QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXXIV, NUMBER 4
DECEMBER 2016, PAGES 705–718
http://dx.doi.org/10.1090/qam/1447
Article electronically published on July 20, 2016

JUNCTION PROBLEM FOR EULER-BERNOULLI AND TIMOSHENKO ELASTIC INCLUSIONS IN ELASTIC BODIES

By

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Abstract. In the paper, we consider an equilibrium problem for a 2D elastic body with thin Euler-Bernoulli and Timoshenko elastic inclusions. It is assumed that inclusions have a joint point, and we analyze a junction problem for these inclusions. Existence of solutions is proved, and different equivalent formulations of the problem are discussed. In particular, junction conditions at the joint point are found. A delamination of the elastic inclusion is also assumed. In this case, inequality type boundary conditions are imposed at the crack faces to prevent a mutual penetration between crack faces. We investigate a convergence to infinity and to zero of a rigidity parameter of the elastic inclusions. Limit problems are analyzed.

1. Introduction. It is known that a quality of composite materials largely depends on properties of inclusions used. In the paper, we consider a junction problem for thin inclusions with different properties located inside an elastic body. In particular, elastic and rigid inclusions are considered, and inclusions of zero rigidity as well. As for elastic thin inclusions, we use two models for describing equilibrium states: Euler-Bernoulli and Timoshenko. This means that thin elastic inclusions are governed by the Euler-Bernoulli and Timoshenko equations with right-hand sides describing an influence of the elastic surrounding media. In fact, the inclusions are considered as thin elastic Euler-Bernoulli and Timoshenko beams incorporated in the elastic body. We should remark that there are different approaches used to describe elastic inclusions in elastic bodies (without delaminations); in particular, we can mention works [24, 25, 31]. Also, we analyze a delamination of the inclusions. In this case we have a crack between the inclusion and the elastic matrix. To model the phenomena, we impose inequality type boundary conditions

Received January 20, 2015.
2010 *Mathematics Subject Classification.* Primary 35Q74, 49J40.
Key words and phrases. Thin inclusion, rigid inclusion, non-penetration condition, crack, variational inequality, junction conditions.
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at the crack faces which guarantees a mutual non-penetration between the faces. It is well known that linear classical models for cracks in deformable bodies are characterized by linear boundary conditions at the crack faces; see, for example, [1], [3], [26]. Note that there are a lot of publications concerning models with the non-penetration conditions at crack faces. In particular, the book [11] contains results for crack models for a wide class of constitutive laws; see also [9, 17, 18, 20, 21, 29]. Existence theorems and qualitative properties of solutions in equilibrium problems for elastic bodies with rigid inclusions can be found in [8, 10], [13], [16], [27], [28], [30]. Elastic behavior of bodies with cracks and rigid inclusions is analyzed in the book [12]. We should remark that cracks also can be seen as inclusions of zero rigidity [14], [15]. Linear crack models are also analyzed, and for scalar second order elliptic equations, see [19].

The paper is organized as follows. In Section 2, an equilibrium problem for an elastic body with Euler-Bernoulli and Timoshenko inclusions is considered. An existence of variational solution is proved, and it is shown that the variational problem formulation is equivalent to a differential setting. We find junction conditions at the joint point. Section 3 is concerned with a case when the Euler-Bernoulli inclusion is delaminated, therefore providing a crack between the inclusion and the elastic matrix. We impose an inequality type boundary condition at the crack faces to exclude a mutual penetration between the crack faces. In this case, contact points between crack faces are unknown, and the model is formulated as a free boundary problem. In Section 4 we analyze a passage of a rigidity parameter of the Euler-Bernoulli inclusion to infinity. It is proved that in the limit, displacements of the Euler-Bernoulli inclusion have a special structure. Also, we find junction conditions at the joint point. A passage of the rigidity parameter to zero is discussed in Section 5.

We refer the reader to [2, 4–6], [22, 23], [32], where different junction problems for beams and plates are discussed.

2. Setting of the problem. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary Γ. Denote $\gamma_b = (0,1) \times \{0\}$, $\gamma_t = (1,2) \times \{0\}$, $\gamma = \gamma_b \cup \gamma_t \cup \{(1,0)\}$ assuming $\bar{\gamma} \subset \Omega$; see Figure 1. Denote by $\nu = (0,1)$ a unit normal vector to γ; $\tau = (0,1)$, $\Omega_\gamma = \Omega \setminus \bar{\gamma}$.

In what follows, the domain Ω_γ represents a region with an elastic material, and γ_b and γ_t are thin elastic Euler-Bernoulli and Timoshenko inclusions, respectively, incorporated in the elastic material. This means that a behavior of the inclusions is described by the Euler-Bernoulli and Timoshenko equations.

Let $B = \{b_{ijkl}\}, i,j,k,l = 1,2,$ be a given elasticity tensor with the usual properties of symmetry and positive definiteness,

$$b_{ijkl} = b_{jikl} = b_{klij}, \;\; i,j,k,l = 1,2, \; b_{ijkl} \in L^\infty(\Omega),$$
$$b_{ijkl}\xi_{ij}\xi_{kl} \ge c_0|\xi|^2 \quad \forall \xi_{ji} = \xi_{ij}, \; c_0 = \text{const} > 0 \; .$$

Summation convention over repeated indices is used; all functions with two lower indices are assumed to be symmetric in these indices. Let $f = (f_1, f_2) \in L^2(\Omega)^2$ be a given function.

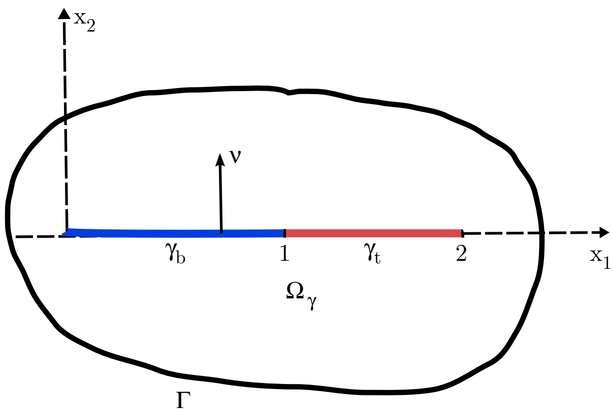

FIG. 1

We first provide a variational formulation of the equilibrium problem for the elastic body with thin inclusions γ_b, γ_t. We introduce a space

