

SHARP LOCAL WELL-POSEDNESS OF KDV TYPE EQUATIONS WITH DISSIPATIVE PERTURBATIONS

BY

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Abstract. In this work, we study the initial value problems associated to some linear perturbations of the KdV equation. Our focus is on the well-posedness issues for initial data given in the L^2 -based Sobolev spaces. We derive a bilinear estimate in a space with weight in the time variable and obtain sharp local well-posedness results.

1. Introduction. In this article, continuing our earlier work [8], we consider the initial value problems (IVPs)

$$\begin{cases} v_t + v_{xxx} + \eta Lv + (v^2)_x = 0, & x \in \mathbb{R}, t \geq 0, \\ v(x, 0) = v_0(x) \end{cases} \quad (1.1)$$

and

$$\begin{cases} u_t + u_{xxx} + \eta Lu + (u_x)^2 = 0, & x \in \mathbb{R}, t \geq 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.2)$$

where $\eta > 0$ is a constant; $u = u(x, t)$, $v = v(x, t)$ are real valued functions; and the linear operator L is defined via the Fourier transform by $\widehat{Lf}(\xi) = -\Phi(\xi)\hat{f}(\xi)$.

The Fourier symbol $\Phi(\xi)$ is of the form

$$\Phi(\xi) = -|\xi|^p + \Phi_1(\xi), \quad (1.3)$$

where $p \in \mathbb{R}^+$ and $|\Phi_1(\xi)| \leq C(1 + |\xi|^q)$ with $0 \leq q < p$. We note that the symbol $\Phi(\xi)$ is a real valued function which is bounded above; i.e., there is a constant C such that

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$\Phi(\xi) < C$ (see Lemma 2.2 below). In our earlier work [8], we considered a particular case of $\Phi(\xi)$ in the form

$$\tilde{\Phi}(\xi) = \sum_{j=0}^n \sum_{i=0}^{2m} c_{i,j} \xi^i |\xi|^j, \quad c_{i,j} \in \mathbb{R}, \quad c_{2m,n} = -1, \tag{1.4}$$

with $p := 2m + n$.

We observe that if u is a solution of (1.2), then $v = u_x$ is a solution of (1.1) with initial data $v_0 = (u_0)_x$. That is why (1.1) is called the derivative equation of (1.2).

In this work, we are interested in investigating the well-posedness results to the IVPs (1.1) and (1.2) for given data in the low regularity Sobolev spaces $H^s(\mathbb{R})$. Recall that, for $s \in \mathbb{R}$, the L^2 -based Sobolev spaces $H^s(\mathbb{R})$ are defined by

$$H^s(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{H^s} < \infty\},$$

where

$$\|f\|_{H^s} := \left(\int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2},$$

and $\hat{f}(\xi)$ is the usual Fourier transform given by

$$\hat{f}(\xi) \equiv \mathcal{F}(f)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx.$$

The factor $\frac{1}{\sqrt{2\pi}}$ in the definition of the Fourier transform does not alter our analysis, so we will omit it.

The notion of well-posedness we use is the standard one. We say that an IVP for given data in a Banach space X is locally well-posed if there exists a certain time interval $[0, T]$ and a unique solution depending continuously upon the initial data and the solution satisfies the persistence property; i.e., the solution describes a continuous curve in X in the time interval $[0, T]$. If the above properties are true for any time interval, we say that the IVP is globally well-posed. If any one of the above properties fails to hold, we say that the problem is ill-posed.

The notion of ill-posedness used in this work is a bit different from the standard one. If one uses the contraction mapping principle, the application data-solution turns out to be always smooth (see for example [18]). There are many works in the literature (see [5], [20], [21], [29] and references therein) where the notion of well-posedness has been strengthened by requiring that the mapping data-solution be smooth. We follow this notion of “well-posedness” and say that the IVP is “ill-posed” if the mapping data-solution fails to be smooth.

The function space in which we work is motivated from the one introduced in [14], where the author proved a sharp local well-posedness for the IVP associated to the Burgers’ equation by showing that the local well-posedness holds for data in $H^s(\mathbb{R})$ if $-\frac{1}{2} < s \leq 0$ and uniqueness fails if $s < -\frac{1}{2}$. The new ingredient in [14] was the use of the function space with time dependent weight. A natural question is whether such a result still holds true if one considers higher order dissipative equation, and what happens if one uses dispersive term v_{xxx} as in (1.1). Recently, analysis in this direction was carried out in [15], where the author considered the Ostrovsky-Stepanyams-Tsimring equation that is a particular case of (1.1) containing a dissipative term with leading order 3 and

obtained a sharp local well-posedness result for data in $H^s(\mathbb{R})$, $-\frac{3}{2} < s \leq 0$. In this work, we introduce suitable function spaces with time dependent weight and prove sharp local well-posedness results for the IVPs (1.1) and (1.2) when the order of the leading dissipative term is bigger than 3. More precisely, for $p > 0$ and $t \in [0, T]$ with $0 \leq T \leq 1$, these spaces are defined with weight in time variable via the norms

$$\|f\|_{X_T^s} := \sup_{t \in [0, T]} \left\{ \|f(t)\|_{H^s} + t^{\frac{|s|}{p}} \|f(t)\|_{L^2} \right\} \tag{1.5}$$

and

$$\|f\|_{Y_T^s} := \sup_{t \in [0, T]} \left\{ \|f(t)\|_{H^s} + t^{\frac{1+|s|}{p}} \|\partial_x f(t)\|_{L^2} \right\}. \tag{1.6}$$

The spaces X_T^s and Y_T^s will be used to prove local well-posedness for the IVPs (1.1) and (1.2) respectively. We use the notation $\langle \cdot \rangle = (1 + |\cdot|)$.

The first main result of this work is about the local well-posedness of the IVP (1.1) and reads as follows.

THEOREM 1.1. Let $\eta > 0$ be fixed and $\Phi(\xi)$ be given by (1.3) with $p > 3$ as the order of the leading term. Then for any data $v_0 \in H^s(\mathbb{R})$, $s > -\frac{p}{2}$ there exist a time $T = T(\|v_0\|_{H^s})$ and a unique solution v to the IVP (1.1) in $C([0, T], H^s(\mathbb{R}))$.

Moreover, the map $v_0 \mapsto v$ is smooth from $H^s(\mathbb{R})$ to $C([0, T]; H^s(\mathbb{R})) \cap X_T^s$ and $v \in C((0, T]; H^\infty(\mathbb{R}))$.

The second result is the same for the IVP (1.2), with low regularity data.

THEOREM 1.2. Let $\eta > 0$ be fixed and $\Phi(\xi)$ be given by (1.3) with $p > 3$ as the order of the leading term. Then for any data $u_0 \in H^s(\mathbb{R})$, $s > 1 - \frac{p}{2}$, there exist a time $T = T(\|u_0\|_{H^s})$ and a unique solution u to the IVP (1.2) in $C([0, T], H^s(\mathbb{R}))$.

Moreover, the map $u_0 \mapsto u$ is smooth from $H^s(\mathbb{R})$ to $C([0, T]; H^s(\mathbb{R})) \cap Y_T^s$ and $u \in C((0, T]; H^\infty(\mathbb{R}))$.

Our next task is to check if the well-posedness results obtained in Theorems 1.1 and 1.2 are optimal. To have an insight into this issue, we analyze it by using a scaling argument. As the regularity requirement for the IVP (1.2) is one more than that for the IVP (1.1), we discuss only the latter. Talking heuristically, semilinear evolution equations like viscous Burgers, Korteweg-de Vries (KdV), nonlinear Schrödinger (NLS) and wave equations are usually expected to be well-posed for given data with Sobolev regularity up to scaling and ill-posed below scaling. However, this is not always true, as can be seen in the KdV case. For $\eta = 0$, the IVP (1.1) turns out to be the KdV equation

$$\begin{cases} v_t + v_{xxx} + (v^2)_x = 0, & x \in \mathbb{R}, t \geq 0, \\ v(x, 0) = v_0(x), \end{cases} \tag{1.7}$$

which satisfies the scaling property. Talking more precisely, if $v(x, t)$ is a solution of the KdV equation with initial data $v_0(x)$, then for $\lambda > 0$, so is $v^\lambda(x, t) = \lambda^2 v(\lambda x, \lambda^3 t)$ with initial data $v^\lambda(x, 0) = \lambda^2 v(\lambda x, 0)$. Note that the homogeneous Sobolev norm of the initial data remains invariant if $s + \frac{3}{2} = 0$. This suggests that the scaling Sobolev regularity for the KdV equation is $-\frac{3}{2}$. But, Kenig et al. [17, 19] proved local well-posedness of the IVP (1.7) for data in $H^s(\mathbb{R})$, $s > -\frac{3}{4}$ is sharp since the flow-map $u_0 \rightarrow u(t)$ is not locally

uniformly continuous from $\dot{H}^s(\mathbb{R})$ to $\dot{H}^s(\mathbb{R})$, $s < -\frac{3}{4}$. This result is far above the critical index suggested by the scaling.

Generally, for the dissipative problem the scaling index is better in the sense that one can lower the regularity requirement on the data to get well-posedness. As can be seen in the proofs of Theorems 1.1 and 1.2 (below), our method depends on the leading order of L . If we discard the third order derivative (dispersive part) and consider only the dissipative operator L with the Fourier symbol $|\xi|^p$, with $p > 0$ in (1.1), i.e.,

$$\begin{cases} v_t + \eta Lv + (v^2)_x = 0, & \widehat{Lv}(\xi) = |\xi|^p \widehat{v}(\xi), \\ v(x, 0) = v_0(x), \end{cases} \tag{1.8}$$

it is easy to check that if $v(x, t)$ solves (1.8) with initial data $v(x, 0)$, then for $\lambda > 0$ so does $v^\lambda(x, t) = \lambda^{p-1}v(\lambda x, \lambda^p t)$ with initial data $v^\lambda(x, 0) = \lambda^{p-1}v(\lambda x, 0)$. Note that

$$\|v^\lambda(0)\|_{\dot{H}^s} = \lambda^{p-\frac{3}{2}+s}\|v(0)\|_{\dot{H}^s}. \tag{1.9}$$

From (1.9) we see that the scaling index for this particular situation is $s_c := \frac{3}{2} - p$. Observe that for $p = 3$ we get $s_c = -\frac{3}{2}$, which coincides with the scaling critical regularity of the KdV equation. In view of this observation, for $p = 3$, one can conclude that the local well-posedness result proved in [15] is up to the Sobolev regularity given by the scaling argument. However, for $p > 3$, as can be seen in Theorems 1.1 and 1.2, the regularity requirement for the local well-posedness is higher than s_c (i.e., $s_c < -\frac{p}{2}$). Since the regularity requirement for the IVP (1.2) is higher by one derivative than that for the IVP (1.1), we see that the scaling index for this case is $s_c + 1$.

