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WAVE INTERACTIONS AND STABILITY OF THE RIEMANN SOLUTION FOR A STRICTLY HYPERBOLIC SYSTEM OF CONSERVATION LAWS

By

ANUPAM SEN (*Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur-2, India*),

T. RAJA SEKHAR (*Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur-2, India*),

AND

V. D. SHARMA (*Department of Mathematics, Indian Institute of Technology Bombay, Mumbai-76, India*)

Abstract. In this article, we study the interaction of delta shock waves for the one-dimensional strictly hyperbolic system of conservation laws with split delta function. We prove that Riemann solutions are stable under local small perturbations of the Riemann initial data. The global structure and large time asymptotic behaviour of the perturbed Riemann solutions are constructed and analyzed case by case.

1. Introduction. In this paper we are concerned with the Riemann problem for the following system that arises in nonlinear elasticity and gasdynamics and the solution of which belongs to some measure space as in Tan et al. [16], Hayes and LeFloch [7], Keyfitz and Kranzer [9], Keyfitz [8], and Nedeljkov [10, 11] and Nedeljkov & Oberguggenberger [12]:

$$\begin{cases} u_t + (u^2)_x = 0, \\ v_t + ((2u+1)v)_x = 0, \ x \in \mathbb{R}, \ t \geq 0, \end{cases} \tag{1.1}$$

with initial data

$$(u, v)(x, 0) = \begin{cases} (u_-, v_-), \ x < 0, \\ (u_+, v_+), \ x > 0, \end{cases} \tag{1.2}$$

where $u_\pm$ and $v_\pm$ are given positive constants.

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E-mail address: sen.anupam123@gmail.com
E-mail address: trajasekhar@maths.iitkgp.ernet.in
E-mail address: vsharma@math.iitb.ac.in

The first equation in (1.1) is just the inviscid Burgers equation and its solution to the Riemann problem is the classical entropy solution. The Dirac function is introduced as a part for v when the characteristic velocity u is discontinuous. Tan et al. [16] considered the Riemann problem of (1.1) and they discovered the form of the standard Dirac delta function supported on a shock wave that was used as a part in their Riemann solution for certain initial data. A delta shock wave is a generalization of an ordinary shock wave; speaking informally, it consists of a discontinuity line $x = x(t)$ plus a distributed Dirac delta function with the discontinuity line as its support. A delta shock wave is over-compressive in the sense that the number of characteristics entering the discontinuity line of the delta shock wave is more than in the case of an ordinary shock wave. Several approaches for constructing δ-shock type solutions are known. The weak asymptotic method was used by Danilov et al. [6] in their study of the delta shock wave-type solution. In [4, 5, 17], it was shown that for some cases the hyperbolic system

$$u_t + (F(u, v))_x = 0,\ v_t + (G(u, v))_x = 0, \tag{1.3}$$

or

$$v_t + (G(u, v))_x = 0,\ (uv)_t + (H(u, v))_x = 0, \tag{1.4}$$

where $F(u, v), G(u, v)$ and $H(u, v)$ are smooth functions and are linear with respect to v, admits "nonclassical" situations, when the Riemann problem does not possess a weak L^∞-solution except for some particular initial data. In contrast to the standard results of existence of weak solutions to strictly hyperbolic systems, here the linear component v of the solution may contain Dirac measures and must be sought in the space of measures, while the nonlinear component u of the solution has bounded variation. In order to solve the Cauchy problem in this nonclassical situation, it is necessary to introduce δ-shock type singularities, which are solutions of the system of conservation laws. Shen and Sun [14] proved that the Riemann solutions are stable for the nonstrictly hyperbolic system of conservation laws under local small perturbations of the Riemann initial data; interaction of delta shock for the chromatography equations and the interaction of rarefaction waves of two-dimensional Euler equations have been studied by Sun [15] and Chen & Zheng [2], respectively.

In this paper, we consider the initial value problem (1.1)-(1.2), and investigate wave interactions between delta shock wave and classical elementary waves such as shock waves, rarefaction waves and contact discontinuities. Further, we study the stability of the Riemann solution of (1.1) and (1.2) under local small perturbations of the Riemann initial data. Here we mainly adopt the method of splitting delta function along a regular curve in $\overline{\mathbb{R}^2_+}$ proposed by Nedeljkov [10]. With the method of splitting delta function, the product of the piecewise smooth function and discontinuity along such a curve makes sense and the differentiation is defined by mapping into the usual Radon measure space. To deal with the interaction of the delta shock wave with the rarefaction wave, we approximate the rarefaction wave by a set of small nonadmissible shocks, like in the wave front tracking algorithm, proposed in [1, 3]. Thus, we consider the initial value problem for (1.1) with the following initial data having two jump discontinuities at $-\varepsilon$ and ε, i.e.,

$$(u,v)(x,0)=\begin{cases}(u_-,v_-), & -\infty<x<-\varepsilon,\\ (u_m,v_m), & -\varepsilon<x<\varepsilon,\\ (u_+,v_+), & \varepsilon<x<\infty,\end{cases} \tag{1.5}$$

where $\varepsilon>0$ is arbitrarily small. Here we restrict ourselves only to consider the situation when $u_{\pm,m}>0$ and $v_{\pm,m}>0$. The initial data (1.5) is a local perturbation of the corresponding Riemann initial data (1.2). We will address the question of determining whether the Riemann solutions of (1.1) and (1.2) are the limits of $(u_\varepsilon,v_\varepsilon)(x,t)$ as $\varepsilon\to 0$, where $(u_\varepsilon,v_\varepsilon)(x,t)$ is the solution of (1.1) and (1.5) for $\varepsilon>0$. We will deal with this problem case by case along with constructing the perturbed solution.

The organization of this paper is as follows. In section 2, we present some preliminaries for the system (1.1) and (1.2) and display the Riemann solution of (1.1) and (1.2) with different possible initial data. In section 3, the interaction of the delta shock waves with classical elementary waves such as shock waves, contact discontinuities and rarefaction waves are discussed for all the possible cases when the initial data have three piecewise constant states. The solutions are constructed globally and the stability of the Riemann solutions are analyzed by letting ε tend to zero. Finally, conclusions are drawn in section 4.