$$W = \{(u, v, w, \varphi) \,|\, u \in H_0^1(\Omega)^2, (v, w) \in H^1(\gamma)^2, v \in H^2(\gamma_b),$$
$$\varphi \in H^1(\gamma_t);\ v = u_\nu,\ w = u_\tau \text{ on } \gamma;\ v_x(1-) + \varphi(1+) = 0\}$$

with the norm

$$\|(u, v, w, \varphi)\|_W^2 = \|u\|^2_{H_0^1(\Omega)^2} + \|(v, w)\|^2_{H^1(\gamma)^2} + \|v\|^2_{H^2(\gamma_b)} + \|\varphi\|^2_{H^1(\gamma_t)}$$

and consider an energy functional on this space,

$$\pi(u, v, w, \varphi) = \frac{1}{2} \int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(u) - \int\limits_{\Omega_\gamma} fu$$
$$+\frac{1}{2} \int\limits_{\gamma_b} v_{xx}^2 + \frac{1}{2} \int\limits_{\gamma} w_x^2 + \frac{1}{2} \int\limits_{\gamma_t} \{\varphi_x^2 + (v_x + \varphi)^2\}.$$

Here $\sigma(u) = \{\sigma_{ij}(u)\},\ \sigma_{ij}(u) = b_{ijkl}\varepsilon_{kl}(u),\ \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), u = (u_1, u_2),\ u_\nu = u\nu,\ u_\tau = u\tau,\ i, j, k, l = 1, 2.$ The standard notation $H_0^1(\Omega),\ H^1(\gamma)$, etc., is used for Sobolev spaces. We identify functions defined on γ with functions of the variable x; $x = x_1,\ h_x = \frac{dh}{dx}, (x_1, x_2) \in \Omega$. For simplicity, we write $\sigma(u)\varepsilon(u) = \sigma_{ij}(u)\varepsilon_{ij}(u), fu = f_i u_i$. In the definition of the functional π, we integrate over Ω_γ. It is done for convenience. In fact, we can integrate over Ω as well.

Consider the minimization problem

$$\text{Find } (u, v, w, \varphi) \in W \text{ such that } \pi(u, v, w, \varphi) = \inf_W \pi. \tag{1}$$

This problem has a unique solution satisfying the identity

$$(u, v, w, \varphi) \in W, \tag{2}$$

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u}) - \int\limits_{\Omega_\gamma} f\bar{u} + \int\limits_{\gamma_b} v_{xx}\bar{v}_{xx} + \int\limits_{\gamma} w_x\bar{w}_x \tag{3}$$

$$+ \int\limits_{\gamma_t} \{(v_x + \varphi)(\bar{v}_x + \bar{\varphi}) + \varphi_x\bar{\varphi}_x\} = 0 \quad \forall\, (\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in W.$$

THEOREM 1. The problem (1) has a solution.

Proof. The functional π is weakly lower semicontinuous; hence, to prove a solution's existence it suffices to check its coercivity on the space W. Let $(u, v, w, \varphi) \in W$. By the Korn inequality, we have

$$\pi(u, v, w, \varphi) \geq c\|u\|^2_{H^1_0(\Omega)^2} - c_1\|u\|_{H^1_0(\Omega)^2} + \frac{1}{2}\int\limits_{\gamma_b} v_{xx}^2 + \frac{1}{2}\int\limits_{\gamma} w_x^2$$

$$+\frac{1}{2}\int\limits_{\gamma_t} \{\varphi_x^2 + (v_x + \varphi)^2\} \pm \alpha \int\limits_{\gamma} (v^2 + w^2),\ \alpha \in \mathbb{R}^2,\ \alpha > 0.$$

According to the boundary conditions $v = u_\nu,\ w = u_\tau$ on γ and imbedding theorems, we have for a small α,

$$\frac{c}{2}\|u\|^2_{H^1_0(\Omega)^2} - \alpha \int\limits_{\gamma} (v^2 + w^2) \geq 0.$$

Then

$$\pi(u, v, w, \varphi) \geq \frac{c}{2}\|u\|^2_{H^1_0(\Omega)^2} - c_1\|u\|_{H^1_0(\Omega)^2} + \frac{1}{2}\int\limits_{\gamma_b} v_{xx}^2 + \frac{1}{2}\int\limits_{\gamma} w_x^2$$

$$+\frac{1}{2}\int\limits_{\gamma_t} \{\varphi_x^2 + (v_x + \varphi)^2\} + \alpha \int\limits_{\gamma} (v^2 + w^2),$$

and taking into account the Lemma below, it follows that

$$\pi(u, v, w, \varphi) \geq \frac{c}{2}\|u\|^2_{H^1_0(\Omega)^2}$$

$$-c_1\|u\|_{H^1_0(\Omega)^2} + c_2\|v\|^2_{H^2(\gamma_b)} + c_3\|w\|^2_{H^1(\gamma)} + c_4\|(v, \varphi)\|^2_{H^1(\gamma_t)^2}.$$

Thus, by the condition $v(1+) = v(1-)$, we obtain

$$\pi(u, v, w, \varphi) \to \infty,\ \|(u, v, w, \varphi)\|_W \to \infty,$$

and therefore a coercivity of the functional π follows. Consequently, a solution of the problem (1) exists. Theorem 1 is proved. □

We formulate the Lemma used to prove Theorem 1.

LEMMA. There exists a constant $c > 0$ such that

$$\int_{\gamma_t} \{v^2 + \varphi_x^2 + (v_x + \varphi)^2\} \geq c\|(v, \varphi)\|^2_{H^1(\gamma_t)^2} \quad \forall (v, \varphi) \in H^1(\gamma_t)^2.$$

A proof of this Lemma can be found in [7].

