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TWO TYPES OF ELECTRIC FIELD ENHANCEMENTS BY INFINITELY MANY CIRCULAR CONDUCTORS ARRANGED CLOSELY IN TWO PARALLEL LINES

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Abstract. In stiff fiber-reinforced composites, high shear stress concentrations occur in narrow regions between neighboring fibers. Electric fields and conductors correspond to shear stresses and cross-sections of fibers respectively in the anti-plane shear model. Due to material failure of composites, there have been intensive studies so far to estimate an electric field in between only a finite number of conductors. Indeed, the composites contain a large number of stiff fibers, and the concentration can be strongly enhanced by some combinations of inclusions. Two types of enhancements in parallel and perpendicular directions by a combination of infinitely many inclusions in a line are the main subjects of this paper. We thus consider the electric fields in between a infinite number of perfectly conducting unit disks arranged closely and regularly in two parallel lines. Asymptotes and optimal blow-up rates for the fields in two kinds of narrow regions are obtained in terms of the distances between conductors. In particular, very strong enhancement in the parallel direction is exhibited to have the blow-up rate substantially different from the existing result in the case of finite inclusions.

1. Introduction. In stiff fiber-reinforced composites, unexpectedly low strengths in longitudinal shear are explained by high shear stress concentrations occurring in between closely spaced neighboring fibers [8]. In the anti-plane shear model, the out-of-plane displacement u satisfies a conductivity equation whose inclusions in the plane are the cross-sections of fibers, and the gradient ∇u implies the shear stress tensor. The problem to estimate ∇u in between inclusions was raised by Babuška in the study of material failure of composites [4]. Many studies on the gradient estimate have been successfully carried out due to such practical significance [7, 16–18]. The generic blow-up rate of $|\nabla u|$ is $\frac{1}{\sqrt{\delta}}$ for small $\delta > 0$ when δ is the distance between two neighboring inclusions

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[2,3,22,23], and moreover, asymptotic behavior of ∇u was also established [1,10,11]. The two dimensional problem has been generalized in various ways including high dimensions [5,6,11–16,19,21,24]. In particular, it has been shown in [20] that the concentration of ∇u can be strongly enhanced by a small inclusion between inclusions. This means that some combinations of inclusions can have strong influence on the concentration. So far, such studies have considered the cases when only a finite number of inclusions exist.

This paper is mainly concerned with two types of enhancements of ∇u in parallel and perpendicular directions by a combination of an infinite number of inclusions arranged closely and regularly in a line, because composites contain a large number of stiff fibers. Dealing with two types of enhancements, we consider two parallel lines where infinitely many circular inclusions are arranged closely and regularly. Asymptotes and optimal blow-up rates for ∇u in two kinds of narrow regions between conductors are obtained in this paper. In particular, the enhancement in the parallel direction is strong enough to provide the blow-up rate $\frac{1}{\delta}$ substantially different from the existing rate $\frac{1}{\sqrt{\delta}}$ in the case of finite number of inclusions, when δ is the distance between neighboring inclusions. But, the other type in the perpendicular direction has the same blow-up rate as the existing one. Simplified derivations of the strong enhancement and asymptotes are presented in Section 2 to help readers understand easily the main results before rigorous proofs.

We set up infinitely many circular perfect conductors arranged closely in two lines. For any integer number n, we choose a pair of unit open disks D_{Rn} and D_{Ln} spaced ϵ apart in the horizontal direction, and moreover the distances between D_{Rn} and D_{Rn+1}, and between D_{Ln} and D_{Ln+1} both are δ in the vertical direction. The open disks D_{Rn} and D_{Ln} are defined as

$$D_{Rn} = \left\{(x,y)|\left(x-\left(1+\frac{\epsilon}{2}\right)\right)^2 + (y-n\,(2+\delta))^2 < 1\right\}$$

and

$$D_{Ln} = \left\{(x,y)|\left(x-\left(-1-\frac{\epsilon}{2}\right)\right)^2 + (y-n\,(2+\delta))^2 < 1\right\}.$$

Then, the domain $\mathbb{R}^2 \setminus \overline{\bigcup_{n=-\infty}^{\infty}(D_{Rn}\cup D_{Ln})}$ has a periodic structure with period $2+\delta$ in the y direction. In this paper, we suppose that ϵ and δ are sufficiently small and positive.

Dealing with the governing equation, let the symbol H denote a harmonic function defined in $\mathbb{R}^2$ whose gradient is a periodic function with period $2+\delta$ in the y direction satisfying

$$\nabla H(x,y) = \nabla H(x, y+2+\delta) \text{ for any } (x,y)\in\mathbb{R}^2. \tag{1.1}$$

For example, every linear function

$$H(x,y) = ax + by$$

is a harmonic function with a periodic gradient described above. For such a harmonic function H, we estimate the gradient ∇u of a solution u to the equation

$$\Delta u = 0 \text{ in } \mathbb{R}^2 \setminus \overline{\bigcup_{n=-\infty}^{\infty}(D_{Ln}\cup D_{Rn})} \tag{1.2}$$

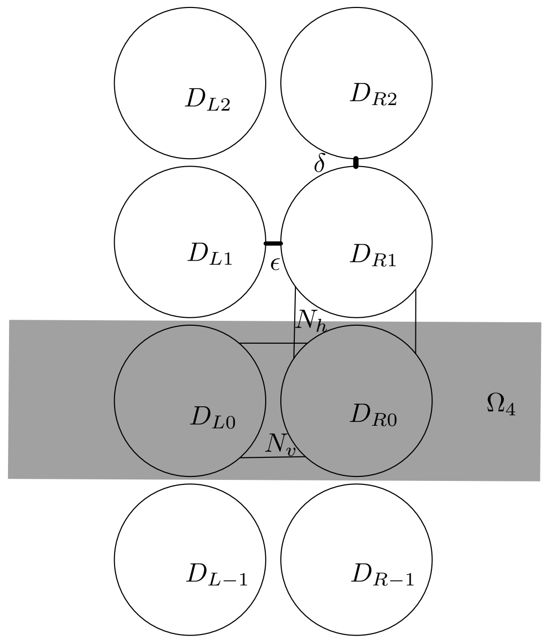

FIG. 1. D_{Ln}, D_{Rn}, δ, ϵ, N_v, N_h and Ω_4.

with the conditions

$$\begin{cases} u = c_L & \text{on } \partial D_{L0}, \\ u = c_R & \text{on } \partial D_{R0}, \\ \displaystyle\int_{\partial D_{L0}} \partial_\nu u ds = \int_{\partial D_{R0}} \partial_\nu u ds = 0, & \\ \displaystyle\int_{\Omega \setminus \overline{D_{L0} \cup D_{R0}}} |\nabla (u - H)|^2 dxdy < \infty, & \\ \nabla u(x,y) = \nabla u(x, y + 2 + \delta) & \text{for } (x,y) \in \mathbb{R}^2 \setminus \overline{\bigcup_{n=-\infty}^{\infty} (D_{Ln} \cup D_{Rn})}. \end{cases} \tag{1.3}$$

Here, the constants c_L and c_R depend on H, ϵ and δ, and the normal unit vector ν points toward the inside of D_{L0} or D_{R0}. The domain Ω denotes a horizontal area as

$$\Omega = \mathbb{R} \times \left(-1 - \frac{\delta}{2},\ 1 + \frac{\delta}{2} \right), \tag{1.4}$$

and the domain Ω_m is defined as

$$\Omega_m = \{(x,y) \mid (x,y) \in \Omega \text{ and } |x| < m\} \tag{1.5}$$

containing D_{L0} and D_{R0} for $m \geq 3$.

By definition, the gradient ∇u is a periodic function with period $2 + \delta$ in the y direction, and the solution u has a constant Dirichlet boundary data on each of boundaries ∂D_{Rn} and ∂D_{Ln} for $n = 0, \pm 1, \pm 2, \cdots$. The existence of the solution u for a harmonic function H can be shown by considering $u(x,y) + u(x,-y)$ and $u(x,y) - u(x,-y)$ in

$\Omega \setminus \overline{(D_{L0} \cup D_{R0})}$. It is worth noting that if u_α and u_β are the solutions for the same harmonic function H, then there is a constant c such that $u_\alpha = u_\beta + c$ and

$$\nabla u_\alpha = \nabla u_\beta$$

in $\mathbb{R}^2 \setminus \overline{\bigcup_{n=-\infty}^{\infty} (D_{Ln} \cup D_{Rn})}$.

In this paper, we establish some asymptotes and optimal blow-up rates for ∇u in two kinds of narrow regions in between D_{Ln} and D_{Rn}, and in between D_{Rn} and D_{Rn+1}. Theorem 1.1 provides asymptotes with a coefficient and an upper bound of the coefficient, and moreover Theorem 1.2 presents a specific asymptote with a lower bound for the coefficient in the case of a linear function

$$H(x,y) = ax + by$$

to get the optimality of gradient estimates.

Theorem 1.1 (Optimal upper bounds). For any harmonic function H with (1.1), let u be a solution to (1.2) with the condition (1.3). Let N_v be a narrow vertical region in between D_{L0} and D_{R0}, and let N_h be a narrow horizontal region in between D_{R0} and D_{R1}, defined as

$$N_v = \left\{ (x,y) \;\middle|\; |x| < 1 + \frac{\epsilon}{2} - \sqrt{1-y^2} \text{ and } |y| < \frac{\sqrt{3}}{2} \right\} \text{ and}$$

$$N_h = \left\{ (x,y) \;\middle|\; \left| y - 1 - \frac{\delta}{2} \right| < 1 + \frac{\delta}{2} - \sqrt{1 - \left(x - 1 - \frac{\epsilon}{2} \right)^2} \text{ and } \left| x - 1 - \frac{\epsilon}{2} \right| < \frac{\sqrt{3}}{2} \right\}.$$

Please refer to Figure 1. Then, we have the following estimates:

- There exists a constant C regardless of ϵ, δ and H such that

$$\|\nabla u\|_{L^\infty(N_h)} \le C \left(\frac{1}{\delta} \big| H(0,1) - H(0,-1) \big| + \|H\|_{L^\infty(\Omega_4)} \right) \tag{1.6}$$

and

$$\|\nabla u\|_{L^\infty(N_v)} \le C \frac{1}{\sqrt{\epsilon}} \|H\|_{L^\infty(\Omega_4)}. \tag{1.7}$$

- More precisely, asymptotic behaviors of ∇u in N_h and in N_v are described as

$$\nabla u(x,y) = \lambda \left(\frac{1}{\delta + \left(x - 1 - \frac{\epsilon}{2}\right)^2} (0,1) + \frac{1}{\sqrt{\delta}} S(x,y)(1,0) \right) + R_1(x,y) \tag{1.8}$$

for any $(x,y) \in N_h$, and

$$\nabla u(x,y) = \mu \frac{\sqrt{\epsilon}}{\epsilon + y^2} (1,0) + R_2(x,y) \text{ for any } (x,y) \in N_v, \tag{1.9}$$

where the constants λ and μ satisfy

$$\lambda = H(0,1) - H(0,-1) \text{ and } |\mu| \le C \|H\|_{L^\infty(\Omega_4)},$$

the function S is defined as $S(x,y) = -2\sqrt{\delta}\ \frac{\left(x-1-\frac{\epsilon}{2}\right)\left(y-1-\frac{\delta}{2}\right)}{\left(\left(x-1-\frac{\epsilon}{2}\right)^2+\delta\right)^2}$ with

$$\|S(x,y)\|_{L^\infty(N_h)} \le 2 \text{ and } S\left(1+\frac{\epsilon}{2}, 1+\frac{\delta}{2}\right) = 0,$$

and the remainder terms R_1 and R_2 are bounded as

$$\|R_1\|_{L^\infty(N_h)} + \|R_2\|_{L^\infty(N_v)} \le C\|H\|_{L^\infty(\Omega_4)}$$

for a constant C regardless of ϵ and δ.