2. Preliminaries. In this section, we describe some results on the Riemann solution to (1.1) which were obtained by Tan et al [16]. The system (1.1) is one of the examples of a strictly hyperbolic system for which the Riemann problem cannot be solved for all combinations of piecewise constant initial states with classical elementary waves such as shock waves, rarefaction waves and contact discontinuities. Such systems arise in nonlinear elasticity and gas dynamics. The eigenvalues of the system (1.1) are $\lambda_1=2u$ and $\lambda_2=(2u+1)$ and the corresponding right eigenvectors are $\overrightarrow{r_1}=(1,-2v)^t$ and $\overrightarrow{r_2}=(0,1)^t$, respectively. Since $\nabla\lambda_1\cdot\overrightarrow{r_1}=2\neq 0$ and $\nabla\lambda_2\cdot\overrightarrow{r_2}=0$, the first characteristic field is genuinely nonlinear and the second one is linearly degenerate. The Riemann invariants associated with the first and second characteristic fields are $w=ve^{2u}$ and $z=u$, respectively.

Let (u_-,v_-) and (u,v) denote, respectively, the left-hand and right-hand states of either classical elementary waves or a delta shock wave. Let us fix (u_-,v_-) in the domain of hyperbolicity and compute the state (u,v) which is connected on the right by either classical elementary waves or a delta shock wave as given below.

The 1-rarefaction wave curves in the phase plane are:

$$R(u_-,v_-)=\begin{cases}\frac{dx}{dt}=\lambda_1=2u,\\ v=v_-e^{2(u_--u)},\\ u_-<u,\end{cases}$$

and 1-shock wave curves in the phase plane are:

$$S(u_-, v_-) = \begin{cases} C = u + u_-, \\ u - u_- = \frac{v_- - v}{v_- + v}, \\ u < u_- < u + 1, \end{cases}$$

where C is the shock speed.

The possible states that can be connected to (u_-, v_-) on the right by a contact discontinuity lie on the curve, which is given as follows:

$$J(u_-, v_-) = \begin{cases} \tau = 2u + 1 = 2u_- + 1, \\ u = u_-, \end{cases}$$

where τ is the speed of contact discontinuity.

Depending on the choice of initial data, there are three possible wave patterns for the solution of Riemann problems (1.1) and (1.2) which are described below.

Case a. If $u_- < u_+$, the Riemann solution consists of a rarefaction wave R followed by a contact discontinuity J

$$(u, v)(x, t) = \begin{cases} (u_-, v_-), & x < \lambda_1(u_-)t, \\ \left(\frac{x}{2t}, v_- e^{2(u_- - \frac{x}{2t})}\right), & \lambda_1(u_-)t \leq x \leq \lambda_1(u_+)t, \\ \left(u_+, v_- e^{2(u_- - u_+)}\right), & \lambda_1(u_+)t < x < \tau t, \\ (u_+, v_+), & x > \tau t. \end{cases}$$

Case b. If $u_+ < u_- < u_+ + 1$, the Riemann problem consists of a shock wave S followed by a contact discontinuity J

$$(u, v)(x, t) = \begin{cases} (u_-, v_-), & x < Ct, \\ (u_+, v_*), & Ct < x < \tau t, \\ (u_+, v_+), & x > \tau t, \end{cases}$$

where $v_* = \frac{v_-(1 + u_- - u_+)}{(1 + u_+ - u_-)}$ and propagation speed of S is $C = (u_- + u_+)$.

Case c. If $u_- \geq (u_+ + 1)$, the solution of the Riemann problem is given in the form of a delta shock wave δS as

$$u(x, t) = \begin{cases} u_-, & x < \sigma t, \\ u_+, & x > \sigma t, \end{cases}, \quad v(x, t) = \left\{ \begin{array}{ll} v_-, & x < \sigma t, \\ v_+, & x > \sigma t, \end{array} \right\} + \beta_-(t) D^- + \beta_+(t) D^+, \tag{2.1}$$

where D^- and D^+ are the left- and right-hand side delta functions with the support on the line $x = \sigma t$, where

$$\sigma = (u_- + u_+), \tag{2.2}$$

$$\beta(t) = \beta_-(t) + \beta_+(t) = (\sigma[v] - [(2u + 1)v])\, t = ((v_+ + v_-)(u_- - u_+) + (v_- - v_+))\, t. \tag{2.3}$$

Now, we briefly review the concept of left- and right-hand side delta functions; indeed, further details can be found in [10, 12].

Let $\overline{\mathbb{R}^2_+}$ be divided into two disjoint open sets Ω_1 and Ω_2 with piecewise smooth boundary curve Γ, which satisfies $\Omega_1 \cap \Omega_2 = \emptyset$ and $\overline{\Omega_1} \cup \overline{\Omega_2} = \overline{\mathbb{R}^2_+}$. Let $C(\Omega_i)$ and $M(\Omega_i)$ be the space of bounded and continuous real-valued functions equipped with the L^∞-norm and the space of measures on Ω_i $(i = 1, 2)$, respectively. Suppose that $C_\Gamma = (C(\Omega_1), C(\Omega_2))$ and $M_\Gamma = (M(\Omega_1), M(\Omega_2))$; the product of $G = (G_1, G_2) \in C_\Gamma$

and $D = (D_1, D_2) \in M_\Gamma$ is defined as an element $GD = (G_1D_1, G_2D_2) \in M_\Gamma$, where G_iD_i $(i = 1, 2)$ is defined as the usual product of a continuous function and a measure. Thus, the product defined as above makes sense.

Every measure on $\overline{\Omega_i}$ can be viewed as a measure on $\overline{\mathbb{R}^2_+}$ with support in $\overline{\Omega_i}$. In view of the above, the mapping $m : M_\Gamma \to M(\overline{\mathbb{R}^2_+})$ can be obtained by taking $m(D) = D_1 + D_2$. Similarly, we have $m(GD) = G_1D_1 + G_2D_2$.

For example, along the piecewise smooth curve $x = \gamma(t)$, the delta function $\delta(x - \gamma(t)) \in M(\overline{\mathbb{R}^2_+})$ can be split in a nonunique way into a left-hand component $D^- \in M(\Omega_1)$ and right-hand one $D^+ \in M(\Omega_2)$ such that $\delta(x - \gamma(t)) = \beta_-(t)D^- + \beta_+(t)D^+ = m(\beta_-(t)D^- + \beta_+(t)D^+)$ with $\beta_-(t) + \beta_+(t) = 1$.