Now we are able to provide a differential formulation of the problem (2)-(3). For given external forces $f = (f_1, f_2)$ acting on the body, we have to find functions $u = (u_1, u_2), v, w, \varphi$, $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, such that

$$-\text{div}\ \sigma = f,\ \sigma - B\varepsilon(u) = 0 \quad \text{in} \quad \Omega_\gamma, \tag{4}$$

$$v_{xxxx} = [\sigma_\nu] \text{ on } \gamma_b,\ -w_{xx} = [\sigma_\tau] \text{ on } \gamma, \tag{5}$$

$$-v_{xx} - \varphi_x = [\sigma_\nu],\ -\varphi_{xx} + v_x + \varphi = 0 \quad \text{on} \quad \gamma_t, \tag{6}$$

$$(u, v, w, \varphi) \in W;\ v_{xxx} = v_{xx} = w_x = 0 \ \text{for}\, x = 0, \tag{7}$$

$$w_x(1-) = w_x(1+);\ v_x + \varphi = \varphi_x = w_x = 0 \ \text{for}\ x = 2, \tag{8}$$

$$-v_{xxx}(1-) = (v_x + \varphi)(1+);\ v_{xx}(1-) = -\varphi_x(1+). \tag{9}$$

Here $\sigma_\nu = \sigma_{ij}\nu_j\nu_i$, $\sigma_\tau = \sigma\nu \cdot \tau$, and $[h] = h^+ - h^-$ is a jump of a function h on γ, where $h^\pm$ are the traces of h on the faces $\gamma^\pm$. The signs $\pm$ correspond to positive and negative directions of ν.

The function $u = (u_1, u_2)$ describes a displacement field of the elastic body; functions w, v fit into displacements of the inclusions γ_b, γ_t along the axis x_1 and axis x_2 respectively; the function φ describes a rotation angle of the inclusion γ_t. Observe that a part of the boundary conditions for functions u, v, w, φ is included in the condition $(u, v, w, \varphi) \in W$.

Relations (4) are the equilibrium equations for the elastic body and Hooke's law; (5)-(6) are the Euler-Bernoulli and Timoshenko equilibrium equations for the inclusions γ_b and γ_t. According to the condition $(u, v, w, \varphi) \in W$, the vertical (along the axis x_2) and tangential (along the axis x_1) displacements of the elastic body coincide with the inclusion displacements at γ. The right-hand sides $[\sigma_\nu], [\sigma_\tau]$ in (5), (6) describe forces acting on γ from the surrounding elastic media.

In what follows we show an equivalence of (4)-(9) and (2)-(3) for smooth solutions.

THEOREM 2. Problem formulations (4)-(9) and (2)-(3) are equivalent provided that the solutions are smooth.

Proof. Let (4)-(9) be fulfilled. Take $(\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in W$ and multiply the first equation from (4) and equations from (5), (6) by $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{\varphi}$ respectively. Integrating over Ω_γ and

γ_b, γ, γ_t, respectively, we get

$$\int\limits_{\Omega_\gamma} (-\text{div}\ \sigma - f)\bar{u} + \int\limits_{\gamma_t} (-v_{xx} - \varphi_x - [\sigma_\nu])\bar{v} - \int\limits_{\gamma} (w_{xx} + [\sigma_\tau])\bar{w} + \int\limits_{\gamma_t} (-\varphi_{xx} + v_x + \varphi)\bar{\varphi} + \int\limits_{\gamma_b} (v_{xxxx} - [\sigma_\nu])\bar{v} = 0.$$

Hence, by the boundary conditions (7)-(9),

$$\int\limits_{\Omega_\gamma} (\sigma(u)\varepsilon(\bar{u}) - f\bar{u}) + \int\limits_{\gamma} [\sigma\nu]\bar{u} + \int\limits_{\gamma_t} (v_x + \varphi)(\bar{v}_x + \bar{\varphi}) + \int\limits_{\gamma} w_x \bar{w}_x - \int\limits_{\gamma_t} [\sigma_\nu]\bar{v} - \int\limits_{\gamma_b} [\sigma_\nu]\bar{v} + \int\limits_{\gamma_b} v_{xx}\bar{v}_{xx} - \int\limits_{\gamma} [\sigma_\tau]\bar{w} + \int\limits_{\gamma_t} \varphi_x \bar{\varphi}_x = 0. \tag{10}$$

We have $[\sigma\nu]\bar{u} = [\sigma_\nu]\bar{u}_\nu + [\sigma_\tau]\bar{u}_\tau$ on γ. To prove (3), by $(\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in W$, it suffices to check that

$$\int\limits_{\gamma} ([\sigma_\nu]\bar{u}_\nu + [\sigma_\tau]\bar{u}_\tau) - \int\limits_{\gamma_t} [\sigma_\nu]\bar{v} - \int\limits_{\gamma_b} [\sigma_\nu]\bar{v} - \int\limits_{\gamma} [\sigma_\tau]\bar{w} = 0. \tag{11}$$

But (11) follows from the conditions $v = u_\nu,\ w = u_\tau$ on γ. Hence from (10) we obtain the identity (3).

Conversely, let (2)-(3) be fulfilled. We take test functions of the form $(\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) = (\psi, 0, 0, 0),\ \psi \in C_0^\infty(\Omega_\gamma)^2$. This gives the equilibrium equation from (4). Next, from (3) it follows that

$$- \int\limits_{\gamma} [\sigma\nu]\bar{u} + \int\limits_{\gamma_b} v_{xxxx}\bar{v} + \int\limits_{\gamma_t} -(v_{xx} + \varphi_x)\bar{v} - \int\limits_{\gamma} w_{xx}\bar{w} - \int\limits_{\gamma_t} \varphi_{xx}\bar{\varphi} + \int\limits_{\gamma_t} (v_x + \varphi)\bar{\varphi} + v_{xx}\bar{v}_x|_0^1 - v_{xxx}\bar{v}|_0^1 + w_x\bar{w}|_0^2 + \varphi_x\bar{\varphi}|_1^2 + (v_x + \varphi)\bar{v}|_1^2 = 0 \ \ \forall (\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in W. \tag{12}$$

Choosing here test functions such that $\bar{v}_x = \bar{v} = 0$ at $x = 0, 1$; $\bar{w} = 0$ at $x = 0, 2$; $\bar{\varphi} = 0$ at $x = 1, 2$; $\bar{v} = 0$ at $x = 2$, and taking into account boundary condition $\bar{u}|_\gamma = (\bar{w}, \bar{v})$, the equilibrium equations (5)-(6) are derived. Going back to (12) we obtain boundary conditions (7)-(9). Hence, the equivalence of (4)-(9) and (2)-(3) is established. Theorem 2 is proved. □

By (9) and the first relations of (7), (8), we can write a full system of junction conditions at the joint point $(1, 0)$:

$$w(1-) = w(1+),\ v(1-) = v(1+),\ v_x(1-) = -\varphi(1+),$$
$$w_x(1-) = w_x(1+),\ -v_{xxx}(1-) = (v_x + \varphi)(1+),\ v_{xx}(1-) = -\varphi_x(1+).$$

