L^2}^2 - C_{\delta,\vartheta}\|V(t)\|_{H^1}^2 - C_\delta\|V(0)\|_{H^2}^2\\ &- C_\delta\vartheta\int_0^t \|(A^{\frac{1}{4}})_{xx}(\tau)\|_{L^2}^2\,\mathrm{d}\tau - C_{\delta,\vartheta}\int_0^t \|(A^{\frac{1}{4}})_x(\tau)\|_{L^2}^2 + \|u(\tau)\|_{H^1}^2\,\mathrm{d}\tau\\ &- C_\delta \sup_{\tau\in[0,t]}\|V(\tau)\|_{H^2}\int_0^t \|V_x(\tau)\|_{H^1}^2 + \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau. \end{aligned} \tag{4.78}$$

Observe that the deviation of ψ_2 and ϕ_2 from $\tilde{\psi}_2$ and $\tilde{\phi}_2$, respectively, is that the former terms contain $\frac{\beta}{2}(z_t - w_t)$ while the latter terms do not. This means that $\tilde{M}_2$ will consist of the same terms as M_2 but without those that stem from $\frac{\beta}{2}(z_t - w_t)$. Thus, crossing out the terms that appear due to the said extra term, a careful analysis in the second step of the proof of Lemma 4.6 shows that

$$\tilde{M}_2 = I_3 + I_4 + I_7 + I_9 + I_{10} + \frac{R_{18}}{\lambda^2\mu^2}$$

where R_{18} is again terms of degree at least 3 containing at least two factors among $z - w, w_x, z_x, z_{xx}, w_{xx}$ and containing at most two among z_{xx}, w_{xx}. Therefore we have, according to Young's inequality,

$$\begin{aligned}\tilde{M}_2 \;&\leq\; -\frac{\theta_c\beta C_e^2}{64}(z_{xx}+w_{xx})^2 - \frac{\theta_c\beta^2 C_e^2}{8}(z_{xx}+w_{xx})(z_x - w_x) + R_{19}\\ &\leq\; -\tilde{C}_3(z_{xx}+w_{xx})^2 + C(z_x - w_x)^2 + R_{19}\end{aligned}$$

for some $\tilde{C}_3 > 0$. With the same explanations as above we have

$$\begin{aligned}\int_0^t\int_0^\ell \tilde{M}_2(\tilde{\psi}_2,\tilde{\phi}_2)\,\mathrm{d}x\,\mathrm{d}\tau \leq -\tilde{C}_3\int_0^t \|(A^{\frac{1}{4}})_{xx}(\tau)\|_{L^2}^2\,\mathrm{d}\tau \\ + C_\delta\Big(\int_0^t \|u_x(\tau)\|_{L^2}^2\,\mathrm{d}\tau + \sup_{\tau\in[0,t]}\|V(\tau)\|_{H^2}\int_0^t \|V_x(\tau)\|_{H^1}^2 + \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau\Big).\end{aligned} \tag{4.79}$$

From (4.75), (4.78), (4.79), choosing $\vartheta > 0$ in Proposition 4.2 small enough and using Lemmas 4.1–4.6, the estimate (4.74) follows. □

5. Proof of the global existence and stability in $H^1 \times H^1 \times \mathbb{R}^2$. An immediate consequence of the results in the previous section is the following estimate for the energy N_2.

COROLLARY 5.1. Let $T > 0$ be such that (1.1) has a solution that belongs to X_T. Then there is a $\delta > 0$ such that if $N_2^2(T) \leq \delta$, then $N_2^2(t) \leq C_\delta(N_2^2(0) + N_2^3(t))$ for all $t \in [0,T]$ and for some $C_\delta > 0$ independent of T. In particular, there exists a $\delta > 0$ such that if $N_2^2(T) \leq \delta$, then $N_2^2(T) \leq \tilde{C}_\delta N_2^2(0)$ for some $\tilde{C}_\delta > 0$ independent of T.