It is worth noting that the constant μ can depend on δ and ϵ in this paper, even though it is bounded regardless of δ and ϵ. The upper bounds (1.6) and (1.7) follow immediately from (1.8) and (1.9) whose proofs are given in Section 4, based on the results in Section 3.

THEOREM 1.2 (Lower bounds for a linear function H). Let N_v, N_h and S be as given in Theorem 1.1. Assume that H is a linear function given as

$$H(x,y) = ax + by$$

for any $(x,y) \in \mathbb{R}^2$ and u is a solution to (1.2) with the condition (1.3) for H. Then, we have the following estimates.

- There exists a constant C regardless of ϵ and δ such that

$$\|\nabla u\|_{L^\infty(N_h)} \ge C\ \frac{1}{\delta}\ |b| \tag{1.10}$$

and

$$\|\nabla u\|_{L^\infty(N_v)} \ge C\ \frac{1}{\sqrt{\epsilon}}\ |a|. \tag{1.11}$$

- More precisely, asymptotic behaviors of ∇u in N_h and in N_v are described as

$$\nabla u(x,y) = 2b\left(\frac{1}{\delta+\left(x-1-\frac{\epsilon}{2}\right)^2}\,(0,1) + \frac{1}{\sqrt{\delta}}S(x,y)(1,0)\right) + R_1(x,y) \tag{1.12}$$

for any $(x,y) \in N_h$, and

$$\nabla u(x,y) = a\mu_0\frac{\sqrt{\epsilon}}{\epsilon+y^2}\,(1,0) + R_2(x,y) \text{ for any } (x,y) \in N_v, \tag{1.13}$$

where the constant μ_0 satisfies

$$\frac{1}{C} < \mu_0 < C \tag{1.14}$$

and the remainder terms R_1 and R_2 are bounded as

$$\|R_1\|_{L^\infty(N_h)} + \|R_2\|_{L^\infty(N_v)} \le C(|a|+|b|)$$

for a constant C regardless of ϵ and δ.

The proof of Theorem 1.2 is presented in Section 5. The bounds (1.10) and (1.11) follow immediately from (1.12) and (1.13) respectively whose proofs are presented in the section.

Remark 1.3 (Optimal blow-up rates). Theorems 1.1 and 1.2 provide the generic blow-up rates of ∇u in between D_{Ln} and D_{Rn}, and in between D_{Ln} and D_{Ln+1} which are

$$\frac{1}{\sqrt{\epsilon}} \text{ and } \frac{1}{\delta},$$

respectively. Especially, the blow-up rate $\frac{1}{\delta}$ in the direction parallel to the lines of disks is substantially different from the existing one $\frac{1}{\sqrt{\delta}}$ in the case of finite inclusions [6, 22].

In case of a single line of disks, the field enhancement in the direction parallel to two lines of disks in (1.6) and the asymptote (1.8) are still available by the same argument. Corollary 1.4 presents analogous results with the same blow-up rate as the case of two lines.

Corollary 1.4 (A single line of circular conductors). For a harmonic function H with a periodic gradient as above, let w be a solution to the equation

$$\Delta w = 0 \text{ in } \mathbb{R}^2 \setminus \overline{\bigcup_{n=-\infty}^{\infty} D_{Rn}}$$

with the conditions

$$\begin{cases} w = c & \text{on } \partial D_{R0}, \\ \displaystyle\int_{\partial D_{R0}} \partial_\nu w ds = 0, & \\ \displaystyle\int_{\Omega\setminus\overline{D_{R0}}} |\nabla(w-H)|^2 dxdy < \infty, & \\ \nabla w(x,y) = \nabla w(x, y+2+\delta) & \text{for any } (x,y) \in \mathbb{R}^2 \setminus \overline{\bigcup_{n=-\infty}^{\infty} D_{Rn}}, \end{cases}$$

where c is a constant depending on H, ϵ. Then, we have the following estimates.

- There exists a positive constant C regardless of ϵ, δ and H such that

$$\begin{aligned} &\frac{1}{C}\,\frac{1}{\delta}\left|H(0,1) - H(0,-1)\right| \\ &\le \|\nabla u\|_{L^\infty(N_h)} \le C\left(\frac{1}{\delta}\big|H(0,1) - H(0,-1)\big| + \|H\|_{L^\infty(\Omega_4)}\right). \end{aligned}$$

  The bound results immediately from the asymptote below.
- An asymptotic behavior of ∇u in N_v is described as

$$\nabla w(x,y) = \lambda\left(\frac{1}{\delta + \left(x - 1 - \frac{\epsilon}{2}\right)^2}(0,1) + \frac{1}{\sqrt{\delta}} S(x,y)(1,0)\right) + R(x,y)$$

  for any $(x,y) \in N_h$, where the constant $\lambda = H(0,1) - H(0,-1)$ and the remainder term R is bounded as

$$\|R\|_{L^\infty(N_h)} \le C\|H\|_{L^\infty(\Omega_4)}$$

  for a constant C regardless of $\delta > 0$.

The proof of Corollary 1.4 is left as an exercise for the readers, since it is much the same as the proof of (1.8) and Proposition 3.1.

REMARK 1.5. We consider the behavior of ∇u in the domain

$$\left(-1-\frac{\epsilon}{2}+\frac{\sqrt{3}}{2},1+\frac{\epsilon}{2}-\frac{\sqrt{3}}{2}\right)\times\left(\frac{\sqrt{3}}{2},2+\delta-\frac{\sqrt{3}}{2}\right)$$

which doesn't belong to N_v and N_h. Theorem 1.1 provides the boundedness of $|\nabla u|$ on its rectangular boundary regardless of ϵ and δ. By the maximum principle, $|\nabla u|$ is bounded in the domian regardless of ϵ and δ. Combined with Theorem 1.1 again, an asymptote for ∇u in

$$\left(\left(-1-\frac{\sqrt{3}}{2},1+\frac{\sqrt{3}}{2}\right)\times\mathbb{R}\right)\setminus\overline{\bigcup_{n=-\infty}^{\infty}(D_{Ln}\cup D_{Rn})}$$

is also obtained, since the gradient ∇u is periodic with period $2+\delta$ in the y direction.

This paper is organized as follows: Section 2 presents brief derivations of the main results to help readers easily understand them before rigorous proofs. Section 3 provides the estimate for potential differences of u between ∂D_{Rn} and ∂D_{Ln}, and between ∂D_{Rn} and ∂D_{Rn+1}. In Section 4, two asymptotes (1.8) and (1.9) for ∇u in Theorem 1.1 result from the potential differences. In Section 5, we establish more descriptive asymptotes of ∇u for $H(x,y)=ax+by$ to prove Theorem 1.2 and to get the optimality of the blow-up rates in Remark 1.3. From now on, the symbols C and C_n denote the constants regardless of small $\epsilon>0$ and $\delta>0$ for $n=1,2,\cdots$.

2. Simplified derivations of strong enhancement and asymptotes. To help readers easily understand the main results, the existence of strong field enhancement in the direction parallel to lines of disks and a simplified derivation of an asymptote for the field from a potential difference are briefly presented before rigorous proofs. The potential differences $u|_{\partial D_{Rn+1}}-u|_{\partial D_{Rn}}$, $u|_{\partial D_{Ln+1}}-u|_{\partial D_{Ln}}$ and $u|_{\partial D_{Rn}}-u|_{\partial D_{Ln}}$ play significant roles in deriving such results.

• **Strong enhancement in the parallel direction.** Two types of field enhancements are considered in this paper. Enhancement in the direction parallel to the lines of disks is much stronger and also has a different nature from one in the perpendicular direction. It results in two different blow-up rates depending on the direction. To give a brief explanation, we suppose that

$$H(x,y)=y.$$

The potential u is constant on each one of the boundaries, and we evaluate the potential differences

$$u|_{\partial D_{Rn+1}}-u|_{\partial D_{Rn}}\text{ and }u|_{\partial D_{Ln+1}}-u|_{\partial D_{Ln}}$$

for any $n=0,\pm1,\pm2,\cdots$. The function $u-H$ is the reflection of H with respect to $\bigcup_{n=-\infty}^{\infty}(D_{Ln}\cup D_{Rn})$ arranged periodically due to the decomposition $u=H+(u-H)$. We can show the periodic property of $u-H$ as

$$(u-H)(x,y+2+\delta)=(u-H)(x,y)$$

for any (x, y) in its domain, since ∇H is also periodic. Then, we have the exact potential differences as

$$\begin{aligned} u|_{\partial D_{Rn+1}} - u|_{\partial D_{Rn}} &= u|_{\partial D_{Ln+1}} - u|_{\partial D_{Ln}} \\ &= H(x_0, y_0 + 2 + \delta) - H(x_0, y_0) = 2 + \delta \end{aligned}$$

by a point (x_0, y_0) on ∂D_{Rn} or ∂D_{Rn}. Please refer to Proposition 3.1 for details. The Mean Value Theorem simply indicates the strong enhancements in the narrow regions. There exists a point (x_1, y_1) in any narrow region between D_{Rn+1} and D_{Rn}, or between D_{Ln+1} and D_{Ln} such that

$$|\nabla u(x_1, y_1)| \geq \frac{2}{\delta},$$

since the distance $d(D_{Rn+1}, D_{Rn}) = d(D_{Ln+1}, D_{Ln}) = \delta$. Moreover, an upper bound and asymptote for $|\nabla u|$ are also derived from the potential difference through more complicated arguments.