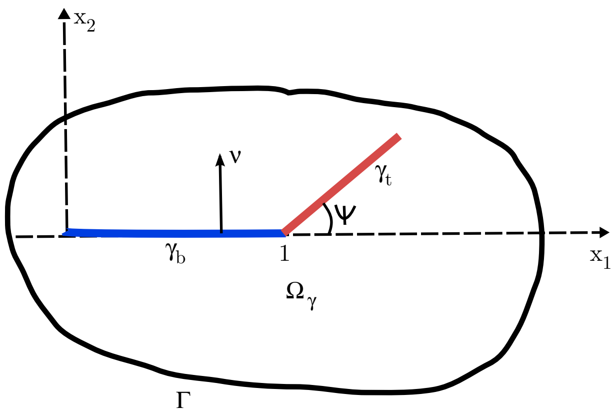

FIG. 2

REMARK. We can formulate an equilibrium problem for the inclusions in a case when an angle between γ_b and γ_t is non-zero at the point $(1,0)$. Denote by ψ an angle between γ_b and γ_t; see Figure 2. Let w_t, v_t and w_b, v_b be tangential and normal displacements of the beams γ_t, γ_b respectively, and let φ be a rotation angle of the beam γ_t. Then kinematical conditions at the junction point $(1,0)$ are as follows:

$$w_t \cos\psi - v_t \sin\psi = w_b,\ w_t \sin\psi + v_t \cos\psi = v_b,\ v_{bx} + \varphi_t = 0. \tag{13}$$

If we formulate an equilibrium problem in the case of Figure 2, these conditions should be included in the definition of the space W. In particular, if $\psi = 0$, the conditions (13) reduce to

$$w_b(1-) = w_t(1+),\ v_b(1-) = v_t(1+),\ v_{bx}(1-) + \varphi_t(1+) = 0$$

fulfilled for the problem (2)-(3).

3. Delaminated elastic inclusion. In this section we assume that the Euler-Bernoulli part γ_b of the inclusion γ is delaminated; see Figure 1. This means that we have a crack located between γ_b and the elastic matrix. To fix the situation, the delamination is assumed to be at the positive side of γ_b. In this case, displacements v, w of the inclusion γ_b should coincide with displacements of the elastic body at γ_b^-. In our model, inequality type boundary conditions are considered at the crack faces to prevent a mutual penetration between the faces.

We denote $\Omega_b = \Omega \setminus \bar{\gamma}_b$ and introduce a set of admissible functions,

$$K = \{(u, v, w, \varphi) \mid u \in H^1_\Gamma(\Omega_b)^2, (v, w) \in H^1(\gamma)^2, v \in H^2(\gamma_b),$$
$$\varphi \in H^1(\gamma_t),\ u|_{\gamma^-} = (w, v),\ [u]\nu \geq 0 \text{ on } \gamma_b;\ v_x(1-) + \varphi(1+) = 0\},$$

where

$$H^1_\Gamma(\Omega_b) = \{\phi \in H^1(\Omega_b) \mid \phi = 0 \text{ on } \Gamma\}.$$

Notice that the inequality $[u]\nu \geq 0$ included in the definition of K provides a mutual non-penetration between the crack faces $\gamma_b^\pm$. An equilibrium problem for the elastic body with the delaminated Euler-Bernoulli inclusion γ_b and the Timoshenko inclusion γ_t can be formulated as follows. We have to find functions $u = (u_1, u_2), v, w, \varphi,\ \sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\text{div}\ \sigma = f,\ \sigma - B\varepsilon(u) = 0 \quad \text{in} \quad \Omega_\gamma, \tag{14}$$

$$v_{xxxx} = [\sigma_\nu] \text{ on } \gamma_b,\ -w_{xx} = [\sigma_\tau] \text{ on } \gamma, \tag{15}$$

$$-v_{xx} - \varphi_x = [\sigma_\nu],\ -\varphi_{xx} + v_x + \varphi = 0 \quad \text{on} \quad \gamma_t, \tag{16}$$

$$(u, v, w, \varphi) \in K;\ v_{xxx} = v_{xx} = w_x = 0 \ \text{for}\, x = 0, \tag{17}$$

$$w_x(1-) = w_x(1+);\ v_x + \varphi = \varphi_x = w_x = 0 \ \text{for}\ x = 2, \tag{18}$$

$$\sigma_\nu^+ \le 0,\ \sigma_\tau^+ = 0,\ \sigma_\nu^+[u]\nu = 0 \text{ on } \gamma_b, \tag{19}$$

$$-v_{xxx}(1-) = (v_x + \varphi)(1+);\ v_{xx}(1-) = -\varphi_x(1+). \tag{20}$$

As before, a part of the boundary conditions is included in the relation $(u, v, w, \varphi) \in K$.

Remark that the problem (14)-(20) admits a variational formulation. Indeed, consider the energy functional

$$\pi_b(u, v, w, \varphi) = \frac{1}{2}\int\limits_{\Omega_b} \sigma(u)\varepsilon(u) - \int\limits_{\Omega_b} fu$$

$$+\frac{1}{2}\int\limits_{\gamma_b} v_{xx}^2 + \frac{1}{2}\int\limits_{\gamma} w_x^2 + \frac{1}{2}\int\limits_{\gamma_t}\{\varphi_x^2 + (v_x + \varphi)^2\}.$$

Then a solution (u, v, w, φ) of the minimization problem,

$$\text{find } (u, v, w, \varphi) \in K \text{ such that } \pi_b(u, v, w, \varphi) = \inf_K \pi_b,$$

satisfies a variational inequality

$$(u, v, w, \varphi) \in K, \tag{21}$$

$$\int\limits_{\Omega_b} \sigma(u)\varepsilon(\bar{u} - u) - \int\limits_{\Omega_b} f(\bar{u} - u) + \int\limits_{\gamma_b} v_{xx}(\bar{v}_{xx} - v_{xx}) \tag{22}$$

$$+\int\limits_{\gamma_t}\{\varphi_x(\bar{\varphi}_x - \varphi_x) + (v_x + \varphi)(\bar{v}_x + \bar{\varphi} - v_x - \varphi)\}$$

$$+\int\limits_{\gamma} w_x(\bar{w}_x - w_x) \ge 0 \quad \forall\, (\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K.$$

This problem has a solution. Indeed, the set K is weakly closed in the space

$$\{(u, v, w, \varphi)\,|\, u \in H^1_\Gamma(\Omega_b)^2, (v, w) \in H^1(\gamma)^2, v \in H^2(\gamma_b),\ \varphi \in H^1(\gamma_t)\},$$

and the functional π_b is coercive on K. A coercivity can be proved as in Section 2.