Proof. According to Lemmas 4.1, 4.3–4.7, there is a $\delta > 0$ such that $N_2^2(t) \leq C_\delta(N_2^2(0) + N_2^3(t))$ for all $t \in [0,T]$ whenever $N_2^2(T) \leq \delta$. In particular, $N_2^2(T) \leq C_\delta(N_2^2(0) + \sqrt{\delta}N_2^2(T))$. Since (4.1) holds, one may choose $\delta > 0$ small enough so that $\tilde{C}_\delta := C_\delta(1 - C_\delta\sqrt{\delta})^{-1} > 0$ and thus $N_2^2(T) \leq \tilde{C}_\delta N_2^2(0)$. □

Proof of Theorem 2.2. The proof is standard; however, we include it here for completeness. According to Corollary 5.1 we have a $\delta > 0$ such that $N_2^2(T) \le \tilde{C}_\delta N_2^2(0)$ for some $\tilde{C}_\delta > 0$ whenever $N_2^2(T) \le \delta$. Take $\delta_0 = \min(\delta/(2\tilde{C}_\delta), \delta/4) > 0$. Suppose that the maximal time of existence $T^* > 0$ is finite. Then either (A, u) leaves every compact subset of $\mathcal{U}$ or $\|(A_x, u_x)(t)\|_{L^\infty[0,\ell]} \to \infty$ as $t \uparrow T^*$. Classical embedding results imply that

$$\|(A, u) - (A_e, 0)\|_{W^{1,\infty}([0,t]\times[0,\ell])^2} \le C_\delta N_2(t).$$

In any case, by continuity there exists $0 < T_1 < T^*$ such that $N_2^2(T_1) = \frac{\delta}{2}$ and $N_2^2(t) > \frac{\delta}{2}$ for all $t \in (T_1, T_1 + \epsilon)$ where $\epsilon > 0$ and $T_1 + \epsilon < T^*$. Because $N_2^2(T_1) < \delta$, there exists $T_2 \in (T_1, T_1 + \epsilon)$ satisfying $N_2^2(T_2) \le \delta$. Corollary 5.1 implies that $N_2^2(T_2) \le \tilde{C}_\delta N_2^2(0) \le \frac{\delta}{2}$, which is a contradiction. Therefore we must have $T^* = +\infty$, and this proves that a global-in-time solution exists. Furthermore, we have the estimate $N_2^2(t) \le \tilde{C}_\delta N_2^2(0)$ for all $t \ge 0$. □

By applying the PDEs, the estimate in Theorem 2.2 implies the following estimate on the time-derivatives of the state.

COROLLARY 5.2. In the situation of Theorem 2.2, there exists a $C_\delta > 0$ such that

$$\begin{aligned}&\sup_{t\ge 0} \left(\|A_t(t)\|_{H^1}^2 + \|A_{tt}(t)\|_{L^2}^2 + \|u_t(t)\|_{H^1}^2 + \|u_{tt}(t)\|_{L^2}^2\right)\\ &+ \int_0^\infty \left(\|A_\tau(\tau)\|_{H^1}^2 + \|A_{\tau\tau}(\tau)\|_{L^2}^2 + \|u_\tau(\tau)\|_{H^1}^2 + \|u_{\tau\tau}(\tau)\|_{L^2}^2 \,\mathrm{d}\tau\right) \le C_\delta E_0.\end{aligned}$$

Now we are ready to prove the following asymptotic behaviour of the solutions.

THEOREM 5.3 (Asymptotic stability). In the framework of Theorem 2.2 we have

$$\lim_{t\to\infty} \left(\|A(t) - A_e\|_{H^1(0,\ell)} + \|u(t)\|_{H^1(0,\ell)} + |h_0(t) - h_{0e}| + |h_\ell(t) - h_{\ell e}|\right) = 0. \tag{5.1}$$

Proof. As functions of time $\|u(\cdot)\|_{H^1(0,\ell)}^2$ and $\|A_x(\cdot)\|_{L^2(0,\ell)}^2$ belong to $W^{1,1}(0,\infty)$ according to Theorem 2.2 and Corollary 5.1 . Hence

$$\lim_{t\to\infty} \left(\|u(t)\|_{H^1(0,\ell)} + \|A_x(t)\|_{L^2(0,\ell)}\right) = 0. \tag{5.2}$$

Using a Gagliardo-Nirenberg-Moser interpolation (see [22]) we have

$$\|A(t) - A_e\|_{L^\infty(0,\ell)} \le C_\ell \|\partial_x A(t)\|_{L^2(0,\ell)}^{1/2} \|A(t) - A_e\|_{L^2(0,\ell)}^{1/2}.$$