A natural question is whether the local results obtained in Theorems 1.1 and 1.2 can be improved up to the regularity given by the scaling argument. The following results provide a negative answer to this question.

THEOREM 1.3. Let $p \geq 2$, $s < -\frac{p}{2}$. Then there does not exist any $T > 0$ such that the IVP (1.1) admits a unique local solution defined in the interval $[0, T]$ such that the flow-map

$$v_0 \mapsto v \tag{1.10}$$

is C^2 -differentiable at the origin from $H^s(\mathbb{R})$ to $C([0, T]; H^s(\mathbb{R}))$.

THEOREM 1.4. Let $p \geq 2$, $s < 1 - \frac{p}{2}$. Then there does not exist any $T > 0$ such that the IVP (1.2) admits a unique local solution defined in the interval $[0, T]$ such that the flow-map

$$u_0 \mapsto u \tag{1.11}$$

is C^2 -differentiable at the origin from $H^s(\mathbb{R})$ to $C([0, T]; H^s(\mathbb{R}))$.

As described earlier, we recall that the contraction mapping argument applied in the proof of the well-posedness theorems shows that the mapping data-solution is always smooth. In light of this observation, Theorems 1.3 and 1.4 show that the well-posedness results for the IVP (1.1) and (1.2) proved respectively in Theorems 1.1 and 1.2 are sharp in the sense that one cannot employ the contraction mapping principle for given data with Sobolev regularity below the one given by these theorems. As in the usual KdV case,

local well-posedness cannot be achieved up to the scaling index (suggested by considering only dissipative term) using a contraction mapping argument if $p > 3$ in (1.3).

As can be seen in the proofs below, our method in this article holds only for $-\frac{p}{2} < s \leq 0$ in Theorem 1.1 and for $1 - \frac{p}{2} < s \leq 0$ in Theorem 1.2 considering $p > 3$. However, for $s > 0$ (Theorem 1.1 and Theorem 1.2) we already have proved local well-posedness in our earlier work [7] in a general setting (see also [8] and [9]). To make this article self-contained, we reproduce the proof of local well-posedness results for $s > 0$, which suits the case considered in the Appendix.

To obtain earlier results we followed the techniques developed by Bourgain [6], Kenig, Ponce and Vega [19] (see also [27]) and Dix [14]. The main ingredients in the proof are estimates in the integral equation associated to an extended IVP that is defined for all $t \in \mathbb{R}$. The main idea in [8] and [9] is to use the usual Bourgain space associated to the KdV equation instead of that associated to the linear part of the IVPs (1.1) and (1.2). For the well-posedness issues in the periodic setting we refer to [11].

In what follows, we present some particular examples that belong to the class considered in (1.1) and (1.2) and discuss the known well-posedness results about them.

The first examples belonging to the classes (1.1) and (1.2) are the Korteweg-de Vries-Burgers (KdV-B) equation

$$\begin{cases} v_t + v_{xxx} - \eta v_{xx} + (v^2)_x = 0, & x \in \mathbb{R}, t \geq 0, \\ v(x, 0) = v_0(x) \end{cases} \tag{1.12}$$

and

$$\begin{cases} u_t + u_{xxx} - \eta u_{xx} + (u_x)^2 = 0, & x \in \mathbb{R}, t \geq 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.13}$$

where $u = u(x, t)$, $v = v(x, t)$ are real-valued functions and $\eta > 0$ is a constant, $\Phi(\xi) = -|\xi|^2$ and $p = 2$. Equation (1.12) has been derived as a model for the propagation of weakly nonlinear dispersive long waves in some physical contexts when dissipative effects occur (see [23]). For motivation, we refer to the recent work of Molinet and Ribaud [21], where the authors proved sharp global well-posedness for $s > -1$ in the framework of the Fourier transform restriction norm spaces introduced by Bourgain [6].

The next examples that fit in (1.1) and (1.2) are

$$\begin{cases} v_t + v_{xxx} - \eta(\mathcal{H}v_x + \mathcal{H}v_{xxx}) + (v^2)_x = 0, & x \in \mathbb{R}, t \geq 0, \\ v(x, 0) = v_0(x) \end{cases} \tag{1.14}$$

and

$$\begin{cases} u_t + u_{xxx} - \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) + (u_x)^2 = 0, & x \in \mathbb{R}, t \geq 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.15}$$

respectively, where \mathcal{H} denotes the Hilbert transform

$$\mathcal{H}g(x) = \text{P.V.} \frac{1}{\pi} \int \frac{g(x - \xi)}{\xi} d\xi,$$

$u = u(x, t)$, $v = v(x, t)$ are real-valued functions and $\eta > 0$ is a constant, $\Phi(\xi) = -|\xi|^3 + |\xi|$ and $p = 3$.

The equation in (1.14) was derived by Ostrovsky et al. [22] to describe the radiational instability of long waves in a stratified shear flow. Recently, Carvajal and Scialom [10] considered the IVP (1.14) and proved the local well-posedness results for given data in H^s , $s \geq 0$. They also obtained an *a priori* estimate for given data in $L^2(\mathbb{R})$, thereby proving the global well-posedness result. The earlier well-posedness results for (1.14) can be found in [1], where for given data in $H^s(\mathbb{R})$, local well-posedness when $s > 1/2$ and global well-posedness when $s \geq 1$ have been proved. In [1], IVP (1.15) is also considered to prove global well-posedness for given data in $H^s(\mathbb{R})$, $s \geq 1$.

Another two models that fit in the classes (1.2) and (1.1) respectively are the Korteweg-de Vries-Kuramoto Sivashinsky (KdV-KS) equation

$$\begin{cases} u_t + u_{xxx} + \eta(u_{xx} + u_{xxxx}) + (u_x)^2 = 0, & x \in \mathbb{R}, t \geq 0, \\ u(x, 0) = u_0(x) \end{cases} \tag{1.16}$$

and its derivative equation

$$\begin{cases} v_t + v_{xxx} + \eta(v_{xx} + v_{xxxx}) + vv_x = 0, & x \in \mathbb{R}, t \geq 0, \\ v(x, 0) = v_0(x), \end{cases} \tag{1.17}$$

where $u = u(x, t)$, $v = v(x, t)$ are real-valued functions and $\eta > 0$ is a constant, $\Phi(\xi) = -|\xi|^4 + |\xi|^2$ and $p = 4$.

The KdV-KS equation arises as a model for long waves in a viscous fluid flowing down an inclined plane and also describes drift waves in a plasma (see [13, 28]). The KdV-KS equation is very interesting in the sense that it combines the dispersive characteristics of the Korteweg-de Vries equation and dissipative characteristics of the Kuramoto-Sivashinsky equation. Also, it is worth noticing that (1.17) is a particular case of the Benney-Lin equation [2, 28], i.e.,

$$\begin{cases} v_t + v_{xxx} + \eta(v_{xx} + v_{xxxx}) + \beta v_{xxxxx} + vv_x = 0, & x \in \mathbb{R}, t \geq 0, \\ v(x, 0) = v_0(x), \end{cases} \tag{1.18}$$

when $\beta = 0$.

The IVPs (1.16) and (1.17) were studied by Biagioni, Bona, Iorio and Scialom [3]. The authors in [3] proved that the IVPs (1.16) and (1.17) are locally well-posed for given data in H^s , $s \geq 1$, with $\eta > 0$. They also constructed appropriate *a priori* estimates and used them to prove the global well-posedness result. The limiting behavior of solutions as the dissipation tends to zero (i.e., $\eta \rightarrow 0$) has also been studied in [3]. The IVP (1.18) associated to the Benney-Lin equation is also widely studied in the literature [2, 4, 28]. Regarding well-posedness issues for the IVP (1.18) the work of Biagioni and Linares [4] is worth mentioning, where they proved global well-posedness for given data in $L^2(\mathbb{R})$. For the sharp well-posedness result for the KdV-KS equation we refer to the recent work of Pilod in [25], where the author proved local well-posedness in $H^s(\mathbb{R})$ for $s > -1$ and ill-posedness for $s < -1$. For recent work on the generalized Benjamin-Ono-Burgers equation we refer to [24], where the author uses Bourgain’s space to obtain local well-posedness for data with low Sobolev regularity.

Now we consider the IVP associated to the linear parts of (1.1) and (1.2),

$$\begin{cases} w_t + w_{xxx} + \eta Lw = 0, & x, t \geq 0, \\ w(0) = w_0. \end{cases} \tag{1.19}$$

The solution to (1.19) is given by $w(x, t) = V(t)w_0(x)$ where the semigroup $V(t)$ is defined as

$$\widehat{V(t)w_0}(\xi) = e^{it\xi^3 + \eta t\Phi(\xi)} \widehat{w_0}(\xi). \tag{1.20}$$

In what follows, without loss of generality, we suppose $\eta = 1$.

This paper is organized as follows. In Section 2, we prove some preliminary estimates. Sections 3 and 4 are dedicated to proving the local well-posedness and ill-posedness results respectively.

2. Preliminary estimates. This section is devoted to obtaining linear and nonlinear estimates that are essential in the proof of the main results. We start with the following estimate that the Fourier symbol defined in (1.3) satisfies.

LEMMA 2.1. There exists $M > 0$ large enough such that for all $|\xi| \geq M$, one has

$$\Phi(\xi) = -|\xi|^p + \Phi_1(\xi) < -1, \tag{2.1}$$

$$\frac{\Phi_1(\xi)}{|\xi|^p} \leq \frac{1}{2}, \tag{2.2}$$

and

$$|\Phi(\xi)| \geq \frac{|\xi|^p}{2}. \tag{2.3}$$

Proof. The inequalities (2.1) and (2.2) are direct consequences of

$$\lim_{\xi \rightarrow \infty} \frac{\Phi_1(\xi) + 1}{|\xi|^p} = 0 \quad \text{and} \quad \lim_{\xi \rightarrow \infty} \frac{\Phi_1(\xi)}{|\xi|^p} = 0,$$

respectively.

