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A MAXIMUM PRINCIPLE FOR FRACTIONAL DIFFUSION DIFFERENTIAL EQUATIONS

By

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Abstract. A weak maximum principle is established for a fractional diffusion equation involving the Riemann-Liouville fractional derivative. As applications, it is used to prove the uniqueness and the continuous dependence of a solution on the initial data.

1. Introduction.

In recent years, problems involving partial differential equations of fractional orders have been used for modeling in engineering, science, economics and other fields (cf. Chechkin, Gorenflo and Sokolov [2], Gorenflo and Mainmardi [3], and Podlubny [8]). In particular, problems of thermal diffusion with subdiffusive and super-diffusive properties are formulated in terms of fractional diffusion equations (cf. Kirk and Olmstead [4], and Olmstead and Roberts [7]). To investigate this type of problems, techniques (such as fixed point theorems, and the method of lower and upper solutions) analogous to the classical diffusion equation are used.

Recently, Ahmad, Alsaedi, Kirane and Mostefaoui [1] studied three types of fractional diffusion equations. For each type, they obtained an upper bound of the sup norm in terms of the integral of the solution. Here, we study a different type of fractional diffusion equation. Similar to the weak maximum principle for the classical diffusion equation, we establish a maximum principle for the solution.

Let a, α, and T be positive real numbers with $0 < \alpha < 1$, and $\mathbb{N}$ be the set of natural numbers. Let us consider the following fractional diffusion equation:

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}D_t^{1-\alpha}u(x,t) + F(x,t) \text{ in } (0,a)\times(0,T], \tag{1.1}$$

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subject to the initial and boundary conditions

$$\left.\begin{aligned} &u(x,0)=\phi(x) \text{ on } [0,a],\\ &u(0,t)=\psi_1(t), u(a,t)=\psi_2(t) \text{ for } 0<t\le T, \end{aligned}\right\} \tag{1.2}$$

where the functions F, ϕ, ψ_1 and ψ_2 are continuous such that $\phi(0)=\psi_1(0)$ and $\phi(a)=\psi_2(0)$, and $D_t^{1-\alpha}u(x,t)$ is the Riemann-Liouville fractional derivative defined as follows: Let $g\in C[0,T]$.

(a) $D_t^p g(t)=\dfrac{1}{\Gamma(1-p)}\dfrac{d}{dt}\displaystyle\int_0^t (t-s)^{-p}g(s)ds$ for $0<p<1$;

(b) $D_t^p g(t)=\dfrac{1}{\Gamma(m+1-p)}\dfrac{d^{m+1}}{dt^{m+1}}\displaystyle\int_0^t (t-s)^{m-p}g(s)ds$ for $m\le p<m+1, m\in\mathbb{N}$;

(c) $D_t^{-p} g(t)=\dfrac{1}{\Gamma(p)}\displaystyle\int_0^t (t-s)^{p-1}g(s)ds$ for any $p>0$.

It follows from a direct computation that for any positive integer k, if g is a non-constant function with $g\in C^k$, then the Riemann-Liouville fractional derivatives of order $k\ge 1$ coincide with the conventional derivatives of order k (cf. Podlubny [8, p. 69]). But for non-zero constant K and non-integer p, the derivative of K becomes

$$D_t^p K=\frac{1}{\Gamma(1-p)}Kt^{-p},$$

which is different from the classical derivative. For ease of reference, we state the following theorem, which summarizes some results about fractional differential equations (cf. Podlubny [8, pp. 72-75]).

THEOREM 1.1. (a) For any real p and $m\in\mathbb{N}$ such that $m-1\le p<m$, $\beta>m-1$,

$$D_t^p t^\beta=\frac{\Gamma(\beta+1)}{\Gamma(-p+\beta+1)}t^{\beta-p}.$$

(b) For $m,n\in\mathbb{N}$ and any real p and q such that $m-1\le p<m$, $n-1\le q<n$,

$$D_t^p(D_t^q g(t))=D_t^{q+p}g(t)-\sum_{j=1}^{n}(D_t^{q-j}g(t))\Big|_{t=0}\frac{t^{-p-j}}{\Gamma(1-p-j)};$$

$$D_t^q(D_t^p g(t))=D_t^{p+q}g(t)-\sum_{j=1}^{m}(D_t^{p-j}g(t))\Big|_{t=0}\frac{t^{-q-j}}{\Gamma(1-q-j)}.$$

Hence $D_t^p(D_t^q g(t))=D_t^q(D_t^p g(t))$ only if $D_t^{p-j}g(0)=0$ and $D_t^{q-j}g(0)=0$ for $j=1,2,\ldots,r$ where $r=\max\{m,n\}$.