Theorem 2.2 implies that $\|A(t) - A_e\|_{L^2(0,\ell)}$ is uniformly bounded in $t \in [0,\infty)$, and thus from (5.2) we get $\|A(t) - A_e\|_{L^\infty(0,\ell)} \to 0$ as $t \to \infty$. In particular, this implies that $\|A(t) - A_e\|_{L^2(0,\ell)} \to 0$, $A(t,0) \to A_e$ and $A(t,\ell) \to A_e$ as $t \to \infty$. The latter two further imply that $h_0(t) \to h_{0e}$ and $h_\ell(t) \to h_{\ell e}$ as $t \to \infty$. Combining these with (5.2) we obtain (5.1). □

The decay rate at which the state converges to the equilibrium can be shown to be exponential, however, if one uses the norm in $L^2(0,\ell)^2 \times \mathbb{R}^2$. This is the goal of the next section.

6. Exponential convergence to the equilibrium in $L^2(0,\ell)^2 \times \mathbb{R}^2$. The exponential stability result for (1.1) is based on linear stability and treating the higher order terms as perturbation of the linearized system. The basic ingredients are the exponential stability derived from semigroup theory, the variation of parameters formula and interpolation estimates. However, care should be taken since the linearization yields a nontrivial kernel, and therefore stability for the linearized problem is only possible in a factor space. The smallness of the data and the order of nonlinearity play an important role in the proof, specifically the applicability of a Gronwall-type estimate. In this way the decay rate for the nonlinear system is the same as the decay rate for the linearized system.

First, we revisit the stability result in [18]. Define the following constants:

$$\alpha = \frac{\kappa^2}{\sqrt{A_e}}, \qquad \gamma = 2b(a_0 + bh_{0e}) = 2b(a_\ell + bh_{\ell e}).$$

Let $\mathcal{X} = L^2(0,\ell)^2 \times \mathbb{R}^2$ be equipped with the weighted norm

$$\|(A,u,h_0,h_\ell)\|_{\mathcal{X}}^2 = \frac{1}{A_e}\|A\|_{L^2(0,\ell)}^2 + \frac{1}{\alpha}\|u\|_{L^2(0,\ell)}^2 + \frac{\gamma A_T}{A_e}(|h_0|^2 + |h_\ell|^2).$$

Consider the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ with domain $\mathcal{D}(\mathcal{A}) = \{(A,u,h_0,h_\ell) \in H^1(0,\ell)^2 \times \mathbb{R}^2 : A(0) = \gamma h_0, A(\ell) = \gamma h_\ell\}$ defined by

$$\mathcal{A}\begin{pmatrix} A \\ u \\ h_0 \\ h_\ell \end{pmatrix} = \begin{pmatrix} -A_e u_x \\ -\alpha A_x - \beta u \\ -\frac{A_e}{A_T}u(0) \\ \frac{A_e}{A_T}u(\ell) \end{pmatrix}.$$

This operator is obtained by linearizing the system (1.1) including its boundary conditions about the equilibrium state $(A_e, 0, h_{0e}, h_{\ell e})$. The operator $\mathcal{A}$ has a nontrivial kernel $\mathcal{N}(\mathcal{A}) = \{c(\gamma,0,1,1) : c \in \mathbb{R}\}$. The orthogonal complement $\mathcal{N}(\mathcal{A})^\perp$ of $\mathcal{N}(\mathcal{A})$ coincides with the kernel of the *volume functional* $\mathcal{V} : \mathcal{X} \to \mathbb{R}$,

$$\mathcal{V}(A,u,h_0,h_\ell) = \int_0^\ell A(x)\,\mathrm{d}x + A_T h_0 + A_T h_\ell.$$

In the following theorem $\sigma(\mathcal{A})$ will denote the spectrum of $\mathcal{A}$, which consists of eigenvalues since the operator is discrete. For the proof and explicit values of σ and k we refer to [18].

THEOREM 6.1. The operator $\mathcal{A}$ is a discrete spectral operator that generates a strongly continuous group $T(t)$, $t \in \mathbb{R}$, on $\mathcal{X}$. If $\beta > 0$, then there exists $M \geq 1$ such that

$$\|T(t)\|_{\mathcal{L}(\mathcal{N}(\mathcal{A})^\perp)} \leq M(1+t^k)e^{-\sigma t}, \qquad t \geq 0,$$

where $\sigma = -\sup_{\lambda\in\sigma(\mathcal{A})} \Re\lambda > 0$ and k is either 0 or 1.

