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BOUNDS AND REPRESENTATIONS OF SOLUTIONS OF PLANAR DIV-CURL PROBLEMS

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Abstract. Estimates and representations of solutions of div-curl systems for planar vector fields are described. Potentials are used to represent solutions as the sum of fields that depend on the source terms and harmonic fields dependent on the boundary data. Sharp 2-norm (energy) bounds for the least energy solutions on bounded regions with Lipschitz boundary are found. Prescribed flux, tangential or mixed flux and tangential boundary conditions require different potentials. The harmonic fields are represented and estimated using Steklov eigenfunctions. Some regularity results are obtained.

1. Introduction. This paper studies properties of solutions of boundary value problems for div-curl systems on bounded regions $\Omega \subset \mathbb{R}^2$ inside closed Lipschitz curves. This is a degenerate elliptic system of two equations in two unknowns that has weak (L^2 or finite energy) solutions subject to conditions on the normal and tangential boundary conditions. Existence, uniqueness and well-posed of solutions of these problems have been studied by many authors since the famous paper of Hermann Weyl [22]. Here attention is focussed on the qualitative properties of the solutions found in Alexander and Auchmuty [1].

In particular representations, bounds on the energy (L^2-norm) and interior regularity, of solutions are obtained. K.O. Friedrichs [16] has called an inequality that bounds the the energy of a field by norms of its divergence and curl, the *main inequality of vector analysis*. Here sharp estimates for this energy are derived in terms of norms of prescribed sources $\rho := \operatorname{div} \mathbf{v}$, $\omega := \operatorname{curl} \mathbf{v}$ and boundary data. The boundary conditions include

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normal data (see section 6), tangential data (section 7) or mixed normal and tangential data (section 8). The normal and tangential problems are underdetermined when the region Ω is not simply connected; in these cases the estimates are for the least energy solution.

The analysis depends on a specific choice of scalar potentials for the vector field that differs from the usual Hodge-Weyl decompositions. This "CGH decomposition" uses potentials that are solutions of zero-Dirichlet problems for Poisson's equation - so they have nice representations using Dirichlet-Laplacian eigenfunctions. Then the harmonic component is determined by the solution of a homogeneous equation subject to nontrivial Neumann boundary data. These harmonic fields have explicit, and orthogonal, representations in terms of the Steklov eigenfunctions of the region and the boundary data.

This system has been used to model many different situations in both fluid mechanics and electromagnetic field theories. This paper only considers the simplest div-curl system so that sharp constants can be identified and spectral representation results outlined. Well-posedness of various boundary value problems for these weak solutions were obtained using variational methods in Alexander and Auchmuty [1]. That paper describes how the existence of unique weak solutions $\mathbf{v} \in L^2(\Omega;\mathbb{R}^2)$ subject to different types of boundary conditions depends on both the prescribed data and also the topology of the region. Many of the results described here may be generalized to the case of general elliptic coefficient matrix $E(x)$, as studied in [1].

The bounds described here are sharp, and very different from those in papers such as that of Křížek and Neittaanmaki [19] or the Schauder estimates of Bolik and von Wahl [9]. Since a major interest is the dependence of solutions on boundary data, this analysis is also quite different from the work of Brezis and Bourgain [11] who studied these problems in $\mathbb{R}^N$ and with periodic boundary conditions. A sophisticated analysis of the prescribed flux and tangential boundary problems was given by Mitrea [20]. She used layer potentials and Besov spaces to study these problems on bounded regions with Lipschitz boundaries.

The methods used here are much more elementary and straightforward than those based on layer potentials - and extend to cover mixed boundary conditions. They appear to be appropriate for both analytic and finite element approximations and computational simulations. Hence results are obtained subject to the boundary regularity requirement (B1) below which just requires that $\partial\Omega$ be Lipschitz. There have been many papers on the numerical analysis and simulation of solutions of these problems by different investigators. For example see Bramble and Pasciak [10] or Monk [21] for an overview.

In section 3, some results about the regularity of orthogonal projections of L^2-vector fields are described. It is shown that, when a field is smooth on an open subset of Ω, so are its potentials. Also the class of harmonic vector fields that may be represented by conjugate harmonic functions is characterized.

In section 4, some orthogonal decompositions of the space $L^2(\Omega;\mathbb{R}^2)$ with the standard inner product are described. There are many other such decompositions - see page 314 of Dautray and Lions [12] for an interesting diagram of possibilities for 3-d regions. There are analogous results for planar regions. Moreover some of the commonly used prescriptions for potentials in the literature are not actually projections. Theorem 4.2 here provides an existence and regularity result for potentials for irrotational or solenoidal vector fields in $L^p(\Omega;\mathbb{R}^2)$. A corollary of this is a proof that harmonic vector fields must be C^∞ on Ω - so it generalizes Weyl's lemma for harmonic functions.

Section 5 collects properties of scalar Laplacian boundary value problems for subsequent use. Theorem 5.2 provides a decomposition for L^1 and L^1_{loc} fields in terms of potentials and a harmonic field provided their divergence and curl are in $L^p,\ p > 1$. A different Hilbert space $H_0(\Delta,\Omega)$ is introduced that enables better properties of the potentials when the divergence and curl are in $L^2(\Omega)$.

Explicit formulae, and estimates, for solutions of Neumann, and other, harmonic boundary value problems in terms of Steklov eigenvalues and eigenfunctions follow from the author's work in [6] and are needed to obtain results about the dependence of solutions on boundary data.

Sections 6 and 7 describe results about the least norm solutions of the prescribed flux and prescribed tangent div-curl boundary value problem respectively when the necessary compatibility conditions hold. When each of the data is L^2, solution estimates depend on the principal Dirichlet and Stekov eigenvalues λ_1 and δ_1 of the Laplacian on Ω. These estimates are sharp. When Ω is not simply connected these solutions are not unique and the extra information required for well-posedness was studied in [1].

When mixed tangential and normal boundary conditions are imposed on $\partial\Omega$, no compatibility conditions on the data are required for the existence of solutions. Under natural assumptions on the data, it is shown how the solutions may be represented using two potentials and estimates of these solutions are found in terms of some different eigenvalues associated with the Laplacian on Ω.

2. Definitions and notation. In this paper, standard definitions as given in Evans [14] or Attouch, Buttazzo and Michaille [2] will generally be used. - specialized to $\mathbb{R}^2$ since this paper only treats planar problems. Cartesian coordinates $x = (x_1, x_2)$ will be used and Euclidean norms and inner products are denoted by $|.|$ and $x \cdot y$. A region is a nonempty, connected, open subset of $\mathbb{R}^2$. Its closure is denoted $\overline{\Omega}$ and its boundary is $\partial\Omega := \overline{\Omega} \setminus \Omega$. Often the position vector x is omitted in formulae for functions and fields and equality should be interpreted as holding a.e. with respect to 2-dimensional Lebesgue measure $d^2x = dx_1\, dx_2$ on Ω.

A number of the results here depend on the differential topology of the region Ω. A curve in the plane is said to be a simple Lipschitz loop if it is a closed, nonself-intersecting curve with at least two distinct points and a uniformly Lipschitz parametrization. Such loops will be compact and have finite, nonzero, length. Arc-length will be denoted $s(.)$ and our standard assumption is

Condition (B1). Ω is a bounded region in $\mathbb{R}^2$ with boundary $\partial\Omega$ the union of a finite number of disjoint simple Lipschitz loops $\{\Gamma_j : 0 \le j \le J\}$.

Here Γ_0 will always be the exterior loop and the other Γ_j will enclose *holes* in the region Ω. The interior region to the loop Γ_0 defined by the Jordan curve theorem will be denoted Ω_0. When $J = 0$, Ω is said to be simply connected and then $\Omega_0 = \Omega$.

The outward unit normal to a region at a point on the boundary is denoted $\nu(z) = (\nu_1(z), \nu_2(z))$. Then $\tau(z) := (-\nu_2(z), \nu_1(z))$ is the positively oriented unit tangent vector at a point $z \in \partial\Omega$. ν, τ are defined *s a.e.* on $\partial\Omega$ when (B1) holds.

In this paper, all functions are assumed to be at least L^1_{loc} and derivatives will be taken in a weak sense. The spaces $W^{1,p}(\Omega), W^{1,p}_0(\Omega)$ are defined as usual for $p \in [1, \infty]$ with standard norms denoted by $\|.\|_{1,p}$. When $p = 2$ the spaces will also be denoted $H^1(\Omega), H^1_0(\Omega)$.

When Ω is bounded, the trace of Lipschitz continuous functions on $\overline{\Omega}$ restricted to $\partial\Omega$ is again Lipschitz continuous. The extension of this linear mapping is a continuous linear mapping of $W^{1,p}(\Omega)$ to $L^p(\partial\Omega, ds)$ for all $p \in [1, \infty]$ when (B1) holds. See [15], Section 4.2 for details. From Morrey's theorem, γ maps $W^{1,p}(\Omega)$ into $C^\alpha(\partial\Omega)$ when $p > 2$ and $\alpha = 1 - 2/p$. Di Benedetto [13, proposition 18.1] shows that when $\varphi \in H^1(\Omega)$ then $\gamma(\varphi) \in L^q(\partial\Omega, ds)$ for all $q \in [1, \infty)$. Also if $\varphi \in W^{1,p}(\Omega)$ with $p \in [1, 2)$, then $\gamma(\varphi) \in L^q(\partial\Omega, ds)$ for all $q \in [1, p_T]$ with $p_T = p/(2 - p)$ under stronger regularity conditions on the boundary.

The region Ω is said to satisfy a *compact trace theorem* provided the trace mapping $\gamma : H^1(\Omega) \to L^2(\partial\Omega, ds)$ is compact. Theorem 1.5.1.10 of Grisvard [18] proves an inequality that implies the compact trace theorem when $\partial\Omega$ satisfies (B1).