• **Asymptotes from potential differences.** To make a brief description, we consider the simplified case when there exist only two unit open disks D_{L0} and D_{R0} spaced ϵ apart, as defined before. For a harmonic function H defined in $\mathbb{R}^2 \setminus \overline{D_{L0} \cup D_{R0}}$, let U be a solution to

$$\Delta U = 0 \text{ in } \mathbb{R}^2 \setminus \overline{D_{L0} \cup D_{R0}}$$

with a constant Dirichlet data on each of D_{L0} and D_{R0} and $(U - H)(x, y) = O\left(\frac{1}{|(x,y)|}\right)$ as $|(x, y)| \to \infty$. Let

$$Q(x, y) = -\frac{1}{2\sqrt{\epsilon} + e_1}\left(\log|(x - \sqrt{\epsilon} - e_2, y)| - \log|(x + \sqrt{\epsilon} + e_2, y)|\right)$$

for any $(x, y) \in \mathbb{R}^2 \setminus \overline{D_{L0} \cup D_{R0}}$, where e_1 and e_2 are proper constants with $|e_1| + |e_2| = O(\epsilon)$ for small $\epsilon > 0$. Then,

$$Q\big|_{\partial D_{L0}} - Q\big|_{\partial D_{R0}} = 1.$$

The solution U can be decomposed into two parts as

$$U(x, y) = \left(U\big|_{\partial D_{R0}} - U\big|_{\partial D_{L0}}\right) Q(x, y) + R(x, y) \text{ in } \mathbb{R}^2 \setminus \overline{D_{L0} \cup D_{R0}},$$

and the function R has no potential difference as

$$R\big|_{\partial D_{L0}} - R\big|_{\partial D_{R0}} = 0$$

implying the boundedness of ∇R in the narrow region regardless of ϵ. Hence,

$$\nabla U(x, y) = \left(U\big|_{\partial D_{R0}} - U\big|_{\partial D_{L0}}\right) \nabla Q(x, y) + O(1)$$

and

$$\nabla Q(x, y) = \left(\frac{\sqrt{\epsilon}}{\epsilon + y^2}, 0\right) + O(1)$$

in a narrow region between D_{L0} and D_{R0}. Therefore, we can obtain an asymptote similar to (1.9) and (1.13) in theorems. This derivation in case of two inclusions was presented in [11], and is modified in this paper dealing with an infinite number of disks.

3. Estimates for potential differences. The potential differences play important roles in establishing the asymptotes and estimates for the gradient ∇u. Once the potential difference is estimated, the methods in [10, 11] are modified to obtain the asymptotes. This section thus provides the estimates for potential differences $u|_{D_{Rn+1}} - u|_{D_{Rn}}$ and $u|_{D_{Rn}} - u|_{D_{Ln}}$.

PROPOSITION 3.1. Let u be a solution to (1.2) with the condition (1.3) for any harmonic function H with a periodic gradient satisfying (1.1). Then, the potential differences between ∂D_{Ln} and ∂D_{Ln+1}, and between ∂D_{Rn} and ∂D_{Rn+1} are obtained as

$$u|_{\partial D_{Rn+1}} - u|_{\partial D_{Rn}} = u|_{\partial D_{Ln+1}} - u|_{\partial D_{Ln}} = H\left(0, 1+\frac{\delta}{2}\right) - H\left(0, -1-\frac{\delta}{2}\right)$$

for any $n = 0, \pm 1, \pm 2, \cdots$.

Proof. We begin by proving that

$$(u-H)(x,y) - (u-H)(x, y-2-\delta) = 0 \tag{3.1}$$

for any $(x,y) \in \mathbb{R}^2 \setminus \overline{\bigcup_{n=-\infty}^{\infty}(D_{Ln} \cup D_{Rn})}$. The gradient $\nabla(u-H)$ is a periodic function with period $2+\delta$ in the y direction as given in (1.1) and (1.3). Then, $\nabla(u-H)(x,y) - \nabla(u-H)(x, y-2-\delta) = (0,0)$ for any $(x,y) \in \mathbb{R}^2 \setminus \overline{\bigcup_{n=-\infty}^{\infty}(D_{Ln} \cup D_{Rn})}$. There exists a constant $d > 0$ such that

$$(u-H)(x,y) - (u-H)(x, y-2-\delta) = d$$

for any $(x,y) \in \mathbb{R}^2 \setminus \overline{\bigcup_{n=-\infty}^{\infty}(D_{Ln} \cup D_{Rn})}$. By the Jensen's inequality, every $x \in [3, \infty)$ has the upper bound for $|d|^2$ as

$$|d|^2 \le \left(\int_{-1-\frac{\delta}{2}}^{1+\frac{\delta}{2}} |\partial_y(u-H)(x,y)| dy\right)^2 \le (2+\delta) \int_{-1-\frac{\delta}{2}}^{1+\frac{\delta}{2}} |\nabla(u-H)(x,y)|^2 dy.$$

Since $\int_{\Omega \setminus \overline{D_{L0} \cup D_{R0}}} |\nabla(u-H)|^2 dxdy < \infty$, it follows from Fubini's theorem that

$$d = 0$$

implying (3.1).

The periodic property (1.1) implies that

$$H(x,y) - H(x, y-2-\delta) = H\left(0, 1+\frac{\delta}{2}\right) - H\left(0, -1-\frac{\delta}{2}\right) \tag{3.2}$$

for any $(x,y) \in \mathbb{R}^2$, since $\nabla(H(x,y) - H(x, y-2-\delta)) = (0,0)$. The equality (3.1) yields this proposition as follows:

$$\begin{aligned}
&H\left(0, 1+\frac{\delta}{2}\right) - H\left(0, -1-\frac{\delta}{2}\right) = H(x,y) - H(x, y-2-\delta) \\
&= H(x,y) - H(x, y-2-\delta) + ((u-H)(x,y) - (u-H)(x, y-2-\delta)) \\
&= u(x,y) - u(x, y-2-\delta)
\end{aligned}$$

for any $(x,y) \in \mathbb{R}^2 \setminus \overline{\bigcup_{n=-\infty}^{\infty}(D_{Ln} \cup D_{Rn})}$. This implies that

$$H\left(0, 1+\frac{\delta}{2}\right) - H\left(0, -1-\frac{\delta}{2}\right) = u|_{\partial D_{Ln+1}} - u|_{\partial D_{Ln}} = u|_{D_{\partial Rn+1}} - u|_{\partial D_{Rn}}. \qquad \square$$

Proposition 3.2. Let u be the solution to (1.2) satisfying (1.3) for

$$H(x,y)=x$$

for any $(x,y)\in\mathbb{R}^2$. Then, the potential difference between ∂D_{Ln} and ∂D_{Rn} is estimated as

$$\frac{1}{C}\sqrt{\epsilon}<u|_{D_{Rn}}-u|_{D_{Ln}}<C\sqrt{\epsilon},$$

and there are no potential differences between ∂D_{Ln} and ∂D_{Ln+1}, and between ∂D_{Rn} and ∂D_{Rn+1}, i.e.,

$$u|_{D_{Ln+1}}-u|_{D_{Ln}}=u|_{D_{Rn+1}}-u|_{R_{Ln}}=0$$

for any $n=0,\pm1,\pm2,\cdots$.

The proof of the proposition is presented in Subsection 3.1 based on Proposition 3.4. The potential difference of u between ∂D_{L0} and ∂D_{R0} can be expressed as an integral containing $\partial_\nu\phi$ in Proposition 3.4 motivated by the method in [22, 23]. The following lemma is used to modify the method for the proposition.

Lemma 3.3. Let h be a harmonic function as

$$\begin{cases}\triangle h=0 & \text{in } (4,\infty)\times\left(1-\frac{\delta}{2},1+\frac{\delta}{2}\right),\\ \partial_y h=0 & \text{on } (4,\infty)\times\left\{1-\frac{\delta}{2},1+\frac{\delta}{2}\right\},\\ \displaystyle\int_{(4,\infty)\times\left(1-\frac{\delta}{2},1+\frac{\delta}{2}\right)}|\nabla h|^2dxdy & <\infty.\end{cases}$$

Then,

$$\sup_{y\in\left(1-\frac{\delta}{2},1+\frac{\delta}{2}\right)}|h(x,y)|=O(1)\text{ and }\sup_{y\in\left(1-\frac{\delta}{2},1+\frac{\delta}{2}\right)}|\nabla h(x,y)|=O\left(\exp\left(-\frac{\pi}{2+\delta}x\right)\right)$$

as $x\to\infty$.

Proof. The function h can be expressed as

$$h(x,y)=\sum_{n=0}^{\infty}a_n\cos\left(\frac{n\pi}{2+\delta}\left(y+1+\frac{\delta}{2}\right)\right)\exp\left(-\frac{n\pi}{2+\delta}x\right).$$

The estimates can be obtained immediately. □

Proposition 3.4. There exists the harmonic function ϕ defined in $\Omega\setminus\overline{(D_{L0}\cup D_{R0})}$ with the following conditions as:

$$\begin{cases}\partial_\nu\phi=0 & \text{on } \partial\Omega=\mathbb{R}\times\left\{y\;\middle|\;y=-1-\frac{\delta}{2}\text{ or }y=1+\frac{\delta}{2}\right\},\\ \phi=c_0 & \text{on } \partial D_{R0},\\ \phi=-c_0 & \text{on } \partial D_{L0},\\ \displaystyle\int_{\partial D_{R0}}\partial_\nu\phi ds=-\int_{\partial D_{L0}} & \partial_\nu\phi ds=1,\\ \displaystyle\int_{\Omega\setminus\overline{D_{L0}\cup D_{R0}}}|\nabla\phi|^2dxdy & <\infty,\end{cases}\tag{3.3}$$

where c_0 is a proper constant depending on ϵ and δ. If u is a solution to (1.2) satisfying the condition (1.3) for any harmonic function H with (1.1), then

$$u\Big|_{\partial D_{R0}} - u\Big|_{\partial D_{L0}} = \int_{\partial D_{L0}\cup\partial D_{R0}} H\partial_\nu \phi ds. \tag{3.4}$$

Proof. First, we prove the existence of ϕ. By the Lax-Milgram theorem, there exists the harmonic function φ_0 defined in $\Omega \setminus \overline{(D_{L0} \cup D_{R0})}$ with conditions:

$$\begin{cases} \partial_\nu \varphi_0 = 0 & \text{on } \mathbb{R} \times \left\{ y \;\middle|\; y = -1 - \frac{\delta}{2} \text{ and } y = 1 + \frac{\delta}{2} \right\}, \\ \varphi_0 = 1 & \text{on } \partial D_{R0}, \\ \varphi_0 = -1 & \text{on } \partial D_{L0}, \\ \int_{\Omega\setminus\overline{D_{L0}\cup D_{R0}}} |\nabla\varphi_0|^2 \, dxdy & < \infty. \end{cases}$$

We can construct a bijective conformal mapping $\Phi : B_1(0,0) \to \Omega$ such that

$$\begin{cases} \triangle\varphi_0(\Phi) = 0 & \text{in } B_1(0,0) \setminus \overline{\Phi^{-1}(D_{L0}) \cup \Phi^{-1}(D_{R0})}, \\ \varphi_0(\Phi) = 1 & \text{on } \partial\Phi^{-1}(D_{R0}), \\ \varphi_0(\Phi) = -1 & \text{on } \partial\Phi^{-1}(D_{L0}), \\ \varphi_0(\Phi) \text{ belongs to } & H^1(B_1(0,0) \setminus \overline{\Phi^{-1}(D_{L0}) \cup \Phi^{-1}(D_{R0})}), \end{cases}$$

since

$$\int_{B_1(0,0)\setminus\overline{\Phi^{-1}(D_{L0})\cup\Phi^{-1}(D_{R0})}} |\nabla(\varphi_0(\Phi))|^2 \, d\xi d\eta = \int_{\Omega\setminus\overline{D_{L0}\cup D_{R0}}} |\nabla\varphi_0|^2 \, dxdy < \infty.$$

By the maximum principle, φ_0 has the maximal value 1 on ∂D_{R0} and also has the minimal value -1 on ∂D_{L0}, since $\varphi_0(\Phi)$ is a harmonic function defined in a bounded domain and $\partial_\nu(\varphi_0(\Phi)) = 0$ on $\partial B_1(0,0)$. By Hopf's lemma,

$$\int_{\partial D_{R0}} \partial_\nu\varphi_0 ds = -\int_{\partial D_{L0}} \partial_\nu\varphi_0 ds > 0,$$

since the normal vector ν points toward the inside of D_{L0} or D_{R0}. Then,

$$\phi = \frac{1}{\int_{\partial D_{R0}} \partial_\nu\varphi_0 ds}\, \varphi_0,$$

which means the existence of ϕ. In addition, it can be easily shown that

$$\phi(x,y) = \phi(x,-y) \tag{3.5}$$

for $(x,y) \in \Omega \setminus \overline{D_{L0} \cup D_{R0}}$.