To use this result for the nonlinear system (1.1), we need further tools. The first one is the following Gronwall-type lemma, whose proof can be found in [6].

LEMMA 6.2. Let $u \in \mathrm{Lip}([0,\infty),\mathbb{R}_+)$ and suppose that for some $C > 0$,

$$u(t) \le C(1+t^k)e^{-\sigma t}u(0) + C\int_0^t (1+(t-s)^k)e^{-\sigma(t-s)}u(s)^{\varrho}\,\mathrm{d}s, \quad t \ge 0,$$

for some $\sigma > 0$, $\varrho > 1$ and nonnegative integer k. Then there exist $\epsilon > 0$ and $C > 0$ such that if $u(0) < \epsilon$, then

$$u(t) \le C(1+t^k)e^{-t\sigma}, \qquad t \ge 0.$$

The next tool is a simple interpolation estimate derived from the well-known Gagliardo-Nirenberg inequality; see [22] for example.

THEOREM 6.3 (Gagliardo-Nirenberg). Let m be a positive integer. There exists $C_\ell > 0$ such that for all $u \in H^m(0,\ell)$ and $j \le m$ we have

$$\|u^{(j)}\|_{L^{2m/j}(0,\ell)} \le C_\ell \|u\|^{1-j/m}_{L^\infty(0,\ell)}\|u\|^{j/m}_{H^m(0,\ell)}.$$

As a consequence, we have the following estimate.

COROLLARY 6.4. There exists $C > 0$ such that for all $u \in H^2(0,\ell)$ it holds that

$$\|u_x\|_{L^\infty(0,\ell)} \le C\|u\|^{7/8}_{H^2(0,\ell)}\|u\|^{1/8}_{L^2(0,\ell)}.$$

Proof. Using the Gagliardo-Nirenberg-Moser estimate in [22], Hölder's inequality and Theorem 6.3 with $m = 2$ and $j = 1$ we have, for generic constants $C > 0$,

$$\begin{aligned}
\|u_x\|_{L^\infty(0,\ell)} &\le C\|u_{xx}\|^{1/2}_{L^2(0,\ell)}\|u_x\|^{1/2}_{L^2(0,\ell)} \\
&\le C\|u_{xx}\|^{1/2}_{L^2(0,\ell)}\|u_x\|^{1/2}_{L^4(0,\ell)} \\
&\le C\|u_{xx}\|^{1/2}_{L^2(0,\ell)}(\|u\|^{1/2}_{L^\infty(0,\ell)}\|u\|^{1/2}_{H^2(0,\ell)})^{1/2} \\
&\le C\|u_{xx}\|^{1/2}_{L^2(0,\ell)}(\|u_x\|^{1/4}_{L^2(0,\ell)}\|u\|^{1/4}_{L^2(0,\ell)}\|u\|^{1/2}_{H^2(0,\ell)})^{1/2}.
\end{aligned}$$

This clearly implies the estimate given in the corollary. □

Now we are in position to prove the following stability result.

THEOREM 6.5 (Exponential stability). Consider the framework of Theorem 2.2. There exists $\delta_0 > 0$ such that if $E_0 \le \delta_0$, then the solution of (1.1) satisfies

$$\|A(t) - A_e\|_{L^2(0,\ell)} + \|u(t)\|_{L^2(0,\ell)} + |h_0(t) - h_{0e}| + |h_\ell(t) - h_{\ell e}| \le C(1+t^k)e^{-\sigma t}$$

for all $t \ge 0$ and for some constant $C = C(E_0) > 0$. The constants k and σ are those of Theorem 6.1.

Proof. Let $z = (B, v, \eta_0, \eta_\ell) = (A - A_e, u, h_0 - h_{0e}, h_\ell - h_{\ell e})$ denote the deviation of the state from the equilibrium. The system (1.1) can be rewritten in terms of the deviations

as

$$\begin{cases} B_t = -A_e v_x - (A - A_e)u_x - uA_x, \\ v_t = -\alpha B_x - \beta v + \alpha A^{-\frac{1}{2}}(A^{\frac{1}{2}} + A_e^{\frac{1}{2}})^{-1}(A - A_e)A_x - uu_x, \\ \eta_0'(t) = -\frac{A_e}{A_T}v(t,0) - \frac{1}{A_T}(A(t,0) - A_e)u(t,0), \\ \eta_\ell'(t) = \frac{A_e}{A_T}v(t,\ell) + \frac{1}{A_T}(A(t,\ell) - A_e)u(t,\ell), \\ B(t,0) = \gamma\eta_0(t) + b^2(h_0(t) - h_{0e})^2, \\ B(t,\ell) = \gamma\eta(t) + b^2(h(t) - h_{\ell e})^2. \end{cases}$$