Problem formulations (21)-(22) and (14)-(20) are equivalent for smooth solutions.

4. Rigidity of Euler-Bernoulli beam goes to infinity. In practice, a solution of the problem (14)-(20) should depend on the rigidity parameters of the elastic inclusions. In the model (14)-(20), these parameters were taken to be equal to 1. In this section we introduce a rigidity parameter $\lambda > 0$ in the Euler-Bernoulli equations of the problem

(14)-(20) and analyze its passage to infinity. For a fixed parameter λ we have to solve the following problem: find $u^\lambda, v^\lambda, w^\lambda, \varphi^\lambda$, $\sigma^\lambda = \{\sigma^\lambda_{ij}\}, i,j=1,2$, such that

$$-\text{div}\ \sigma^\lambda = f,\ \sigma^\lambda - B\varepsilon(u^\lambda) = 0 \quad \text{in} \quad \Omega_\gamma, \tag{23}$$

$$\lambda v^\lambda_{xxxx} = [\sigma^\lambda_\nu] \text{ on } \gamma_b,\ -\text{div}(a^\lambda w^\lambda_x) = [\sigma^\lambda_\tau] \text{ on } \gamma, \tag{24}$$

$$-v^\lambda_{xx} - \varphi^\lambda_x = [\sigma^\lambda_\nu],\ -\varphi^\lambda_{xx} + v^\lambda_x + \varphi^\lambda = 0 \quad \text{on} \quad \gamma_t, \tag{25}$$

$$(u^\lambda, v^\lambda, w^\lambda, \varphi^\lambda) \in K;\ v^\lambda_{xxx} = v^\lambda_{xx} = w^\lambda_x = 0 \text{ for } x = 0, \tag{26}$$

$$\lambda w^\lambda_x(1-) = w^\lambda_x(1+);\ v^\lambda_x + \varphi^\lambda = \varphi^\lambda_x = w^\lambda_x = 0 \text{ for } x = 2, \tag{27}$$

$$\sigma^{\lambda+}_\nu \le 0,\ \sigma^{\lambda+}_\tau = 0,\ \sigma^{\lambda+}_\nu [u^\lambda]\nu = 0 \text{ on } \gamma_b, \tag{28}$$

$$-\lambda v^\lambda_{xxx}(1-) = (v^\lambda_x + \varphi^\lambda)(1+);\ \lambda v^\lambda_{xx}(1-) = -\varphi^\lambda_x(1+). \tag{29}$$

Here $a^\lambda(x) = 1$ on γ_t, and $a^\lambda(x) = \lambda$ on γ_b. Notice that from the second equation of (24) it follows that

$$-\lambda w^\lambda_{xx} = [\sigma^\lambda_\tau] \text{ on } \gamma_b;\ -w^\lambda_{xx} = [\sigma^\lambda_\tau] \text{ on } \gamma_t.$$

The problem (23)-(29) admits an equivalent variational formulation. There exists a unique solution of the variational inequality (with $\sigma(u^\lambda) = \sigma^\lambda$)

$$(u^\lambda, v^\lambda, w^\lambda, \varphi^\lambda) \in K, \tag{30}$$

$$\begin{aligned} &\int_{\Omega_b} \sigma(u^\lambda)\varepsilon(\bar u - u^\lambda) - \int_{\Omega_b} f(\bar u - u^\lambda) + \lambda \int_{\gamma_b} v^\lambda_{xx}(\bar v_{xx} - v^\lambda_{xx}) \\ &+ \int_{\gamma_t} \{\varphi^\lambda_x(\bar\varphi_x - \varphi^\lambda_x) + (v^\lambda_x + \varphi^\lambda)(\bar v_x + \bar\varphi - v^\lambda_x - \varphi^\lambda)\} \\ &+ \int_\gamma a^\lambda w^\lambda_x(\bar w_x - w^\lambda_x) \ge 0 \quad \forall\, (\bar u, \bar v, \bar w, \bar\varphi) \in K. \end{aligned} \tag{31}$$

Our aim is to justify a passage to the limit $\lambda \to \infty$ in the problem (30)-(31).

From (30)-(31) it follows that

$$\int_{\Omega_b} \sigma(u^\lambda)\varepsilon(u^\lambda) - \int_{\Omega_b} f u^\lambda + \lambda \int_{\gamma_b} (v^\lambda_{xx})^2 + \int_{\gamma_t} \{(\varphi^\lambda_x)^2 + (v^\lambda_x + \varphi^\lambda)^2\} + \int_\gamma a^\lambda (w^\lambda_x)^2 = 0. \tag{32}$$

Taking into account the arguments used to prove a coercivity of the functional π in Section 2, from (32) we derive a uniform estimate for $\lambda \ge \lambda_0 > 0:$

$$\|u^\lambda\|_{H^1_\Gamma(\Omega_b)^2} + \|(v^\lambda, w^\lambda)\|_{H^1(\gamma)^2} + \|v^\lambda\|_{H^2(\gamma_b)} + \|\varphi^\lambda\|_{H^1(\gamma_t)} \le c, \tag{33}$$

and, moreover, we have uniformly for $\lambda > 0$,

$$\lambda \int_{\gamma_b} \{(v^\lambda_{xx})^2 + (w^\lambda_x)^2\} \le c. \tag{34}$$

Hence, we can assume that as $\lambda \to +\infty$,

$$u^\lambda \to u \quad \text{weakly in } \; H^1_\Gamma(\Omega_b)^2, \tag{35}$$

$$v^\lambda \to v \text{ weakly in } H^1(\gamma), \text{ weakly in } H^2(\gamma_b), \, v_{xx} = 0 \text{ in } \gamma_b, \tag{36}$$

$$w^\lambda \to w \quad \text{weakly in } \; H^1(\gamma), \; w_x = 0 \; \text{ in } \; \gamma_b, \tag{37}$$

$$\varphi^\lambda \to \varphi \quad \text{weakly in } \; H^1(\gamma_t). \tag{38}$$

In particular,

$$v(x) = c_0 + c_1 x, \; w(x) = b_0, \; x \in (0,1); \; b_0, c_0, c_1 - \text{const.} \tag{39}$$