We will generally use the following equivalent inner product on $H^1(\Omega)$:

$$[\varphi, \psi]_\partial := \int_\Omega \nabla\varphi \cdot \nabla\psi \; d^2x + \int_{\partial\Omega} \gamma(\varphi)\, \gamma(\psi)\; ds. \tag{2.1}$$

The associated norm is denoted $\|\varphi\|_\partial$. The proof that this norm is equivalent to the usual $(1, 2)-$norm on $H^1(\Omega)$ when (B1) holds is Corollary 6.2 of [4] and also is part of theorem 21A of [23]. The inner product on $H^1_0(\Omega)$ is the restriction of this inner product.

When Ω satisfies (B1), then the *Gauss-Green* theorem holds in the forms

$$\int_\Omega D_j\varphi(x)\, d^2x \;=\; \int_{\partial\Omega} \gamma(\varphi)(z)\; \nu_j(z)\, ds(z) \qquad \text{for all} \quad \varphi \in W^{1,1}(\Omega) \tag{2.2}$$

and

$$\int_\Omega \varphi(x) D_j\, \psi(x)\; dx = \int_{\partial\Omega} \gamma(\varphi)\, \gamma(\psi)\, \nu_j\, ds - \int_\Omega \psi(x) D_j\, \varphi(x)\; dx \quad \text{for each } j \tag{2.3}$$

and all $\varphi,\ \psi$ in $W^{1,p}(\Omega)$ with $p \geq 4/3$. Often the trace operator will be implicit in boundary integrals

When $\varphi \in W^{1,1}(\Omega)$ is weakly differentiable, then the *gradient* and *Curl* of φ are the vector fields

$$\nabla\varphi(x) := (D_1\varphi(x), D_2\varphi(x)) \quad \text{and} \quad \nabla^\perp\varphi(x) := (D_2\varphi(x), -D_1\varphi(x)). \tag{2.4}$$

Here $D_j\varphi$ or $\varphi_{,j}$ denotes the weak j-th derivative.

A function $\rho \in L^1_{loc}$ is defined to be the Laplacian of φ provided

$$\int_\Omega \varphi\; \Delta v\, d^2x \;=\; \int_\Omega \rho\; v\;\; d^2x \qquad \text{for all} \quad v \in C^2_c(\Omega).$$

A function $\varphi \in W^{1,1}(\Omega)$ is said to be harmonic on Ω provided

$$\int_\Omega \nabla\varphi \cdot \nabla\chi \; d^2x \;=\; 0 \qquad \text{for all} \quad \chi \in C_c^2(\Omega). \tag{2.5}$$

The subspace of all harmonic functions in $H^1(\Omega)$ will be denoted $\mathcal{H}(\Omega)$ and it is straightforward to observe that $H^1(\Omega) = H_0^1(\Omega) \oplus_\partial \mathcal{H}(\Omega)$ and that $\mathcal{H}(\Omega)$ is isomorphic to the trace space $H^{1/2}(\partial\Omega)$. Later use will be made of the analysis in [5] where this is described and $\partial-$orthogonal bases of the space $\mathcal{H}(\Omega)$ are found that involve the Steklov eigenfunctions of the Laplacian on Ω.

3. Projections and potentials in $L^2(\Omega;\mathbb{R}^2)$. For $p \in [1,\infty]$, $L^p(\Omega;\mathbb{R}^2)$ is the space of planar vector fields $\mathbf{v}(x) = (v_1(x), v_2(x))$ on Ω whose (Cartesian) components are L^p-functions on Ω. In particular $L^2(\Omega;\mathbb{R}^2)$ is the real Hilbert space of L^2-vector fields on Ω with inner product

$$\langle \mathbf{v}, \mathbf{w}\rangle \;:=\; \int_\Omega \mathbf{v}\cdot\mathbf{w} \;\, d^2x. \tag{3.1}$$

Throughout this paper *orthogonal* means L^2-*orthogonal* unless a different adjective is present. Our analysis is based on the use of *scalar potentials* φ, ψ, defined by L^2-projections, with the property that

$$\mathbf{v}(x) \;=\; \nabla^\perp\psi(\mathrm{x}) \,-\, \nabla\varphi(\mathrm{x}) \qquad \text{on} \quad \Omega. \tag{3.2}$$

Often ψ is called a stream function and this representation is called a Helmholtz decomposition. The choice of signs in (3.2) is commonly used in applications as it provides some mathematical symmetries. Different prescriptions for the φ, ψ have been used in the literature. Here the essential requirement is that the two components be defined by projections. The specific choice is determined by the boundary conditions and the fields will have L^2-orthogonality properties on Ω. Later in (4.7) a harmonic representation is introduced and the harmonic component again has prescribed potentials so (3.2) continues to hold.

Let $G(\Omega), G_0(\Omega), Curl(\Omega), Curl_0(\Omega)$ be the subspaces of gradients and Curls with potentials φ in $H^1(\Omega), H_0^1(\Omega)$ respectively. They are closed subspaces of $L^2(\Omega;\mathbb{R}^2)$ from the variational characterization of projections based on Riesz' projection theorem as described in section 3 of Auchmuty [3].

The best approximation of a field $\mathbf{v} \in L^2(\Omega;\mathbb{R}^2)$ by gradient fields is found by minimizing the functional $\mathcal{E}_v$ defined by

$$\mathcal{E}_v(\varphi) \;:=\; \int_\Omega \; [\,|\nabla\varphi|^2 \,+\, 2\,\mathbf{v}\cdot\nabla\varphi\,] \;\, \mathrm{d}^2\mathrm{x} \tag{3.3}$$

on $H^1(\Omega), H_0^1(\Omega)$ respectively. $H^1(\Omega)$ will be replaced by the space $H_m^1(\Omega)$ of all potentials with mean value $\overline{\varphi} = 0$ to ensure uniqueness. The inner product on both $H_m^1(\Omega)$ and $H_0^1(\Omega)$ is $\langle\varphi,\chi\rangle_\nabla \;:=\; \langle\nabla\varphi, \nabla\chi\rangle$. These variational principles define orthogonal projections with properties that may be summarized via the next two theorems.

THEOREM 3.1. Assume Ω obeys (B1) and $\mathbf{v} \in L^2(\Omega;\mathbb{R}^2)$. Then there is a unique $\varphi_v \in H^1_m(\Omega)$ that minimizes $\mathcal{E}_v$ on $H^1_m(\Omega)$. φ_v satisfies

$$\int_\Omega (\nabla\varphi + \mathbf{v}) \cdot \nabla\chi \ \mathrm{d}^2\mathrm{x} \ = \ 0 \qquad \text{for all} \quad \chi \in \mathrm{H}^1(\Omega) \tag{3.4}$$

with $\| \nabla\varphi_v\| \ \leq \ \|\mathbf{v}\|$. If $\mathbf{v} = -\nabla\varphi$ for another φ, then there is a constant c such that $\varphi = \varphi_v + c$ on Ω.

Proof. When (B1) holds there is a $\lambda_m > 0$ such that

$$\int_\Omega |\nabla\varphi|^2 \ d^2x \ \geq \ \lambda_m \int_\Omega \varphi^2 \ d^2x \qquad \text{for all} \quad \varphi \in H^1_m(\Omega). \tag{3.5}$$

Hence $\mathcal{E}_v$ is continuous, strictly convex and coercive on $H^1_m(\Omega)$ so there is a unique minimizer of $\mathcal{E}_v$. This functional is Gateaux differentiable and the minimization condition is (3.4). Choose $\chi = \nabla\varphi_v$ in (3.4); then the inequality follows from Cauchy-Schwarz. The last statement holds as each function in $H^1(\Omega)$ has a unique decomposition of the form $\varphi = \varphi_v + c$ with $\varphi_v \in H^1_m(\Omega)$ and $c = \overline{\varphi}$. □

Define $P_G : L^2(\Omega;\mathbb{R}^2) \to L^2(\Omega;\mathbb{R}^2)$ by $P_G\,\mathbf{v} \ := \ -\nabla\varphi_v$. This result implies that $G(\Omega)$ is a closed subspace of $L^2(\Omega;\mathbb{R}^2)$ from corollary 3.3 of [3] and that P_G is the projection of $L^2(\Omega;\mathbb{R}^2)$ onto $G(\Omega)$. The extremality condition (3.4) implies that φ_v is a weak solution of the Neumann problem

$$-\Delta\varphi \ = \ \mathrm{div}\,\mathbf{v} \qquad \text{on} \quad \Omega \quad \text{and} \quad \mathrm{D}_\nu\varphi \ = \ -\,\mathbf{v}\cdot\nu \qquad \text{on} \quad \partial\Omega. \tag{3.6}$$

The complementary projection $Q_G := I - P_G$ is the planar Leray projection of fluid mechanics.

THEOREM 3.2. Assume Ω obeys (B1) and $\mathbf{v} \in L^2(\Omega;\mathbb{R}^2)$. Then there is a unique $\varphi_0 \in H^1_0(\Omega)$ that minimizes $\mathcal{E}_v$ on $H^1_0(\Omega)$. φ_0 satisfies

$$\int_\Omega (\nabla\varphi + \mathbf{v}) \cdot \nabla\chi \ \mathrm{d}^2\mathrm{x} \ = \ 0 \qquad \text{for all} \quad \chi \in \mathrm{H}^1_0(\Omega) \quad \text{and} \quad \| \nabla\varphi_0\| \ \leq \ \|\mathbf{v}\|. \tag{3.7}$$

Proof. When (B1) holds there is a $\lambda_1 > 0$ such that

$$\int_\Omega |\nabla\varphi|^2 \ d^2x \ \geq \ \lambda_1 \int_\Omega \varphi^2 \ d^2x \qquad \text{for all} \quad \varphi \in H^1_0(\Omega). \tag{3.8}$$

Hence $\mathcal{E}_v$ is continuous, strictly convex and coercive on $H^1_0(\Omega)$ so there is a unique minimizer φ_0 of $\mathcal{E}_v$. This functional is Gateaux differentiable and the minimization condition is (3.7). Choose $\chi = \nabla\varphi_0$ in (3.7); then the last part follows from Cauchy-Schwarz. □

The extremality condition (3.7) says that $\varphi_0 \in H^1_0(\Omega)$ is a weak solution of the Dirichlet problem for

$$-\Delta\varphi \ = \ \mathrm{div}\,\mathbf{v} \qquad \text{on} \quad \Omega. \tag{3.9}$$

Define $P_{G0} : L^2(\Omega;\mathbb{R}^2) \to L^2(\Omega;\mathbb{R}^2)$ by $P_{G0}\,\mathbf{v} \ := \ -\nabla\varphi_0$. This result implies that $G_0(\Omega)$ is a closed subspace of $L^2(\Omega;\mathbb{R}^2)$ from corollary 3.3 of [3] and shows that P_{G0} is the projection of $L^2(\Omega;\mathbb{R}^2)$ onto $G_0(\Omega)$. The complementary projection $Q_{G0} := I - P_{G0}$ is the projection onto the null space of the divergence operator - see Theorem 4.2 of the next section.