In order to use the results for abstract homogeneous linear time-invariant systems via semigroup theory, we consider a new state variable $w := z - (\phi, 0, 0, 0)$ where

$$\phi(t,x) = \frac{\ell - x}{\ell}b^2(h_0(t) - h_{0e})^2 + \frac{x}{\ell}b^2(h_\ell(t) - h_{\ell e})^2.$$

This is introduced in order to compensate for the nonlinearity in the boundary conditions. It is easy to see that $w(t) \in \mathcal{D}(\mathcal{A})$ for all $t \geq 0$ and it satisfies the system

$$\dot{w}(t) = \mathcal{A}w(t) + F(t), \qquad t > 0, \tag{6.1}$$

where

$$F(t) = \begin{pmatrix} -(A(t) - A_e)u_x(t) - u(t)A_x(t) - \phi_t(t) \\ \alpha A(t)^{-\frac{1}{2}}(A(t)^{\frac{1}{2}} + A_e^{\frac{1}{2}})^{-1}(A(t) - A_e)A_x(t) - u(t)u_x(t) - \alpha\phi_x(t) \\ -\frac{1}{A_T}(A(t,0) - A_e)u(t,0) \\ \frac{1}{A_T}(A(t,\ell) - A_e)u(t,\ell) \end{pmatrix}.$$

Because $u \in C^1([0,\infty); H^1(0,\ell))$ it follows that $uu_x \in C^1([0,\infty); L^2(0,\ell))$. Using the regularity of A, u, h_0 and h_ℓ stated in Theorem 2.2 together with a similar argument as in the previous statement, one can show that $F \in C^1([0,\infty); \mathcal{X})$. A standard result in semigroup theory (see [16, Section 4.2] for example) shows that (6.1) has a unique solution in $\mathcal{X}$ and it is given by the variation of parameters formula

$$w(t) = T(t)w(0) + \int_0^t T(t-s)F(s)\,\mathrm{d}s. \tag{6.2}$$

By uniqueness, this function w must coincide with the function $z - (\phi, 0, 0, 0)$ above.

Since the semigroup $T(t)$ is exponentially stable only in $\mathcal{N}(\mathcal{A})^\perp$, we will decompose the solution w into two parts. First decompose F as a sum $F = F_1 + (F_2)_t$ where $F_2 = (-\phi, 0, 0, 0)$. By construction, $F_1(s) \in \mathcal{N}(\mathcal{A})^\perp$ for all $s \geq 0$. This can be easily seen since $F_1(s)$ lies in the kernel of $\mathcal{V}$ for all $s \geq 0$. Let $\Pi : \mathcal{X} \to \mathcal{N}(\mathcal{A})$ be the orthogonal projection of $\mathcal{X}$ onto $\mathcal{N}(\mathcal{A})$. Conservation of volume implies that $\mathcal{V}(A_0^0, u^0, h_0^0, h_\ell^0) = \mathcal{V}(A_e, 0, h_{0e}, h_{\ell e})$ or equivalently $z(0) \in \mathcal{N}(\mathcal{A})^\perp$. Furthermore, we have $F_1(s) + (I - \Pi)(F_2)_t(s) \in \mathcal{N}(\mathcal{A})^\perp$ for all $s \geq 0$. We write

$$w(t) = w_1(t) + w_2(t)$$

where

$$w_1(t) = T(t)(z(0) + (I - \Pi)F_2(0)) + \int_0^t T(t-s)(F_1(s) + (I-\Pi)(F_2)_t(s))\,\mathrm{d}s$$
$$w_2(t) = T(t)\Pi F_2(0) + \int_0^t T(t-s)\Pi(F_2)_t(s)\,\mathrm{d}s.$$