The estimate (2.3) follows from (2.1) and (2.2). In fact, for $|\xi| > M$,

$$|\Phi(\xi)| = |\xi|^p - \Phi_1(\xi) \geq \frac{|\xi|^p}{2}, \tag{2.4}$$

and this concludes the proof of (2.3). □

LEMMA 2.2. The Fourier symbol $\Phi(\xi)$ given by (1.3) is bounded from above and the following estimate holds true:

$$\|e^{t\Phi(\xi)}\|_{L^\infty} \leq e^{TC_M}. \tag{2.5}$$

Proof. From Lemma 2.1, there is $M > 1$ large enough such that for $|\xi| \geq M$ one has $\Phi(\xi) < -1$. Consequently, $e^{t\Phi(\xi)} \leq e^{-t} \leq 1$. Now for $|\xi| < M$, it is easy to get $\Phi(\xi) < C_M$, so that $e^{t\Phi(\xi)} \leq e^{TC_M}$. Therefore, in any case

$$\|e^{t\Phi(\xi)}\|_{L^\infty} \leq e^{TC_M},$$

as required. □

The following result is an elementary fact from calculus.

LEMMA 2.3. Let $f(t) = t^a e^{tb}$ with $a > 0$ and $b < 0$. Then for all $t \geq 0$ one has

$$f(t) \leq \left(\frac{a}{|b|}\right)^a e^{-a}. \tag{2.6}$$

LEMMA 2.4. Let $V(t)$ be as defined in (1.20) and $v_0 \in H^s(\mathbb{R})$. Then

$$V(\cdot)v_0 \in C([0, \infty); H^s(\mathbb{R})) \cap C((0, \infty); H^\infty(\mathbb{R})).$$

Proof. It is sufficient to prove that $V(t)v_0 \in H^{s'}(\mathbb{R})$ for $s' > s$. Now,

$$\begin{aligned} \|V(t)v_0\|_{H^{s'}} &= \|\langle \xi \rangle^{s'} e^{it\xi^3 + t\Phi(\xi)} \widehat{v_0}(\xi)\|_{L^2} \\ &= \|\langle \xi \rangle^s \widehat{v_0}(\xi) \langle \xi \rangle^{s'-s} e^{t\Phi(\xi)}\|_{L^2} \\ &\leq \|\langle \xi \rangle^{s'-s} e^{t\Phi(\xi)}\|_{L^\infty} \|v_0\|_{H^s}. \end{aligned} \tag{2.7}$$

Let $M \gg 1$ be as in Lemma 2.1. Then we have

$$\begin{aligned} \|\langle \xi \rangle^{s'-s} e^{t\Phi(\xi)}\|_{L^\infty} &\leq \|\langle \xi \rangle^{s'-s} e^{t\Phi(\xi)}\|_{L^\infty(|\xi| \leq M)} + \|\langle \xi \rangle^{s'-s} e^{t\Phi(\xi)}\|_{L^\infty(|\xi| > M)} \\ &\leq C_M + \|e^{-\frac{|\xi|^p}{2}t} \langle \xi \rangle^{s'-s}\|_{L^\infty} < \infty, \quad t \in \mathbb{R}^+. \end{aligned} \tag{2.8}$$

The continuity follows using the dominated convergence theorem. □

LEMMA 2.5. Let $0 < T \leq 1$ and $t \in [0, T]$. Then for all $s \in \mathbb{R}$, we have

$$\|V(t)u_0\|_{X_T^s} \lesssim e^{C_M T} \|u_0\|_{H^s}, \tag{2.9}$$

where the constant C_M depends on M with M as in Lemma 2.1.

Proof. We start by estimating the first component of the X_T^s -norm. We have that

$$\|V(t)u_0\|_{H^s} = \|\langle \xi \rangle^s e^{t\Phi(\xi)} \widehat{u_0}(\xi)\|_{L^2} \leq \|e^{t\Phi(\xi)}\|_{L^\infty} \|u_0\|_{H^s}. \tag{2.10}$$

Using (2.5) in (2.10), we get

$$\|V(t)u_0\|_{H^s} \leq e^{TC_M} \|u_0\|_{H^s}. \tag{2.11}$$

Now, we move to estimate the second component of the X_T^s -norm. The case $s \geq 0$ is quite easy, so we consider only the case when $s < 0$. Using the Plancherel identity, we have

$$\begin{aligned} t^{\frac{|s|}{p}} \|V(t)u_0\|_{L^2} &= t^{\frac{|s|}{p}} \|e^{t\Phi(\xi)} \widehat{u_0}\|_{L^2} \\ &= t^{\frac{|s|}{p}} \|\langle \xi \rangle^{-s} e^{t\Phi(\xi)} \langle \xi \rangle^s \widehat{u_0}\|_{L^2} \\ &\leq t^{\frac{|s|}{p}} \|\langle \xi \rangle^{|s|} e^{t\Phi(\xi)}\|_{L^\infty} \|u_0\|_{H^s}. \end{aligned} \tag{2.12}$$

Since $\langle \xi \rangle^{|s|} \lesssim 1 + |\xi|^{|s|}$, from (2.12), one obtains

$$t^{\frac{|s|}{p}} \|V(t)u_0\|_{L^2} \leq t^{\frac{|s|}{p}} \left[\|e^{t\Phi(\xi)}\|_{L^\infty} + \|\langle \xi \rangle^{|s|} e^{t\Phi(\xi)}\|_{L^\infty} \right] \|u_0\|_{H^s}. \tag{2.13}$$

From (2.5), we have $\|e^{t\Phi(\xi)}\|_{L^\infty} \leq e^{TC_M}$. To estimate $\|\langle \xi \rangle^{|s|} e^{t\Phi(\xi)}\|_{L^\infty}$ we proceed as follows:

$$\|\langle \xi \rangle^{|s|} e^{t\Phi(\xi)}\|_{L^\infty} \leq \|\langle \xi \rangle^{|s|} e^{t\Phi(\xi)} \chi_{\{|\xi| \leq M\}}\|_{L^\infty} + \|\langle \xi \rangle^{|s|} e^{t\Phi(\xi)} \chi_{\{|\xi| > M\}}\|_{L^\infty}. \tag{2.14}$$

For the low-frequency part, it is easy to get

$$\|\xi\|^{|s|} e^{t\Phi(\xi)} \chi_{\{|\xi| \leq M\}} \|_{L^\infty} \leq M^{|s|} e^{C_M T}. \quad (2.15)$$

Now, we move to estimate the high-frequency part $\|\xi\|^{|s|} e^{t\Phi(\xi)} \chi_{\{|\xi| > M\}} \|_{L^\infty}$ in (2.14). For this, we make use of the time weight in the definition of X_T^s -norm. Define for $|\xi| > M$, $g(t, \xi) := t^{\frac{|s|}{p}} |\xi|^{|s|} e^{t\Phi(\xi)}$. Using the estimate (2.6) from Lemma 2.3, we get

$$g(t, \xi) \leq \left(\frac{|s|}{p|\Phi(\xi)|} \right)^{\frac{|s|}{p}} e^{-\frac{|s|}{p} |\xi|^{|s|}}. \quad (2.16)$$

Since $M > 1$ is large, an application of the estimate (2.3) from Lemma 2.1 into (2.16) yields

$$g(t, \xi) \leq \left(\frac{2|s|}{p|\xi|^p} \right)^{\frac{|s|}{p}} e^{-\frac{|s|}{p} |\xi|^{|s|}} \leq \left(\frac{2|s|}{p} \right)^{\frac{|s|}{p}} e^{-\frac{|s|}{p}}. \quad (2.17)$$

In light of the estimate (2.17), one obtains that

$$t^{\frac{|s|}{p}} \|\xi\|^{|s|} e^{t\Phi(\xi)} \chi_{\{|\xi| > M\}} \|_{L^\infty} \leq \left(\frac{2|s|}{p} \right)^{\frac{|s|}{p}} e^{-\frac{|s|}{p}}. \quad (2.18)$$

Inserting estimates (2.5), (2.15) and (2.18) into (2.13), we get

$$t^{\frac{|s|}{p}} \|V(t)u_0\|_{L^2} \leq \left(\frac{2|s|}{p} \right)^{\frac{|s|}{p}} e^{-\frac{|s|}{p}} \|u_0\|_{H^s} \lesssim \|u_0\|_{H^s}. \quad (2.19)$$

The conclusion of the lemma follows from (2.11) and (2.19). \square

LEMMA 2.6. Let $0 < T \leq 1$ and $t \in [0, T]$. Then for all $s \in \mathbb{R}$, we have

$$\|V(t)u_0\|_{Y_T^s} \lesssim e^{TC_M} \|u_0\|_{H^s}, \quad (2.20)$$

where the constant C_M depends on M with M as in Lemma 2.1.

Proof. The estimate for the first component of the Y_T^s -norm has already been obtained in (2.11). In what follows, we estimate the second component of the Y_T^s -norm. We only consider the case when $s < 0$. In the case when $s \geq 0$ the estimates follow easily. Using the Plancherel identity, we have

$$\begin{aligned} t^{\frac{1+|s|}{p}} \|\partial_x V(t)u_0\|_{L^2} &= t^{\frac{1+|s|}{p}} \|\xi e^{t\Phi(\xi)} \widehat{u_0}\|_{L^2} \\ &= t^{\frac{1+|s|}{p}} \|\xi \langle \xi \rangle^{-s} e^{t\Phi(\xi)} \langle \xi \rangle^s \widehat{u_0}\|_{L^2} \\ &\leq t^{\frac{1+|s|}{p}} \|\xi \langle \xi \rangle^{|s|} e^{t\Phi(\xi)}\|_{L^\infty} \|u_0\|_{H^s}. \end{aligned} \quad (2.21)$$

Now,

$$\begin{aligned} t^{\frac{1+|s|}{p}} \|\xi \langle \xi \rangle^{|s|} e^{t\Phi(\xi)}\|_{L^\infty} &\leq t^{\frac{1+|s|}{p}} \|\xi \langle \xi \rangle^{|s|} e^{t\Phi(\xi)} \chi_{\{|\xi| \leq M\}}\|_{L^\infty} + t^{\frac{1+|s|}{p}} \|\xi \langle \xi \rangle^{|s|} e^{t\Phi(\xi)} \chi_{\{|\xi| > M\}}\|_{L^\infty} \\ &=: J_1 + J_2. \end{aligned} \quad (2.22)$$