Second, we prove the equality (3.4). From definition, u is constant on each of the boundaries ∂D_{R0} and ∂D_{L0}, and $\int_{\partial D_{R0}} \partial_\nu\phi ds = -\int_{\partial D_{L0}} \partial_\nu\phi ds = 1$. Thus,

$$\begin{aligned} u\Big|_{\partial D_{R0}} - u\Big|_{\partial D_{L0}} &= \int_{\partial D_{L0}\cup\partial D_{R0}} u\partial_\nu\phi ds \\ &= \int_{\partial D_{L0}\cup\partial D_{R0}} H\partial_\nu\phi ds + \int_{\partial D_{L0}\cup\partial D_{R0}} (u - H)\partial_\nu\phi ds. \end{aligned}$$

We shall use the divergence theorem to prove that

$$\int_{\partial D_{L0}\cup\partial D_{R0}}(u-H)\partial_\nu\phi=0.$$

This immediately results in the desirable equality (3.4). To use the divergence theorem, we define $\tilde{u}$ and $\tilde{H}$ as even functions with respect to y as $\tilde{u}(x,y)=\frac{1}{2}\left(u(x,y)+u(x,-y)\right)$ and $\tilde{H}(x,y)=\frac{1}{2}\left(H(x,y)+H(x,-y)\right)$. By the periodic property of ∇u, $\tilde{u}-\tilde{H}$ has zero Neumann data on two horizontal boundaries $\partial\Omega$ so that

$$\partial_y\left(\tilde{u}-\tilde{H}\right)\left(x,1+\frac{\delta}{2}\right)=\partial_y\left(\tilde{u}-\tilde{H}\right)\left(x,-1-\frac{\delta}{2}\right)=0$$

for any $x\in\mathbb{R}$. From definition of u,

$$\int_{\Omega\setminus\overline{D_{L0}\cup D_{R0}}}\left|\nabla\left(\tilde{u}-\tilde{H}\right)\right|^2dxdy<\infty.$$

By Lemma 3.3,

$$\tilde{u}(x,y)-\tilde{H}(x,y)=O(1)\text{ and }\nabla\left(\tilde{u}(x,y)-\tilde{H}(x,y)\right)=O\left(\exp-|x|\right)$$

as $|x|\to\infty$, and ϕ and $\nabla\phi$ also show the same behaviors as $|x|\to\infty$. Thus, we can use the divergence theorem so that by (3.5),

$$\begin{aligned}&\int_{\partial D_{L0}\cup\partial D_{R0}}(u-H)\partial_\nu\phi ds\\&=\int_{\partial D_{L0}\cup\partial D_{R0}}(\tilde{u}-\tilde{H})\partial_\nu\phi ds\\&=\int_{\partial\left(\Omega\setminus\overline{D_{L0}\cup D_{R0}}\right)}(\tilde{u}-\tilde{H})\partial_\nu\phi ds\\&=\int_{\partial(D_{L0}\cup D_{R0})}\phi\partial_\nu(\tilde{u}-\tilde{H})ds\\&=\phi|_{\partial D_{L0}}\int_{\partial D_{L0}}\partial_\nu(\tilde{u}-\tilde{H})ds+\phi|_{\partial D_{R0}}\int_{\partial D_{R0}}\partial_\nu(\tilde{u}-\tilde{H})ds=0.\end{aligned}$$

Thus, we have done it. □

3.1. *Proof of Proposition* 3.2. In this subsection, we suppose that

$$H(x,y)=x$$

for any $(x,y)\in\mathbb{R}^2$. The function u is the solution to (1.2) satisfying (1.3) for H.

The integral equation (3.4) is mainly used to estimate the potential difference of u between ∂D_{L0} and ∂D_{R0}. To do so, we use the function ϕ constructed in (3.12) as a series of ϕ_n whose property has been well known, and which is also given explicitly as in (3.10).

First, some maximum principles related to ϕ are presented in Lemmas 3.5 and 3.7 before constructing ϕ. Let Ω_R be the right-hand side of Ω as

$$\Omega_R=(0,\infty)\times\left(-1-\frac{\delta}{2},\ 1+\frac{\delta}{2}\right).\tag{3.6}$$

LEMMA 3.5. There exists a harmonic function ϕ_R defined on $\Omega_R\backslash\overline{D_{R0}}$ with the conditions

$$\begin{cases} \partial_\nu \phi_R = 0 & \text{on } (0,\infty)\times\left\{y \;\middle|\; y=-1-\frac{\delta}{2} \text{ or } y=1+\frac{\delta}{2}\right\}, \\ \phi_R = 0 & \text{on } \{0\}\times\left(-1-\frac{\delta}{2},\ 1+\frac{\delta}{2}\right), \\ \phi_R = 1 & \text{on } \partial D_{R0}, \\ \displaystyle\int_{\Omega_R\backslash\overline{D_{R0}}} |\nabla\phi_R|^2\, dxdy & < \infty. \end{cases} \tag{3.7}$$

Meanwhile, the function ϕ_R has the extreame values only on the boundary so that

$$0 < \phi_R < 1 \text{ in } \Omega_R \setminus \overline{D_{R0}}.$$

REMARK 3.6. It is obvious that

$$\phi_R = \frac{1}{c_0}\phi$$

in $\Omega_R \setminus \overline{D_{R0}}$, where the constant c_0 is given in Proposition 3.4.

Proof. By the Lax-Milgram theorem, there exists the unique harmonic function ϕ_R defined on $\Omega_R \setminus \overline{D_{R0}}$ with the boundary condition (3.7). As mentioned in the remark above, the existence of ϕ_R results immediately from ϕ given in Proposition 3.4 due to $\phi_R = \frac{1}{c_0}\phi$.

We use a conformal map to prove that $0 < \phi_R < 1$ in $\Omega_R \setminus \overline{D_{R0}}$. Let

$$B_1^+(0,0) = \left\{(\xi,\eta) \mid \xi^2+\eta^2 < 1 \text{ and } \xi > 0\right\}.$$

There exists a bijective conformal mapping $\varPhi_R : B_1^+(0,0) \to \Omega_R$ such that

$$\begin{cases} \triangle\phi_R(\varPhi_R) = 0 & \text{in } B_1^+(0,0)\setminus\overline{\varPhi_R^{-1}(D_{R0})}, \\ \phi_R(\varPhi_R) = 1 & \text{on } \partial\varPhi_R^{-1}(D_{R0}), \\ \phi_R(\varPhi_R) = 0 & \text{on } \xi^2+\eta^2 = 1 \text{ and } \xi > 0, \\ \partial_\nu\left(\phi_R(\varPhi_R)\right)(0,\eta) = 0 & \text{on } |\eta| < 1, \\ \phi_R(\varPhi_R) \text{ belongs to} & H^1(B_1(0,0)\setminus\overline{\varPhi_R^{-1}(D_{R0})}). \end{cases}$$

By the maximal principle, $0 < \phi_R(\varPhi_R) < 1$ in the bounded domain $B_1^+(0,0)\setminus\overline{\varPhi_R^{-1}(D_{R0})}$. This implies that

$$0 < \phi_R < 1 \text{ in } \Omega_R \setminus \overline{D_{R0}}.$$

□

The following lemma is derived easily by an argument analogous to Lemma 3.5 and by the same function $\varPhi_R : B_1^+(0,0) \to \Omega_R$. Hence, the proof is omitted.

LEMMA 3.7. Let ρ be a harmonic function defined in $\Omega_R \setminus \overline{D_{R0}}$ with the boundary conditions:

$$\begin{cases} \partial_\nu \rho = 0 & \text{on } (0,\infty) \times \left\{ y \;\middle|\; y = -1 - \frac{\delta}{2} \text{ or } y = 1 + \frac{\delta}{2} \right\}, \\ \rho = 0 & \text{on } \left\{ x \;\middle|\; x = 0 \right\} \times \left(-1 - \frac{\delta}{2},\ 1 + \frac{\delta}{2}\right), \\ \rho > 0 & \text{on } \partial D_{R0}, \end{cases} \tag{3.8}$$

and also satisfying

$$\int_{\Omega_R \setminus \overline{D_{R0}}} |\nabla \rho|^2 \, dxdy < \infty.$$

Then,

$$0 < \rho \text{ in } \Omega_R \setminus \overline{D_{R0}}.$$

Second, we use a series of functions ϕ_n to express ϕ, given in Proposition 3.4. Here, ϕ_n is the harmonic function satisfying

$$\begin{cases} \Delta \phi_n = 0 & \text{in } \mathbb{R}^2 \setminus \overline{D_{Ln} \cup D_{Rn}}, \\ \phi_n = \text{ a constant} & \text{on } \partial D_{Ln}, \\ \phi_n = -\phi_n\Big|_{\partial D_{Ln}} & \text{on } \partial D_{Rn}, \\ \displaystyle\int_{\partial D_{Rn}} \partial_\nu \phi_n ds = -\int_{\partial D_{Ln}} \partial_\nu \phi_n ds & = 1, \\ \phi_n(\mathbf{x}) = O\left(\dfrac{1}{|\mathbf{x}|}\right) & \text{as } \mathbf{x} \to \infty, \end{cases} \tag{3.9}$$

for any integer n. Then, the function ϕ_n is expressed explicitly as

$$\phi_n = \frac{1}{2\pi}\big(\log|\mathbf{x} - (-p, n(2+\delta))| - \log|\mathbf{x} - (p, n(2+\delta))|\big) \tag{3.10}$$

where

$$p = \sqrt{\epsilon} + O(\epsilon)$$

for small $\epsilon > 0$. Refer to [11, 19] for details. We define the sum $\widetilde{\phi}$ of the series of ϕ_n in the manner as

$$\widetilde{\phi} = \lim_{N \to \infty} \sum_{n=-N}^{N} \phi_n.$$

The function $\widetilde{\phi}$ is well defined in $\Omega \setminus \overline{D_{L0} \cup D_{R0}}$ by the help of a neutralization reaction between ϕ_n and ϕ_{-n}, and satisfies

$$\begin{cases} \triangle \widetilde{\phi} = 0 & \text{in } \Omega \setminus \overline{D_{L0} \cup D_{R0}}, \\ \partial_\nu \widetilde{\phi} = 0 & \text{on } \partial\Omega = \mathbb{R} \times \left\{ y \;\middle|\; y = -1 - \frac{\delta}{2} \text{ or } y = 1 + \frac{\delta}{2} \right\}, \\ \displaystyle\int_{\partial D_{R0}} \partial_\nu \widetilde{\phi} ds = -\int_{\partial D_{L0}} & \partial_\nu \widetilde{\phi} ds = 1, \\ \displaystyle\int_{\Omega \setminus \overline{D_{L0} \cup D_{R0}}} \left|\nabla \widetilde{\phi}\right|^2 dxdy & < \infty. \end{cases}$$