The solutions concept used in this paper can be described as follows: carry out the multiplication and composition in the space M_Γ and then take the mapping $m : M_\Gamma \to M(\overline{\mathbb{R}^2_+})$ before differentiation in the space of distributions.

To define the measure solutions, the two-dimensional weighted δ measure $\beta(s)\delta_\Gamma$ that has support on a smooth curve $\Gamma = \{(x(s), t(s)) : a < s < b\}$ can be defined by:

$\langle \beta(.)\delta_\Gamma, \psi(., .)\rangle = \int_a^b \beta(s)\psi(x(s), t(s))\sqrt{x'(s)^2 + t'(s)^2}ds$, for every test function $\psi \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+)$.

With the above definition, a family of δ-measure solutions (u, v) of (1.1) and (1.2) with parameter σ, in case $u_- \geq (u_+ + 1)$, can be expressed as:

$u(x,t) = u_- + [u]H(x - \sigma t)$, $v(x,t) = v_- + [v]H(x - \sigma t) + \beta(t)\delta_\Gamma$, where $\Gamma = \{(\sigma t, t) : 0 \leq t < \infty\}$, $\beta(t) = (\sigma[v] - [(2u + 1)v])\, t = ((v_+ + v_-)(u_- - u_+) + (v_- - v_+))\, t$ and $H(x)$ is the Heaviside function.

The measure solution (2.1) satisfies the generalized Rankine-Hugoniot jump conditions

$$\begin{cases} \frac{dx(t)}{dt} = \sigma, \\ \frac{d\beta(t)}{dt} = (\sigma[v] - [(2u + 1)v]), \\ \sigma[u] = [u^2], \end{cases}$$

where $[v] = v\,(x(t) + 0) - v\,(x(t) - 0)$ denotes the jump in v across the discontinuity $x = x(t)$.

In order to ensure the uniqueness, the Delta entropy conditions $\lambda_1(u_+) \leq \sigma \leq \lambda_1(u_-)$ and $\lambda_2(u_+) \leq \sigma \leq \lambda_2(u_-)$ are imposed, which means that all the characteristics on both sides of the δ-shock waves are incoming.

3. Interactions of a delta shock wave with classical elementary waves. The interactions of a delta shock wave with the classical elementary waves such as shock wave, rarefaction wave and contact discontinuity, obtained from the solution of the Riemann problem, give rise to new emerging waves (classical and nonclassical). We define the initial function with two jump discontinuities at $-\varepsilon$ and ε as in (1.5), with an appropriate choice of (u_m, v_m) and (u_+, v_+) in terms of (u_-, v_-) and arbitrary $\varepsilon > 0$. Hence, with the initial data (1.5), we have two Riemann problems locally. The wave which is coming from the first Riemann problem may interact with the wave of the second Riemann problem, and a new Riemann problem is formed at the time of interaction. Further, during the process of interaction, the strength of a delta shock wave is computed completely. We face the question of determining whether the Riemann solution of (1.1) and (1.2) is the limit of $(u_\varepsilon, v_\varepsilon)$(x, t) as $\varepsilon \to 0$, where $(u_\varepsilon, v_\varepsilon)$(x, t) is the solution of (1.1) and (1.5). In order to cover all the cases completely, our discussion is divided into two parts according to whether the delta shock wave appears or not. If the delta shock wave does not appear in the solution of Cauchy problems (1.1) and (1.5), then the interactions only involve classical elementary waves [13].

According to the different combinations, we have five possibilities for an interaction with a delta shock wave to occur from $(-\varepsilon, 0)$ and $(\varepsilon, 0)$ as follows:

(i) δS and S+J, (ii) S+J and δS, (iii) δS and δS, (iv) δS and R+J, (v) R+J and δS.

Case I: Interaction of a delta shock wave with a shock wave followed by contact discontinuity (δS and $S + J$). We consider that (u_-, v_-) is connected to (u_m, v_m) by a delta shock wave, δS_1, of the first Riemann problem and (u_m, v_m) is connected to (u_+, v_+) by a shock wave, S, followed by contact discontinuity, J, of the second Riemann problem. In other words, for a given (u_-, v_-), we choose (u_m, v_m) and (u_+, v_+) in such a way that $u_- \geq u_m + 1$ and $u_+ < u_m < u_+ + 1$. Since the speed, σ_1=$(u_- + u_m)$, of a delta shock wave of the first Riemann problem is greater than the speed, C=$(u_m + u_+)$, of shock S of the second Riemann problem, delta shock δS_1 overtakes S and interaction will take place at, say, (x_1, t_1) (see Figure 1).

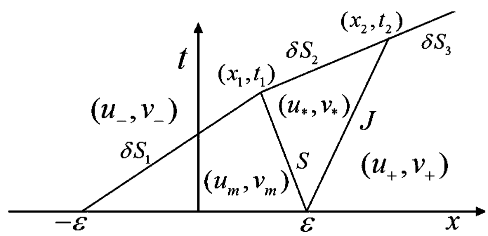

Fig. 1. When $u_- \geq u_m + 1$ and $u_+ < u_m < u_+ + 1$

The interaction point (x_1, t_1) can be calculated as follows:

$$\begin{cases} (x_1 + \varepsilon) = (u_- + u_m)t_1, \\ (x_1 - \varepsilon) = (u_+ + u_m)t_1, \end{cases}$$

which implies that

$$(x_1, t_1) = \left(\frac{\varepsilon(u_+ + u_- + 2u_m)}{(u_- - u_+)}, \frac{2\varepsilon}{(u_- - u_+)} \right). \tag{3.1}$$

The strength of δS_1 at (x_1, t_1) is given by

$$\beta(t_1) = ((v_m + v_-)(u_- - u_m) + (v_- - v_m))\, t_1. \tag{3.2}$$

At the point of interaction (x_1, t_1), new initial data is formed as follows:

$$u|_{t=t_1} = \begin{cases} u_-, \ x < x_1, \\ u_*, \ x > x_1, \end{cases}, \ v|_{t=t_1} = \left. \begin{cases} v_-, \ x < x_1, \\ v_*, \ x > x_1, \end{cases} \right\} + \beta(t_1)\delta_{(x_1,t_1)},$$

where $(u_*, v_*) = \left(u_+, \frac{v_m(1+u_m-u_+)}{(1+u_+-u_m)}\right)$ is the state between S and J.