Since $\langle \xi \rangle^{|s|} \lesssim 1 + |\xi|^{|s|}$, and $t \in [0, T]$ with $0 \leq T \leq 1$, we have

$$J_1 \lesssim C_M t^{\frac{1+|s|}{p}} \leq C_M. \quad (2.23)$$

Now, we move to estimate the high-frequency part J_2 . For this, we use the estimate (2.6) from Lemma 2.3 with $b = \Phi(\xi) < 0$ and $a = \frac{1+|s|}{p}$ to get

$$e^{t\Phi(\xi)} \leq \left(\frac{ae^{-1}}{|\Phi(\xi)|} \right)^a \frac{1}{t^a}. \tag{2.24}$$

Since $M > 1$ is large, $\langle \xi \rangle^{|s|} \lesssim |\xi|^{|s|}$, an application of the estimate (2.3) from Lemma 2.1 into (2.24), yields

$$t^{\frac{1+|s|}{p}} |\xi|^{1+|s|} \left(\frac{ae^{-1}}{|\Phi(\xi)|} \right)^a \frac{1}{t^a} \lesssim t^{\frac{1+|s|}{p}-a} |\xi|^{1+|s|-ap} \lesssim C_M, \tag{2.25}$$

and consequently

$$J_2 \lesssim C_M. \tag{2.26}$$

The conclusion of the lemma follows from (2.11), (2.21), (2.22), (2.23) and (2.26). \square

LEMMA 2.7. Let $-\frac{p}{2} < s, p > 3$ and $\tau \in (0, 1]$. Then we have

$$\|\xi \langle \xi \rangle^s e^{\tau\Phi(\xi)}\|_{L_\xi^2} \lesssim \frac{1}{\tau^{\frac{1}{2} + \frac{s}{p}}} \tag{2.27}$$

and

$$\|\xi e^{\tau\Phi(\xi)}\|_{L_\xi^2} \lesssim \frac{1}{\tau^{\frac{3+}{2p}}}. \tag{2.28}$$

Proof. In order to prove (2.27), let M be as in Lemma 2.1, and decompose the integral

$$\|\xi \langle \xi \rangle^s e^{\tau\Phi(\xi)}\|_{L_\xi^2}^2 = \int_{|\xi| \leq M} \xi^2 \langle \xi \rangle^{2s} e^{2\tau\Phi(\xi)} d\xi + \int_{|\xi| \geq M} \xi^2 \langle \xi \rangle^{2s} e^{2\tau\Phi(\xi)} d\xi =: I_1 + I_2. \tag{2.29}$$

In the first integral, since $1 + 2\frac{s}{p} > 0$ and $\tau \in (0, 1]$, we have

$$I_1 \leq \int_{|\xi| \leq M} M^2 e^{2C\tau} d\xi \leq 2M^3 e^{2C\tau} \leq \frac{2M^3 e^{2C}}{\tau^{1+2\frac{s}{p}}}. \tag{2.30}$$

Now, we consider the second integral in (2.29). For sufficiently large M , if we take $b = 2\Phi(\xi) < 0$ (see Lemma 2.1) and $a = 1 + 2\frac{s}{p} > 0$, then using the estimates (2.6) and (2.3), we get

$$I_2 \lesssim \frac{1}{\tau^a} \int_{|\xi| \geq M} \xi^2 \langle \xi \rangle^{2s} \frac{1}{|\Phi(\xi)|^a} d\xi \leq \frac{1}{\tau^{1+2\frac{s}{p}}} \int_{|\xi| \geq M} \frac{1}{|\xi|^{-2-2s+p(1+2\frac{s}{p})}} d\xi \lesssim \frac{1}{\tau^{1+2\frac{s}{p}}},$$

where in the last inequality the fact that $-2 - 2s + p(1 + 2s/p) = p - 2 > 1$ has been used, and this proves (2.27).

The proof of (2.28) is very similar. Again we consider M as in Lemma 2.1 and decompose the integral

$$\|\xi e^{\tau\Phi(\xi)}\|_{L_\xi^2}^2 = \int_{|\xi| \leq M} \xi^2 e^{2\tau\Phi(\xi)} d\xi + \int_{|\xi| \geq M} \xi^2 e^{2\tau\Phi(\xi)} d\xi =: J_1 + J_2. \tag{2.31}$$

Since $\frac{p}{3} > 0$ and $\tau \in (0, 1]$, we have

$$J_1 \leq \int_{|\xi| \leq M} M^2 e^{2C\tau} d\xi \leq 2M^3 e^{2C\tau} \leq \frac{2M^3 e^{2C}}{\tau^{\frac{3+}{p}}}. \tag{2.32}$$

Similarly as in the case of I_2 , using (2.6) with $b = 2\Phi(\xi) < 0$ and $a = \frac{3^+}{p} > 0$, and estimate (2.3), we obtain

$$J_2 \lesssim \frac{1}{\tau^a} \int_{|\xi| \geq M} \xi^2 \frac{1}{|\Phi(\xi)|^a} d\xi \leq \frac{1}{\tau^{\frac{3^+}{p}}} \int_{|\xi| \geq M} \frac{1}{|\xi|^{-2+p(\frac{3^+}{p})}} d\xi \lesssim \frac{1}{\tau^{\frac{3^+}{p}}},$$

where in the last inequality the fact that $-2+p(\frac{3^+}{p}) > 1$ has been used, and this proves (2.28). \square

PROPOSITION 2.8. Let $-\frac{p}{2} < s \leq 0$, $p > 3$, $0 < T \leq 1$ and $t \in [0, T]$. Then we have

$$\left\| \int_0^t V(t-t') \partial_x(uv)(t') dt' \right\|_{X_T^s} \lesssim T^\alpha \|u\|_{X_T^s} \|v\|_{X_T^s}, \tag{2.33}$$

where $\alpha = \frac{2s+p}{2p} > 0$.

Proof. Using the definition of $V(t)$ and Minkowski's inequality, we have

$$\begin{aligned} \left\| \int_0^t V(t-t') \partial_x(uv)(t') dt' \right\|_{H^s} &\leq \int_0^t \|\xi \langle \xi \rangle^s e^{(t-t')\Phi(\xi)} (\widehat{u(t')} * \widehat{v(t')}) dt'\|_{L_\xi^2} \\ &\leq \int_0^t \|\xi \langle \xi \rangle^s e^{(t-t')\Phi(\xi)}\|_{L_\xi^2} \|(\widehat{u(t')} * \widehat{v(t')})(\xi)\|_{L_\xi^\infty} dt'. \end{aligned} \tag{2.34}$$

Young's inequality, the Plancherel identity and the definition of X_T^s norm yield

$$\|(\widehat{u(t')} * \widehat{v(t')})(\xi)\|_{L_\xi^\infty} \leq t'^{-\frac{2|s|}{p}} \|u\|_{X_T^s} \|v\|_{X_T^s}. \tag{2.35}$$

Combining inequalities (2.34), (2.35) and inequality (2.27) in Lemma 2.7, we get

$$\left\| \int_0^t V(t-t') \partial_x(uv)(t') dt' \right\|_{H^s} \lesssim \|u\|_{X_T^s} \|v\|_{X_T^s} \int_0^t \frac{1}{|t-t'|^{\frac{1}{2}+\frac{s}{p}} |t'|^{\frac{2|s|}{p}}} dt'. \tag{2.36}$$

Using a change of variable $t' = t\tau$, we get

$$\begin{aligned} \left\| \int_0^t V(t-t') \partial_x(uv)(t') dt' \right\|_{H^s} &\lesssim t^{\frac{p+2s}{2p}} \|u\|_{X_T^s} \|v\|_{X_T^s} \int_0^1 \frac{1}{|1-\tau|^{\frac{1}{2}+\frac{s}{p}} |\tau|^{\frac{2|s|}{p}}} d\tau \\ &\lesssim t^{\frac{p+2s}{2p}} \|u\|_{X_T^s} \|v\|_{X_T^s}. \end{aligned} \tag{2.37}$$

Similarly, inequality (2.28) in Lemma 2.7 and (2.35) give

$$t^{\frac{|s|}{p}} \left\| \int_0^t V(t-t') \partial_x(uv)(t') dt' \right\|_{L^2} \lesssim \|u\|_{X_T^s} \|v\|_{X_T^s} t^{\frac{|s|}{p}} \int_0^1 \frac{1}{|t-t'|^{\frac{3^+}{2p}} |t'|^{\frac{2|s|}{p}}} d\tau. \tag{2.38}$$

Again, performing a change of variable $t' = t\tau$, one has

$$\begin{aligned} t^{\frac{|s|}{p}} \left\| \int_0^t V(t-t') \partial_x(uv)(t') dt' \right\|_{L^2} &\lesssim t^{\frac{2p+2s-3^+}{2p}} \|u\|_{X_T^s} \|v\|_{X_T^s} \int_0^1 \frac{1}{|1-\tau|^{\frac{3^+}{2p}} |\tau|^{\frac{2|s|}{p}}} d\tau \\ &\lesssim t^{\frac{2p+2s-3^+}{2p}} \|u\|_{X_T^s} \|v\|_{X_T^s}. \end{aligned} \tag{2.39}$$

\square

We also need the following estimate.

LEMMA 2.9. Let $1 - \frac{p}{2} < s, p > 3$ and $\tau \in (0, 1]$. Then we have

$$\|\langle \xi \rangle^s e^{\tau\Phi(\xi)}\|_{L^2_\xi} \lesssim \frac{1}{\tau^{\frac{p-2+2s}{2p}}}. \tag{2.40}$$

Proof. For M large as in Lemma 2.1, we have

$$\|\langle \xi \rangle^s e^{\tau\Phi(\xi)}\|_{L^2}^2 = \int_{|\xi| \leq M} \langle \xi \rangle^{2s} e^{2\tau\Phi(\xi)} d\xi + \int_{|\xi| > M} \langle \xi \rangle^{2s} e^{2\tau\Phi(\xi)} d\xi =: A + B. \tag{2.41}$$

Now, for $\tau \in (0, 1]$ and $a = \frac{p-2+2s}{p} > 0$, one has

$$A \leq C_M e^{TC_M} \lesssim \frac{1}{\tau^a}. \tag{2.42}$$

To obtain an estimate for the high frequency part B , we use estimate (2.6) with $a = \frac{p-2+2s}{p} > 0$ and $b = 2\Phi(\xi) < 0$ to obtain

$$B \leq \int_{|\xi| > M} \frac{|\xi|^{2s} (ae^{-1})^a}{\tau^a |\Phi(\xi)|^a} d\xi \lesssim \int_{|\xi| > M} \frac{1}{|\xi|^{pa-2s}} \frac{1}{\tau^a} d\xi \lesssim \frac{1}{\tau^a}, \tag{2.43}$$

where in the last inequality $pa - 2s > 1$ has been used. □

PROPOSITION 2.10. Let $1 - \frac{p}{2} < s \leq 0, p > 3, 0 < T \leq 1$ and $t \in [0, T]$. Then we have

$$\left\| \int_0^t V(t-t')(u_x v_x)(t') dt' \right\|_{Y_T^s} \lesssim T^\theta \|u\|_{Y_T^s} \|v\|_{Y_T^s}, \tag{2.44}$$

where $\theta = \frac{p-2+2s}{2p} > 0$.