(c) For any non-negative real numbers p and q, $D_t^p(D_t^{-q}g(t))=D_t^{p-q}g(t)$, and

$$D^{-p}(D_t^q g(t))=D_t^{q-p}g(t)-\sum_{j=1}^{n}(D_t^{q-j}g(t))\Big|_{t=0}\frac{t^{p-j}}{\Gamma(1+p-j)},$$

where $0\le n-1\le q<n$.

The fractional diffusion differential operator given by (1.1) was studied by Olmstead and Roberts [6]. Its Green's function was given explicitly there. To illustrate that a problem of the type (1.1) and (1.2) has a classical solution, let us consider the problem

$$\frac{\partial}{\partial t}u\left(x,t\right)-\frac{\partial^2}{\partial x^2}D_t^{1-\frac{1}{2}}u\left(x,t\right)=\left(2t^{1/2}+\sqrt{\pi}t\right)\sin x \text{ in } \left(0,\pi\right)\times\left(0,T\right],$$
$$u\left(x,0\right)=0 \text{ for } 0\leq x\leq\pi;\; u\left(0,t\right)=0=u\left(\pi,t\right) \text{ for } 0<t\leq T.$$

To verify that it has a classical solution

$$u\left(x,t\right)=\frac{4}{3}t^{3/2}\sin x,$$

we note that it satisfies the initial and boundary conditions

$$\frac{\partial}{\partial t}u\left(x,t\right)=2t^{1/2}\sin x$$

and

$$D_t^{1-\frac{1}{2}}t^{3/2}=\frac{\Gamma\left(\frac{3}{2}+1\right)}{\Gamma\left(-\frac{1}{2}+\frac{3}{2}+1\right)}t^{3/2-1/2}=\Gamma\left(\frac{5}{2}\right)t=\frac{3\sqrt{\pi}}{4}t,$$

which gives

$$\frac{\partial^2}{\partial x^2}D_t^{1-\frac{1}{2}}u\left(x,t\right)=-\sqrt{\pi}t\sin x.$$

In the above example, $\left(2t^{1/2}+\sqrt{\pi}t\right)\sin x>0$ in $\left(0,\pi\right)\times\left(0,T\right]$, and u attains its minimum on the parabolic boundary. Thus, this example also illustrates the maximum principle established in Section 2.

The problem (1.1) and (1.2) having a solution implies $u_t\left(x,t\right)$ exists. Thus for any $0<\alpha<1$, $D_t^{1-\alpha}u\left(x,t\right)$ exists for $t>0$. Hence, a solution $u\left(x,t\right)$ of the problem (1.1) and (1.2) in the region $[0,a]\times[0,T]$ is a (classical) solution in $C\left([0,a]\times[0,T]\right)\cap C^{2,1}\left((0,a)\times(0,T]\right)$.

Maximum principles were given by Al-Refai and Luchko [9, 10] for the types of fractional diffusion equations different from (1.1). For the maximum principles given in [9] to hold, existence of a classical solution (with existence of a continuous u_t on the closed time interval $[0,T]$) is assumed. In [10], the assumption of a solution with existence of a continuous u_t in $(0,T]$ such that $u_t\in L^1\left[0,T\right]$ is made.

2. Main results. Luchko [5] showed that the Riemann-Liouville derivative of a function at its local extreme value may not be zero. The following results give the bound of the derivative of the function at its extreme values.

LEMMA 2.1. Let $g(t)\in C[0,T]$. Assume that $g'(t)$ exists and is continuous for $t\in(0,T]$.