A new delta shock will be generated after interaction and we denote it by δS_2, which can be expressed as

$$\begin{cases} u(x,t) = u_- + (u_* - u_-)H, \\ v(x,t) = v_- + (v_* - v_-)H + \beta_-(t)D^- + \beta_+(t)D^+, \end{cases} \tag{3.3}$$

where H is the Heaviside function and $\beta(t)D{=}\beta_-(t)D^- + \beta_+(t)D^+$ is a split delta function. All these functions H, D^+ and D^- have support on the same line $x = x_1{+}\sigma_2(t{-}t_1)$, namely they are functions of $x - x_1 - \sigma_2(t - t_1)$ where $\sigma_2 = (u_- + u_+)$ is the propagation speed of δS_2. It may be noticed that D^- is the delta measure on the set $\overline{\mathbb{R}^2}_+ \cap \{(x,t) : x \le x_1 + \sigma_2(t - t_1)\}$ and D^+ is the delta measure on the set $\overline{\mathbb{R}^2}_+ \cap \{(x,t) : x \ge x_1 + \sigma_2(t - t_1)\}$.

It follows from (3.3) that

$$v_t(x,t) = -\sigma_2(v_* - v_-)\delta + (\beta'_-(t) + \beta'_+(t))\delta - \sigma_2(\beta_-(t) + \beta_+(t))\delta' \tag{3.4}$$

$$\begin{aligned} ((2u+1)v)_x(x,t) = {} & ((2u_* + 1)v_* - (2u_- + 1)v_-)\delta + ((2u_* + 1)\beta_+(t) \\ & + (2u_- + 1)\beta_-(t))\delta'. \end{aligned} \tag{3.5}$$

Substituting (3.4) and (3.5) into the second equation of (1.1) and equating the coefficients of δ and δ', we obtain

$$-\sigma_2(v_* - v_-) + (2u_* + 1)v_* - (2u_- + 1)v_- + \beta'_-(t) + \beta'_+(t) = 0, \tag{3.6}$$

$$-\sigma_2(\beta_-(t) + \beta_+(t)) + (2u_* + 1)\beta_+(t) + (2u_- + 1)\beta_-(t) = 0. \tag{3.7}$$

With initial condition (3.2), it follows from (3.6) that

$$\beta(t) = \beta(t_1) + \{(u_- - u_*)(v_* + v_-) + (v_- - v_*)\}(t - t_1), \tag{3.8}$$

where $\beta(t)$ denotes the strength of δS_2 when $t > t_1$ (i.e., after interaction of δS_1 and S). Obviously, $\beta_-(t)$ and $\beta_+(t)$ can be derived explicitly from (3.7) and (3.8). Since the propagation speed, $\sigma_2{=}(u_- + u_+)$, of δS_2 is greater than the speed, $\tau{=}(2u_+ + 1)$, of contact discontinuity J, it follows that the delta shock δS_2 overtakes J at, say, (x_2, t_2).

The intersecting point (x_2, t_2) can be calculated from the following equations:

$$\begin{cases} (x_2 - x_1) = \sigma_2(t_2 - t_1), \\ (x_2 - \varepsilon) = \tau t_2, \end{cases}$$

which imply that

$$(x_2, t_2) = \left(\frac{\tau(\sigma_2 t_1 - x_1) + \sigma_2\varepsilon}{(\sigma_2 - \tau)}, \frac{(\sigma_2 t_1 - x_1) + \varepsilon}{(\sigma_2 - \tau)} \right). \tag{3.9}$$

After time t_2, the delta shock wave will pass through J with the same speed as before and its strength changes due to the difference between v_* and v_+. We denote it by δS_3 after time t_2 and its strength is

$$\beta(t) = \beta(t_2) + \{(v_+ + v_-)(u_- - u_+) + (v_- - v_+)\}(t - t_2),$$

where $\beta(t_2)$ can be calculated from (3.8).

It is easy to see from (3.1) and (3.9) that both (x_1, t_1) and (x_2, t_2) tend to (0, 0) as $\varepsilon \to 0$. Moreover, we have $\beta(t_2) \to 0$ as $\varepsilon \to 0$ from (3.8). Thus the limit of the solution

(1.1) and (1.5) is still a single delta shock wave, which is exactly the corresponding Riemann solution of (1.1) and (1.2) in this case.

CASE II: INTERACTION OF A SHOCK WAVE FOLLOWED BY CONTACT DISCONTINUITY WITH A DELTA SHOCK WAVE ($S+J$ AND δS). We consider that (u_-, v_-) is connected to (u_m, v_m) by a shock wave, S, followed by contact discontinuity, J, of the first Riemann problem and (u_m, v_m) is connected to (u_+, v_+) by a delta shock wave, δS_1, of the second Riemann problem. In other words, for a given (u_-, v_-), we choose (u_m, v_m) and (u_+, v_+) in such a way that $u_m < u_- < u_m+1$ and $u_m \geq u_+ + 1$. Since the speed, $\sigma_1{=}(u_+ + u_m)$, of a delta shock wave of the second Riemann problem is less than the speed, $\tau{=}(2u_m+1)$, of contact discontinuity, J, of the first Riemann problem, the contact discontinuity J overtakes δS_1 and the interaction will take place at, say, (x_1, t_1) (see Figure 2).

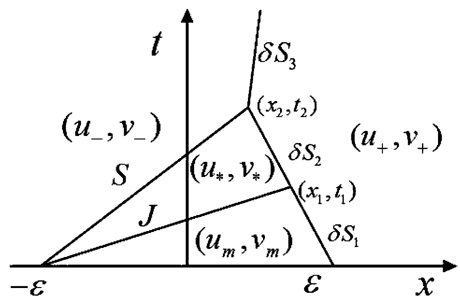

FIG. 2. When $u_m < u_- < u_m + 1$ and $u_m \geq u_+ + 1$

The interaction point is

$$(x_1, t_1) = \left(\frac{\varepsilon(3u_m + u_+ + 1)}{u_m - u_+ + 1}, \frac{2\varepsilon}{u_m - u_+ + 1} \right), \tag{3.10}$$