Proof. We start considering the H^s part of the Y_T^s -norm. Using the definition of $V(t)$ and Minkowski’s inequality, we have

$$\begin{aligned} \left\| \int_0^t V(t-t')(u_x v_x)(t') dt' \right\|_{H^s} &\leq \int_0^t \|\langle \xi \rangle^s e^{(t-t')\Phi(\xi)} (\widehat{u_x(t')} * \widehat{v_x(t')}) dt'\|_{L^2_\xi} \\ &\leq \int_0^t \|\langle \xi \rangle^s e^{(t-t')\Phi(\xi)}\|_{L^2_\xi} \|(\widehat{u_x(t')} * \widehat{v_x(t')})(\xi)\|_{L^\infty} dt'. \end{aligned} \tag{2.45}$$

Young’s inequality, the Plancherel identity and the definition of Y_T^s norm yield

$$\|(\widehat{u_x(t')} * \widehat{v_x(t')})(\xi)\|_{L^\infty} \leq t'^{-\frac{2(1+|s|)}{p}} \|u\|_{Y_T^s} \|v\|_{Y_T^s}. \tag{2.46}$$

Using (2.40) and (2.46) in (2.45), we get

$$\left\| \int_0^t V(t-t')(u_x v_x)(t') dt' \right\|_{H^s} \lesssim \|u\|_{Y_T^s} \|v\|_{Y_T^s} \int_0^t \frac{1}{|t-t'|^{\frac{\alpha}{2}} |t'|^{\frac{2(1+|s|)}{p}}} dt'. \tag{2.47}$$

Making a change of variable $t' = t\tau$, one obtains

$$\left\| \int_0^t V(t-t')(u_x v_x)(t') dt' \right\|_{H^s} \lesssim t^{1-\frac{\alpha}{2}-\frac{2(1+|s|)}{p}} \|u\|_{Y_T^s} \|v\|_{Y_T^s} \int_0^1 \frac{1}{|1-\tau|^{\frac{\alpha}{2}} |\tau|^{\frac{2(1+|s|)}{p}}} d\tau. \tag{2.48}$$

For our choice of $a = \frac{p-2+2s}{p}$ and $1/2 > s > 1 - \frac{p}{2}$ the integral in the RHS of (2.48) is finite, so we deduce that

$$\left\| \int_0^t V(t-t')(u_x v_x)(t') dt' \right\|_{H^s} \lesssim t^{\frac{p-2+2s}{2p}} \|u\|_{Y_T^s} \|v\|_{Y_T^s}. \tag{2.49}$$

Now, we move to estimate the second part of the Y_T^s -norm. Note that

$$\begin{aligned} \left\| \int_0^t \partial_x V(t-t')(u_x v_x)(t') dt' \right\|_{L^2} &\leq \int_0^t \|\xi e^{(t-t')\Phi(\xi)} \widehat{u_x(t')} * \widehat{v_x(t')}\|_{L^2} dt' \\ &\leq \int_0^t \|\widehat{u_x(t')} * \widehat{v_x(t')}\|_{L^\infty} \|\xi e^{(t-t')\Phi(\xi)}\|_{L^2} dt'. \end{aligned} \tag{2.50}$$

We have $\|\widehat{u_x(t)} * \widehat{v_x(t)}\|_{L^\infty} \leq t^{-\frac{2(1+|s|)}{p}} \|u\|_{Y_T^s} \|v\|_{Y_T^s}$. Taking $a = \frac{3^+}{2p}$, from (2.28) one gets $\|\xi e^{\tau\Phi(\xi)}\|_{L^2} \lesssim \frac{1}{\tau^a}$. So, from (2.50) one can deduce that

$$\left\| \int_0^t \partial_x V(t-t')(u_x v_x)(t') dt' \right\|_{L^2} \lesssim \|u\|_{Y_T^s} \|v\|_{Y_T^s} \int_0^t \frac{1}{|t-t'|^a |t'|^{\frac{2(1+|s|)}{p}}} dt'. \tag{2.51}$$

Making a change of variable $t' = t\tau$, one obtains from (2.51)

$$t^{\frac{(1+|s|)}{p}} \left\| \int_0^t \partial_x V(t-t')(u_x v_x)(t') dt' \right\|_{L^2} \lesssim t^{1-\frac{(1+|s|)}{p}-a} \|u\|_{Y_T^s} \|v\|_{Y_T^s} \int_0^1 \frac{1}{|1-\tau|^a |\tau|^{\frac{2(1+|s|)}{p}}} d\tau. \tag{2.52}$$

For our choice of $a = \frac{3^+}{2p}$ and $s > 1 - \frac{p}{2}$ the integral in the RHS of (2.52) is finite. Therefore, from (2.52), we obtain

$$\begin{aligned} t^{\frac{1+|s|}{p}} \left\| \int_0^t \partial_x V(t-t')(u_x v_x)(t') dt' \right\|_{L^2} &\lesssim t^{\frac{2p+2s-5^+}{2p}} \|u\|_{Y_T^s} \|v\|_{Y_T^s} \\ &\lesssim t^{\frac{p-2+2s}{2p}} \|u\|_{Y_T^s} \|v\|_{Y_T^s}. \end{aligned} \tag{2.53}$$

Combining (2.49) and (2.53) we get the required estimate (2.44). □

The following results deal with the gain of regularity of the nonlinear part.

PROPOSITION 2.11. Let $-\frac{p}{2} < s, p > 3$ and $0 \leq \mu < \frac{p}{2}$. If

$$\|f\|_{Z_T^s} := \sup_{t \in (0, T]} \left\{ \|f(t)\|_{H^s} + t^{\frac{|s|}{p}} \|f(t)\|_{L^2} \right\} < \infty, \tag{2.54}$$

then the application

$$t \mapsto \mathcal{L}(f)(t) := \int_0^t V(t-t') \partial_x (f^2)(t') dt', \quad 0 \leq t \leq T \leq 1, \tag{2.55}$$

is continuous from $[0, T]$ to $H^{s+\mu}$.

Proof. We start by proving that $\mathcal{L}(f)(t) \in H^{s+\mu}(\mathbb{R})$ for all f such that $\|f\|_{Z_T^s} < \infty$. We consider two different cases.

CASE I ($s \geq 0$). Let $0 < t \leq T \leq 1$. Since $\langle \xi \rangle^s \leq \langle \xi - x \rangle^s \langle x \rangle^s$ and $f \in Z_T^s$, we have

$$\begin{aligned} \|\mathcal{L}(f)(t)\|_{H^{s+\mu}} &= \|\langle \xi \rangle^{s+\mu} \int_0^t \left(e^{(t-t')\Phi(\xi)} \right) i\xi \widehat{f} * \widehat{f}(\xi, t') dt'\|_{L^2} \\ &\leq \int_0^t \|\langle \xi \rangle^\mu \xi \left(e^{(t-t')\Phi(\xi)} \right)\|_{L^2} \sup_{t' \in (0, T)} \|f(t')\|_{H^s}^2 dt' \quad (2.56) \\ &\lesssim \|f\|_{Z_T^s}^2 \int_0^t \frac{1}{|t-t'|^{\frac{1}{2} + \frac{\mu}{p}}} dt' < \infty, \end{aligned}$$

where the definition of Z_T^s -norm, Minkowski's inequality and inequality (2.27) from Lemma 2.7 are used.

CASE II ($s \leq 0$). Similarly as in the proof of Proposition 2.8, we obtain

$$\begin{aligned} \|\mathcal{L}(f)(t)\|_{H^{s+\mu}} &= \|\langle \xi \rangle^{s+\mu} \int_0^t \left(e^{(t-t')\Phi(\xi)} \right) i\xi \widehat{f} * \widehat{f}(\xi, t') dt'\|_{L^2} \\ &\leq \int_0^t \|\langle \xi \rangle^{s+\mu} \xi \left(e^{(t-t')\Phi(\xi)} \right)\|_{L^2} \frac{1}{|t'|^{\frac{2|s|}{p}}} \|f\|_{Z_T^s}^2 dt' \quad (2.57) \\ &\lesssim \|f\|_{Z_T^s}^2 \int_0^t \frac{1}{|t-t'|^{\frac{1}{2} + \frac{s+\mu}{p}}} dt' < \infty. \end{aligned}$$

Now we move to prove the continuity. Let $t_0 \in [0, T]$ be fixed and let f be such that $\|f\|_{Z_T^s} < \infty$. We will show that

$$\lim_{t \rightarrow t_0} \|\mathcal{L}(f)(t) - \mathcal{L}(f)(t_0)\|_{H^{s+\mu}} = 0. \quad (2.58)$$

Let us use (2.55) and the additive property of the integral to get for $t \in [0, T]$ that

$$\begin{aligned} \|\mathcal{L}(f)(t) - \mathcal{L}(f)(t_0)\|_{H^{s+\mu}} &= \left\| \int_0^{t_0} V(t_0 - t') \partial_x(f^2)(t') d\tau - \int_0^t V(t-t') \partial_x(f^2)(t') dt' \right\|_{H^{s+\mu}} \\ &\leq \left\| \int_0^t (V(t_0 - t') - V(t - t')) \partial_x(f^2)(t') dt' \right\|_{H^{s+\mu}} + \left\| \int_t^{t_0} V(t-t') \partial_x(f^2)(t') dt' \right\|_{H^{s+\mu}} \\ &=: I_1(t, t_0) + I_2(t, t_0). \quad (2.59) \end{aligned}$$

Consider the first term

$$I_1(t, t_0) = \|\langle \xi \rangle^{s+\mu} \int_0^t \left(e^{(t_0-t')\Phi(\xi)} - e^{(t-t')\Phi(\xi)} \right) i\xi \widehat{f} * \widehat{f}(\xi, t') dt'\|_{L^2}. \quad (2.60)$$

As $(e^{(t_0-t')\Phi(\xi)} - e^{(t-t')\Phi(\xi)}) \rightarrow 0$ for $t \rightarrow t_0$, using Lebesgue's Dominated Convergence Theorem we have that

$$I_1(t, t_0) \rightarrow 0, \quad \text{whenever } t \rightarrow t_0.$$

Analogously, as

$$\int_0^{t_0} \|V(t-t') \partial_x(f^2)(t') dt'\|_{H^{s+\mu}} < \infty$$

we also have

$$I_2(t, t_0) \rightarrow 0, \quad \text{whenever } t \rightarrow t_0,$$

and this completes the proof. □

The next result follows by using (2.40) from Lemma 2.9 and the procedure applied in Proposition 2.10.