Here, $\widetilde{\phi}$ is not constant on each of ∂D_{L0} and ∂D_{R0}, and Ω is as given at (1.4). There exists a harmonic function $\widetilde{v}$ defined in $\Omega \setminus \overline{(D_{L0} \cup D_{R0})}$ with conditions:

$$\begin{cases} \partial_\nu \widetilde{v} = 0 & \text{on } \partial\Omega = \mathbb{R} \times \left\{ y \,\middle|\, y = -1 - \frac{\delta}{2} \text{ or } y = 1 + \frac{\delta}{2} \right\}, \\ \widetilde{\phi} + \widetilde{v} = \text{ a constant } \widetilde{c} & \text{on } \partial D_{R0}, \\ \widetilde{\phi} + \widetilde{v} = -\widetilde{c} & \text{on } \partial D_{L0}, \\ \displaystyle\int_{\partial D_{R0}} \partial_\nu \widetilde{v} ds = \int_{\partial D_{L0}} & \partial_\nu \widetilde{v} ds = 0, \\ \displaystyle\int_{\Omega \setminus \overline{D_{L0} \cup D_{R0}}} |\nabla \widetilde{v}|^2 \, dxdy & < \infty, \end{cases} \tag{3.11}$$

for a proper constant $\tilde{c}$. The existence of $\tilde{v}$ is derived in the same way as u to (1.2) for a given H in the introduction. In another way, the existence of $\tilde{v}$ is also derived from the existence of ϕ shown in Proposition 3.4, since $\widetilde{\phi} + \widetilde{v}$ satisfies all conditions of ϕ so that

$$\phi = \widetilde{\phi} + \widetilde{v}. \tag{3.12}$$

The function ϕ can be decomposed into three functions as

$$\phi = \alpha\phi + \left(\widetilde{\phi} - \alpha\phi\right) + \widetilde{v},$$

where the positive constant α is defined as

$$\alpha = \frac{\widetilde{\phi}(\frac{\epsilon}{2}, 0) - \widetilde{\phi}(-\frac{\epsilon}{2}, 0)}{\phi\Big|_{\partial D_{R0}} - \phi\Big|_{\partial D_{L0}}}.$$

LEMMA 3.8. There is a constant C such that

$$0 < 1 - \alpha \le C\sqrt{\epsilon}$$

for small $\epsilon > 0$.

Proof. First, we show that

$$\begin{aligned} 0 &< \left(\widetilde{\phi} - \alpha\phi + \widetilde{v}\right)\Big|_{\partial D_{R0}} - \left(\widetilde{\phi} - \alpha\phi + \widetilde{v}\right)\Big|_{\partial D_{L0}} \\ &= \int_{\partial D_{L0} \cup \partial D_{R0}} \left(\widetilde{\phi} - \alpha\phi\right) \partial_\nu \phi \; ds \\ &\le C_1 \sqrt{\epsilon} \int_{\partial D_{L0} \cup \partial D_{R0}} x \partial_\nu \phi \; ds \end{aligned} \tag{3.13}$$

for a positive constant C_1. In the same way as Proposition 3.4, the integration by parts yields

$$\begin{aligned} &\int_{\partial D_{L0} \cup \partial D_{R0}} \tilde{v} \partial_\nu \phi ds = \int_{\partial D_{L0} \cup \partial D_{R0}} \phi \partial_\nu \tilde{v} ds \\ &= \left(\phi\Big|_{\partial D_{L0}}\right) \int_{\partial D_{L0}} \partial_\nu \tilde{v} ds + \left(\phi\Big|_{\partial D_{R0}}\right) \int_{\partial D_{R0}} \partial_\nu \tilde{v} ds = 0. \end{aligned}$$

We thus have the equality

$$\left(\widetilde{\phi}-\alpha\phi+\widetilde{v}\right)\Big|_{\partial D_{R0}}-\left(\widetilde{\phi}-\alpha\phi+\widetilde{v}\right)\Big|_{\partial D_{L0}}=\int_{\partial D_{L0}\cup\partial D_{R0}}\left(\widetilde{\phi}-\alpha\phi+\tilde{v}\right)\partial_\nu\phi\ ds$$
$$=\int_{\partial D_{L0}\cup\partial D_{R0}}\left(\widetilde{\phi}-\alpha\phi\right)\partial_\nu\phi\ ds.$$

Meanwhile, calculationg $\widetilde{\phi}(x,y)$ directly from (3.10), there exists a positive constant C_2 regardless of ϵ and δ such that

$$0<\widetilde{\phi}(x,y)-\alpha\phi(x,y)\le C_2\sqrt{\epsilon}x \quad \text{for } (x,y)\in\partial D_{R0}\setminus\{(\epsilon/2,0)\},$$
$$C_2\sqrt{\epsilon}x\le\widetilde{\phi}(x,y)-\alpha\phi(x,y)<0 \quad \text{for } (x,y)\in\partial D_{L0}\setminus\{(-\epsilon/2,0)\},$$

and $\widetilde{\phi}\left(\frac{\epsilon}{2},0\right)-\alpha\phi\left(\frac{\epsilon}{2},0\right)=\widetilde{\phi}\left(-\frac{\epsilon}{2},0\right)-\alpha\phi\left(-\frac{\epsilon}{2},0\right)=0$. Thus,

$$0<\int_{\partial D_{L0}\cup\partial D_{R0}}\left(\widetilde{\phi}-\alpha\phi\right)\partial_\nu\phi\ ds\le C_2\sqrt{\epsilon}\int_{\partial D_{L0}\cup\partial D_{R0}}x\partial_\nu\phi\ ds,$$

since $\partial_\nu\phi>0$ on ∂D_{R0} and $\partial_\nu\phi<0$ on ∂D_{L0} by the Hopf lemma. This implies (3.13).

Second, the positivity of $1-\alpha$ is derived simply from (3.13). We note that

$$\left(\widetilde{\phi}-\alpha\phi+\widetilde{v}\right)\Big|_{\partial D_{R0}}=-\left(\widetilde{\phi}-\alpha\phi+\widetilde{v}\right)\Big|_{\partial D_{L0}}=(1-\alpha)\phi\Big|_{\partial D_{R0}}$$

and $\phi\Big|_{\partial D_{R0}}>0$. It follows immediately from (3.13) that

$$1-\alpha>0.$$

Third, we consider $\int_{\partial D_{L0}\cup\partial D_{R0}}x\partial_\nu\phi\ ds$ to estimate $1-\alpha$. The inequality (3.13) implies

$$\int_{\partial D_{L0}\cup\partial D_{R0}}x\partial_\nu\phi\ ds>0.$$

The integral can be decomposed into two terms as

$$\int_{\partial D_{L0}\cup\partial D_{R0}}x\partial_\nu\phi\ ds=\alpha\int_{\partial D_{L0}\cup\partial D_{R0}}x\partial_\nu\phi\ ds+\int_{\partial D_{L0}\cup\partial D_{R0}}x\partial_\nu\left(\widetilde{\phi}-\alpha\phi+\widetilde{v}\right)\ ds. \tag{3.14}$$

By Lemma 3.3, the boundedness of $\int_{\Omega\setminus\overline{D_{L0}\cup D_{R0}}}\left|\nabla\left(\widetilde{\phi}-\alpha\phi+\widetilde{v}\right)\right|^2dxdy$ implies the existence of a constant c_1 such that $\left(\widetilde{\phi}-\alpha\phi+\widetilde{v}\right)(x,y)$ converges to a constant c_1, or $-c_1$, as x approaches ∞ or $-\infty$, respectively, and also show that $\partial_x\left(\widetilde{\phi}-\alpha\phi+\widetilde{v}\right)(x,y)$ shrinks exponentially fast, to 0, as $|x|$ approaches ∞. The integration by parts yields

$$\left|\int_{\partial D_{L0}\cup\partial D_{R0}}x\partial_\nu\left(\widetilde{\phi}-\alpha\phi+\widetilde{v}\right)\ ds\right|$$
$$=\left|\lim_{l\to\infty}\int_{\partial D_{L0}\cup\partial D_{R0}\cup(\{x|x=\pm l\}\times(-1-\delta/2,\ 1+\delta/2))}\partial_\nu x\left(\widetilde{\phi}-\alpha\phi+\widetilde{v}\right)\ ds\right|$$
$$=\lim_{l\to\infty}\left|\int_{\{l\}\times(-1-\delta/2,\ 1+\delta/2)}\widetilde{\phi}-\alpha\phi+\widetilde{v}\ ds-\int_{\{-l\}\times(-1-\delta/2,\ 1+\delta/2)}\widetilde{\phi}-\alpha\phi+\widetilde{v}\ ds\right|,$$

since $\widetilde{\phi}-\alpha\phi+\widetilde{v}$ is constant on ∂D_{L0} and on ∂D_{R0}, respectively. Note that $\widetilde{\phi}-\alpha\phi+\widetilde{v} = (1-\alpha)\phi$, $(1-\alpha)>0$, $\phi(x,y)=-\phi(-x,y)$ and $\phi=\frac{1}{c_0}\phi_R$ for $(x,y)\in\Omega_R\setminus\overline{D_{R0}}$, where ϕ_R and $c_0>0$ are defined in Lemma 3.5 and Proposition 3.4, respectively. The maximum principle in Lemma 3.5 and (3.13) yield

$$\begin{aligned}
&\left|\int_{\partial D_{L0}\cup\partial D_{R0}} x\partial_\nu\left(\widetilde{\phi}-\alpha\phi+\widetilde{v}\right)\,ds\right| \\
&= 2\lim_{l\to\infty}\left|\int_{\{l\}\times(-1-\delta/2,\;1+\delta/2)}\widetilde{\phi}-\alpha\phi+\widetilde{v}\;ds\right| \\
&\le (2+\delta)\left(\left(\widetilde{\phi}-\alpha\phi+\widetilde{v}\right)\Big|_{\partial D_{R0}}-\left(\widetilde{\phi}-\alpha\phi+\widetilde{v}\right)\Big|_{\partial D_{L0}}\right) \\
&\le 3\int_{\partial D_{L0}\cup\partial D_{R0}}\left(\widetilde{\phi}-\alpha\phi\right)\partial_\nu\phi\;ds \\
&\le 3C_2\sqrt{\epsilon}\int x\partial_\nu\phi ds,
\end{aligned}$$

since $\delta<1$. Applying this bound to the decompostion (3.14),

$$0<(1-\alpha)\int_{\partial D_{L0}\cup\partial D_{R0}} x\partial_\nu\phi\;ds \le 3C_2\sqrt{\epsilon}\int_{\partial D_{L0}\cup\partial D_{R0}} x\partial_\nu\phi\;ds.$$