and strength of δS_1 at (x_1, t_1) is

$$\beta(t_1) = ((v_+ + v_m)(u_m - u_+) + (v_m - v_+))\, t_1. \tag{3.11}$$

After interaction we denote δS_1 as δS_2. Now the speed of δS_2 is $\sigma_2{=}(u_+ + u_m)$ and the speed of shock wave S is C$=(u_m + u_-)$, so S overtakes δS_2 in a finite time. The intersection point is

$$(x_2, t_2) = \left(\frac{C(x_1 - \sigma_2 t_1) + \sigma_2 \varepsilon}{(C - \sigma_2)}, \frac{x_1 - \sigma_2 t_1 + \varepsilon}{(C - \sigma_2)} \right), \tag{3.12}$$

and strength of δS_2 at (x_2, t_2) is

$$\beta(t_2) = \beta(t_1) + \{(v_+ + v_*)(u_* - u_+) + (v_* - v_+)\}(t_2 - t_1), \tag{3.13}$$

where $(u_*, v_*) = \left(u_m, \frac{v_-(1+u_- - u_m)}{(1+u_m - u_-)}\right)$ is the state between S and J.

At the point (x_2, t_2) a new initial data is formed as follows:

$$u|_{t=t_2} = \begin{cases} u_-, & x < x_2, \\ u_+, & x > x_2, \end{cases} \quad v|_{t=t_2} = \begin{cases} v_-, & x < x_2, \\ v_+, & x > x_2, \end{cases} + \beta(t_2)\delta_{(x_2,t_2)}.$$

A new delta shock wave δS_3 is generated after interaction of S with δS_2 at (x_2, t_2), which can be expressed as

$$\begin{cases} u(x,t) = u_- + (u_+ - u_-)H, \\ v(x,t) = v_- + (v_+ - v_-)H + \beta_-(t)D^- + \beta_+(t)D^+, \end{cases} \tag{3.14}$$

where H is the Heaviside function and $\beta(t)D{=}\beta_-(t)D^- + \beta_+(t)D^+$ is a split delta function. All these functions H, D^+ and D^- have support on the same line $x = x_2{+}\sigma_3(t{-}t_2)$, namely they are functions of $x - x_2 - \sigma_3(t - t_2)$ where $\sigma_3 = (u_- + u_+)$ is the propagation speed of δS_3. It may be noticed that D^- is the delta measure on the set $\overline{\mathbb{R}^2}_+ \cap \{(x,t) : x \le x_2 + \sigma_3(t - t_2)\}$ and D^+ is the delta measure on the set $\overline{\mathbb{R}^2}_+ \cap \{(x,t) : x \ge x_2 + \sigma_3(t - t_2)\}$.

It follows from (3.14) that

$$v_t(x,t) = -\sigma_3(v_+ - v_-)\delta + (\beta'_-(t) + \beta'_+(t))\delta - \sigma_3(\beta_-(t) + \beta_+(t))\delta', \tag{3.15}$$

$$\begin{aligned} ((2u+1)v)_x(x,t) = ((2u_+ + 1)v_+ - (2u_- + 1)v_-)\delta + ((2u_+ + 1)\beta_+(t) \\ + (2u_- + 1)\beta_-(t))\delta'. \end{aligned} \tag{3.16}$$

Substituting (3.15) and (3.16) into the second equation of (1.1) and equating the coefficients of δ and δ', we obtain

$$-\sigma_3(v_+ - v_-) + (2u_+ + 1)v_+ - (2u_- + 1)v_- + \beta'_-(t) + \beta'_+(t) = 0, \tag{3.17}$$

$$-\sigma_3(\beta_-(t) + \beta_+(t)) + (2u_+ + 1)\beta_+(t) + (2u_- + 1)\beta_-(t) = 0. \tag{3.18}$$

With initial condition (3.13), it follows from (3.17) that

$$\beta(t) = \beta(t_2) + \{(u_- - u_+)(v_+ + v_-) + (v_- - v_+)\}(t - t_2), \tag{3.19}$$

where $\beta(t)$ denotes the strength of δS_3 when $t > t_2$ (i.e., after interaction of δS_2 and S). Obviously, $\beta_-(t)$ and $\beta_+(t)$ can be derived explicitly from (3.18) and (3.19).

Hence, the result of an interaction of a shock wave followed by a contact discontinuity and a delta shock wave is a delta shock wave. From (3.10) and (3.12), it is easy to see that both (x_1, t_1) and (x_2, t_2) tend to $(0, 0)$ as $\varepsilon \to 0$; moreover it follows from (3.11) and (3.13) that $\beta(t_1) \to 0$ and $\beta(t_2) \to 0$ as $\varepsilon \to 0$. Thus, the limit of the solution of (1.1) and (1.5) is still a single delta shock wave, which is exactly the corresponding Riemann solution of (1.1) and (1.2).

Case III: Interaction of two delta shock waves (δS_1 and δS_2). We consider that (u_-, v_-) is connected to (u_m, v_m) by a delta shock wave, δS_1, of the first Riemann problem and also, (u_m, v_m) is connected to (u_+, v_+) by a delta shock wave, δS_2, of the second Riemann problem. In other words, for a given (u_-, v_-), we choose (u_m, v_m) and (u_+, v_+) such that $u_- \ge u_m + 1$ and $u_m \ge u_+ + 1$. Since the speed, $\sigma_1{=}(u_- + u_m)$, of a delta shock wave of the first Riemann problem is greater than the speed, $\sigma_2{=}(u_m + u_+)$, of a delta shock wave of the second Riemann problem, we infer that δS_1 overtakes δS_2 and interaction takes place at, say, (x_1, t_1) (see Figure 3).