These characterizations of these projections allow the proof that they preserve interior regularity. A vector field $\mathbf{v}$ is said to be H^m on an open subset $\mathcal{O}$ provided each component v_j is of class H^m on $\mathcal{O}$. Specifically the following holds.

THEOREM 3.3. Suppose Ω obeys (B1) and $\mathcal{O}$ is open with $\overline{\mathcal{O}} \subset \Omega$. If $\mathbf{v} \in \mathrm{H}^{\mathrm{m}}(\mathcal{O})$ with $m \geq 1$, then $P_G\mathbf{v}$, $P_{G0}\mathbf{v}$, $Q_G\mathbf{v}$, and $Q_{G0}\mathbf{v}$ are in $H^m_{loc}(\mathcal{O})$.

Proof. When $\mathbf{v} \in H^m$ on $\mathcal{O}$, then $\operatorname{div}\mathbf{v}$ is H^{m-1} and thus φ_v is H^{m+1}_{loc} from standard elliptic regularity results for solutions of (3.6) as in Evans, [14, chapter 6], or elsewhere. Hence the gradient is H^m_{loc} so the results hold for $P_G\mathbf{v}$ and $\mathbf{v}$ The result for $P_{G0}\mathbf{v}, \mathrm{Q}_{\mathrm{G0}}\mathbf{v}$ is proved in the same way since the potentials now are solutions of (3.9). □

Analogous analyses hold for projections onto spaces of Curls. First note that

$$\|\mathbf{v} - \nabla^\perp\psi\|^2 - \|\mathbf{v}\|^2 = \int_\Omega [\,|\nabla\psi|^2 - 2\,\mathbf{v}\wedge\nabla\psi\,]\; d^2x$$

where $\wedge$ denotes the 2-d vector product. Consider the variational problems of minimizing the functional

$$\mathcal{C}_v(\psi) := \int_\Omega [\,|\nabla\psi|^2 - 2\,\mathbf{v}\wedge\nabla\psi\,]\; d^2x \tag{3.10}$$

on $H^1(\Omega), H^1_0(\Omega)$ respectively. Results about these variational principles may be summarized as follows.

THEOREM 3.4. Assume Ω obeys (B1) and $\mathbf{v} \in L^2(\Omega;\mathbb{R}^2)$. Then there is a unique $\psi_v \in H^1_m(\Omega)$ that minimizes $\mathcal{C}_v$ on $H^1_m(\Omega)$. ψ_v satisfies

$$\int_\Omega (\nabla^\perp\psi - \mathbf{v})\cdot\nabla^\perp\chi\ \mathrm{d^2x} = 0 \qquad \text{for all} \quad \chi \in \mathrm{H}^1(\Omega) \tag{3.11}$$

and (i) $\|\nabla^\perp\psi_v\| \leq \|\mathbf{v}\|$, (ii) if $\mathbf{v} = \nabla^\perp\psi$, then there is a constant c such that $\psi_v - \psi \equiv c$ on Ω and (iii) if $\mathcal{O}$ is an open subset of Ω with $\overline{\mathcal{O}} \subset \Omega$ and $\mathbf{v} \in \mathrm{H}^{\mathrm{m}}(\mathcal{O})$, then $\nabla^\perp\psi_v \in H^m_{loc}(\mathcal{O})$.

Proof. This proof just requires appropriate modifications to those of Theorems 3.1 and 3.3 □

The extremality condition (3.11) says that ψ_v is a weak solution of the Neumann problem

$$-\Delta\psi = \operatorname{curl}\mathbf{v} \quad \text{on} \quad \Omega \quad \text{and} \quad \mathrm{D}_\nu\psi = -\,\mathbf{v}\cdot\tau \quad \text{on} \quad \partial\Omega. \tag{3.12}$$

Define $P_C : L^2(\Omega;\mathbb{R}^2) \to L^2(\Omega;\mathbb{R}^2)$ by $P_C\,\mathbf{v} := \nabla^\perp\psi_v$. The theorem implies that $Curl(\Omega)$ is a closed subspace of $L^2(\Omega;\mathbb{R}^2)$ from corollary 3.3 of [3] and that P_C is the projection of $L^2(\Omega;\mathbb{R}^2)$ onto $Curl(\Omega)$. The orthogonal complement of this projection is $Q_C := I - P_C$ and Q_C and P_C are orthogonal projections from (3.11).

Similarly, the problem of minimizing the functional $\mathcal{C}_v$ on $H^1_0(\Omega)$ has solutions that satisfy the following. The proof is similar to that of Theorems 3.2 and 3.3.

THEOREM 3.5. Assume Ω obeys (B1) and $\mathbf{v} \in L^2(\Omega;\mathbb{R}^2)$. Then there is a unique $\psi_{v0} \in H^1_0(\Omega)$ that minimizes $\mathcal{C}_v$ on $H^1_0(\Omega)$. ψ_{v0} satisfies

$$\int_\Omega (\nabla^\perp\psi - \mathbf{v})\cdot\nabla^\perp\chi\ \mathrm{d^2x} = 0 \qquad \text{for all} \quad \chi \in \mathrm{H}^1_0(\Omega) \tag{3.13}$$

with (i) $\| \nabla^{\perp}\psi_{v0}\| \leq \|\mathbf{v}\|$, (ii) if $\mathbf{v} = \nabla^{\perp}\psi$ with $\psi \in H_0^1(\Omega)$, then $\psi_{v0} = \psi$ and (iii) if $\mathcal{O}$ is an open subset of Ω with $\overline{\mathcal{O}} \subset \Omega$ and $\mathbf{v} \in \mathrm{H^m}(\mathcal{O})$, then $\nabla^{\perp}\psi_{v0} \in H^m_{loc}(\mathcal{O})$.

Define $P_{C0} : L^2(\Omega; \mathbb{R}^2) \to L^2(\Omega; \mathbb{R}^2)$ by $P_{C0}\,\mathbf{v} := \nabla^{\perp}\psi_{v0}$. This result implies that $Curl_0(\Omega)$ is a closed subspace of $L^2(\Omega; \mathbb{R}^2)$ from corollary 3.3 of [3] and (ii) shows that P_{C0} is the projection of $L^2(\Omega; \mathbb{R}^2)$ onto $Curl_0(\Omega)$. The extremality condition (3.13) says that $\psi_{v0} \in H_0^1(\Omega)$ is a weak solution of the Dirichlet problem for

$$- \Delta\varphi = \mathrm{curl}\,\mathbf{v} \qquad \text{on} \quad \Omega. \tag{3.14}$$

The complementary projection $Q_{C0} := I - P_{C0}$ is the projection onto the null space of the curl operator as described in the next section. These results may be combined to yield the following result that has been central in the study of 2-dimensional perfect fluids and much classical study of vector fields.

THEOREM 3.6. Assume Ω satisfies (B1) and $\mathbf{v} \in G(\Omega) \cap Curl(\Omega)$, then the potentials φ_v, ψ_v are conjugate harmonic functions on Ω and $\mathbf{v}$ is C^{∞} on Ω.

Proof. The assumption is that there are functions φ_v, ψ_v in $H^1(\Omega)$ such that $\mathbf{v} = \nabla\varphi_v = \nabla^{\perp}\psi_v$ on Ω. These are the Cauchy-Riemann equations. Thus φ_v, ψ_v are harmonic functions satisfying equation (2.5) from the commutativity of weak derivatives. So they, and $\mathbf{v}$, are C^{∞} on Ω. $\square$

4. Div, curl and orthogonality of planar vector fields. In many classical field theoretic problems. vector fields are found from information about just their divergence and curl. That is, one need not know all four components of the derivative matrix $D\,\mathbf{v}(x) := (v_{j,k}(x))$ of $\mathbf{v}$ to determine the field $\mathbf{v}$. It is sufficient to know the two linear combinations $\mathrm{curl}\,\mathbf{v}$, $\mathrm{div}\,\mathbf{v}$. Moreover these operators may be defined without requiring that individual components of $\mathbf{v}$ be $W^{1,1}(\Omega)$.