We introduce a set of infinitesimal rigid displacements,

$$R(\gamma_b) = \{\rho = (\rho_1, \rho_2) \mid \rho(x_1, x_2) = d(-x_2, x_1) + (d^1, d^2),$$
$$(x_1, x_2) \in \gamma_b\}, \; d, d^1, d^2 - \text{const},$$

and a set of admissible functions,

$$K_r = \{(u, v, w, \varphi) \mid u \in H^1_\Gamma(\Omega_b)^2, (v, w, \varphi) \in H^1(\gamma_t)^3, \, [u]\nu \geq 0 \text{ on } \gamma_b;$$
$$u|_{\gamma_t} = (w, v), \; u|_{\gamma_b^-} = (\rho_1, \rho_2) \in R(\gamma_b), \; \rho_{2x}(1) + \varphi(1) = 0\}.$$

By (35)-(38), we can pass to the limit in (30)-(31) as $\lambda \to \infty$ and get

$$(u, v, w, \varphi) \in K_r, \tag{40}$$

$$\int\limits_{\Omega_b} \sigma(u)\varepsilon(\bar{u} - u) - \int\limits_{\Omega_b} f(\bar{u} - u) \tag{41}$$
$$+ \int\limits_{\gamma_t} \{\varphi_x(\bar{\varphi}_x - \varphi_x) + (v_x + \varphi)(\bar{v}_x + \bar{\varphi} - v_x - \varphi)$$
$$+ w_x(\bar{w}_x - w_x)\} \geq 0 \quad \forall\, (\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K_r.$$

Problem (40)-(41) admits an equivalent differential formulation: find displacements $u = (u_1, u_2)$, v, w, a rotation angle φ, a stress tensor $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, and $\rho^0 \in R(\gamma_b)$ such that

$$-\text{div } \sigma = f, \, \sigma - B\varepsilon(u) = 0 \quad \text{in} \quad \Omega_\gamma, \tag{42}$$

$$-v_{xx} - \varphi_x = [\sigma_\nu], \, -\varphi_{xx} + v_x + \varphi = 0, \, -w_{xx} = [\sigma_\tau] \text{ on } \gamma_t, \tag{43}$$

$$(u, v, w, \varphi) \in K_r; \; u = \rho^0 \text{ on } \gamma_b^-, \tag{44}$$

$$\sigma_\nu^+ \leq 0, \, \sigma_\tau^+ = 0, \, \sigma_\nu^+[u]\nu = 0 \; \text{ on } \gamma_b, \tag{45}$$

$$v_x + \varphi = \varphi_x = w_x = 0 \; \text{ for } x = 2, \tag{46}$$

$$\int\limits_{\gamma_b} [\sigma\nu]\rho + (w_x \rho_1)(1) - (\varphi_x \rho_{2x})(1) \tag{47}$$
$$+ ((v_x + \varphi)\rho_2)(1) = 0 \;\; \forall \rho = (\rho_1, \rho_2) \in R(\gamma_b).$$

Thus we have proved the following statement.

THEOREM 3. The solutions of the problem (30)-(31) converge to the solution of (40)-(41) in the sense of (35)-(38) as $\lambda \to \infty$.

The model (42)-(47), or (40)-(41), describes an equilibrium state for the elastic body with the rigid inclusion γ_b and elastic Timoshenko inclusion γ_t. The identity (47) provides equilibrium conditions for the rigid inclusion γ_b; i.e. a principal vector of forces and a principal vector of moments acting on γ_b are equal to zero. Indeed, denoting $(\sigma\nu)^{\pm}$ by $(\sigma^1, \sigma^2)^{\pm}$ on $\gamma_b^{\pm}$, the condition (47) can be rewritten in the following form:

$$\int\limits_{\gamma_b} [\sigma^1] = -w_x(1), \ \int\limits_{\gamma_b} [\sigma^2] = -(v_x + \varphi)(1), \tag{48}$$

$$\int\limits_{\gamma_b} ([\sigma^2]x_1 - [\sigma^1]x_2) = \varphi_x(1). \tag{49}$$

We can also write junction conditions included in the definition of K_r:

$$w(1) = \rho_1^0(1), \ v(1) = \rho_2^0(1), \ \varphi(1) + \rho_{2x}^0(1) = 0 \tag{50}$$

and consider (48)-(50) as a complete system of junction conditions at the joint point $(1, 0)$. Consequently, non-local condition (47) can be seen as a part of junction conditions at the joint point $(1, 0)$.

5. Rigidity of Euler-Bernoulli beam goes to zero. As in the previous section, consider the delaminated Euler-Bernoulli inclusion γ_b with the rigidity parameter $\lambda > 0$ and the Timoshenko inclusion γ_t inside the elastic body; see Figure 1. For any fixed $\lambda > 0$, we have the following equilibrium problem:

$$(u^\lambda, v^\lambda, w^\lambda, \varphi^\lambda) \in K, \tag{51}$$

$$\begin{aligned} &\int\limits_{\Omega_b} \sigma(u^\lambda)\varepsilon(\bar{u} - u^\lambda) - \int\limits_{\Omega_b} f(\bar{u} - u^\lambda) + \lambda \int\limits_{\gamma_b} v_{xx}^\lambda(\bar{v}_{xx} - v_{xx}^\lambda) \\ &+ \int\limits_{\gamma_t} \{\varphi_x^\lambda(\bar{\varphi}_x - \varphi_x^\lambda) + (v_x^\lambda + \varphi^\lambda)(\bar{v}_x + \bar{\varphi} - v_x^\lambda - \varphi^\lambda)\} \\ &+ \int\limits_{\gamma} a^\lambda w_x^\lambda(\bar{w}_x - w_x^\lambda) \geq 0 \quad \forall\, (\bar{u}, \bar{v}, \bar{w}, \bar{\varphi}) \in K. \end{aligned} \tag{52}$$

It is interesting to analyze a passage to the limit in the problem (51)-(52) as $\lambda \to 0$. From (51)-(52) we have

$$\begin{aligned} &\int\limits_{\Omega_b} \sigma(u^\lambda)\varepsilon(u^\lambda) - \int\limits_{\Omega_b} f u^\lambda + \lambda \int\limits_{\gamma_b} (v_{xx}^\lambda)^2 \\ &+ \int\limits_{\gamma_t} \{(\varphi_x^\lambda)^2 + (v_x^\lambda + \varphi^\lambda)^2\} + \int\limits_{\gamma} a^\lambda (w_x^\lambda)^2 = 0. \end{aligned} \tag{53}$$