Because $T(t)\Pi = \Pi$ and $\Pi(F_2)_t(s) = (\Pi F_2(s))_t$ we actually have $w_2(t) = \Pi F_2(t)$. Using (6.2) and Theorem 6.1 we have

$$\begin{aligned}\|w(t)\|_{\mathcal{X}} \;\le\; & M(1+t^k)e^{-\sigma t}\|z(0) + (I-\Pi)F_2(0)\|_{\mathcal{X}} + \|\Pi F_2(t)\|_{\mathcal{X}} \\ & + M\int_0^t (1+(t-s)^k)e^{-\sigma(t-s)}\|F_1(s) + (I-\Pi)(F_2)_t(s)\|_{\mathcal{X}}\,\mathrm{d}s. \qquad (6.3)\end{aligned}$$

The next task is to estimate each term of (6.3) in terms of the norm $\|z(t)\|_{\mathcal{X}}$ of the deviation $z(t)$. Since $\|I-\Pi\|_{\mathcal{L}(\mathcal{X})} \le 1$ it holds that for all $t \ge 0$,

$$\|(I-\Pi)F_2(t)\|_{\mathcal{X}} \le C\|\phi(t)\|_{L^2(0,\ell)} \le C\|z(t)\|_{\mathcal{X}}^2 \le CE_0^{1/2}\|z(t)\|_{\mathcal{X}} \qquad (6.4)$$

for some $C > 0$ independent of E_0. Similarly, for all $t \ge 0$,

$$\|w(t)\|_{\mathcal{X}} = \|z(t) + F_2(t)\|_{\mathcal{X}} \ge (1 - CE_0^{1/2})\|z(t)\|_{\mathcal{X}}. \qquad (6.5)$$

From Corollary 6.4 we obtain

$$\|u(t)u_x(t)\|_{L^2} \le \|u(t)\|_{L^2}\|u_x(t)\|_{L^\infty} \le C\|u(t)\|_{H^2}^{7/8}\|u(t)\|_{L^2}^{9/8} \le CE_0^{7/16}\|z(t)\|_{L^2}^{9/8}.$$

The other terms in the first and second rows of F_1 can be estimated similarly. Now we estimate the third and fourth rows of F_1. By Sobolev embedding we have

$$|(A(t,y) - A_e)u(t,y)| \le C(\|(A(t)-A_e)u(t)\|_{L^2(0,\ell)} + \|[(A(t)-A_e)u(t)]_x\|_{L^2(0,\ell)}),$$

for $y = 0, \ell$. Expanding the term $[(A(t)-A_e)u(t)]_x = A_x(t)u(t) + (A(t)-A_e)u_x(t)$, it can be seen that each term can be estimated in the same manner as we estimated $u(t)u_x(t)$ above. For the first term, we apply the Gagliardo-Nirenberg-Moser interpolation once more to get

$$\begin{aligned}\|(A(t)-A_e)u(t)\|_{L^2(0,\ell)} \;&\le\; \|A(t)-A_e\|_{L^2(0,\ell)}\|u(t)\|_{L^\infty(0,\ell)} \\ &\le\; C\|A(t)-A_e\|_{L^2(0,\ell)}\|u_x(t)\|_{L^2(0,\ell)}^{1/2}\|u(t)\|_{L^2(0,\ell)}^{1/2} \\ &\le\; C(E_0)\|z(t)\|_{\mathcal{X}}^{3/2} \;\le\; C(E_0)\|z(t)\|_{\mathcal{X}}^{9/8}.\end{aligned}$$

Combining all of our estimates yields

$$\|F_1(t)\|_{\mathcal{X}} \le C(E_0)\|z(t)\|_{\mathcal{X}}^{9/8}. \qquad (6.6)$$

The next step is to estimate $\|(1-\Pi)(F_2)_t(t)\|_{\mathcal{X}}$. Using the differential boundary conditions, the derivative of ϕ with respect to t is given by

$$\begin{aligned}\phi_t(t,x) \;=\; & -\,2A_T b^2\ell^{-1}(\ell - x)(h_0(t) - h_{0e})A(t,0)u(t,0) \\ & +\,2A_T b^2\ell^{-1}x(h_\ell(t) - h_{\ell e})A(t,\ell)u(t,\ell)\end{aligned}$$

and by interpolation we can estimate its L^2-norm by

$$\begin{aligned}\|\phi_t(t)\|_{L^2(0,\ell)} &\le C(|h_0(t)-h_{0e}|+|h_\ell(t)-h_{\ell e}|)\|A(t)\|_{L^\infty(0,\ell)}\|u(t)\|_{L^\infty(0,\ell)}\\ &\le CE_0^{1/2}(|h_0(t)-h_{0e}|+|h_\ell(t)-h_{\ell e}|)\|A(t)\|^{1/2}_{L^2(0,\ell)}\|u(t)\|^{1/2}_{L^2(0,\ell)}\\ &\le C(E_0)\|z(t)\|^{9/8}_{\mathcal{X}}.\end{aligned}$$