In the following only fields whose curl and div are locally integrable functions on Ω will be considered. When $\mathbf{v} \in \mathrm{L}^1_{loc}(\Omega; \mathbb{R}^2)$, define $\mathrm{curl}\,\mathbf{v} := \omega \in \mathrm{L}^1_{loc}(\Omega)$ provided ω satisfies

$$\int_{\Omega} \nabla^{\perp}\psi \cdot \mathbf{v} \; d^2x = \int_{\Omega} \omega\,\psi \; d^2x \qquad \text{for all} \quad \psi \in C^1_c(\Omega). \tag{4.1}$$

Similarly $\mathrm{div}\,\mathbf{v} := \rho \in \mathrm{L}^1_{loc}(\Omega)$ provided ρ satisfies

$$\int_{\Omega} \nabla\varphi \cdot \mathbf{v} \; d^2x = -\int_{\Omega} \rho\,\varphi \; d^2x \qquad \text{for all} \quad \varphi \in C^1_c(\Omega). \tag{4.2}$$

When the components of a planar vector field $\mathbf{v}$ are in $W^{1,1}(\Omega)$, then the derivative of the field is the matrix valued function whose entries are L^1 functions on Ω and one has

$$\mathrm{div}\,\mathbf{v} = \mathrm{v}_{1,1} + \mathrm{v}_{2,2} \qquad \text{and} \qquad \mathrm{curl}\,\mathbf{v} = \mathrm{v}_{2,1} - \mathrm{v}_{1,2}. \tag{4.3}$$

A field $\mathbf{v} \in \mathrm{L}^1_{loc}(\Omega; \mathbb{R}^2)$ is *irrotational*, or *solenoidal*, respectively provided

$$\int_{\Omega} \nabla^{\perp}\varphi \cdot \mathbf{v} \; d^2x = 0 \qquad \text{or} \qquad \int_{\Omega} \nabla\varphi \cdot \mathbf{v} \; d^2x = 0 \qquad \text{for all} \quad \varphi \in C^1_c(\Omega). \tag{4.4}$$

A field $\mathbf{v} \in \mathrm{L}^1_{loc}(\Omega;\mathbb{R}^2)$ is *harmonic* if it is both irrotational and solenoidal on Ω. Let $\mathcal{H}^0(\Omega,\mathbb{R}^2)$ be the closed subspace of all harmonic vector fields in $L^2(\Omega;\mathbb{R}^2)$. Observe that the space $H_{GC}(\Omega) := G(\Omega) \cap Curl(\Omega)$ of vector fields that are both gradients and curls is a space of harmonic fields.

Let $N(\text{curl}), N(\text{div})$ be the subspaces of irrotational, solenoidal vector fields in $L^2(\Omega;\mathbb{R}^2)$ respectively. Note that fields in $G(\Omega)$ are in $N(\text{curl})$ and fields in $Curl(\Omega)$ are in $N(\text{div})$ from the commutativity of weak differentiation. A first result is the following decomposition which is independent of the differential topology of the region Ω.

THEOREM 4.1. Assume Ω satisfies (B1); then

$$(a) \qquad L^2(\Omega;\mathbb{R}^2) = Curl_0(\Omega) \oplus N(\text{curl}) = G_0(\Omega) \oplus N(\text{div}), \quad \text{and} \tag{4.5}$$

$$(b) \qquad L^2(\Omega;\mathbb{R}^2) = Curl_0(\Omega) \oplus G_0(\Omega) \oplus \mathcal{H}^0(\Omega,\mathbb{R}^2). \tag{4.6}$$

Proof. (a) follows from the definition of $N(\text{curl})$ and $N(\text{div})$ since $C^1_c(\Omega)$ is dense in $H^1_0(\Omega)$. Then (b) follows as $\mathbf{v} \in \mathcal{H}^0(\Omega,\mathbb{R}^2)$ iff it is orthogonal to both $Curl_0(\Omega)$ and $G_0(\Omega)$. □

This theorem implies that the projections Q_{G0}, Q_{C0} of the preceding section are the projections onto the subspaces $N(\text{div}), N(\text{curl})$ respectively since P_{G0}, P_{C0} are the projections onto their orthogonal complements.

(4.6) will be called the *CGH decomposition* of L^2-vector fields and will be used henceforth. It differs from Hodge-Weyl decompositions where zero boundary conditions are imposed on only one of the potentials. As in the preceding section, our sign convention will be.

$$\mathbf{v} = \nabla^\perp\psi - \nabla\varphi + \mathbf{h} \qquad \text{with} \quad \psi,\ \varphi \in \mathrm{H}^1_0(\Omega) \quad \text{and} \quad \mathbf{h} \in \mathcal{H}^0(\Omega,\mathbb{R}^2). \tag{4.7}$$

Let $\mathcal{O} := I_1 \times I_2$ be an open rectangle in $\mathbb{R}^2$. Poincaré's lemma provides explicit formulae for the potentials φ_p, ψ_p of continuous fields $\mathbf{v}$ on $\mathcal{O}$. Given a point $P \in \mathcal{O}$ and a piecewise C^1- curve Γ_x joining P to $x = (x_1, x_2) \in \mathcal{O}$, define

$$\varphi_p(x) := \int_{\Gamma_x} v_1\,dx_1 + v_2\,dx_2 \quad \text{and} \quad \psi_p(x) := \int_{\Gamma_x} v_1\,dx_2 - v_2\,dx_1. \tag{4.8}$$

These are C^1- functions on Ω. When $\mathbf{v}$ is irrotational, $\nabla\varphi_p = \mathbf{v}$ and when $\mathbf{v}$ is solenoidal then $\nabla^\perp\psi_p = \mathbf{v}$ on $\mathcal{O}$. See Dautray and Lions, [12, Chapter IX, section 1, lemma 3], for a proof in the case where Ω is a block in $\mathbb{R}^3$. The proof there is easily modified for this 2-dimensional case.

The line integrals in (4.8) are not well defined when the field $\mathbf{v}$ is only L^p on Ω. Nevertheless, potentials may be proved to exist using a density argument. The following result is known for such fields with $p = 2$ and $\mathcal{O} \subset \mathbb{R}^3$; see Girault - Raviart [17] or Monk, [21, theorem 3.37].

THEOREM 4.2. Assume Ω satisfies (B1) and $p \in [1,\infty)$. If $\mathbf{v} \in L^p(\Omega;\mathbb{R}^2)$ is irrotational, then there is a $\varphi \in W^{1,p}(\Omega)$ such that $\mathbf{v} = \nabla\varphi$ on Ω. If $\mathbf{v} \in L^p(\Omega;\mathbb{R}^2)$ is solenoidal, then there is a $\psi \in W^{1,p}(\Omega)$ such that $\mathbf{v} = \nabla^\perp\psi$ on Ω.

Proof. First assume that Ω is convex; then Poincaré's lemma implies this result holds when $\mathbf{v}$ is C^1 on Ω. To prove this holds for any L^p field introduce a C^1- mollifier Φ and consider fields on the open convex neighborhood Ω_1 of points within distance 1 of Ω. The sequence of C^1- fields $\mathbf{v}^{(m)}$ defined by convolution $\mathbf{v}^{(m)} := \Phi_{\mathrm{m}} \star \mathbf{v}$ converges to the zero extension of $\mathbf{v}$ to Ω_1 in $L^p(\Omega, R^2)$. Each of these $\mathbf{v}^{(m)} = \nabla^{\perp}\psi^{(m)}$ on Ω_1 from Poincaré's lemma as in equation (4.8). Normalize the $\psi^{(m)}$ to have mean value zero. Since these fields are a Cauchy sequence in $L^p(\Omega, R^2)$, the $\psi^{(m)}$ are Cauchy in $W^{1,p}_m(\Omega)$, so they converge to a limit $\tilde{\psi}$. Taking limits, $\nabla^{\perp}\tilde{\psi}$ is the zero extension of $\mathbf{v}$ to Ω_1, so the result for solenoidal fields holds. The result for irrotational fields and scalar potentials holds by modifying the above argument appropriately.

When Ω is not convex, choose Ω_1 in the above proof to be the neighborhood of distance 1 from the convex hull of Ω. Then the same arguments yield the statement of the theorem. □

Note that the preceding proof extends to 3-dimensional vector fields and regions, with the usual modifications, as the construction of Poincaré's lemma is valid there - and the other ingredients are independent of dimension. A corollary is the following vector-valued version of Weyl's lemma - and also extends to 3-d vector fields.

Corollary 4.3. Assume Ω satisfies (B1) and $p \in [1, \infty)$. If $\mathbf{v} \in L^p(\Omega; \mathbb{R}^2)$ is a harmonic vector field, then it is C^∞ on Ω.

Proof. Since $\mathbf{v}$ is irrotational, there is a $\varphi \in W^{1,p}(\Omega)$ such that $\mathbf{v} = \nabla\varphi$ on Ω. As $\mathbf{v}$ also is solenoidal, φ is a weak solution of Laplace's equation. Thus, from Weyl's lemma, φ is C^∞ on Ω, and thus $\mathbf{v}$ is also. □

It should be noted that the above results do not require any topological conditions on the region Ω. The theorem implies that $G(\Omega)^{\perp} \subset Curl(\Omega)$ and $Curl(\Omega)^{\perp} \subset G(\Omega)$ for any region Ω satisfying (B1) - and it is well known that these are strict inclusions when Ω is not simply connected.

5. Div-curl and Laplacian boundary value problems. The div-curl boundary value problem is to find a vector field $\mathbf{v}$ defined on a bounded region $\Omega \subset \mathbb{R}^2$ that satisfies

$$\text{div } \mathbf{v}(x) = \rho(x) \qquad \text{and} \qquad \text{curl } \mathbf{v}(x) = \omega(x) \qquad \text{for} \quad x \in \Omega \tag{5.1}$$

subject to prescribed boundary conditions on $\partial\Omega$.

Generally either the normal component $\mathbf{v} \cdot \nu$, or the tangential component $\mathbf{v} \cdot \tau$, of the field at the boundary are prescribed in applications. When the normal component is prescribed everywhere on the boundary we have a *normal Div-Curl boundary value problem* that will be analyzed in the next section. Problems where the tangential component is prescribed everywhere are called *tangential Div-Curl boundary value problems* and are studied in section 7. When normal components are prescribed on part of the boundary and tangential components on the complementary subset, they are called *mixed Div-Curl boundary value problems.*

To obtain bounds and representation results for these problems, some properties of solutions of Laplacian boundary value problems on regions obeying (B1) are required. While these are standard second order elliptic boundary problems, the author does not know of an accessible reference for many of the results on Lipschitz regions so proofs are provided here. The use of Steklov eigenfunctions and eigenvalues to describe the dependence of solutions on the boundary conditions provides a much simpler description than the use of Layer potentials used by many authors. Stronger regularity results are well known when the boundary $\partial\Omega$ is C^k with $k \geq 1$ or solutions are sought in various Schauder spaces.