PROPOSITION 2.12. Let $1 - \frac{\mu}{2} < s, p > 3$ and $0 \leq \mu < \frac{p}{2}$. If

$$\|f\|_{\tilde{Z}_T^s} := \sup_{t \in (0, T]} \left\{ \|f(t)\|_{H^s} + t^{\frac{1+|s|}{p}} \|\partial_x f(t)\|_{L^2} \right\} < \infty, \tag{2.61}$$

then the application

$$t \mapsto \tilde{\mathcal{L}}(f)(t) := \int_0^t V(t-t')(f_x)^2(t') dt', \quad 0 \leq t \leq T \leq 1, \tag{2.62}$$

is continuous from $[0, T]$ to $H^{s+\mu}$.

3. Proof of the well-posedness result. This section is devoted to providing proofs of the local well-posedness results stated in Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let $s \leq 0$ (for $s > 0$, see Appendix). Consider the IVP (1.1) in its equivalent integral form

$$v(t) = V(t)v_0 - \int_0^t V(t-t')(v^2)_x(t') dt', \tag{3.1}$$

where $V(t)$ is the semigroup associated with the linear part given by (1.20).

Define an application

$$\Psi(v)(t) = V(t)v_0 - \int_0^t V(t-t')(v^2)_x(t') dt'. \tag{3.2}$$

For $-\frac{p}{2} \leq s \leq 0, r > 0$ and $0 < T \leq 1$, let us define a ball

$$B_r^T = \{f \in X_T^s; \|f\|_{X_T^s} \leq r\}.$$

We will prove that there exists $r > 0$ and $0 < T \leq 1$ such that the application Ψ maps B_r^T into B_r^T and is a contraction. Let $v \in B_r^T$. By using Lemma 2.5 and Proposition 2.8, we get

$$\|\Psi(v)\|_{X_T^s} \leq c\|v_0\|_{H^s} + cT^\alpha \|v\|_{X_T^s}^2, \tag{3.3}$$

where $\alpha = \frac{2s+p}{2p} > 0$.

Now, using the definition of B_r^T , one obtains

$$\|\Psi(v)\|_{X_T^s} \leq \frac{r}{4} + cT^\alpha r^2 \leq \frac{r}{2}, \tag{3.4}$$

where we have chosen $r = 4c\|v_0\|_{H^s}$ and $T > 0$ such that $cT^\alpha r = 1/4$. Therefore, from (3.4) we see that the application Ψ maps B_r^T into itself. A similar argument proves that Ψ is a contraction. Hence Ψ has a fixed point v which is a solution of the IVP (1.1) such that $v \in C([0, T], H^s(\mathbb{R}))$. The smoothness of the solution map $v_0 \mapsto v$ is a consequence of the contraction mapping principle using the Implicit Function Theorem (for details see [18]).

For the regularity part, we have from Lemma 2.4 that the linear part is in

$$C([0, \infty); H^s(\mathbb{R})) \cap C((0, \infty); H^\infty(\mathbb{R})).$$

Proposition 2.11 shows that the nonlinear part is in $C((0, T]; H^{s+\mu}(\mathbb{R}))$, $\mu > 0$. Combining this information, we have $v \in C([0, T]; H^s(\mathbb{R})) \cap C((0, T]; H^{s+\mu}(\mathbb{R}))$. The rest of the proof follows a standard argument, so we omit the details. \square

Proof of Theorem 1.2. The proof of this theorem is similar to the one presented for Theorem 1.1. Here, we will use the estimates from Lemma 2.6 and Proposition 2.10, so, we omit the details. \square

4. Ill-posedness result. In this section we will use the ideas presented in [21] to prove the ill-posedness result stated in Theorems 1.3 and 1.4. The idea is to prove that there are no spaces X_T^s and Y_T^s that are continuously embedded in $C([0, T]; H^s(\mathbb{R}))$ on which a contraction mapping argument can be applied. We start with the following result.

PROPOSITION 4.1. Let $p \geq 2$, $s < -\frac{p}{2}$ and $T > 0$. Then there does not exist a space X_T^s continuously embedded in $C([0, T]; H^s(\mathbb{R}))$ such that

$$\|V(t)v_0\|_{X_T^s} \lesssim \|v_0\|_{H^s} \tag{4.1}$$

and

$$\left\| \int_0^t V(t-t')\partial_x(v(t'))^2 dt' \right\|_{X_T^s} \lesssim \|v\|_{X_T^s}^2 \tag{4.2}$$

hold true.

Proof. The proof follows a contradiction argument. If possible, suppose that there exists a space X_T^s that is continuously embedded in $C([0, T]; H^s(\mathbb{R}))$ such that the estimates (4.1) and (4.2) hold true. If we consider $v = V(t)v_0$, then from (4.1) and (4.2) we get

$$\left\| \int_0^t V(t-t')\partial_x[V(t')v_0]^2 dt' \right\|_{H^s} \lesssim \|v_0\|_{H^s}^2. \tag{4.3}$$

The main idea to complete the proof is to find appropriate initial data v_0 for which the estimate (4.3) fails to hold whenever $s < -\frac{p}{2}$.

Let $N \gg 1$, $0 < \gamma \ll 1$, $I_N := [N, N + 2\gamma]$ and define initial data via the Fourier transform

$$\widehat{v}_0(\xi) := N^{-s}\gamma^{-\frac{1}{2}} [\chi_{\{I_N\}}(\xi) + \chi_{\{-I_N\}}(\xi)]. \tag{4.4}$$

A simple calculation shows that $\|v_0\|_{H^s} \sim 1$.

Now, we move to calculate the H^s norm of $f(x, t)$, where

$$f(x, t) := \int_0^t V(t-t')\partial_x[V(t')v_0]^2 dt'. \tag{4.5}$$

Taking the Fourier transform in the space variable x , we get

$$\begin{aligned} \widehat{f}(t)(\xi) &= \int_0^t e^{i(t-t')\xi^3+(t-t')\Phi(\xi)} i\xi \left(\widehat{V(t')v_0} * \widehat{V(t')v_0} \right) (\xi) dt' \\ &= \int_0^t e^{i(t-t')\xi^3+(t-t')\Phi(\xi)} i\xi \int_{\mathbb{R}} \widehat{v_0}(\xi - \xi_1) \widehat{v_0}(\xi_1) e^{it'\xi_1^3+t'\Phi(\xi_1)+it'(\xi-\xi_1)^3+t'\Phi(\xi-\xi_1)} d\xi_1 dt' \\ &= i\xi e^{it\xi^3+t\Phi(\xi)} \int_{\mathbb{R}} \widehat{v_0}(\xi - \xi_1) \widehat{v_0}(\xi_1) \int_0^t e^{it'[-\xi^3+\xi_1^3+(\xi-\xi_1)^3]+t'[\Phi(\xi_1)-\Phi(\xi)+\Phi(\xi-\xi_1)]} dt' d\xi_1. \end{aligned} \tag{4.6}$$

We have that

$$\int_0^t e^{it'[3\xi\xi_1(\xi_1-\xi)]+t'[\Phi(\xi_1)-\Phi(\xi)+\Phi(\xi-\xi_1)]} dt' = \frac{e^{it3\xi\xi_1(\xi_1-\xi)+t[\Phi(\xi_1)-\Phi(\xi)+\Phi(\xi-\xi_1)]} - 1}{\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1) + i3\xi\xi_1(\xi_1 - \xi)}. \tag{4.7}$$

Now, inserting (4.7) into (4.6), one obtains

$$\widehat{f}(t)(\xi) = i\xi e^{it\xi^3} \int_{\mathbb{R}} \widehat{v_0}(\xi - \xi_1) \widehat{v_0}(\xi_1) \frac{e^{it3\xi\xi_1(\xi_1-\xi)+t\Phi(\xi_1)+t\Phi(\xi-\xi_1)} - e^{t\Phi(\xi)}}{\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1) + i3\xi\xi_1(\xi_1 - \xi)} d\xi_1. \tag{4.8}$$

Therefore,

$$\|f\|_{H^s}^2 \gtrsim \int_{-\gamma/2}^{\gamma/2} \langle \xi \rangle^{2s} \frac{\xi^2}{N^{4s}\gamma^2} \left| \int_K \frac{e^{3it\xi\xi_1(\xi_1-\xi)+t\Phi(\xi_1)+t\Phi(\xi-\xi_1)} - e^{t\Phi(\xi)}}{\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1) + 3i\xi\xi_1(\xi_1 - \xi)} d\xi_1 \right|^2 d\xi, \tag{4.9}$$

where

$$K = \{\xi_1; \quad \xi - \xi_1 \in I_N, \xi_1 \in -I_N\} \cup \{\xi_1; \quad \xi_1 \in I_N, \xi - \xi_1 \in -I_N\}.$$

We have that $|K| \geq \gamma$ and

$$|3\xi\xi_1(\xi_1 - \xi)| \approx N^2\gamma. \tag{4.10}$$

In order to estimate (4.9) we consider two cases:

Case 1: $\xi - \xi_1 \in I_N, \xi_1 \in -I_N$. In this case

$$\begin{aligned} |\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1)| &= | -(-\xi_1)^p + |\xi|^p - (\xi - \xi_1)^p + \Phi_1(\xi_1) - \Phi_1(\xi) + \Phi_1(\xi - \xi_1) | \\ &\leq | -2(-1)^p \xi_1^p | + |(\xi^p - p\xi^{p-1}\xi_1 + \dots + p(-1)^{p-1}\xi\xi_1^{p-1}) \\ &\quad + |\xi|^p + \Phi_1(\xi_1) - \Phi_1(\xi) + \Phi_1(\xi - \xi_1)|. \end{aligned} \tag{4.11}$$

Therefore,

$$|\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1)| \leq C(N^p + N^r) \leq 2CN^p, \quad r < p. \tag{4.12}$$

Similarly we obtain $|\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1)| \geq C(N^p - N^r) \gtrsim N^p, r < p$.