Consequently,

$$\|(1-\Pi)(F_2)_t(t)\|_{\mathcal{X}} \le C(E_0)\|z(t)\|^{9/8}_{\mathcal{X}}. \tag{6.7}$$

Using (6.4), (6.5), (6.6), (6.7) in (6.3) we have

$$\begin{aligned}\|z(t)\|_{\mathcal{X}} \le \frac{MC(E_0)}{1-CE_0^{1/2}}\Bigg((1+t^k)e^{-\sigma t}\|z(0)\|_{\mathcal{X}}\\ + \int_0^t (1+(t-s)^k)e^{-\sigma(t-s)}\|z(s)\|^{9/8}_{\mathcal{X}}\,\mathrm{d}s\Bigg)\end{aligned} \tag{6.8}$$

whenever $CE_0^{1/2} \le C\delta_0^{1/2} < 1$.

Finally, we check the Lipschitz continuity of the map $t \mapsto \|z(t)\|_{\mathcal{X}}$. From the continuity equation, it holds that

$$\begin{aligned}&|\|A(t)-A_e\|_{L^2(0,\ell)} - \|A(s)-A_e\|_{L^2(0,\ell)}|\\ &\quad\le \|A(t)-A(s)\|_{L^2(0,\ell)}\\ &\quad\le \left\|\int_t^s u(\tau)A_x(\tau)+A(\tau)u_x(\tau)\,\mathrm{d}\tau\right\|_{L^2(0,\ell)}\\ &\quad\le |t-s|\max_{\tau\ge 0}\|u(\tau)A_x(\tau)+A(\tau)u_x(\tau)\|_{L^2(0,\ell)}\\ &\quad\le C|t-s|\max_{\tau\ge 0}(\|u(\tau)\|_{H^1(0,\ell)}\|A_x(\tau)\|_{L^2(0,\ell)}+\|A(\tau)\|_{H^1(0,\ell)}\|u_x(\tau)\|_{L^2(0,\ell)})\\ &\quad\le C(E_0)|t-s|\end{aligned}$$

for all $s,t \ge 0$. The same estimate can be obtained for u and h_0, h_ℓ using the momentum equation and the ODE boundary conditions, respectively. Therefore $\|z(\cdot)\|_{\mathcal{X}} \in \mathrm{Lip}([0,\infty),\mathbb{R}_+)$. The result now easily follows from (6.8) and the Gronwall-type estimate Lemma 6.2. □

References

[1] Karine Beauchard and Enrique Zuazua, *Large time asymptotics for partially dissipative hyperbolic systems*, Arch. Ration. Mech. Anal. **199** (2011), no. 1, 177–227, DOI 10.1007/s00205-010-0321-y. MR2754341 (2012e:35146)

[2] Alfio Borzì and Georg Propst, *Numerical investigation of the Liebau phenomenon*, Z. Angew. Math. Phys. **54** (2003), no. 6, 1050–1072, DOI 10.1007/s00033-003-1108-x. MR2022157 (2004k:76138)

[3] Sunčica Čanić and Eun Heui Kim, *Mathematical analysis of the quasilinear effects in a hyperbolic model blood flow through compliant axi-symmetric vessels*, Math. Methods Appl. Sci. **26** (2003), no. 14, 1161–1186, DOI 10.1002/mma.407. MR2002976 (2004g:76139)

[4] Jean-François Coulombel and Thierry Goudon, *The strong relaxation limit of the multidimensional isothermal Euler equations*, Trans. Amer. Math. Soc. **359** (2007), no. 2, 637–648 (electronic), DOI 10.1090/S0002-9947-06-04028-1. MR2255190 (2007h:35215)

[5] Constantine M. Dafermos, *Hyperbolic conservation laws in continuum physics*, 3rd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 325, Springer-Verlag, Berlin, 2010. MR2574377 (2011i:35150)

[6] Klemens Fellner and Gaël Raoul, *Stability of stationary states of non-local equations with singular interaction potentials*, Math. Comput. Modelling **53** (2011), no. 7-8, 1436–1450, DOI 10.1016/j.mcm.2010.03.021. MR2782822