The solutions of these boundary value problems will be found by introducing appropriate potentials φ_0, ψ_0 in the CGH decomposition of Theorem 4.1(b). The potentials φ_0, ψ_0 are solutions of Poisson's equation with zero Dirichlet boundary data and are characterized by variational principles.

Given $\rho \in L^p(\Omega)$, consider the problem of minimizing the functional $\mathcal{D}$ defined by

$$\mathcal{D}(\varphi) \ := \ \int_\Omega \ [\,|\nabla\varphi|^2 \ - \ 2\,\rho\,\varphi\,] \ d^2x \tag{5.2}$$

on $H_0^1(\Omega)$. The essential results about this classical problem may be summarized as follows

Theorem 5.1. Suppose (B1) holds, $\rho \in L^p(\Omega)$ and $1 < p \leq \infty$. Then there is a unique minimizer $\varphi_0 := \mathcal{G}_D\rho$ of $\mathcal{D}$ on $H_0^1(\Omega)$ that satisfies

$$\int_\Omega [\,\nabla\varphi\cdot\nabla\chi \ - \ \rho\chi\,] \ = \ 0 \qquad \text{for all} \quad \chi \in H_0^1(\Omega). \tag{5.3}$$

$\mathcal{G}_D$ is linear, 1-1 and a compact map of $L^p(\Omega)$ into $H_0^1(\Omega)$.

Proof. When Ω is bounded then the imbedding $i : H_0^1(\Omega) \to L^q(\Omega)$ is continuous for all $q \in [1,\infty)$ from the Sobolev imbedding theorem. Thus the linear term in $\mathcal{D}$ is weakly continuous. The existence of a unique minimizer then holds as $\lambda_1 > 0$ in (3.8), so $\mathcal{D}$ is continuous, convex and coercive on $H_0^1(\Omega)$.

$\mathcal{D}$ is G-differentiable on $H_0^1(\Omega)$ and the extremality condition for a minimizer is (5.3). Write the minimizer as $\mathcal{G}_D\rho$; then $\mathcal{G}_D$ is a linear mapping that satisfies

$$\|\nabla\varphi\|_2 \ \leq \ C_{p'} \ \|\rho\|_p \tag{5.4}$$

with p' conjugate to p and C_q the imbedding constant for $H_0^1(\Omega)$ into $L^q(\Omega)$. Thus $\mathcal{G}_D$ is continuous. It is 1-1 from properties of harmonic functions.

To prove $\mathcal{G}_D$ is compact let $\{\rho_m : m \geq 1\}$ be a weakly convergent sequence in $L^p(\Omega)$. The imbedding of $L^p(\Omega)$ into $H^{-1}(\Omega)$ is compact for $p \in (1,\infty)$ by duality to the Kondratchev theorem, so the sequence $\{\rho_m\}$ is strongly convergent in $H^{-1}(\Omega)$. A standard result is that $\mathcal{G}_D$ is a continuous linear map of $H^{-1}(\Omega)$ to $H_0^1(\Omega)$ so it is a compact linear mapping of $L^p(\Omega)$ to $H_0^1(\Omega)$ by composition when $p > 1$. □

It is worth noting that this result enables proofs of many of the results about the approximation of solutions of (5.3) by eigenfunction expansions in terms of the eigenfunctions of the zero-Dirichlet Laplacian eigenproblem. It is well known that there are orthonormal bases of $H_0^1(\Omega)$ consisting of such eigenfunctions. Since $\mathcal{G}_D$ is compact, finite rank approximations using these eigenfunctions will converge to the solution $\mathcal{G}_D\rho$ for all ρ in these $L^p(\Omega)$ and this solution has the standard spectral representation arising from the spectral theorem for compact self-adjoint maps on $L^2(\Omega)$.

This result enables a generalization of the harmonic decomposition of L^2 fields to fields in $L^1(\Omega;\mathbb{R}^2)$ with divergence and curl in L^p as follows.

THEOREM 5.2 (L^p-harmonic decomposition). Suppose (B1) holds, $\mathbf{v} \in L^1_{loc}(\Omega;\mathbb{R}^2)$, (or $L^1(\Omega;\mathbb{R}^2)$), with $\operatorname{div}\mathbf{v}$, $\operatorname{curl}\mathbf{v} \in \mathrm{L}^p(\Omega)$ for some $p > 1$. Then there are $\varphi_0, \psi_0 \in H^1_0(\Omega)$ and a harmonic field $\mathbf{h} \in L^1_{loc}(\Omega;\mathbb{R}^2)$, (or $L^1(\Omega;\mathbb{R}^2)$) such that

$$\mathbf{v} \;=\; \nabla^{\perp}\psi_0 \;-\; \nabla\varphi_0 + \mathbf{h} \qquad \text{on} \quad \Omega. \tag{5.5}$$

Proof. Let $\rho := \operatorname{div}\mathbf{v}$, $\omega := \operatorname{curl}\mathbf{v}$ and φ_0 is the solution of (5.3), ψ_0 is the solution with ω in place of ρ. They exist and are in $H^1_0(\Omega)$ from Theorem 5.1. Then $\mathbf{h} := \mathbf{v} - \nabla^{\perp}\psi_0 + \nabla\varphi_0$ is a harmonic field that is in $L^1_{loc}(\Omega;\mathbb{R}^2)$, (or $L^1(\Omega;\mathbb{R}^2)$), respectively when $\mathbf{v}$ is. □

Note that the three components in this decomposition remain L^2- orthogonal. When stronger regularity conditions are imposed on the field, or its divergence or curl, then $\mathbf{h}$ and the potentials φ_0, ψ_0 will have further regularity

When ρ or ω are L^2 then better information is obtained by using the space $H_0(\Delta,\Omega)$ of functions in $H^1_0(\Omega)$ whose Laplacians are in $L^2(\Omega)$. This is a real Hilbert space with respect to the inner product

$$\langle\varphi,\chi\rangle_{\Delta} \;:=\; \int_{\Omega} \left[\,\Delta\varphi\,\Delta\chi \,+\, \nabla\varphi\cdot\nabla\chi\,\right]\; d^2x. \tag{5.6}$$

When Ω satisfies (B1) and $\partial\Omega$ is C^1, then it is well known that $H_0(\Delta,\Omega) = H^1_0(\Omega) \cap H^2(\Omega)$. See Evans chapter 8 for example. This need not hold when only (B1) is required. Such issues have been studied by Grisvard [18], Jerison, Koenig and others. In this case Theorem 5.1 may be improved as follows with constants that can be characterized - and known for many regions Ω.

THEOREM 5.3. Assume (B1) holds and λ_1 is the constant of (3.8). Then the operator $\mathcal{G}_D$ is a homeomorphism of $L^2(\Omega)$ and $H_0(\Delta,\Omega)$ with $\varphi_0 = \mathcal{G}_D\rho$ satisfying

$$\|\varphi_0\|_2 \;\le\; \frac{1}{\lambda_1}\,\|\rho\|_2, \quad \|\nabla\varphi_0\|_2 \;\le\; \frac{1}{\sqrt{\lambda_1}}\,\|\rho\|_2 \quad \text{and} \quad \|D_\nu\varphi_0\|_{2,\partial\Omega} \;\le\; C_0\,\|\rho\|_2 \tag{5.7}$$

where $C_0 > 0$ depends only on Ω.

Proof. The first two inequalities here follow from the spectral representation of $\mathcal{G}_D$ in terms of the Dirichlet eigenfunctions of the Laplacian on Ω. The inequality for $D_\nu\varphi_0$ is theorem 3.2 of [8]. □

Note also that if $\mathcal{O}$ is an open subset of Ω with $\overline{\mathcal{O}} \subset \Omega$ and ρ is H^m on $\mathcal{O}$, then the solution φ_0 of (5.3) will be of class H^{m+1}_{loc} on $\mathcal{O}$ from the usual interior regularity analysis. When $\rho, \omega \in L^p(\Omega)$, let $\varphi_0 = \mathcal{G}_D\rho$, $\psi_0 := \mathcal{G}_D\omega$ be solutions of (5.3) and consider

$$\mathbf{h}(x) \;:=\; \mathbf{v} \;-\; \nabla^{\perp}\psi_0\,(\mathrm{x}) \;+\; \nabla\varphi_0\,(\mathrm{x}). \tag{5.8}$$

Substituting in (5.1), one sees that $\mathbf{h}$ will be a harmonic field with

$$\mathbf{h}\cdot\nu \;=\; \mathbf{v}\cdot\nu + \mathrm{D}_\nu\varphi_0 \quad \text{and} \quad \mathbf{h}\cdot\tau \;=\; \mathbf{v}\cdot\tau + \mathrm{D}_\nu\psi_0 \qquad \text{on} \quad \partial\Omega. \tag{5.9}$$

That is, the solvability of this div-curl system is decomposed into zero-Dirichlet boundary value problems involving the source terms and a boundary value problem for a harmonic field. So the following sections will concentrate on issues regarding different types of boundary value problems for harmonic vector fields.

Some related results about the Neumann problem for the Laplacian will also be needed later. The harmonic components of solutions of our problems involve potentials $\chi \in H^1(\Omega)$ that satisfy

$$\int_\Omega \nabla\chi \cdot \nabla\xi \, d^2x - \int_{\partial\Omega} \eta\xi \; ds \;=\; 0 \qquad \text{for all} \quad \xi \in H^1(\Omega). \tag{5.10}$$

This is the weak form of Laplace's equation subject to $D_\nu \chi = \eta$ on $\partial\Omega$.