The interaction point of δS_1 and δS_2 is obtained as

$$(x_1, t_1) = \left(\frac{\varepsilon(u_- + u_+ + 2u_m)}{(u_- - u_+)}, \frac{2\varepsilon}{(u_- - u_+)} \right), \tag{3.20}$$

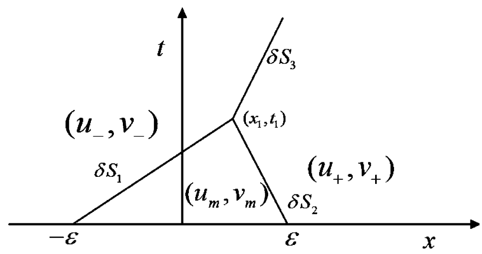

FIG. 3. When $u_- \geq (u_m + 1)$ and $u_m \geq (u_+ + 1)$

and the strength at (x_1, t_1) is

$$\beta(t_1) = \{(v_m + v_-)(u_- - u_m) + (v_m + v_+)(u_m - u_+) + (v_- - v_+)\}t_1, \tag{3.21}$$

which is exactly the strength of δS_1 together with that of δS_2 at the time $t = t_1$. Now the new initial value problem is formulated at (x_1, t_1) which can be dealt with in a manner similar to the case I. A new delta shock wave will be generated after the interaction of δS_1 and δS_2 at (x_1, t_1) which is denoted by δS_3. The speed of δS_3 is $\sigma_3 = (u_- + u_+)$ and its strength is

$$\beta(t) = \beta(t_1) + \{(v_+ + v_-)(u_- - u_+) + (v_- - v_+)\}(t - t_1).$$

Hence, the result of an interaction of two delta shock waves is still a single delta shock wave. It is easy to see from (3.20) that (x_1, t_1) tends to (0, 0) as $\varepsilon \to 0$. Moreover, $\beta(t_1) \to 0$ as $\varepsilon \to 0$ from (3.21). Thus, the limit of the solution of (1.1) and (1.5) is still a single delta shock wave, which is exactly the corresponding Riemann solution of (1.1) and (1.2).

CASE IV: INTERACTION OF A DELTA SHOCK WAVE WITH A RAREFACTION WAVE FOLLOWED BY CONTACT DISCONTINUITY (δS AND $R + J$). We consider that (u_-, v_-) is connected to (u_m, v_m) by a delta shock wave, δS_1, of the first Riemann problem and (u_m, v_m) is connected to (u_+, v_+) by a rarefaction wave, R, followed by the contact discontinuity, J, of the second Riemann problem. In other words, for a given (u_-, v_-), we choose (u_m, v_m) and (u_+, v_+) in such a way that $u_- \geq u_m + 1$ and $u_+ > u_m$. Since the speed, $\sigma_1 = (u_- + u_m)$, of a delta shock wave of the first Riemann problem is greater than the speed, $\xi_b = 2u_m$, of the left edge of the rarefaction wave of the second Riemann problem, it follows that δS_1 intersects R after a finite time and the interaction takes place at, say, (x_1, t_1).

The intersection point (x_1, t_1) is determined by

$$\begin{cases} (x_1 + \varepsilon) = (u_- + u_m)t_1, \\ (x_1 - \varepsilon) = 2u_m t_1, \end{cases}$$

implying thereby that

$$(x_1, t_1) = \left(\frac{\varepsilon(u_- + 3u_m)}{(u_- - u_m)}, \frac{2\varepsilon}{(u_- - u_m)} \right).$$

The strength of δS_1 at (x_1, t_1) is

$$\beta(t_1) = ((v_m + v_-)(u_- - u_m) + (v_- - v_m))\, t_1. \tag{3.22}$$

Now, let us approximate the rarefaction wave by a set of nonphysical shock waves supported by the lines $x = \varepsilon + 2\,(u_m + n\eta)\, t$, $0 < \eta \ll 1, n \in \mathbb{N}$ [12].

At least in the beginning, until $u_- \geq (u_m + n\eta) + 1$, the result of successive iterations of the delta shock wave with the nonphysical shock waves are delta shock waves with increasing speed. The values of u and v on the left-hand side of the delta shock wave are u_- and v_-, respectively and the right-hand side values are determined with the help of a rarefaction wave. We use $\Gamma_0 : \{(f(t), t) : t \geq t_1\}$ with $f(t_1) = x_1$ to express the curve δS_2. The values of u and v on the right-hand side of Γ_0 are as follows:

$$u = \left(\frac{x - \varepsilon}{2t}\right) = \left(\frac{f(t) - \varepsilon}{2t}\right), v = v_m e^{(2u_m - \frac{x-\varepsilon}{t})}.$$

The propagation speed of δS_2 can be determined by using the Rankine-Hugoniot jump condition that follows from (1.1), i.e.,

$$f'(t) = \frac{df(t)}{dt} = \frac{[u^2]}{[u]} = \left(\frac{x - \varepsilon}{2t} + u_-\right),\ f(t_1) = x_1. \tag{3.23}$$

The unique solution of (3.23) can be written as

$$f(t) = 2tu_- - 2\sqrt{2\varepsilon(u_- - u_m)t} + \varepsilon,\ t \geq t_1. \tag{3.24}$$

Now we construct a delta shock wave having support on the curve Γ_0 as follows:

$$\begin{cases} u(x,t) = \begin{cases} u_-, & x < f(t), \\ \frac{x-\varepsilon}{2t}, & x > f(t), \end{cases} \\ v(x,t) = \left\{ \begin{array}{ll} v_-, & x < f(t), \\ v_m e^{\left(2u_m - \left(\frac{x-\varepsilon}{t}\right)\right)}, & x > f(t), \end{array} \right\} + \beta_-(t) D^-_{\Gamma_0} + \beta_+(t) D^+_{\Gamma_0}, \end{cases} \tag{3.25}$$

where $\beta(t) D_{\Gamma_0} = \beta_-(t) D^-_{\Gamma_0} + \beta_+(t) D^+_{\Gamma_0}$ is a split delta function having support on Γ_0 and $\beta(t) = \beta_-(t) + \beta_+(t)$ is the strength of δS_2 at the time t.

Substituting (3.23) and (3.25) into the second equation of (1.1) and equating the coefficients of δ and δ', we obtain

$$-f'(t)\,(v(t) - v_-) + \beta'_-(t) + \beta'_+(t) + \left(\left(\frac{f(t) - \varepsilon}{t} + 1\right) v(t) - (2u_- + 1)\, v_-\right) = 0, \tag{3.26}$$
$$-f'(t)\,(\beta_-(t) + \beta_+(t)) + \left(\left(\frac{f(t) - \varepsilon}{t} + 1\right) \beta_+(t) + (2u_- + 1)\beta_-(t)\right) = 0,$$

where $v(t) = v_m e^{\left(2u_m - \left(\frac{f(t)-\varepsilon}{t}\right)\right)}$ denotes the value of $v|_{\Gamma_0}$ in the rarefaction wave.

From (3.26), with initial condition (3.22), one can calculate $\beta(t)$ for $t \geq t_1$ in the rarefaction wave.