[7] Miguel Ángel Fernández, Vuk Milišić, and Alfio Quarteroni, *Analysis of a geometrical multiscale blood flow model based on the coupling of ODEs and hyperbolic PDEs*, Multiscale Model. Simul. **4** (2005), no. 1, 215–236 (electronic), DOI 10.1137/030602010. MR2164715 (2006c:76124)

[8] B. Hanouzet and R. Natalini, *Global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy*, Arch. Ration. Mech. Anal. **169** (2003), no. 2, 89–117, DOI 10.1007/s00205-003-0257-6. MR2005637 (2004h:35135)

[9] Shuichi Kawashima, *Large-time behaviour of solutions to hyperbolic-parabolic systems of conservation laws and applications*, Proc. Roy. Soc. Edinburgh Sect. A **106** (1987), no. 1-2, 169–194, DOI 10.1017/S0308210500018308. MR899951 (89d:35022)

[10] Ta Tsien Li, *Global classical solutions for quasilinear hyperbolic systems*, RAM: Research in Applied Mathematics, vol. 32, Masson, Paris; John Wiley & Sons, Ltd., Chichester, 1994. MR1291392 (95m:35115)

[11] Tatsien Li and Libin Wang, *Global propagation of regular nonlinear hyperbolic waves*, Progress in Nonlinear Differential Equations and their Applications, 76, Birkhäuser Boston, Inc., Boston, MA, 2009. MR2524566 (2012b:35002)

[12] P.-L. Lions, B. Perthame, and E. Tadmor, *Kinetic formulation of the isentropic gas dynamics and p-systems*, Comm. Math. Phys. **163** (1994), no. 2, 415–431. MR1284790 (95c:76093)

[13] Hailiang Liu, Jinghua Wang, and Tong Yang, *Stability of a relaxation model with a nonconvex flux*, SIAM J. Math. Anal. **29** (1998), no. 1, 18–29, DOI 10.1137/S003614109629903X. MR1617172 (99k:39055)

[14] Takaaki Nishida, *Nonlinear hyperbolic equations and related topics in fluid dynamics*, Publications Mathématiques d'Orsay, No. 78-02, Département de Mathématique, Université de Paris-Sud, Orsay, 1978. MR0481578 (58 #1690)

[15] J. T. Ottesen, *Valveless pumping in a fluid-filled closed elastic tube-system: one-dimensional theory with experimental validation*, J. Math. Biol. **46** (2003), no. 4, 309–332, DOI 10.1007/s00285-002-0179-1. MR1980846 (2004d:76044)

[16] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983. MR710486 (85g:47061)

[17] Gilbert Peralta and Georg Propst, *Local well-posedness of a class of hyperbolic PDE-ODE systems on a bounded interval*, J. Hyperbolic Differ. Equ. **11** (2014), no. 4, 705–747, DOI 10.1142/S0219891614500222. MR3312050

[18] Gilbert Peralta and Georg Propst, *Stability and boundary controllability of a linearized model of flow in an elastic tube*, ESAIM Control Optim. Calc. Var. **21** (2015), no. 2, 583–601. MR3348413

[19] H. J. Rath and I. Teipel, *Der Fördereffekt in ventillosen, elastischen Leitungen*, Z. Angew. Math. Phys. **29** (1978), 123–133.

[20] Weihua Ruan, M. E. Clark, Meide Zhao, and Anthony Curcio, *Global solution to a hyperbolic problem arising in the modeling of blood flow in circulatory systems*, J. Math. Anal. Appl. **331** (2007), no. 2, 1068–1092, DOI 10.1016/j.jmaa.2006.09.034. MR2313701 (2008c:35175)

[21] Yasushi Shizuta and Shuichi Kawashima, *Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation*, Hokkaido Math. J. **14** (1985), no. 2, 249–275, DOI 10.14492/hokmj/1381757663. MR798756 (86k:35107)

[22] Michael E. Taylor, *Partial differential equations. III*, Nonlinear equations. Applied Mathematical Sciences, vol. 117, Springer-Verlag, New York, 1997. MR1477408 (98k:35001)

[23] Wen-An Yong, *Entropy and global existence for hyperbolic balance laws*, Arch. Ration. Mech. Anal. **172** (2004), no. 2, 247–266, DOI 10.1007/s00205-003-0304-3. MR2058165 (2005c:35195)