(a) If $g(t)$ attains its maximum value over $[0,T]$ at $t_0\in(0,T]$, then for $0<\alpha<1$,

$$D_t^{1-\alpha}g(t_0)\geq\frac{t_0^{\alpha-1}}{\Gamma(\alpha)}g(t_0).$$

(b) If $g(t)$ attains its minimum value over $[0,T]$ at $t_0\in(0,T]$, then for $0<\alpha<1$,

$$D_t^{1-\alpha}g(t_0)\leq\frac{t_0^{\alpha-1}}{\Gamma(\alpha)}g(t_0).$$

The proof of Lemma 2.1 (a) can be found in that of Theorem 2.1 of Al-Refai and Luchko [9]. By applying a similar argument to $-g(t)$, we obtain Lemma 2.1 (b).

Next, we give the result on the positivity of the solution.

THEOREM 2.2. If $u(x,t)$ satisfies (1.1), $u(x,0) = \phi(x) \geq 0$, $u(0,t) = 0 = u(a,t)$, and $F(x,t) \geq 0$ for $(x,t) \in (0,a) \times (0,T]$, then $u(x,t) \geq 0$ for $(x,t) \in [0,a] \times [0,T]$.

Proof. For any $\epsilon > 0$ and a fixed positive number $\beta < 1-\alpha$, let $v(x,t) = u(x,t) + \epsilon t^{\beta}$. Then $v_t = u_t + \epsilon\beta t^{\beta-1}$, $v(0,t) = v(a,t) = \epsilon t^{\beta} > 0$ for $t > 0$, and $v(x,0) = \phi(x)$ for $x \in [0,a]$. Since

$$D_t^{1-\alpha}v(x,t) = D_t^{1-\alpha}u(x,t) + D_t^{1-\alpha}\epsilon t^{\beta} = D_t^{1-\alpha}u(x,t) + \frac{\epsilon\Gamma(1+\beta)t^{\alpha+\beta-1}}{\Gamma(\beta+\alpha)},$$

we get

$$\frac{\partial^2}{\partial x^2}D_t^{1-\alpha}v(x,t) = \frac{\partial^2}{\partial x^2}D_t^{1-\alpha}u(x,t).$$

Hence, $v(x,t)$ satisfies the equation

$$\frac{\partial}{\partial t}v(x,t) = \frac{\partial^2}{\partial x^2}D_t^{1-\alpha}v(x,t) + F(x,t) + \epsilon\beta t^{\beta-1} \text{ in } (0,a) \times (0,T].$$

Suppose that there exists some $(x,t) \in [0,a]\times[0,T]$ such that $v(x,t) < 0$. Since $v(x,t) \geq 0$ for $(x,t) \in \{0,a\} \times [0,T] \cup [0,a] \times \{0\}$, there is $(x_0,t_0) \in (0,a) \times (0,T]$ such that $v(x_0,t_0)$ is the negative minimum of v over $[0,a] \times [0,T]$. It follows from Lemma 2.1 (b) that

$$D_t^{1-\alpha}v(x_0,t_0) \leq \frac{t_0^{\alpha-1}}{\Gamma(\alpha)}v(x_0,t_0) < 0. \tag{2.1}$$

Let $w(x,t) = D_t^{1-\alpha}v(x,t)$. It follows from Theorem 1.1 (b) that $D_t^{\alpha}w(x,t) = D_t^{\alpha+1-\alpha}v(x,t) = v_t(x,t)$. Since $v(x,t)$ is bounded in $[0,a] \times [0,T]$, we have

$$D_t^{-\alpha}v(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(x,s)ds \to 0 \text{ as } t \to 0.$$

It follows from Theorem 1.1 (c) that for $(x,t) \in (0,a) \times (0,T)$,

$$\begin{aligned} D_t^{\alpha-1}w(x,t) &= D_t^{\alpha-1}\left(D_t^{1-\alpha}v(x,t)\right) \\ &= D_t^{\alpha-1+1-\alpha}v(x,t) - D_t^{-\alpha}v(x,t)\big|_{t=0}\frac{t^{-\alpha}}{\Gamma(1-\alpha)} \\ &= v(x,t). \end{aligned}$$

From Theorem 1.1 (c),

$$\frac{\partial}{\partial t}v(x,t) = D_t^1 v = D_t^1\left(D_t^{\alpha-1}w(x,t)\right) = D_t^{1+\alpha-1}w(x,t) = D_t^{\alpha}w(x,t).$$