Hence

$$|\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1)| \sim N^p. \tag{4.13}$$

Case 2: $\xi_1 \in I_N, \xi - \xi_1 \in -I_N$. In this case

$$\begin{aligned} |\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1)| &= | -\xi_1^p + |\xi|^p - (-1)^p(\xi - \xi_1)^p + \Phi_1(\xi_1) - \Phi_1(\xi) + \Phi_1(\xi - \xi_1) | \\ &\leq | -2\xi_1^p | + |(-1)^p(\xi^p - p\xi^{p-1}\xi_1 + \dots + p(-1)^{p-1}\xi\xi_1^{p-1}) \\ &\quad + |\xi|^p + \Phi_1(\xi_1) - \Phi_1(\xi) + \Phi_1(\xi - \xi_1)|. \end{aligned} \tag{4.14}$$

In this way,

$$|\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1)| \leq C(N^p + N^r) \leq 2CN^p, \quad r < p, \tag{4.15}$$

and analogously $|\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1)| \gtrsim N^p$. Therefore,

$$|\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1)| \sim N^p. \tag{4.16}$$

Similarly for any $\xi_1 \in K$, we get, for large N ,

$$\begin{aligned} \Phi(\xi_1) + \Phi(\xi - \xi_1) &= -|\xi_1|^p - |\xi - \xi_1|^p + \Phi_1(\xi_1) + \Phi_1(\xi - \xi_1) \leq -2N^p + CN^r \\ &\leq -N^p, \end{aligned} \tag{4.17}$$

$r < p$.

Let

$$\frac{f}{g} := \frac{e^{3it\xi\xi_1(\xi_1-\xi)+t\Phi(\xi_1)+t\Phi(\xi-\xi_1)} - e^{t\Phi(\xi)}}{\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1) + 3i\xi\xi_1(\xi_1 - \xi)};$$

then

$$\left| \operatorname{Re} \left\{ \frac{f}{g} \right\} \right| = \frac{|\operatorname{Re} f \operatorname{Re} g + \operatorname{Im} f \operatorname{Im} g|}{|g|^2} \geq \frac{|\operatorname{Re} f \operatorname{Re} g|}{|g|^2} - \frac{|\operatorname{Im} f \operatorname{Im} g|}{|g|^2}. \tag{4.18}$$

For $\gamma \ll 1$, one can obtain

$$\begin{aligned} \operatorname{Re} f &= \operatorname{Re} \left\{ e^{3it\xi\xi_1(\xi_1-\xi)+t\Phi(\xi_1)+t\Phi(\xi-\xi_1)} - e^{t\Phi(\xi)} \right\} \leq e^{t\Phi(\xi_1)+t\Phi(\xi-\xi_1)} - e^{-t\gamma^p/2} \\ &\leq e^{-tN^p} - e^{-t\gamma^p/2} \\ &\leq \frac{-e^{-t\gamma^p/2}}{2} \end{aligned} \tag{4.19}$$

and also

$$\begin{aligned} \operatorname{Im} f &= \operatorname{Im} \left\{ e^{3it\xi\xi_1(\xi_1-\xi)+t\Phi(\xi_1)+t\Phi(\xi-\xi_1)} - e^{t\Phi(\xi)} \right\} \leq e^{t\Phi(\xi_1)+t\Phi(\xi-\xi_1)} \\ &\leq e^{-tN^p}. \end{aligned} \tag{4.20}$$

From (4.10), (4.13) and (4.16) we conclude that for any $\xi_1 \in K$, one has

$$|\operatorname{Re} g| = |\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1)| \sim N^p, \quad |\operatorname{Im} g| = 3|\xi\xi_1(\xi_1 - \xi)| \sim N^2\gamma. \tag{4.21}$$

Using (4.21), (4.19), (4.20) and (4.18) considering N very large, it follows that

$$\left| \operatorname{Re} \left\{ \frac{f}{g} \right\} \right| \gtrsim \frac{-e^{-t\gamma^p/2}N^p}{N^{2p} + \gamma^2N^4} - \frac{e^{-tN^p}}{N^p + \gamma N^2} \gtrsim \frac{-e^{-t\gamma^p/2}N^p}{N^{2p} + \gamma^2N^4}. \tag{4.22}$$

Combining (4.9), (4.21) and (4.19), using that $|z| \geq -\operatorname{Re}z$, we arrive at

$$\|f\|_{H^s}^2 \gtrsim \gamma^{-2}N^{-4s}\gamma\langle\gamma\rangle^{2s}\gamma^2 \frac{e^{-t\gamma^p}N^{2p}}{(N^{2p} + \gamma^2N^4)^2}\gamma. \tag{4.23}$$

Taking $\gamma \sim 1$ and N very large, we obtain

$$\|f\|_{H^s}^2 \gtrsim \begin{cases} N^{-4s-2p}, & \text{if } p \geq 2, \\ N^{-4s+2p-8}, & \text{if } p \leq 2, \end{cases} \tag{4.24}$$

and this is a contradiction if $-4s - 2p > 0$ for $p \geq 2$ and if $-4s + 2p - 8 > 0$ for $p \leq 2$, or equivalently $s < -\frac{p}{2}$ for $p \geq 2$ and $s < \frac{p}{2} - 2$ for $0 \leq p \leq 2$. \square

Proof of Theorem 1.3. For $v_0 \in H^s(\mathbb{R})$, consider the Cauchy problem

$$\begin{cases} v_t + v_{xxx} + \eta Lv + (v^2)_x = 0, & x \in \mathbb{R}, t \geq 0, \\ v(x, 0) = \epsilon v_0(x), \end{cases} \tag{4.25}$$

where $\epsilon > 0$ is a parameter. The solution $v^\epsilon(x, t)$ of (4.25) depends on the parameter ϵ . We can write (4.25) in the equivalent integral equation form as

$$v^\epsilon(t) = \epsilon V(t)v_0 - \int_0^t V(t-t')(v^2)_x(t')dt', \tag{4.26}$$

where $V(t)$ is the unitary group describing the solution of the linear part of the IVP (4.25).

Differentiating $v^\epsilon(x, t)$ in (4.26) with respect to ϵ and evaluating at $\epsilon = 0$, we get

$$\left. \frac{\partial v^\epsilon(x, t)}{\partial \epsilon} \right|_{\epsilon=0} = V(t)v_0(x) =: v_1(x) \tag{4.27}$$

and

$$\left. \frac{\partial^2 v^\epsilon(x, t)}{\partial \epsilon^2} \right|_{\epsilon=0} = 2 \int_0^t V(t-t')\partial_x(v_1^2(x, t'))dt' =: v_2(x). \tag{4.28}$$

If the flow-map is C^2 at the origin from $H^s(\mathbb{R})$ to $C([-T, T]; H^s(\mathbb{R}))$, we must have

$$\|v_2\|_{L_T^\infty H^s(\mathbb{R})} \lesssim \|v_0\|_{H^s(\mathbb{R})}^2. \tag{4.29}$$

But from Proposition 4.1 we have seen that the estimate (4.29) fails to hold for $s < -\frac{p}{2}$ if we consider v_0 given by (4.4), and this completes the proof of the theorem. \square

Now, we move to prove an ill-posedness result to the IVP (1.2).

PROPOSITION 4.2. Let $p \geq 2$, $s < 1 - \frac{p}{2}$ and $T > 0$. Then there does not exist a space Y_T^s continuously embedded in $C([0, T]; H^s(\mathbb{R}))$ such that

$$\|V(t)u_0\|_{Y_T^s} \lesssim \|u_0\|_{H^s} \tag{4.30}$$

and

$$\left\| \int_0^t V(t-t')(u_x(t'))^2 dt' \right\|_{Y_T^s} \lesssim \|u\|_{Y_T^s}^2 \tag{4.31}$$

hold true.

Proof. Analogously as in the proof of Proposition 4.1 we consider the same v_0 as defined in (4.4). We consider $u_0 := v_0$ and calculate the H^s norm of $g(x, t)$, where

$$g(x, t) := \int_0^t V(t-t')[\partial_x V(t')u_0]^2 dt'. \tag{4.32}$$

We have

$$\widehat{g(t)}(\xi) = i\xi e^{it\xi^3} \int_{\mathbb{R}} (\xi - \xi_1)\widehat{v_0}(\xi - \xi_1)\xi_1\widehat{u_0}(\xi_1) \frac{e^{it3\xi\xi_1(\xi_1 - \xi)} + t\Phi(\xi_1) + t\Phi(\xi - \xi_1) - e^{t\Phi(\xi)}}{\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1) + i3\xi\xi_1(\xi_1 - \xi)} d\xi_1$$

and

$$\|g\|_{H^s}^2 \gtrsim \int_{-\gamma/2}^{\gamma/2} \langle \xi \rangle^{2s} \frac{\xi^2}{N^{4s}\gamma^2} \left| \int_{K_\epsilon} \frac{\xi_1(\xi - \xi_1)e^{3it\xi\xi_1(\xi_1 - \xi)} + t\Phi(\xi_1) + t\Phi(\xi - \xi_1) - e^{t\Phi(\xi)}}{\Phi(\xi_1) - \Phi(\xi) + \Phi(\xi - \xi_1) + 3i\xi\xi_1(\xi_1 - \xi)} d\xi_1 \right|^2 d\xi, \tag{4.33}$$

where

$$K_\epsilon = \{\xi_1; \quad \xi - \xi_1 \in I_N, \xi_1 \in -I_N\} \cup \{\xi_1; \quad \xi_1 \in I_N, \xi - \xi_1 \in -I_N\}.$$

Similarly as in the proof of Proposition 4.1, we obtain

$$\|g\|_{H^s}^2 \gtrsim \gamma^{-2} N^{-4s} \gamma \langle \gamma \rangle^{2s} \gamma^2 \frac{N^4 e^{-t\gamma^p}}{(N^p + \gamma N^2)^2} \gamma. \tag{4.34}$$

Taking $p \geq 2$, $\gamma \sim 1$ and N very large, one can get

$$\|g\|_{H^s}^2 \gtrsim N^{-4s-2p+4},$$

and this is a contradiction if $-4s - 2p + 4 > 0$ or equivalently if $s < 1 - \frac{p}{2}$. □

Proof of Theorem 1.4. For $v_0 \in H^s(\mathbb{R})$, consider the Cauchy problem

$$\begin{cases} u_t + u_{xxx} + \eta Lu + (u_x)^2 = 0, & x \in \mathbb{R}, t \geq 0, \\ u(x, 0) = \epsilon u_0(x), \end{cases} \tag{4.35}$$

where $\epsilon > 0$ is a parameter. The solution $u^\epsilon(x, t)$ of (4.35) depends on the parameter ϵ . We can write (4.35) in the equivalent integral equation form as

$$u^\epsilon(t) = \epsilon V(t)u_0 - \int_0^t V(t-t')(u_x)^2(t')dt', \tag{4.36}$$

where $V(t)$ is the unitary group describing the solution of the linear part of the IVP (4.35).