Since $\int_{\partial D_{L0}\cup\partial D_{R0}} x\partial_\nu\phi\;ds>0$, we are done. □

Now, we take the last step to prove Proposition 3.2. Calculationg $\widetilde{\phi}(x,y)$ directly from (3.10), there exists a positive constant C_* regardless of ϵ and δ such that

$$\frac{1}{C_*}\sqrt{\epsilon}\le\tilde{\phi}(x,y)\le C_*\sqrt{\epsilon}$$

for any $(x,y)\in\partial D_{R0}$ containing $\left(\frac{\epsilon}{2},0\right)$. The definition and symmetric property imply $\alpha=\frac{\tilde{\phi}(\frac{\epsilon}{2},0)}{\phi|_{\partial D_{R0}}}$, and Lemma 3.8 yields $\frac{1}{2}\le\alpha\le 2$. Thus, there is a constant $C_{**}>0$ regardless of ϵ and δ such that

$$\frac{1}{C_{**}}\widetilde{\phi}\le\phi\le C_{**}\widetilde{\phi}\quad\text{ on }\partial D_{R0}.$$

By Lemma 3.7, this inequality on the boundary can be extended into $\Omega_R\setminus\overline{D_{R0}}$ so that

$$\frac{1}{C_{**}}\widetilde{\phi}\le\phi\le C_{**}\widetilde{\phi}\quad\text{ in }\Omega_R\setminus\overline{D_{R0}}.$$

By the divergence theorem,

$$\int_{\partial D_{R0}} x\partial_\nu\phi ds=\lim_{l\to\infty}\int_{-1-\frac{\delta}{2}}^{1+\frac{\delta}{2}}\partial_\nu x\phi(l,y)dy\;=\;\lim_{l\to\infty}\int_{-1-\frac{\delta}{2}}^{1+\frac{\delta}{2}}\phi(l,y)dy$$

whose value is intermediate between

$$\frac{1}{C_{**}}\lim_{l\to\infty}\int_{-1-\frac{\delta}{2}}^{1+\frac{\delta}{2}}\widetilde{\phi}(l,y)dy\text{ and }C_{**}\lim_{l\to\infty}\int_{-1-\frac{\delta}{2}}^{1+\frac{\delta}{2}}\widetilde{\phi}(l,y)dy.$$

By Lemma 3.7,

$$\frac{1}{C_{**}}(2+\delta)\inf_{(x,y)\in\partial D_{R0}}\widetilde{\phi}(x,y)\le\int_{\partial D_{R0}} x\partial_\nu\phi ds\le C_{**}(2+\delta)\sup_{(x,y)\in\partial D_{R0}}\widetilde{\phi}(x,y).$$

Thus,

$$(2+\delta)\frac{1}{C_*C_{**}}\sqrt{\epsilon} \le \int_{\partial D_{R0}} x\partial_\nu \phi ds \le (2+\delta)C_*C_{**}\sqrt{\epsilon}.$$

By the symmetric property, $\int_{\partial D_{L0}} x\partial_\nu\phi ds = \int_{\partial D_{R0}} x\partial_\nu \phi ds$. Therefore, the equality (3.4) implies the desirable result in Proposition 3.2. □

4. Proof of Theorem 1.1. The proofs (1.8) and (1.9) only are presented in this proof, since the upper bounds (1.6) and (1.7) follow immediately from (1.8) and (1.9). Now, we begin by defining the domains as

$$\widetilde{\Omega} = \mathbb{R}\times\left(-1-\frac{1}{2}\delta, 3+\frac{3}{2}\delta\right), \qquad \widetilde{\Omega}_4 = (-4,4)\times\left(-1-\frac{1}{2}\delta, 3+\frac{3}{2}\delta\right),$$

$$\widetilde{\Omega}_{4R} = (0,4)\times\left(-1-\frac{1}{2}\delta, 3+\frac{3}{2}\delta\right), \qquad \widetilde{\Omega}_{4L} = (-4,0)\times\left(-1-\frac{1}{2}\delta, 3+\frac{3}{2}\delta\right),$$

which are used in this proof. We assume that H is a harmonic function with a periodic gradient as (1.1) and u is a solution to (1.2) with the condition (1.3).

LEMMA 4.1. There exists a constant C regardless of δ and ϵ such that

$$\left|u\big|_{\partial D_{R0}} - u\big|_{\partial D_{L0}}\right| = \left|u\big|_{\partial D_{R1}} - u\big|_{\partial D_{L1}}\right| \le C\|H\|_{L^\infty(\Omega_4)}\sqrt{\epsilon}.$$

Proof. Let $\widetilde{u}$ and r be defined by

$$\widetilde{u}(x,y) = \frac{1}{2}\left(u(x,y)+u(x,-y)\right) \text{ and } r(x,y) = \frac{1}{2}\left(u(x,y)-u(x,-y)\right)$$

and let

$$\widetilde{H}(x,y) = \frac{1}{2}\left(H(x,y)+H(x,-y)\right).$$

Since $u = \widetilde{u}+r$ and $r|_{\partial D_{L0}} = r|_{\partial D_{R0}} = 0$, Proposition 3.4 implies

$$\begin{aligned} u|_{\partial D_{R0}} - u|_{\partial D_{L0}} &= \widetilde{u}|_{\partial D_{R0}} - \widetilde{u}|_{\partial D_{L0}} \\ &= \int_{\partial D_{L0}\cup\partial D_{R0}} \widetilde{H}\partial_\nu\phi ds \\ &= \int_{\partial D_{L0}\cup\partial D_{R0}} \left(\widetilde{H}(x,y)-\widetilde{H}(0,0)\right)\partial_\nu\phi ds. \end{aligned}$$

It follows from $\partial_y \widetilde{H}(0,0) = 0$ that

$$|\widetilde{H}(x,y)-\widetilde{H}(0,0)| \le C\left(\|\nabla H\|_{L^\infty(\Omega_3)} + \|D^2H\|_{L^\infty(\Omega_3)}\right)|x|$$

for any $(x,y)\in\partial D_{L0}\cup\partial D_{R0}$. Thus,

$$|u|_{\partial D_{R0}} - u|_{\partial D_{L0}}| \le C\left(\|\nabla H\|_{L^\infty(\Omega_3)} + \|D^2H\|_{L^\infty(\Omega_3)}\right)\int_{\partial D_{L0}\cup\partial D_{R0}} x\partial_\nu\phi ds.$$

Here, the standard gradient estimate for harmonic functions implies that

$$\|\nabla H\|_{L^\infty(\Omega_3)} + \|D^2H\|_{L^\infty(\Omega_3)} \le C\|H\|_{L^\infty((-4,4)\times(-3,3))}$$

for some $C>0$, since the domain Ω_3 has nonzero distance from $\partial((-4,4)\times(-3,3))$, and the periodic property of ∇H or (3.2) imply

$$\|H\|_{L^\infty((-4,4)\times(-3,3))} \le 2\|H\|_{L^\infty(\Omega_4)}.$$

Thus, we are done. □

The following lemma provides some maximal principles more general than Lemma 3.7.

LEMMA 4.2. Let Ω_R be as defined in the proof of Proposition 3.2. Assume that ρ_{00}, ρ_{10}, ρ_{01} and ρ_{11} are harmonic functions defined on $\Omega_R\setminus\overline{D_{R0}}$ with the boundary conditions:

$$\begin{cases}\partial_\nu^i\rho_{ij}=0 & \text{on } (0,\infty)\times\left\{y \;\middle|\; y=-1-\frac{\delta}{2} \text{ or } y=1+\frac{\delta}{2}\right\},\\ \partial_\nu^j\rho_{ij}=0 & \text{on } \{0\}\times\left(-1-\frac{\delta}{2},\ 1+\frac{\delta}{2}\right),\end{cases}$$

and also satisfying

$$\int_{\Omega_R\setminus\overline{D_{R0}}} |\nabla\rho_{ij}|^2\,dxdy<\infty$$

for any $i,\ j=0,\ 1$, where $\partial_\nu^0 u=u$ and $\partial_\nu^1 u=\partial_\nu u$. Then,

$$\|\rho_{ij}\|_{L^\infty(\Omega_R\setminus\overline{D_{R0}})} \le \|\rho_{ij}\|_{L^\infty(\partial D_{R0})}.$$

This lemma can also be derived easily by the same function $\Phi_R : B_1^+(0,0)\to\Omega_R$ used in Lemmas 3.5 and 3.7, the maximum principle and the Hopf lemma. The proof of this lemma is left as an exercise for the readers.

LEMMA 4.3. Let a_* be the constant defined as

$$a_* = u\left(-\frac{\epsilon}{2},0\right) - H\left(-\frac{\epsilon}{2},0\right) = u\big|_{\partial D_{L0}} - H\left(-\frac{\epsilon}{2},0\right).$$

Then,

$$\|u-H-a_*\|_{L^\infty\left(\widetilde{\Omega}\setminus\overline{D_{L0}\cup D_{L1}\cup D_{R0}\cup D_{R1}}\right)} \le C\|H\|_{L^\infty(\Omega_4)}$$

and

$$\|u-u\big|_{\partial D_{L0}}\|_{L^\infty\left(\widetilde{\Omega}_4\setminus\overline{D_{L0}\cup D_{L1}\cup D_{R0}\cup D_{R1}}\right)} \le C\|H\|_{L^\infty(\Omega_4)}.$$

Proof. We define the notation $(\cdot)_e$ and $(\cdot)_o$ as follows:

$$(v)_e(x,y)=\frac{1}{2}\left(v(x,y)+v(x,-y)\right)$$

and

$$(v)_o(x,y)=\frac{1}{2}\left(v(x,y)-v(x,-y)\right)$$

for a function v defined in $\overline{\Omega}\setminus(D_{L0}\cup D_{R0})$. Then,

$$u-H=(u-H)_e+(u-H)_o.$$

First, we estimate $\|(u-H)_e - a_*\|_{L^\infty(\Omega\setminus\overline{D_{L0}\cup D_{R0}})}$. For any $(x,y)\in\partial D_{L0}$,

$$\begin{aligned}&(u-H)_e(x,y)-a_*\\&=(u-H)_e(x,y)-u\big|_{\partial D_{L0}}+H\left(-\frac{\epsilon}{2},0\right)\\&=(u)_e(x,y)-u\big|_{\partial D_{L0}}-(H)_e(x,y)+(H)_e\left(-\frac{\epsilon}{2},0\right)\\&=-(H)_e(x,y)+(H)_e\left(-\frac{\epsilon}{2},0\right),\end{aligned}$$

and for any $(x,y)\in\partial D_{R0}$,

$$\begin{aligned}&(u-H)_e(x,y)-a_*\\&=(u-H)_e(x,y)-u\big|_{\partial D_{L0}}+H\left(-\frac{\epsilon}{2},0\right)\\&=u\big|_{\partial D_{R0}}-u\big|_{\partial D_{L0}}-(H)_e(x,y)+(H)_e\left(-\frac{\epsilon}{2},0\right).\end{aligned}$$