To study the existence of solutions of this problem, consider the problem of minimizing the functional $\mathcal{N} : H^1(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{N}(\chi) \;:=\; \int_\Omega \; |\nabla\chi|^2 \;\; d^2x \;-\; 2\int_{\partial\Omega} \; \eta \; \chi \;\; ds \tag{5.11}$$

with $\eta \in L^2(\partial\Omega, ds)$. A necessary condition for the existence of a solution of this problem is that

$$\int_{\partial\Omega} \; \eta \;\; ds \;=\; 0. \tag{5.12}$$

To prove the existence of a solution of this problem we need the fact that there is a $\delta_1 > 0$ such that

$$\int_\Omega \; |\nabla\varphi|^2 \;\; d^2x \;\geq\; \delta_1 \int_{\partial\Omega} \; \varphi^2 \;\; ds \qquad \text{for all} \quad \varphi \in H^1(\Omega) \text{ that satisfy (5.12).} \tag{5.13}$$

This δ_1 is the first nonzero harmonic Steklov eigenvalue for the region Ω.

When (B1) and (5.12) hold then a standard variational argument says that there is a unique minimizer $\chi = \mathcal{B}\eta$ of $\mathcal{N}$ on $H^1_m(\Omega)$ that satisfies (5.10).

This solution operator $\mathcal{B}$ may be regarded as an integral operator that maps functions from $L^2(\partial\Omega, ds)$ to $\mathcal{H}(\Omega) \subset H^1(\Omega)$. In particular it has a nice expression in terms of the harmonic Steklov eigenfunctions of Ω. See Auchmuty [4] and [6] for a discussion of the Steklov eigenproblem for the Laplacian on bounded regions and [8] for further results about $H_0(\Delta, \Omega)$.

A function $s_j \in \mathcal{H}(\Omega)$ is a Steklov eigenfunction for the Laplacian on Ω provided it is a nontrivial solution of the system

$$\int_\Omega \nabla s_j \cdot \nabla\xi \, d^2x \;=\; \delta \int_{\partial\Omega} s_j \, \xi \, ds \qquad \text{for all} \quad \xi \in H^1(\Omega). \tag{5.14}$$

Here $\delta \in \mathbb{R}$ is the associated Steklov eigenvalue. Let $\Lambda := \{\delta_j ; j \geq 0\}$ be the set of Steklov eigenvalues repeated according to multiplicity and with δ_j an increasing sequence. The first eigenvalue is $\delta_0 = 0$ and the corresponding eigenfunctions are constants on $\overline{\Omega}$. It is a simple eigenvalue and the next eigenvalue is $\delta_1 > 0$ appearing in (5.13). Normalize an associated set of Steklov eigenfunctions $\mathcal{S} := \{s_j : j \geq 0\}$ to be L^2-orthonormal on $\partial\Omega$. Then

$$\int_\Omega \nabla s_j \cdot \nabla s_k \, d^2x \;=\; \delta_j \qquad \text{when} \quad j = k \quad \text{and} \quad 0 \quad \text{otherwise.} \tag{5.15}$$

Theorem 4.1 of [5] says that this sequence can be chosen to be an orthonormal basis of $L^2(\partial\Omega, ds)$ with the usual inner product. If $\eta \in L^2(\partial\Omega, ds)$ satisfies the compatibility condition (5.12), then it has the representation

$$\eta(z) \;=\; \sum_{j=1}^{\infty} \hat{\eta}_j \, s_j(z) \qquad \text{on} \quad \partial\Omega \qquad \text{with} \quad \hat{\eta}_j = \int_{\partial\Omega} \eta \, s_j \, ds. \tag{5.16}$$

Define boundary integral operators $\mathcal{B}_M : L^2(\partial\Omega, ds) \to \mathcal{H}(\Omega)$ by

$$\mathcal{B}_M \, \eta(x) \;:=\; \int_{\partial\Omega} B_M(x,z)\, \eta(z)\, ds \qquad \text{with} \quad B_M(x,y) \;:=\; \sum_{j=1}^{M} \delta_j^{-1}\, s_j(x)\, s_j(z) \tag{5.17}$$

for $M \geq 1$. These are finite rank operators and the solution operator $\mathcal{B}$ introduced above has the following properties.

Theorem 5.4. Assume (B1) holds, Λ is the set of harmonic Steklov eigenvalues on Ω repeated according to multiplicity and $\mathcal{S}$ is a $\partial-$orthogonal set of harmonic Steklov eigenfunctions and an orthonormal basis of $L^2(\partial\Omega, ds)$. When $\eta \in L^2(\partial\Omega, ds)$ satisfies (5.12), then the sequence $\{\mathcal{B}_M \, \eta\}$ converges strongly to a function $\chi := \mathcal{B}\,\eta$ in $H^1(\Omega)$. χ is C^∞ on Ω and $\mathcal{B}$ is a continuous linear transformation of $L^2(\partial\Omega, ds)$ to $\mathcal{H}(\Omega)$ with $\|\nabla\chi\|_2 \;\leq\; {\delta_1}^{-1} \; \|\eta\|_{2,\partial\Omega}$.

Proof. The first part of this theorem is proved in [5] where it is shown that $\mathcal{S}$ is an orthonormal basis of $L^2(\partial\Omega, ds)$. Thus (5.16) holds. When χ is a solution of equation (5.10), then $\chi(x) \;=\; \sum_{j=1}^{\infty} \hat{\chi}_j \, s_j(x)$ on Ω as $\mathcal{S}$ is a maximal orthogonal set in $\mathcal{H}(\Omega)$. Take $\xi = s_j$ in (5.10); then the coefficients $\hat{\chi} = \hat{\eta}_j/\delta_j$ for $j \geq 1$, and the sequence converges strongly in the $\partial-$norm of $\mathcal{H}(\Omega)$. The function χ is C^∞ as it is harmonic and the bound on $\|\nabla\chi\|_2$ follows from the orthogonality of $\mathcal{S}$. $\square$

Solutions of (5.10) and (5.12) are unique up to a constant – so their gradients and Curls are unique. For each η above there will be a unique c_m such that $\chi + c_m \in H^1_m(\Omega)$. χ will be H^1 when the Neumann data is in $H^{-1/2}(\partial\Omega)$ and more generally will be in the space $\mathcal{H}^s(\Omega)$ when $\eta \in H^{s-3/2}(\partial\Omega)$, $s \geq 0$, with the definitions of [6].

6. The normal div-curl boundary value problem. The normal div-curl boundary value problem is to find a field $\mathbf{v} \in \mathrm{L}^2(\Omega; \mathbb{R}^2)$ that solves (5.1) subject to

$$\mathbf{v}(z) \cdot \nu(z) \;=\; \eta_\nu(\mathrm{z}) \qquad \text{on} \quad \partial\Omega \tag{6.1}$$

with $\eta_\nu \in L^2(\partial\Omega, ds)$. From the divergence theorem, a necessary condition for (5.1) - (6.1) to have a solution is the compatibility condition

$$\int_\Omega \rho \; d^2x \;=\; \int_{\partial\Omega} \eta_\nu(z) \; ds(z). \tag{6.2}$$

Assume that the solution has the form (4.7); then the potentials are solutions of (5.3) with $\psi_0 = \mathcal{G}_D\omega,\; \varphi_0 = \mathcal{G}_D\rho$. Since $\psi_0 \equiv 0$ on $\partial\Omega$, the harmonic component satisfies

$$\mathbf{h}(z) \cdot \nu(z) \;=\; \eta_\nu(z) \,+\, D_\nu\varphi_0(z) \qquad \text{for } z \in \partial\Omega. \tag{6.3}$$

Consider the problem of finding a gradient field that solves this problem. If $h = \nabla\chi$ is a solution of this problem, then χ is a harmonic function that satisfies (5.10) with $\eta(z)$ given by the right hand side of (6.3). From Theorem 5.3, this problem has a solution and the following holds.

THEOREM 6.1. Assume (B1), $\rho, \omega \in L^2(\Omega)$, $\eta_\nu \in L^2(\partial\Omega, ds)$ and (6.2) holds. Let $\varphi_0 = \mathcal{G}_D\rho$, $\psi_0 = \mathcal{G}_D\omega$. Then there is a unique $\chi \in H^1_m(\Omega)$ such that $\mathbf{h}(x) = \nabla\chi(x)$ is a harmonic field satisfying (5.10) with η given by the right hand side of (6.3). Then $\mathbf{v} := \nabla^\perp\psi_0 - \nabla\varphi_0 + \nabla\chi$ is a solution of (5.1) - (6.1) with

$$\|\mathbf{v}\|_2 \le \frac{1}{\sqrt{\lambda_1}} [\, \|\rho\|_2 + \|\omega\|_2 \,] + \frac{1}{\sqrt{\delta_1}} [\, \|\eta_\nu\|_{2,\partial\Omega} + C_0\|\rho\|_2 \,]. \tag{6.4}$$

Proof. Given $\rho, \omega \in L^2(\Omega)$, Theorem 5.3 yields the first two terms in the inequality (6.4). Note that (6.2) implies the compatibility condition (5.12), so there is a unique $\chi \in H^1_m(\Omega)$ that is harmonic on Ω and satisfies the boundary condition (6.3) from Theorem 5.4. The three fields in this representation of $\mathbf{v}$ are L^2-orthogonal so it only remains to bound $\|\nabla\chi\|_2$. This bound now follows from the last parts of Theorems 5.3 and 5.4. □

It is worth noting that the constants in this inequality are best possible in that there are choices of ρ, ω and η_ν for which the right hand side equals the 2-norm of a solution of the problem. If $\rho, \omega \in L^p(\Omega)$ for some $p > 1$, then (5.2) implies that

$$\|\mathbf{v} - \nabla\chi\|_2 \le \mathrm{C_{p'}}\, [\, \|\rho\|_{\mathrm{p}} + \|\omega\|_{\mathrm{p}} \,].$$

Also the regularity of the potentials φ_0, ψ_0 here depends on the regularity of ρ, ω and the boundary $\partial\Omega$. They are independent of the boundary data. The boundary data η_ν only influences the harmonic component $\nabla\chi$. Moreover χ is very smooth ($C^\infty \cap H^1$) on Ω as it is a finite energy solution of Laplace's equation.