Hence, taking into account the Lemma and coercivity arguments of Section 2, we obtain uniform in $\lambda > 0$ estimates

$$\|u^\lambda\|_{H^1_\Gamma(\Omega_b)^2} \leq c, \ \|(v^\lambda, w^\lambda, \varphi^\lambda)\|_{H^1(\gamma_t)^3} \leq c. \tag{54}$$

By (54) and imbedding theorems, we have

$$\int_{\gamma_b}(v^\lambda)^2 = \int_{\gamma_b}(u_\nu^{\lambda -})^2 \le c, \ \int_{\gamma_b}(w^\lambda)^2 = \int_{\gamma_b}(u_\tau^{\lambda -})^2 \le c. \tag{55}$$

Hence, in view of (54) and the Lemma, the following estimate holds for $\lambda \le \lambda_0$:

$$\lambda\|v^\lambda\|^2_{H^2(\gamma_b)} + \lambda\|w^\lambda\|^2_{H^1(\gamma_b)} \le c.$$

We can assume that as $\lambda \to 0$,

$$u^\lambda \to u \quad \text{weakly in } \ H^1_\Gamma(\Omega_b)^2, \tag{56}$$

$$(v^\lambda, w^\lambda, \varphi^\lambda) \to (v, w, \varphi) \quad \text{weakly in } \ H^1(\gamma_t)^3, \tag{57}$$

$$(\sqrt{\lambda}v^\lambda, \sqrt{\lambda}w^\lambda) \to (\tilde v, \tilde w) \quad \text{weakly in } \ H^2(\gamma_b) \times H^1(\gamma_b). \tag{58}$$

We introduce a set of admissible displacements for a limit problem:

$$K_0 = \{(u, v, w, \varphi) \mid u \in H^1_\Gamma(\Omega_b)^2, (v, w, \varphi) \in H^1(\gamma_t)^3, \\ [u]\nu \ge 0 \text{ on } \gamma_b;\ v = u_\nu,\ w = u_\tau \text{ on } \gamma_t\}.$$

By (56)-(58), we can pass to the limit in (52) as $\lambda \to 0$. It gives

$$(u, v, w, \varphi) \in K_0, \tag{59}$$

$$\int_{\Omega_b}\sigma(u)\varepsilon(\bar u - u) - \int_{\Omega_b} f(\bar u - u) \tag{60}$$

$$+\int_{\gamma_t}\{\varphi_x(\bar\varphi_x - \varphi_x) + (v_x + \varphi)(\bar v_x + \bar\varphi - v_x - \varphi)$$

$$+w_x(\bar w_x - w_x)\} \ge 0.$$

The inequality (60) holds for all functions $(\bar u, \bar v, \bar w, \bar\varphi) \in K$. It is interesting to check that it holds for all $(\bar u, \bar v, \bar w, \bar\varphi) \in K_0$. But it is still an open problem. On the other hand, we can write a variational inequality formally corresponding to $\lambda = 0$ in (51)-(52) (see also differential formulation (23)-(29)):

$$(u, v, w, \varphi) \in K_0, \tag{61}$$

$$\int_{\Omega_b}\sigma(u)\varepsilon(\bar u - u) - \int_{\Omega_b} f(\bar u - u) \tag{62}$$

$$+\int_{\gamma_t}\{\varphi_x(\bar\varphi_x - \varphi_x) + (v_x + \varphi)(\bar v_x + \bar\varphi - v_x - \varphi)$$

$$+w_x(\bar w_x - w_x)\} \ge 0 \quad \forall\, (\bar u, \bar v, \bar w, \bar\varphi) \in K_0.$$

The problem (61)-(62) admits an equivalent differential formulation: find functions $u = (u_1, u_2), v, w, \varphi$, $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, such that

$$-\text{div}\ \sigma = f,\ \sigma - B\varepsilon(u) = 0 \quad \text{in} \quad \Omega_\gamma, \tag{63}$$

$$-v_{xx} - \varphi_x = [\sigma_\nu],\ -\varphi_{xx} + v_x + \varphi = 0,\ -w_{xx} = [\sigma_\tau] \text{ on } \gamma_t, \tag{64}$$

$$u = 0 \text{ on } \Gamma;\ [u]\nu \geq 0 \text{ on } \gamma_b;\ v = u_\nu,\ w = u_\tau \text{ on } \gamma_t, \tag{65}$$

$$\sigma_\nu^\pm \leq 0,\ [\sigma_\nu] = 0,\ \sigma_\tau^\pm = 0,\ \sigma_\nu^+[u]\nu = 0 \text{ on } \gamma_b, \tag{66}$$

$$v_x + \varphi = \varphi_x = w_x = 0 \text{ for } x = 1, 2. \tag{67}$$

We can show that problem formulations (61)-(62) and (63)-(67) are equivalent. Details are omitted.

It is possible to provide a minimization problem corresponding to (61)-(62). Indeed, define an energy functional

$$\pi_0(u, v, w, \varphi) = \frac{1}{2}\int\limits_{\Omega_b} \sigma(u)\varepsilon(u) - \int\limits_{\Omega_b} fu$$
$$+\frac{1}{2}\int\limits_{\gamma_t} \{w_x^2 + \varphi_x^2 + (v_x + \varphi)^2\}.$$

Then a solution (u, v, w, φ) of the minimization problem

$$\text{find } (u, v, w, \varphi) \in K_0 \text{ such that } \pi_0(u, v, w, \varphi) = \inf_{K_0} \pi_0$$

exists and satisfies (61)-(62).

Acknowledgement. This work was supported by the Russian Science Foundation (grant 15-11-10000).

References

[1] Blaise Bourdin, Gilles A. Francfort, and Jean-Jacques Marigo, *The variational approach to fracture*, Springer, New York, 2008. Reprinted from J. Elasticity **91** (2008), no. 1-3 [MR2390547], with a foreword by Roger Fosdick. MR2473620

[2] Philippe G. Ciarlet, Hervé Le Dret, and Robert Nzengwa, *Junctions between three-dimensional and two-dimensional linearly elastic structures* (English, with French summary), J. Math. Pures Appl. (9) **68** (1989), no. 3, 261–295. MR1025905

[3] Gianni Dal Maso and Rodica Toader, *A model for the quasi-static growth of brittle fractures: existence and approximation results*, Arch. Ration. Mech. Anal. **162** (2002), no. 2, 101–135, DOI 10.1007/s002050100187. MR1897378

[4] Antonio Gaudiello and Elvira Zappale, *Junction in a thin multidomain for a fourth order problem*, Math. Models Methods Appl. Sci. **16** (2006), no. 12, 1887–1918, DOI 10.1142/S0218202506001753. MR2287334

[5] Antonio Gaudiello and Ali Sili, *Asymptotic analysis of the eigenvalues of an elliptic problem in an anisotropic thin multidomain*, Proc. Roy. Soc. Edinburgh Sect. A **141** (2011), no. 4, 739–754, DOI 10.1017/S0308210510000521. MR2819710

[6] Antonio Gaudiello and Elvira Zappale, *A model of joined beams as limit of a 2D plate*, J. Elasticity **103** (2011), no. 2, 205–233, DOI 10.1007/s10659-010-9281-6. MR2772851

[7] H. Itou and A.M. Khludnev, *On delaminated thin Timoshenko inclusions inside elastic bodies*, Math. Meth. Appl. Sciences, DOI 10.1002/mma.3279.