Thus, the equality (3.4) implies that

$$\|(u-H)_e-a_*\|_{L^\infty(\partial D_{L0}\cup\partial D_{R0})}\le 4\|H\|_{L^\infty(\partial D_{L0}\cup\partial D_{R0})}. \tag{4.1}$$

Note that $(u-H)_e-a_*$ is a harmonic function in $\Omega\setminus\overline{D_{L0}\cup D_{R0}}$ with

$$\begin{cases}\partial_\nu\left((u-H)_e-a_*\right)=0 & \text{on } (-\infty,\infty)\times\left\{y\;\middle|\; y=-1-\frac{\delta}{2}\text{ or } y=1+\frac{\delta}{2}\right\},\\ \displaystyle\int_{\Omega_R\setminus\overline{D_{L0}\cup D_{R0}}}\left|\nabla\left((u-H)_e-a_*\right)\right|^2 & dxdy<\infty,\end{cases}$$

due to a periodic property of ∇u and ∇H. Applying $(u-H)_e-a_*$ to $\rho_{10}+\rho_{11}$ in Lemma 4.2, the bound (4.1) implies

$$\|(u-H)_e-a_*\|_{L^\infty(\Omega\setminus\overline{D_{L0}\cup D_{R0}})}\le 4\|H\|_{L^\infty(\partial D_{L0}\cup\partial D_{R0})}.$$

Second, we estimate $\|(u-H)_o\|_{L^\infty(\Omega\setminus\overline{D_{L0}\cup D_{R0}})}$. On $\partial D_{L0}\cup\partial D_{R0}$,

$$(u-H)_o=-(H)_o,$$

since u is constant on ∂D_{L0} and ∂D_{R0}, respectively. Meanwhile, the equality (3.1) in the proof of Proposition 3.1 means that the harmonic function $(u-H)_o$ satisfies

$$(u-H)_o(x,y)=0 \text{ on } (-\infty,\infty)\times\left\{y\;\middle|\; y=-1-\frac{\delta}{2}\text{ or } y=1+\frac{\delta}{2}\right\}.$$

Since $(u-H)_o=-(H)_o$ on $\partial D_{L0}\cup\partial D_{R0}$ and $\int_{\Omega_R\setminus\overline{D_{L0}\cup D_{R0}}}|\nabla(u-H)_o|^2\,dxdy<\infty$, the results on ρ_{00} and ρ_{01} in Lemma 4.2 yield

$$\|(u-H)_o\|_{L^\infty(\Omega\setminus\overline{D_{L0}\cup D_{R0}})}\le\|(u-H)_o\|_{L^\infty(\partial D_{L0}\cup\partial D_{R0})}\le\|H\|_{L^\infty(\partial D_{L0}\cup\partial D_{R0})}.$$

Combining the first and second cases,

$$\|u-H-a_*\|_{L^\infty(\Omega\setminus\overline{D_{L0}\cup D_{R0}})}\le 5\|H\|_{L^\infty(\partial D_{L0}\cup\partial D_{R0})}.$$

The first inequality in this lemma can be derived by (3.1) as follows:

$$\begin{aligned}&\|u - H - a_*\|_{L^\infty(\widetilde{\Omega}\setminus\overline{D_{L0}\cup D_{L1}\cup D_{R0}\cup D_{R1}})}\\&= \|u - H - a_*\|_{L^\infty(\Omega\setminus\overline{D_{L0}\cup D_{R0}})}\\&\le 5\|H\|_{L^\infty(\partial D_{L0}\cup\partial D_{R0})},\end{aligned}$$

where $\widetilde{\Omega}$ is defined in the beginning of this section. The second inequality also follows immediately so that

$$\begin{aligned}&\|u - u|_{\partial D_{L0}}\|_{L^\infty(\widetilde{\Omega}_4\setminus\overline{D_{L0}\cup D_{L1}\cup D_{R0}\cup D_{R1}})}\\&\le \|u - H - a_*\|_{L^\infty(\widetilde{\Omega}\setminus\overline{D_{L0}\cup D_{L1}\cup D_{R0}\cup D_{R1}})} + \left\|H - H\left(-\frac{\epsilon}{2}, 0\right)\right\|_{L^\infty(\widetilde{\Omega}_4\setminus\overline{D_{L0}\cup D_{L1}\cup D_{R0}\cup D_{R1}})}\\&\le 5\|H\|_{L^\infty(\partial D_{L0}\cup\partial D_{R0})} + 2\|H\|_{L^\infty(\widetilde{\Omega}_4)}\\&\le 7\|H\|_{L^\infty(\widetilde{\Omega}_4)}\\&\le 14\|H\|_{L^\infty(\Omega_4)},\end{aligned}$$

due to (3.2). We are done. □

4.1. *Proof of* (1.8). The potential difference $u|_{D_{R1}} - u|_{D_{R0}}$ was evaluated exactly in Proposition 3.1. The value has a very different nature from the cases of finite number of inclusions, and also results in much stronger concentration than finite cases. In this proof, we establish an asymptote of ∇u from the potential difference. Indeed, a nice method to get an asymptote was already introduced by Kang, Lim and Yun in the case of two circular inclusions in [11], and Bao, Li and Yin in [5] showed the boundedness of the gradient in the case of no potential difference. In this proof, we modify these methods to apply to our problem, and obtain an asymptote describing the stronger concentration. Hence, the potential difference evaluated in Proposition 3.1 plays the most important role in the result.

To establish the asymptote, we consider the decomposition of ∇u into two terms as

$$\nabla u = \alpha_h \nabla\phi_h + \nabla u_h,$$

where α_h, ϕ_h and u_h are defined below. The function ϕ_h has a high concentration in between D_{R0} and D_{R1}, and is also easy to handle. In this proof, we estimate the coefficient α_h and show that ∇u_h is bounded regardless of ϵ and δ. Thus, we can establish the desirable asymptote (1.8).

We define α_h, ϕ_h and u_h and set the decomposition up. Let $\phi_h(x, y)$ be the unique solution to

$$\begin{cases}\Delta\phi_h = 0 & \text{in } \mathbb{R}^2\setminus\overline{D_{R0}\cup D_{R1}},\\ \phi_h = \text{ a constant} & \text{on } \partial D_{R1},\\ \phi_h = -\phi_h\Big|_{\partial D_{R1}} & \text{on } \partial D_{R0},\\ \displaystyle\int_{\partial D_{R1}}\partial_\nu\phi_h ds = -\int_{\partial D_{R0}}\partial_\nu\phi_h ds & = \frac{2\pi}{\sqrt{\delta}},\\ \phi_h(\mathbf{x}) = O\left(\dfrac{1}{|\mathbf{x}|}\right) & \text{as } |\mathbf{x}|\to\infty,\end{cases} \tag{4.2}$$

in the same as (3.9) and (3.10). The solution can be expressed as

$$\phi_h(x,y) = \frac{1}{\sqrt{\delta}}\left(\log\left|(x,y)-\left(1+\frac{\epsilon}{2}, 1+\frac{\delta}{2}-p_h\right)\right| - \log\left|(x,y)-\left(1+\frac{\epsilon}{2}, 1+\frac{\delta}{2}+p_h\right)\right|\right)$$

where

$$p_h = \sqrt{\delta} + O(\delta)$$

for small δ. Let α_h be the constant as

$$\alpha_h = \frac{u|_{\partial D_{R1}} - u|_{\partial D_{R0}}}{\phi_h|_{\partial D_{R1}} - \phi_h|_{\partial D_{R0}}},$$

and we define a harmonic function u_h as $u_h = u - \alpha_h\phi_h - (u-\alpha_h\phi_h)|_{\partial D_{R0}}$. The solution u is decomposed into $\alpha_h\phi_h + (u-\alpha_h\phi_h)|_{\partial D_{R0}}$ and u_h as follows:

$$u = (\alpha_h\phi_h + (u-\alpha_h\phi_h)|_{\partial D_{R0}}) + u_h.$$

Hence,

$$\nabla u = \alpha_h \nabla\phi_h + \nabla u_h.$$

From the defintion of α_h, two functions $\alpha_h\phi_h$ and u have the same potential difference between ∂D_{R1} and ∂D_{R0}, and u_h has no difference between the boundaries so that $\alpha_h\phi_h|_{\partial D_{R1}} - \alpha_h\phi_h|_{\partial D_{R0}} = u|_{\partial D_{R1}} - u|_{\partial D_{R0}}$ and $u_h|_{\partial D_{R1}} - u_h|_{\partial D_{R0}} = 0$. Indeed, this means that ∇u is dominated by $\alpha_h\nabla\phi_h$. By direct calculation and the definition of N_h, there is a constant C regardless of ϵ and δ such that

$$\left|\nabla\phi_h - 2\left(-2\,\frac{\left(x-1-\frac{\epsilon}{2}\right)\left(y-1-\frac{\delta}{2}\right)}{\left(\left(x-1-\frac{\epsilon}{2}\right)^2+\delta\right)^2}, \frac{1}{\delta+\left(x-1-\frac{\epsilon}{2}\right)^2}\right)\right| \le C$$

and

$$\left|2\sqrt{\delta}\,\frac{\left(x-1-\frac{\epsilon}{2}\right)\left(y-1-\frac{\delta}{2}\right)}{\left(\left(x-1-\frac{\epsilon}{2}\right)^2+\delta\right)^2}\right| = |S(x,y)|$$

$$\le \left|\frac{y-1-\frac{\delta}{2}}{\left(x-1-\frac{\epsilon}{2}\right)^2+\delta}\right| \le 2\left|\frac{\left(x-1-\frac{\epsilon}{2}\right)^2}{\left(x-1-\frac{\epsilon}{2}\right)^2+\delta}\right| \le 2$$

in N_h. By Proposition 3.1, direct calculation of ϕ_h implies

$$\left|\alpha_h - \frac{1}{2}\left(H(0,1)-H(0,-1)\right)\right| \le C\delta\|\nabla H\|_{L^\infty(\Omega_3)}.$$

By (3.2), a standard gradient estimate for harmonic functions yields $\|\nabla H\|_{L^\infty(\Omega_3)} \le C\|H\|_{L^\infty(\widetilde{\Omega}_4)} \le 3C\|H\|_{L^\infty(\Omega_4)}$ in the same way as the proof of Lemma 4.1. Hence,

$$\left|\alpha_h - \frac{1}{2}\left(H(0,1)-H(0,-1)\right)\right| \le C\delta\|H\|_{L^\infty(\Omega_4)}. \tag{4.3}$$