From Theorem 1.1 (a), we get for any $t > 0$,

$$D_t^{1-\alpha}v(x,t) = D_t^{1-\alpha}u(x,t) + D_t^{1-\alpha}\epsilon t^{\beta} = D_t^{1-\alpha}u(x,t) + \frac{\epsilon\Gamma(1+\beta)t^{\alpha+\beta-1}}{\Gamma(\beta+\alpha)}.$$

It follows from a direct computation that

$$D_t^{1-\alpha}u(x,t) = \frac{1}{\Gamma(\alpha)}\left[t^{\alpha-1}\phi(x) + \int_0^t (t-s)^{\alpha-1}u_s(x,s)ds\right] \quad \text{for } t > 0. \tag{2.2}$$

Since the left-hand side of (2.2) and the first term of the right-hand side of (2.2) exist, it follows that the second term on the right-hand side exists and tends to 0 as $t \to 0^+$. As $t \to 0^+$, $t^{\alpha-1}\phi(x) \geq 0$. Therefore, $D_t^{1-\alpha}u(x,t) > 0$. Hence, we obtain

$$w(x,t) = D_t^{1-\alpha}v(x,t) = D_t^{1-\alpha}u(x,t) + \frac{\epsilon\Gamma(1+\beta)t^{\alpha+\beta-1}}{\Gamma(\beta+\alpha)} > 0 \text{ as } t \to 0^+.$$

Furthermore, it follows from the boundary condition of $v(x,t)$ that

$$D_t^{1-\alpha}v(0,t) = D_t^{1-\alpha}v(a,t) = D_t^{1-\alpha}\epsilon t^\beta = \frac{\epsilon\Gamma(1+\beta)t^{\alpha+\beta-1}}{\Gamma(\beta+\alpha)} > 0 \text{ for } t > 0.$$

Therefore, $w(x,t)$ satisfies the problem

$$\begin{aligned}
&D_t^\alpha w(x,t) = \frac{\partial^2}{\partial x^2}w(x,t) + F(x,t) + \epsilon\beta t^{\beta-1} \text{ in } (0,a)\times(0,T],\\
&w(x,0) > 0 \text{ on } [0,a],\\
&w(0,t) > 0, w(a,t) > 0 \text{ for } 0 < t \leq T.
\end{aligned}$$

From (2.1), we have $w(x_0,t_0) < 0$. Since $w(x,t) > 0$ on the boundary, there exists $(x_1,t_1) \in (0,a)\times(0,T]$ such that $w(x_1,t_1)$ is the negative minimum of $w(x,t)$ in $[0,a]\times[0,T]$. It follows from Lemma 2.1 (b) that

$$D_t^\alpha w(x_1,t_1) \leq \frac{t_1^\alpha}{\Gamma(1-\alpha)}w(x_1,t_1) < 0.$$

Since $w(x_1,t_1)$ is a local minimum, we obtain $\partial^2/\partial x^2 w(x_1,t_1) \geq 0$. Therefore at (x_1,t_1),

$$0 > D_t^\alpha w(x_1,t_1) = \frac{\partial^2}{\partial x^2}w(x_1,t_1) + F(x_1,t_1) + \epsilon\beta t_1^{\beta-1} > 0.$$

This contradiction shows that $v(x,t) \geq 0$ on $[0,a]\times[0,T]$, and this implies that $u(x,t) \geq -\epsilon t^\beta$ on $[0,a]\times[0,T]$ for any $\epsilon > 0$. Since ϵ is arbitrary, we have $u(x,t) \geq 0$ on $[0,a]\times[0,T]$. □

A similar result can be obtained for the negativity of the solution $u(x,t)$ by considering $-u(x,t)$ when $\phi(x) \leq 0$ and $F(x,t) \leq 0$.

THEOREM 2.3. If $u(x,t)$ satisfies (1.1), $u(x,0) = \phi(x) \leq 0$, $u(0,t) = 0 = u(a,t)$, and $F(x,t) \leq 0$ for $(x,t) \in [0,a]\times(0,T]$, then $u(x,t) \leq 0$ for $(x,t) \in [0,a]\times[0,T]$.

The results in Theorems 2.2 and 2.3 can be extended to obtain the next two theorems.