This result implies that these solutions of this problem can be written as series expansions involving the Dirichlet and Steklov eigenfunctions of the Laplacian as $\mathcal{G}_D$ and $\mathcal{B}$ from (5.15) have representations with respect to eigenfunction bases of $H^1_0(\Omega)$ and $\mathcal{H}(\Omega)$ respectively.

The following corollary is of the same type as estimates studied in [9]. Let $H(\mathrm{curl}, \Omega)$, $H_\partial(\mathrm{curl}, \Omega)$, $H(\mathrm{div}, \Omega)$ and $H_\partial(\mathrm{div}, \Omega)$ be the spaces of vector fields in $L^2(\Omega; \mathbb{R}^2)$ that also have, respectively, $\mathrm{curl}\,\mathbf{v}, \mathrm{curl}\,\mathbf{v}$ and $\mathbf{v}\cdot\tau$, $\mathrm{div}\,\mathbf{v}, \mathrm{div}\,\mathbf{v}$ and $\mathbf{v}\cdot\nu$ in L^2.

COROLLARY 6.2. Suppose (B1) holds, $\mathrm{curl}\,\mathbf{v}, \mathrm{div}\,\mathbf{v} \in \mathrm{L}^2(\Omega)$, $\mathbf{v}\cdot\nu \in \mathrm{L}^2(\partial\Omega, \mathrm{ds})$; then $\mathbf{v} \in \mathrm{H}_\partial(\mathrm{div}, \Omega) \cap \mathrm{H}(\mathrm{curl}, \Omega)$ and there is a $C > 0$ such that

$$\|\mathbf{v}\|_2^2 \le \mathrm{C} \left[\, \|\,\mathrm{curl}\,\mathbf{v}\|_2^2 + \|\,\mathrm{div}\,\mathbf{v}\,\|_2^2 + \|\mathbf{v}\cdot\nu\|^2_{2,\partial\Omega} \,\right]. \tag{6.5}$$

Proof. The inequality (6.4) implies (6.5) for an appropriate choice of C. Since (6.5) holds, $\mathbf{v}$ is in both $H_\partial(\mathrm{div}, \Omega)$ and $H(\mathrm{curl}, \Omega)$ provided the region satisfies (B1). □

When Ω is not simply connected this boundary value problem has further solutions. These were studied in [1] where the well-posed problem was shown to require the circulations around each hole be further specified for uniqueness. The above solution is the least energy (2-norm) solution of the problem.

7. The tangential div-curl boundary value problem. The tangential div-curl boundary value problem is to find a vector field $\mathbf{v} \in \mathrm{L}^2(\Omega;\mathbb{R}^2)$ that satisfies (5.1) subject to

$$\mathbf{v}(z)\cdot\tau(z) \;=\; \eta_\tau(z) \qquad \text{on} \quad \partial\Omega \tag{7.1}$$

with $\eta_\tau \in L^2(\partial\Omega, ds)$. From the divergence theorem, a necessary condition for the existence of solutions is that

$$\int_\Omega \omega \;\, d^2x \;=\; \int_{\partial\Omega} \eta_\tau(z)\; ds. \tag{7.2}$$

When a solution has the CGH form (4.7), the potentials φ_0, ψ_0 are solutions of (5.3) given by $\psi_0 = \mathcal{G}_D\omega,\ \varphi_0 = \mathcal{G}_D\rho$. Then the harmonic component satisfies

$$-\,\mathbf{h}(z)\cdot\tau(z) \;=\; \eta_\tau(z) \,+\, D_\nu\psi_0(z) \qquad \text{on} \quad \partial\Omega \tag{7.3}$$

since $\varphi \equiv 0$ on $\partial\Omega$. Suppose that this harmonic field has a representation $\mathbf{h} \;=\; -\nabla^\perp\chi$. Then χ is a harmonic function that satisfies (5.10) with $\eta(z)$ given by the right hand side of (7.3). This problem has a solution of the form (5.8) with this harmonic potential χ.

THEOREM 7.1. Assume $\rho, \omega \in L^2(\Omega)$, $\eta_\tau \in L^2(\partial\Omega, ds)$ and (B1), (7.2) hold. Let $\varphi_0 = \mathcal{G}_D\rho, \psi_0 = \mathcal{G}_D\omega$. Then there is a unique $\chi \in H^1_m(\Omega)$ such that $\mathbf{h}(x) \;=\; -\nabla^\perp\chi(x)$ is a harmonic field satisfying (7.3) on $\partial\Omega$. The field $\mathbf{v} = \nabla^\perp\psi_0 \,-\, \nabla\varphi_0 \,-\, \nabla^\perp\chi$ is a solution of (5.1) - (7.1) with

$$\|\mathbf{v}\|_2 \;\le\; \frac{1}{\sqrt{\lambda_1}}\,[\;\|\rho\|_2 \,+\, \|\omega\|_2\;] \,+\, \frac{1}{\sqrt{\delta_1}}\,[\;\|\eta_\nu\|_{2,\partial\Omega} \,+\, \mathrm{C}_0\|\omega\|_2\;]\,. \tag{7.4}$$

Proof. This proof is essentially the same as that of Theorem 6.1. The compatibility condition (7.2) implies that compatibility condition for the solvability of (5.10) with η given by the right hand side of (7.3) holds. The estimates now follow as in Theorem 6.1. □

In a similar manner to Corollary 6.2 of the last section one finds

COROLLARY 7.2. Suppose (B1) holds and $\operatorname{curl}\mathbf{v}, \operatorname{div}\mathbf{v} \in \mathrm{L}^2(\Omega)$, $\mathbf{v}\cdot\tau \in \mathrm{L}^2(\partial\Omega, \mathrm{ds})$; then $\mathbf{v} \in \mathrm{H}(\mathrm{div},\Omega) \cap \mathrm{H}_\partial(\mathrm{curl},\Omega)$ and there is a $C > 0$ such that

$$\|\mathbf{v}\|_2^2 \;\le\; \mathrm{C}\;\left[\;\|\operatorname{curl}\mathbf{v}\|_2^2 \,+\, \|\operatorname{div}\mathbf{v}\|_2^2 \,+\, \|\mathbf{v}\cdot\tau\|_{2,\partial\Omega}^2\;\right]. \tag{7.5}$$

When the region Ω has holes (that is, its boundary has more than one connected component), then the solution of this boundary value problem is nonunique. There are nonzero harmonic vector fields associated with potential differences between different components of the boundary. This was studied in [1] where the well-posed problem was described and the solution described in the above theorem is the least energy (2-norm) solution of the problem.

8. Mixed boundary conditions. In electromagnetic field theory, problems where given flux conditions are prescribed on part of the boundary and tangential boundary data is prescribed on the complementary part need to be solved. The well-posedness and uniqueness of solutions of these problems was studied in sections 12 - 15 of Alexander and Auchmuty [1]. Here our primary interest is in obtaining 2-norm bounds on solutions in terms of the data. The constant in the relevant estimate will be the value of a natural optimization problem that is related to an eigenvalue in the case where the data is L^2.

The analysis of such a problem differs considerably from that for the normal and tangential boundary value problems. First no compatibility conditions on the data are required for L^2-solvability. In addition the two potentials are each found directly by solving similar variational problems on appropriate closed subspaces of $H^1(\Omega)$ determined by the topology of the boundary data. Here L^2–bounds on these solutions in terms of the data will be obtained.

The mixed div-curl boundary value problem is to find vector fields $\mathbf{v} \in \mathrm{L}^2(\Omega;\mathbb{R}^2)$ that satisfy (5.1) subject to

$$\mathbf{v}(z)\cdot\tau(z) \;=\; \eta_\tau(\mathrm{z}) \qquad \text{on} \quad \Gamma_\tau \quad \text{and} \quad \mathbf{v}(z)\cdot\nu(z) \;=\; \eta_\nu(\mathrm{z}) \qquad \text{on} \quad \Gamma_\nu. \tag{8.1}$$

Here Γ_τ, Γ_ν are nonempty open subsets of $\partial\Omega$ whose union is dense in $\partial\Omega$.

Condition (B2). Γ is an nonempty open subset of $\partial\Omega$ with a finite number of disjoint components $\{\gamma_1,\ldots,\gamma_L\}$ and there is a finite distance d_0 such that $d(\gamma_j,\gamma_k)\geq d_0$ when $j\neq k$.

When Γ satisfies (B2), define $H^1_{\Gamma 0}(\Omega)$ to be the subspace of $H^1(\Omega)$ of functions whose traces are zero on the set $\Gamma\subset\partial\Omega$. This is a closed subspace of $H^1(\Omega)$ from lemma 12.1 of [1]. When (B1) and (B2) hold then $H^1_{\Gamma 0}(\Omega)$ is a real Hilbert space with the $\partial-$inner product (2.1). Note that this inner product reduces to

$$\langle\varphi,\psi\rangle_{\partial,\Gamma} \;:=\; \int_\Omega \nabla\varphi\cdot\nabla\psi \;\, d^2x \,+\, \int_{\tilde{\Gamma}} \varphi\,\psi\; ds \tag{8.2}$$

where $\tilde{\Gamma}$ is the complement of Γ in $\partial\Omega$.

Lemma 8.1. Suppose $\Omega, \partial\Omega, \Gamma$ satisfy (B1) and (B2). Then
(i) $\langle\varphi,\psi\rangle_1 \;:=\; \langle\nabla\varphi,\nabla\psi\rangle$ is an equivalent inner product on $H^1_{\Gamma 0}(\Omega)$ to the $\partial-$inner product.
(ii) For $q\in(1,\infty)$, there is an $M_q(\Gamma)$ such that

$$\|\varphi\|_q^q \,+\, \|\varphi\|_{q,\partial\Omega}^q \;\leq\; M_q(\Gamma)\;\|\nabla\varphi\|_2^q \qquad \text{for all} \quad \varphi\in H^1_{\Gamma 0}(\Omega). \tag{8.3}$$

(iii) If $\Gamma_1\supset\Gamma$, then $M_q(\Gamma_1)\;\leq\; M_q(\Gamma)$.