[8] Alexander Khludnev, *Contact problems for elastic bodies with rigid inclusions*, Quart. Appl. Math. **70** (2012), no. 2, 269–284, DOI 10.1090/S0033-569X-2012-01233-3. MR2953103

[9] A. M. Khludnev, *Singular invariant integrals for elastic body with delaminated thin elastic inclusion*, Quart. Appl. Math. **72** (2014), no. 4, 719–730, DOI 10.1090/S0033-569X-2014-01355-9. MR3291824

[10] Alexander Khludnev and Atusi Tani, *Overlapping domain problems in the crack theory with possible contact between crack faces*, Quart. Appl. Math. **66** (2008), no. 3, 423–435, DOI 10.1090/S0033-569X-08-01118-7. MR2445521

[11] A.M. Khludnev and V.A. Kovtunenko, *Analysis of cracks in solids*, Southampton-Boston, WIT Press, 2000.

[12] A.M. Khludnev, *Elasticity problems in non-smooth domains*, Moscow, Fizmatlit, 2010.

[13] A. M. Khludnev, *On a crack located at the boundary of a rigid inclusion in elastic plate*, Izvestiya RAS, Mech. of Solids (2010), no. 5, 98–110.

[14] Alexander Khludnev and Günter Leugering, *On elastic bodies with thin rigid inclusions and cracks*, Math. Methods Appl. Sci. **33** (2010), no. 16, 1955–1967, DOI 10.1002/mma.1308. MR2744613

[15] A.M. Khludnev and M. Negri, *Crack on the boundary of a thin elastic inclusion inside an elastic body*, Z. Angew. Math. Mech. **92** (2012), no. 5, 341–354.

[16] A. Khludnev and M. Negri, *Optimal rigid inclusion shapes in elastic bodies with cracks*, Z. Angew. Math. Phys. **64** (2013), no. 1, 179–191, DOI 10.1007/s00033-012-0220-1. MR3023082

[17] A. M. Khludnev, V. A. Kovtunenko, and A. Tani, *On the topological derivative due to kink of a crack with non-penetration. Anti-plane model*, J. Math. Pures Appl. (9) **94** (2010), no. 6, 571–596, DOI 10.1016/j.matpur.2010.06.002. MR2737389

[18] Victor A. Kovtunenko, *Shape sensitivity of curvilinear cracks on interface to non-linear perturbations*, Z. Angew. Math. Phys. **54** (2003), no. 3, 410–423, DOI 10.1007/s00033-003-0143-y. MR2048661

[19] P. A. Krutitskii, *The Neumann problem for the equation $\Delta u - k^2 u = 0$ in the exterior of non-closed Lipschitz surfaces*, Quart. Appl. Math. **72** (2014), no. 1, 85–91, DOI 10.1090/S0033-569X-2013-01319-4. MR3185133

[20] N. P. Lazarev and E. M. Rudoy, *Shape sensitivity analysis of Timoshenko's plate with a crack under the nonpenetration condition*, ZAMM Z. Angew. Math. Mech. **94** (2014), no. 9, 730–739, DOI 10.1002/zamm.201200229. MR3259385

[21] N. Lazarev, *Shape sensitivity analysis of the energy integrals for the Timoshenko-type plate containing a crack on the boundary of a rigid inclusion*, Z. Angew. Math. Phys. **66** (2015), no. 4, 2025–2040, DOI 10.1007/s00033-014-0488-4. MR3377729

[22] H. Le Dret, *Modeling of the junction between two rods* (English, with French summary), J. Math. Pures Appl. (9) **68** (1989), no. 3, 365–397. MR1025910

[23] H. Le Dret, *Modeling of a folded plate*, Computational Mechanics **5** (1990), 401–416.

[24] P. K. Mallick, *Fiber-Reinforced Composites. Materials, Manufacturing, and Design*, Marcel Dekker, Inc., 1993.

[25] M. Nasser and A. Hassen, *Embedded beam under equivalent load induced from a surface moving load*, Acta Mechanica **67** (1987), 237-247.

[26] Sergey A. Nazarov and Boris A. Plamenevsky, *Elliptic problems in domains with piecewise smooth boundaries*, de Gruyter Expositions in Mathematics, vol. 13, Walter de Gruyter & Co., Berlin, 1994. MR1283387

[27] N. V. Neustroeva, *Unilateral contact of elastic plates with a rigid inclusion*, Vestnik of Novosibirsk State University (math., mech., informatics) **9** (2009), no. 4, 51–64.

[28] T. A. Rotanova, *On unilateral contact between two plates one of which has a rigid inclusion*, Vestnik of Novosibirsk State University (math., mech., informatics) **11** (2011), no. 1, 87–98.

[29] E. M. Rudoy, *Differentiation of energy functionals in the problem for a curvilinear crack with possible contact between crack faces*, Izvestiya RAS, Mech. of Solids **6** (2007), 113–127.

[30] E. M. Rudoy, *Asymptotics of energy functional for elastic body with a crack and rigid inclusion. 2D case*, Appl. Math. Mechs. **75** (2011), no. 2, 719–729.

[31] Giuseppe Saccomandi and Millard F. Beatty, *Universal relations for fiber-reinforced elastic materials*, Math. Mech. Solids **7** (2002), no. 1, 95–110, DOI 10.1177/1081286502007001226. MR1900936

[32] Isabelle Titeux and Evariste Sanchez-Palencia, *Junction of thin plates*, Eur. J. Mech. A Solids **19** (2000), no. 3, 377–400, DOI 10.1016/S0997-7538(00)00175-3. MR1763831