THEOREM 2.4. Suppose $u(x,t)$ satisfies (1.1), $u(x,0) = \phi(x)$ on $[0,a]$, $u(0,t) = g_1$, and $u(a,t) = g_2$, where g_1 and g_2 are given real numbers. If $F(x,t) \geq 0$ for $(x,t) \in [0,a]\times[0,T]$, then $u(x,t) \geq \min_{[0,a]}\{g_1, g_2, \phi(x)\}$ for $(x,t) \in [0,a]\times[0,T]$.

Proof. Let $M = \min_{[0,a]}\{g_1,g_2,\phi(x)\}$ and $\bar{u}(x,t) = u(x,t) - M$. Then, $\bar{u}(0,t) = g_1 - M \geq 0$, $\bar{u}(a,t) = g_2 - M \geq 0$ for $t \in [0,T]$, and $\bar{u}(x,0) = \phi(x) - M \geq 0$ for $x \in [0,a]$. Since

$$\frac{\partial}{\partial t}\bar{u} = \frac{\partial}{\partial t}u, \qquad \frac{\partial^2}{\partial x^2}D_t^{1-\alpha}\bar{u}(x,t) = \frac{\partial^2}{\partial x^2}D_t^{1-\alpha}u(x,t),$$

it follows that $\bar{u}(x,t)$ satisfies (1.1). Thus, it follows from an argument similar to the proof of Theorem 2.2 that $\bar{u}(x,t) \geq 0$ on $[0,a] \times [0,T]$. That is,

$$u(x,t) \geq \min_{[0,a]}\{g_1, g_2, \phi(x)\} \text{ for } (x,t) \in [0,a] \times [0,T].$$

□

By using $\tilde{u}(x,t) = -u(x,t)$, a proof similar to that of Theorem 2.4 gives the following result.

THEOREM 2.5. Suppose $u(x,t)$ satisfies (1.1), $u(x,0) = \phi(x)$ on $[0,a]$, $u(0,t) = g_1$, and $u(a,t) = g_2$, where g_1 and g_2 are given real numbers. If $F(x,t) \leq 0$ for $(x,t) \in [0,a] \times [0,T]$, then $u(x,t) \leq \max_{[0,a]}\{g_1, g_2, \phi(x)\}$ for $(x,t) \in [0,a] \times [0,T]$.

Theorems 2.4 and 2.5 are similar to the weak maximum principle for the heat equation.

Similar to the classical case, the fractional version of the weak maximum principle can be used to prove the uniqueness of a solution.

THEOREM 2.6. The problem (1.1)-(1.2) has at most one solution.

Proof. Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of the problem (1.1)-(1.2). Then,

$$\frac{\partial}{\partial t}(u_1(x,t) - u_2(x,t)) = \frac{\partial^2}{\partial x^2} D_t^{1-\alpha}(u_1(x,t) - u_2(x,t)),$$

with zero initial condition and zero boundary conditions for $u_1(x,t) - u_2(x,t)$. It follows from Theorems 2.4 and 2.5 that $u_1(x,t) - u_2(x,t) = 0$ on $[0,a] \times [0,T]$. We have a contradiction. The result then follows. □

Theorems 2.4 and 2.5 can be used to show that a solution $u(x,t)$ of the problem (1.1)-(1.2) depends continuously on the initial data $\phi(x)$.

THEOREM 2.7. Suppose $u(x,t)$ and $\tilde{u}(x,t)$ are the solutions of the problem (1.1)-(1.2) corresponding to the initial data $\phi(x)$ and $\tilde{\phi}(x)$ respectively. If $\max_{[0,a]}\{|\phi(x) - \tilde{\phi}(x)|\} \leq \epsilon$, then $|u(x,t) - \tilde{u}(x,t)| \leq \epsilon$.

Proof. The function $v(x,t) = u(x,t) - \tilde{u}(x,t)$ satisfies the problem

$$\frac{\partial}{\partial t} v(x,t) = \frac{\partial^2}{\partial x^2} D_t^{1-\alpha} v(x,t),$$

with initial condition $v(x,0) = \phi(x) - \tilde{\phi}(x)$ and zero boundary conditions. It follows from Theorems 2.4 and 2.5 that $|v(x,t)| \leq \max_{[0,a]}\{|\phi(x) - \tilde{\phi}(x)|\}$. The result then follows. □

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