In the following, we see that the delta shock wave having support on Γ_0 is an overcompressive wave only up to the point $(x_2, t_2) = (\varepsilon\,(1 + 4(u_- - 1)(u_- - u_m))\,, 2\varepsilon(u_- - u_m))$, where $x_2 = \varepsilon + 2t_2(u_- - 1)$.

We distinguish the following two subcases based on the relation between u_- and u_+.

Subcase 3.1. If $u_- \geq (u_+ + 1)$, the curve Γ_0 meets the line $x = \varepsilon + 2tu_+$ before time t_2. Thus the overcompressibility condition is satisfied throughout the rarefaction wave (see Figure 4). The intersection point of Γ_0 and the line $x = \varepsilon + 2tu_+$, the leading edge

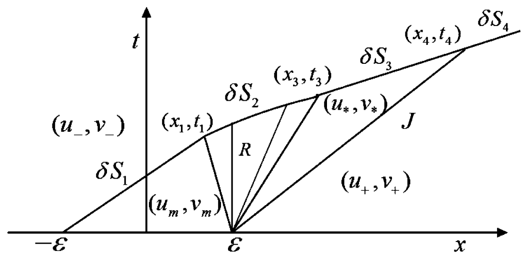

Fig. 4. When $u_- \geq (u_+ + 1)$

of rarefaction, is

$$(x_3, t_3) = \left(\frac{\varepsilon(u_+ + u_-)^2 - 4\varepsilon u_+ u_m}{(u_- - u_+)^2}, \frac{2\varepsilon(u_- - u_m)}{(u_- - u_+)^2} \right).$$

After interaction of the wave front of rarefaction with δS_2, we denote δS_2 by δS_3. The propagation speed of δS_3 is $\sigma_3 = (u_+ + u_-)$ and the speed of contact discontinuity is $\tau = (2u_+ + 1)$. It is easy to see that δS_3 overtakes J after a finite time. After interaction of δS_3 with J, we denote δS_3 by δS_4, which has the same speed as before; just the strength varies due to the difference between v_* and v_+. It is clear to see that the limit of the solution of (1.1) and (1.5) is just a delta shock wave. Similar to the analysis in case I, the Riemann solution of (1.1) and (1.2) is also stable with respect to the local perturbation of the Riemann initial data in this case.

Subcase 3.2. When $u_- < (u_+ + 1)$, the termination of an overcompressibility condition of a delta shock wave takes place within the rarefaction fan and it splits into a delta contact discontinuity [12, 14] having support on a curve Γ_1 and a shock wave having support on some other curve Γ_2, where Γ_1 and Γ_2 are to be determined. The curve Γ_1 is a characteristic curve of the equation $v_t + (2u + 1)v_x = 0$, which passes through (x_2, t_2). Using the fact that $u = \left(\frac{f_1(t) - \varepsilon}{2t}\right)$ on $\Gamma_1 = \{(f_1(t), t) : t > t_2\}$, it is easy to find the function $f_1(t)$ by solving the initial value problem

$$f_1'(t) = \left(\frac{f_1(t) - \varepsilon}{t} \right) + 1, \; f_1(t_2) = x_2. \tag{3.27}$$

The unique solution of (3.27) is

$$f_1(t) = \varepsilon + t \left(\log t - \log(2\varepsilon(u_- - u_m)) + 2(u_- - 1) \right). \tag{3.28}$$

From the first equation of (1.1) and applying the Rankine-Hugoniot jump condition, the curve $\Gamma_2 : \{(f_2(t), t) : t > t_2\}$ is uniquely determined by solving the initial value problem

$$f_2'(t) = \left(\frac{f_2(t) - \varepsilon}{2t} \right) + u_-, \; f_2(t_2) = x_2. \tag{3.29}$$

The solution of (3.29) is

$$f_2(t) = 2tu_- - 2\sqrt{2\varepsilon(u_- - u_m)t} + \varepsilon, \tag{3.30}$$

which is equal to the function $f(t)$. From (3.28) and (3.30), we find that Γ_2 is strictly left to the curve Γ_1, i.e., $f_2(t) > f_1(t)$ for $t > t_2$ (see Figure 5).

Let $\Re$ be the region between Γ_1 and Γ_2 for $t > t_2$. The value of u inside $\Re$ is $u(x,t)=\frac{x-\varepsilon}{2t}$ which satisfies the first equation of (1.1). Now we find the value of v in this area. Let $v_1(t)$ denote the value of v on the right-hand side of Γ_2. The value of u on the right-hand side of Γ_2 is given by

$$u|_{\Gamma_2} = \frac{f_2(t) - \varepsilon}{2t} = u_- - \sqrt{\frac{2\varepsilon(u_- - u_m)}{t}}.$$

The values of u and v on the left-hand side of Γ_2 are u_- and v_-, respectively. The Rankine-Hugoniot condition for the second equation of (1.1) gives

$$-f_2'(t)\,(v_- - v_1(t)) + (2u_- + 1)\,v_- - \left(\frac{f_2(t) - \varepsilon}{t} + 1\right) v_1(t) = 0. \tag{3.31}$$

Solving the equation (3.31), we obtain

$$v_1(t) = \left(\frac{\sqrt{t} + B}{\sqrt{t} - B}\right) v_-,$$

where $B = \sqrt{t_2} = \sqrt{2\varepsilon(u_- - u_m)}$. Let $w(x,t)$ be the value of v inside $\Re$. Then w is a solution of the linear partial differential equation

$$w_t + \left(\frac{x - \varepsilon}{t} + 1\right) w_x = 0,\ \ w|_{\Gamma_2} = v_1(t). \tag{3.32}$$

The solution of the equation (3.32) is constant along the characteristic curves

$$\gamma : \frac{dx}{dt} = \frac{x - \varepsilon}{t} + 1$$

initiating along Γ_2 and a solution exists at each point between the curves Γ_1 and Γ_2. In particular, w is a locally integrable function because $v_1(t) = \mathcal{O}\left(\frac{1}{(\sqrt{t}-\sqrt{t_2})}\right)$ as $t \to t_2$. Next we look for the existence of the delta contact discontinuity supported by Γ_1 and a shock wave supported by Γ_2 through the rarefaction wave.