Proof. (i) Let $\lambda_1(\Gamma)$ be the least eigenvalue of the Laplacian on $H^1_{\Gamma 0}(\Omega)$ so that

$$\int_\Omega \;|\nabla\varphi|^2\, d^2x \;\geq\; \lambda_1(\Gamma)\, \int_\Omega \;\varphi^2\; d^2x \qquad \text{for all} \quad \varphi\in H^1_{\Gamma 0}(\Omega). \tag{8.4}$$

This exists and is positive when (B2) holds as $\sigma(\Gamma) > 0$; see proposition 13.2 in [1] for a proof. Since the $\partial-$ norm and the standard norm on $H^1(\Omega)$ are equivalent, there is a $C>0$ such that

$$\|\varphi\|_\partial^2 \;\leq\; C\,\|\varphi\|_{1,2}^2 \;\leq\; C\,\left(1\,+\,\lambda_1(\Gamma)^{-1}\right)\,\|\nabla\varphi\|_2^2.$$

Thus the norm from (i) is equivalent to the ∂-norm.

(ii) Consider the functional $\mathcal{G}_q(\varphi) \;:=\; \|\varphi\|_q^q + \|\varphi\|_{q,\partial\Omega}^q$ on $H^1_{\Gamma 0}(\Omega)$. This functional is convex and weakly continuous as the imbedding of $H^1(\Omega)$ into $L^q(\Omega)$ and $L^q(\partial\Omega, ds)$ are compact for any $q\geq 1$ when (B1) holds. Let B_1 be the unit ball in $H^1_{\Gamma 0}(\Omega)$ with respect to the inner product of (i). Define $M_q(\Gamma) := \sup_{\varphi \in B_1} \mathcal{G}_q(\varphi)$. This sup is finite and (8.3) follows upon scaling.

(iii) When $\Gamma_1 \supset \Gamma$, then $H^1_{\Gamma_1 0}(\Omega) \subset H^1_{\Gamma 0}(\Omega)$, so the associated unit ball is smaller and thus $M_q(\Gamma_1) \leq M_q(\Gamma)$. □

It appears that the value of $M_q(\Gamma)$ increases to ∞ as $\sigma(\Gamma)$ decreases to zero. It would be of interest to estimate or quantify this dependence. When $q = 2$ the constant $M_2(\Gamma)$ is related to the least eigenvalue of an eigenvalue problem for the Laplacian where the eigenvalue appears in both the equation and the boundary condition. Suppose that $\lambda_1(\Omega, \Gamma)$ is the least eigenvalue of

$$-\Delta u = \lambda u \quad \text{on} \quad \Omega \quad \text{with} \quad u = 0 \quad \text{on} \quad \Gamma, \quad D_\nu u = \lambda u \quad \text{on} \quad \tilde{\Gamma}, \tag{8.5}$$

then $M_2(\Gamma) = \lambda_1(\Omega, \Gamma)^{-1}$.

The conditions required here are that Γ_τ, Γ_ν are proper subsets of $\partial\Omega$ satisfying the following.

Condition (B3). Γ_ν and Γ_τ are disjoint, satisfy (B2) and have union dense in $\partial\Omega$.

Let $G_{\Gamma_\tau}(\Omega), Curl_{\Gamma_\nu}(\Omega)$ be the spaces of gradients of functions in $H^1_{\Gamma_\tau 0}(\Omega)$ and curls of functions in $H^1_{\Gamma_\nu 0}(\Omega)$ respectively. These spaces are L^2- orthogonal as fields in $L^2(\Omega; \mathbb{R}^2)$. The vector field $\mathbf{v} := \nabla^\perp \psi - \nabla\varphi$ will satisfy the boundary condition (8.1) in a weak sense provided $\varphi \in H^1_{\Gamma_\nu 0}(\Omega)$ and $D_\nu \varphi + \eta_\nu = 0$ on Γ_ν and $\psi \in H^1_{\Gamma_\tau 0}(\Omega)$ with $D_\nu \psi + \eta_\tau = 0$ on Γ_τ.

As described in [1] there are variational principles for the potentials in this representation and the field $\mathbf{v}$ will be a solution of (5.1) - (8.1) provided $\varphi \in H^1_{\Gamma_\tau 0}(\Omega)$ is a solution of

$$\int_\Omega [\nabla\varphi \cdot \nabla\chi - \rho\chi] \, d^2x + \int_{\Gamma_\nu} \eta_\nu \chi \, ds = 0 \qquad \text{for all} \quad \chi \in H^1_{\Gamma_\tau 0}(\Omega). \tag{8.6}$$

Similarly $\psi \in H^1_{\Gamma_\nu 0}(\Omega)$ is a solution of

$$\int_\Omega [\nabla\psi \cdot \nabla\chi - \omega\chi] \, d^2x + \int_{\Gamma_\tau} \eta_\tau \chi \, ds = 0 \qquad \text{for all} \quad \chi \in H^1_{\Gamma_\nu 0}(\Omega). \tag{8.7}$$

Note that these equations are of the same type; they differ only in that Γ_τ, Γ_ν are interchanged from one to the other. They can be written as a problem of finding $\varphi \in H^1_{\Gamma 0}(\Omega)$ satisfying

$$\int_\Omega \nabla\varphi \cdot \nabla\chi \, d^2x = \mathcal{F}(\chi) \qquad \text{for all} \quad \chi \in H^1_{\Gamma 0}(\Omega). \tag{8.8}$$

Here $\mathcal{F}(\chi)$ is the linear functional defined by $\mathcal{F}(\chi) = \int_\Omega \rho\,\chi \, d^2x - \int_{\partial\Omega} \eta\,\chi \, ds$. For notational convenience the functions η_ν, η_τ are extended to all of $\partial\Omega$ by zero.

The general result about this problem may be described as follows.

Theorem 8.2. Assume that Ω, Γ satisfy (B1)-(B2) with $\rho \in L^q(\Omega), \eta \in L^q(\partial\Omega, ds), q > 1$. Then there is a unique solution $\tilde{\varphi} \in H^1_{\Gamma 0}(\Omega)$ of (8.8) and it satisfies

$$\|\nabla\tilde{\varphi}\|_2^q \leq M_{q'}(\Gamma)^{q-1} \left[\|\rho\|_q^q + \|\eta\|_{q,\tilde{\Gamma}}^q \right]. \tag{8.9}$$

Proof. When $\tilde{\varphi}$ is a solution of (8.8) and $|\mathcal{F}(\chi)| \leq C \, \|\nabla\chi\|_2$ for all $\chi \in H^1_{\Gamma 0}(\Omega)$, then $\|\nabla\tilde{\varphi}\|_2 \leq C$. So the result just requires an appropriate estimate of $\mathcal{F}(\chi)$. Two applications of Hölder's inequality to the definition of $\mathcal{F}$ yield that

$$|\mathcal{F}(\chi)| \leq \left[\|\rho\|_q^q + \|\eta\|_{q,\partial\Omega}^q \right]^{1/q} \cdot \left[\|\chi\|_{q'}^{q'} + \|\chi\|_{q',\partial\Omega}^{q'} \right]^{1/q'}$$

for all $\chi \in H^1_{\Gamma 0}(\Omega)$. Then (8.2) yields

$$|\mathcal{F}(\chi)| \leq \left[\|\rho\|_q^q + \|\eta\|_{q,\partial\Omega}^q \right]^{1/q} M_{q'}(\Gamma)^{1/q'} \|\nabla\chi\|_2.$$

This inequality yields (8.9). □

Corollary 8.3. Assume that $\Omega, \Gamma_\nu, \Gamma_\tau$ satisfy (B1)-(B3) with $\rho, \omega \in L^q(\Omega), \eta_\nu, \eta_\tau \in L^q(\partial\Omega, ds)$ and $q > 1$. Then there is a solution $\tilde{\mathbf{v}} = \nabla^\perp\tilde{\psi} - \nabla\tilde{\varphi}$ of (5.1) - (8.1) with

$$\| \tilde{\mathbf{v}} \|_2^2 \leq C_q(\Gamma_\tau) \left[\|\rho\|_q^q + \|\eta_\nu\|_{q,\Gamma_\nu}^q \right]^{2/q} + C_q(\Gamma_\nu) \left[\|\omega\|_q^2 + \|\eta_\tau\|_{q,\Gamma_\tau}^2 \right]^{2/q}. \tag{8.10}$$

If $\mathbf{v}$ is any solution of this mixed div-curl system, then $\|\mathbf{v}\|_2 \geq \| \tilde{\mathbf{v}} \|_2$.

Proof. Let $\tilde{\varphi}, \tilde{\psi}$ be the solutions of (8.6) - (8.7) respectively. Then their orthogonality implies that $\| \tilde{\mathbf{v}} \|_2^2 = \|\nabla\tilde{\varphi}\|_2^2 + \|\nabla\tilde{\psi}\|_2^2$. Theorem 8.2 implies that there is a constant such that

$$\|\nabla\tilde{\varphi}\|_2^2 \leq C_q(\Gamma_\tau) \left[\| \rho \|_q^q + \| \eta_\nu \|_{q,\Gamma_\nu}^q \right]^{2/q}.$$

Similarly the other mixed boundary value problem has solution $\tilde{\psi}$ with

$$\|\nabla\tilde{\psi}\|_2^2 \leq C_q(\Gamma_\nu) \left[\| \omega \|_q^q + \| \eta_\tau \|_{q,\Gamma_\tau}^q \right]^{2/q}.$$

Adding these two expressions leads to the inequality of (8.10). □

In general there is an affine subspace of solutions of (5.1) - (8.1) as described in section 14 of [1]. To find a well-posed problem certain linear functionals of the solutions must be further specified and the energy of the solution depends on these extra imposed conditions.

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