Differentiating $u^\epsilon(x, t)$ in (4.36) with respect to ϵ and evaluating at $\epsilon = 0$ we get

$$\frac{\partial u^\epsilon(x, t)}{\partial \epsilon} \Big|_{\epsilon=0} = V(t)u_0(x) =: u_1(x) \tag{4.37}$$

and

$$\frac{\partial^2 u^\epsilon(x, t)}{\partial \epsilon^2} \Big|_{\epsilon=0} = 2 \int_0^t V(t-t')(\partial_x u_1(x, t'))^2 dt' =: u_2(x). \tag{4.38}$$

If the flow-map is C^2 at the origin from $H^s(\mathbb{R})$ to $C([-T, T]; H^s(\mathbb{R}))$, we must have

$$\|u_2\|_{L_T^\infty H^s(\mathbb{R})} \lesssim \|u_0\|_{H^s(\mathbb{R})}^2. \tag{4.39}$$

But from Proposition 4.2 we have that the estimate (4.39) fails to hold for $s < 1 - \frac{p}{2}$ if we consider $u_0 := v_0$ given by (4.4), and this completes the proof. □

Appendix A. Our objective here is to prove local well-posedness results for the IVPs (1.1) and (1.2) for given data in $H^s(\mathbb{R})$, $s > 0$. However, our method of proof shows that this can be achieved by proving for $s > -1$ and $s > 0$ respectively.

Let $s \in \mathbb{R}$, $0 < t \leq T \leq 1$, $p > \frac{5}{2}$. To deal with the IVP (1.1), we define

$$\|f\|_{\mathcal{X}_T^s} := \sup_{t \in [0, T]} \left\{ \|f(t)\|_{H^s(\mathbb{R})} + t^{\frac{s}{4p}} \left(\|f(t)\|_{L_x^4} + \|\partial_x f(t)\|_{L_x^4} + \|D_x^s \partial_x f(t)\|_{L_x^4} \right) \right\} \tag{A.1}$$

and introduce a Banach space

$$\mathcal{X}_T^s := \{f \in C([0, T]; H^s(\mathbb{R})) : \|f\|_{\mathcal{X}_T^s} < \infty\}. \tag{A.2}$$

To deal with the IVP (1.2), we define

$$\|f\|_{\mathcal{Y}_T^s} := \sup_{t \in [0, T]} \left\{ \|f(t)\|_{H^s} + t^{\frac{5}{4p}} \left(\|\partial_x f(t)\|_{L_x^4} + \|D_x^s \partial_x f(t)\|_{L_x^4} \right) \right\} \tag{A.3}$$

and introduce the next Banach space

$$\mathcal{Y}_T^s := \{f \in C([0, T]; H^s(\mathbb{R})) : \|f\|_{\mathcal{Y}_T^s} < \infty\}. \tag{A.4}$$

The local well-posedness results read as follows.

THEOREM A.1. Let $\eta > 0$ be fixed and $\Phi(\xi)$ be given by (1.3) with $p \geq \frac{5}{2}$ as the order of the leading term. Then the IVP (1.1) is locally well-posed for any data $v_0 \in H^s(\mathbb{R})$ whenever $s > -1$. Moreover, the map $v_0 \mapsto v$ is smooth from $H^s(\mathbb{R})$ to $C([0, T]; H^s(\mathbb{R})) \cap \mathcal{X}_T^s$.

THEOREM A.2. Let $\eta > 0$ be fixed and $\Phi(\xi)$ be given by (1.3) with $p \geq \frac{5}{2}$ as the order of the leading term. Then the IVP (1.2) is locally well-posed for any data $u_0 \in H^s(\mathbb{R})$ whenever $s > 0$. Moreover, the map $u_0 \mapsto u$ is smooth from $H^s(\mathbb{R})$ to $C([0, T]; H^s(\mathbb{R})) \cap \mathcal{Y}_T^s$.

Before proving these theorems, we record some linear and nonlinear estimates. We start with the linear estimates whose proofs follow in Lemmas 2.5 and 2.6.

LEMMA A.3. Let $V(t)$ be as defined in (1.20), $0 < t \leq T \leq 1$ and $p > \frac{5}{2}$. Then for all $s \in \mathbb{R}$, we have

$$\|V(t)w_0\|_{\mathcal{X}_T^s} \leq C\|w_0\|_{H^s} \tag{A.5}$$

and

$$\|V(t)w_0\|_{\mathcal{Y}_T^s} \leq C\|w_0\|_{H^s}. \tag{A.6}$$

Now we prove nonlinear estimates.

LEMMA A.4. Let $V(t)$ be as defined in (1.20), $0 < t \leq T \leq 1$ and $p > \frac{5}{2}$. Then the following estimates hold true:

$$\left\| \int_0^t V(t-\tau) \partial_x(v^2)(\tau) d\tau \right\|_{\mathcal{X}_T^s} \leq T^\omega \|v\|_{\mathcal{X}_T^s}^2, \quad \forall s \geq -1, \tag{A.7}$$

and

$$\left\| \int_0^t V(t-\tau) (\partial_x u)^2(\tau) d\tau \right\|_{\mathcal{Y}_T^s} \leq T^\omega \|u\|_{\mathcal{Y}_T^s}^2, \quad \forall s \geq 0, \tag{A.8}$$

where $\omega = \frac{2p-5}{2p} > 0$.

Proof. By using (A.5) from Lemma A.3, we get

$$\begin{aligned} \left\| \int_0^t V(t-\tau) \partial_x(v^2)(\tau) d\tau \right\|_{\mathcal{X}_T^s} &\leq \int_0^t \|\partial_x(v^2)(\tau)\|_{H^s} d\tau \\ &\leq C \int_0^T (\|\partial_x(v^2)\|_{L^2} + \|D_x^s \partial_x(v^2)\|_{L^2}) d\tau \\ &\leq C \int_0^T \|\partial_x(v^2)\|_{L^2} d\tau + \int_0^T \|D_x^{\tilde{s}}(v^2)\|_{L^2} d\tau \\ &=: I_1 + I_2, \end{aligned} \tag{A.9}$$

where $\tilde{s} = 1 + s$. In what follows, we will obtain an estimate for I_2 . Now, using the fractional chain rule (see Tao [26], (A.15), page 338), for $\tilde{s} \geq 0$, i.e., for $s \geq -1$, we have

$$\begin{aligned}
 I_2 &\leq C \int_0^T \|v\|_{L^4} \|D_x^{\tilde{s}} v\|_{L^4} d\tau \\
 &\leq C \|v\|_{\mathcal{X}_T^s}^2 \int_0^T \frac{1}{\tau^{\frac{5}{4p}}} \frac{1}{\tau^{\frac{5}{4p}}} d\tau \\
 &\leq C \|v\|_{\mathcal{X}_T^s}^2 \int_0^T \frac{1}{\tau^{\frac{5}{2p}}} d\tau \\
 &\leq C_p \|v\|_{\mathcal{X}_T^s}^2 T^\omega.
 \end{aligned}
 \tag{A.10}$$

The estimate for I_1 will follow from that of I_2 considering $s = 0$. In fact,

$$I_1 \leq C \int_0^T \|v\|_{L^4} \|\partial_x v\|_{L^4} d\tau \leq C \|v\|_{\mathcal{X}_T^s}^2 \int_0^T \frac{1}{\tau^{\frac{5}{2p}}} d\tau \leq C_p \|v\|_{\mathcal{X}_T^s}^2 T^\omega.
 \tag{A.11}$$

Inserting estimates (A.10) and (A.11) into (A.9) we obtain the required estimate (A.7).

Now, we move to prove the estimate (A.8). By using (A.6) from Lemma A.3 and the fractional chain rule as in (A.10), for $s \geq 0$, we get

$$\begin{aligned}
 \left\| \int_0^t V(t - \tau)(\partial_x u)^2(\tau) d\tau \right\|_{\mathcal{Y}_T^s} &\leq C \int_0^T (\|(u_x)^2\|_{L^2} + \|D_x^s(u_x)^2\|_{L^2}) d\tau \\
 &\leq C \int_0^T (\|u_x\|_{L^4}^2 + \|u_x\|_{L^4} \|D_x^s(u_x)\|_{L^4}) d\tau \\
 &\leq C \|u\|_{\mathcal{Y}_T^s}^2 \int_0^T \frac{1}{\tau^{\frac{10}{4p}}} d\tau + C \|u\|_{\mathcal{Y}_T^s}^2 \int_0^T \frac{1}{\tau^{\frac{5}{4p}}} \frac{1}{\tau^{\frac{5}{4p}}} d\tau \\
 &\leq C \|u\|_{\mathcal{Y}_T^s}^2 \int_0^T \frac{1}{\tau^{\frac{5}{2p}}} d\tau \\
 &\leq C_p \|u\|_{\mathcal{Y}_T^s}^2 T^\omega,
 \end{aligned}
 \tag{A.12}$$

as required. □

Using the linear and nonlinear estimates from Lemmas A.3 and A.4, the proofs of Theorems A.1 and A.2 follow using the same argument as in the proofs of Theorems 1.1 and 1.2 respectively. So we omit the details.

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