The curve $\Gamma_1 : x = f_1(t)$ intersects the line $x = \varepsilon + 2tu_+$, which is a wave front of rarefaction, at the point (x_4, t_4), as $u_- < u_+ + 1$, where t_4 is greater than t_2 which is determined by solving the equation $2tu_+ - 2(u_- - 1)t - t\log(\frac{t}{B^2}) = 0$.

When $u_- > u_+$, the curve $\Gamma_2 : x = f_2(t)$ intersects the line $x = \varepsilon + 2tu_+$ at (x_5, t_5); after interaction we denote Γ_2 by Γ_4 (see Figure 5). The interaction point (x_5, t_5) is

$$(x_5, t_5) = \left(\varepsilon + \frac{2t_2u_+}{(u_- - u_+)^2}, \frac{t_2}{(u_- - u_+)^2}\right).$$

When $u_- < u_+$, the curve $\Gamma_2 : x = f_2(t)$ will not intersect the line x=$\varepsilon + 2tu_+$ (see Figure 6). In this case the shock wave having support on Γ_2 stays inside the rarefaction fan since $\Gamma_2 \cap \{(x,t) : x = \varepsilon + 2tu_+\} = \emptyset$. Actually, $f_2(t)$ has the line $x = \varepsilon + 2tu_+$ as an asymptote as $t \to \infty$.

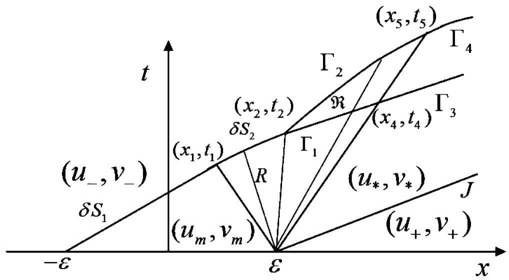

FIG. 5. When $u_+ < u_- < u_+ + 1$

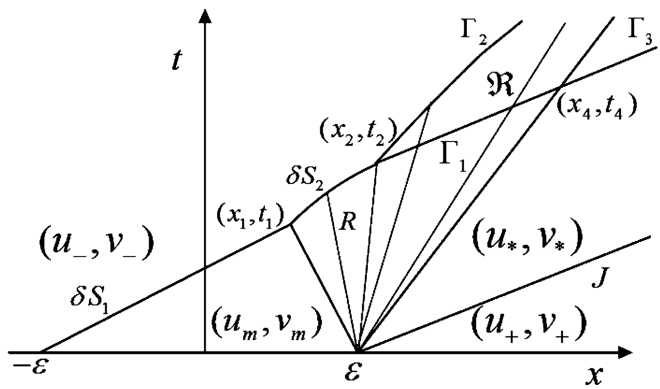

FIG. 6. When $u_- < u_+$

In both the cases $\Gamma_1 : x = f_1(t)$ intersects the line $x = \varepsilon + 2tu_+$. After interaction, we denote Γ_1 by Γ_3. The solution of Riemann problem (1.1) in the region $x > (\varepsilon + 2tu_+)$ is a delta contact discontinuity along the curve Γ_3. We solve the initial value problem (1.1) with initial data prescribed on the line $x = \varepsilon + 2tu_+$ as follows:

$$u = u_+ \text{ on } x = \varepsilon + 2tu_+,$$
$$v = \left\{ \begin{array}{ll} v_*, & x = \varepsilon + 2u_+t,\ t < t_4 \\ w(x, \frac{x-\varepsilon}{2u_+}), & x = \varepsilon + 2u_+t,\ t > t_4 \end{array} \right\} + \beta(t_4)\delta_{(x_4,\ t_4)},$$

.

where $v_* = v_m e^{2(u_m - u_+)}$, $\beta(t_4)\delta_{(x_4,\ t_4)}$ is the delta function with strength $\beta(t_4)$ at the point (x_4, t_4), and w is constant along the characteristics of

$$w_t + \left(\frac{x-\varepsilon}{t} + 1 \right) w_x = 0,\ \gamma : \frac{dx}{dt} = \frac{x-\varepsilon}{t} + 1.$$

Along the line $x = \varepsilon + 2tu_+$, the slope of the characteristic curve is $(2u_+ + 1)$. Thus, we may continue the solution to the right of $x = \varepsilon + 2tu_+$ as a delta contact discontinuity having support on the line $\Gamma_3 : x = x_4 + (2u_+ + 1)(t - t_4)$ as follows:

$$u(x,t) = u_+,$$
$$v(x,t) = \left\{ \begin{array}{ll} v_*, & x - x_4 < (2u_+ + 1)(t - t_4), \\ w(x,t), & x - x_4 > (2u_+ + 1)(t - t_4), \end{array} \right\} + \beta(t)\delta_{\Gamma_3}.$$

There is no further interaction with the original contact discontinuity along the (parallel) line $x = \varepsilon + (2u_+ + 1)t$.

When $u_+ < u_- < u_+ + 1$ the solution consists of shock and contact discontinuity, which is exactly the corresponding Riemann solution of (1.1) and (1.2).

Case V: Interaction of a rarefaction wave followed by contact discontinuity with a delta shock wave ($R + J$ and δS). The analysis and computation can be done similarly to the case IV when we consider the interaction between a rarefaction wave, R, followed by contact discontinuity, J, of the first Riemann problem and a delta shock wave, δS, of the second Riemann problem. The occurrence of this case depends on the conditions $u_m > u_-$ and $u_m \geq u_+ + 1$.

4. Conclusion. By letting $\varepsilon \to 0$, it is easy to see that the Riemann solutions are stable under the local small perturbation of the Riemann initial data (1.2) in the above typical five cases when the delta shock wave is included. Otherwise, when the delta shock wave is not involved, the conclusion is obviously true and can be dealt with in a similar manner. So far we have finished the discussion for all kinds of interactions, and the solutions for the perturbed initial value problems (1.1) and (1.5) have been constructed globally. We summarize our results in the following theorem.

Theorem 4.1. *The limits of the perturbed Riemann solutions of (1.1) and (1.5) are exactly the corresponding Riemann solutions of (1.1) and (1.2) as $\varepsilon \to 0$, and the asymptotic behaviour of the perturbed Riemann solution is governed completely by the states $(u_\pm, v_\pm)$, implying thereby that the Riemann solutions of (1.1) and (1.2) are stable with respect to such a local small perturbation (1.5) of the Riemann initial data (1.2).*

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