The remainder of the proof is deduced only to prove the boundedness of ∇u_h such that

$$\|\nabla u_h\|_{L^\infty(N_h)} \le C\|H\|_{L^\infty(\Omega_4)}$$

for some constant C. Then, we obtain the desirable (1.8). Some properties of u_h are considered before proving the boundedness. From the defintion of α_h,

$$u_h|_{\partial D_{R1}} - u_h|_{\partial D_{R0}} = 0,$$

$$\alpha_h\phi_h|_{\partial D_{R1}} - \alpha_h\phi_h|_{\partial D_{R0}} = u|_{\partial D_{R1}} - u|_{\partial D_{R0}},$$

and $|\alpha_h| \le 2\|H\|_{L^\infty(\Omega_4)}$ by (4.3). Lemma 4.3 implies that

$$\begin{aligned}
&\|u_h\|_{L^\infty\left(\widetilde{\Omega}_{4R}\setminus\overline{D_{R0}\cup D_{R1}}\right)} \\
&\le \|u - u|_{\partial D_{R0}}\|_{L^\infty\left(\widetilde{\Omega}_{4R}\setminus\overline{D_{R0}\cup D_{R1}}\right)} + \left|\alpha_h\phi_h\big|_{\partial D_{R0}}\right| + \|\alpha_h\phi_h\|_{L^\infty\left(\widetilde{\Omega}_{4R}\setminus\overline{D_{R0}\cup D_{R1}}\right)} \\
&\le C\|H\|_{L^\infty(\Omega_4)}.
\end{aligned} \tag{4.4}$$

From definition,

$$u_h = 0 \text{ on } \partial D_{R0} \cup \partial D_{R1}. \tag{4.5}$$

Dealing with the boundedness of ∇u_h, we decompose u_h into two functions u_+ and u_- as

$$u_h = u_+ + u_-,$$

where u_+ and u_- are the harmonic functions given as

$$\begin{cases}
\triangle u_+ = \triangle u_- = 0 & \text{in } \widetilde{\Omega}_{4R}\setminus\overline{D_{R0}\cup D_{R1}}, \\
u_+\big|_{\partial D_{R0}\cup\partial D_{R1}} = u_-\big|_{\partial D_{R0}\cup\partial D_{R1}} = 0, & \\
u_+(x,y) = \max\{u_h(x,y),0\} \ge 0 & \text{for any } (x,y)\in\partial\widetilde{\Omega}_{4R}, \\
u_-(x,y) = \min\{u_h(x,y),0\} \le 0 & \text{for any } (x,y)\in\partial\widetilde{\Omega}_{4R}.
\end{cases}$$

Then,

$$u_+ \ge 0 \text{ and } u_- \le 0 \text{ in } \widetilde{\Omega}_{4R}\setminus\overline{D_{R0}\cup D_{R1}}.$$

In order to derive the boundedness of ∇u_+ in N_h, we estimate ∇u_+ on the boundary ∂N_h which consists of four curves $\partial D_{R1}\cap\partial N_h$, $\partial D_{R0}\cap\partial N_h$, $\left\{1+\frac{\sqrt{3}}{2}+\frac{\epsilon}{2}\right\}\times[\frac{1}{2},\frac{3}{2}+\delta]$ and $\left\{1-\frac{\sqrt{3}}{2}+\frac{\epsilon}{2}\right\}\times[\frac{1}{2},\frac{3}{2}+\delta]$. We use u_{+0} and u_{+1} defined as

$$\begin{cases}
\triangle u_{+0} = 0 & \text{in } \widetilde{\Omega}_{4R}\setminus\overline{D_{R0}}, \\
u_{+0} = u_+ & \text{on } \partial\widetilde{\Omega}_{4R}, \\
u_{+0} = 0 & \text{on } \partial D_{R0},
\end{cases}
\quad\text{and}\quad
\begin{cases}
\triangle u_{+1} = 0 & \text{in } \widetilde{\Omega}_{4R}\setminus\overline{D_{R1}}, \\
u_{+1} = u_+ & \text{on } \partial\widetilde{\Omega}_{4R}, \\
u_{+1} = 0 & \text{on } \partial D_{R1}.
\end{cases}$$

It follows from definitions and (4.4) that

$$\begin{aligned}
&\|u_{+0}\|_{L^\infty(\widetilde{\Omega}_{4R}\setminus\overline{D_{R0}})} + \|u_{+1}\|_{L^\infty(\widetilde{\Omega}_{4R}\setminus\overline{D_{R1}})} \\
&\le 2\|u_h\|_{L^\infty(\widetilde{\Omega}_{4R}\setminus\overline{D_{R0}\cup D_{R1}})} \le C\|H\|_{L^\infty(\Omega_4)}.
\end{aligned} \tag{4.6}$$

Since $u_{+0} - u_+ = 0$ on $\partial\widetilde{\Omega}_{4R}\cup\partial D_{R0}$ and $u_{+0} - u_+ \ge 0$ on ∂D_{R1},

$$0 \le u_+ \le u_{+0} \ \text{ in } \widetilde{\Omega}_{4R}\setminus\overline{D_{R0}\cup D_{R1}}. \tag{4.7}$$

Since $u_{+0} - u_+ = u_+ = 0$ on ∂D_{R0}, the functions $u_{+0} - u_+$ and u_+ attain the minimal value 0 on ∂D_{R0}. Hopf's lemma thus implies that $0 \ge \partial_\nu u_+ \ge \partial_\nu u_{+0}$. Thus,

$$0 \le |\nabla u_+| \le |\partial_\nu u_{+0}| \ \text{ on } \partial D_{R0}, \tag{4.8}$$

and similarly

$$0 \le |\nabla u_+| \le |\partial_\nu u_{+1}| \text{ on } \partial D_{R1}. \tag{4.9}$$

Since $u_{+0} = 0$ on ∂D_{R0} and $u_{+1} = 0$ on ∂D_{R1}, the Kelvin transform can extend the functions u_{+0}, and u_{+1}, into harmonic functions $\widetilde{u}_{+0}$, and $\widetilde{u}_{+1}$, defined open sets containing ∂D_{R0}, and ∂D_{R1}, respectively. For any $(x_0, y_0) \in \partial D_{R0} \cap \partial N_h$, the extended function $\tilde{u}_{+0}$ is defined in $B_{\frac{1}{8}}(x_0, y_0)$. A gradient estimate for harmonic functions and (4.6) yield

$$|\nabla u_{+0}(x_0, y_0)| = |\nabla \widetilde{u}_{+0}(x_0, y_0)| \le C_1 \left(\frac{1}{8}\right)^{-1} \sup_{B_{\frac{1}{8}}(x_0,y_0)} |\tilde{u}_{+0}(x, y)|$$
$$\le C_2 \|u_{+0}\|_{L^\infty(\widetilde{\Omega}_{4R} \setminus \overline{D_{R0}})} \le C_3 \|H\|_{L^\infty(\Omega_4)}.$$

Thus, (4.8) implies

$$\|\nabla u_+\|_{L^\infty(\partial D_{R0} \cap \partial N_h)} \le C \|H\|_{L^\infty(\Omega_4)}, \tag{4.10}$$

and in the same way, (4.6) and (4.9) yield

$$\|\nabla u_+\|_{L^\infty(\partial D_{R1} \cap \partial N_h)} \le C \|H\|_{L^\infty(\Omega_4)}. \tag{4.11}$$

Hence, we have the upper bounds for $|\nabla u_+|$ on each boundary $\partial D_{R1} \cap \partial N_h$ and $\partial D_{R0} \cap \partial N_h$ as above. Meanwhile, we estimate $|\nabla u_+|$ on two vertical line segments $\left\{1 + \frac{\sqrt{3}}{2} + \frac{\epsilon}{2}\right\} \times [\frac{1}{2}, \frac{3}{2} + \delta]$ and $\left\{1 - \frac{\sqrt{3}}{2} + \frac{\epsilon}{2}\right\} \times [\frac{1}{2}, \frac{3}{2} + \delta]$ which are the remainder boundaries of ∂N_h. Since $u_+ = 0$ on $\partial D_{R0} \cup \partial D_{R1}$, the Kelvin transform extends u_+ to a harmonic function $\tilde{u}_+$ defined in an open set containing $\partial(D_{R0} \cup D_{R1})$ as well as $\widetilde{\Omega}_{4R} \setminus \overline{(D_{R0} \cup D_{R1})}$. For any point (x_0, y_0) on the vertical line segments above, the extended harmonic function $\tilde{u}_+$ is defined in the open disk $B_{\frac{1}{8}}(x_0, y_0)$. A gradient estimate for harmonic functions and (4.7) thus yield

$$|\nabla u_+(x_0, y_0)| = |\nabla \tilde{u}_+(x_0, y_0)| \le C_1 \left(\frac{1}{8}\right)^{-1} \sup_{B_{\frac{1}{8}}(x_0,y_0)} |\tilde{u}_+(x, y)|$$
$$\le C_2 \|u_+\|_{L^\infty(\widetilde{\Omega}_{4R} \setminus \overline{D_{R0} \cup D_{R1}})} \le C_2 \|u_{+0}\|_{L^\infty(\widetilde{\Omega}_{4R} \setminus \overline{D_{R0}})}. \tag{4.12}$$

By the definitions of u_{+0} and u_{+1}, and by (4.4), $\|u_{+0}\|_{L^\infty(\widetilde{\Omega}_{4R} \setminus \overline{D_{R0}})} + \|u_{+1}\|_{L^\infty(\widetilde{\Omega}_{4R} \setminus \overline{D_{R1}})} \le C_3 \|H\|_{L^\infty(\Omega_4)}$. Hence, (4.10), (4.11) and (4.12) result in a gradient estimate on the boundary ∂N_h as

$$\|\nabla u_+\|_{L^\infty(\partial N_h)} \le C_4 \|H\|_{L^\infty(\Omega_4)}.$$

By the maximal principle,

$$\|\nabla u_+\|_{L^\infty(N_h)} \le C_4 \|H\|_{L^\infty(\Omega_4)}.$$

In the same way, we also get

$$\|\nabla u_-\|_{L^\infty(N_h)} \le C_5 \|H\|_{L^\infty(\Omega_4)}.$$

Therefore, we obtain

$$\|\nabla u_h\|_{L^\infty(N_h)} \le C_6 \|H\|_{L^\infty(\Omega_4)}.$$

We are done. □

4.2. *Proof of* (1.9). An estimate for the potential difference $u|_{\partial D_{R0}} - u|_{\partial D_{L0}}$ was obtained in Lemma 4.1. We repeat the same method as the proof of (1.8) to establish the asymptote from the potential difference. Thus, this proof also begins at the decomposition as

$$u = \beta_v \nabla \phi_v + \nabla u_v.$$

Here, ϕ_v is the unique solution to

$$\begin{cases} \Delta \phi_v = 0 & \text{in } \mathbb{R}^2 \setminus \overline{D_{L0} \cup D_{R0}}, \\ \phi_v = \text{ a constant} & \text{on } \partial D_{R0}, \\ \phi_v = -\phi_v\Big|_{\partial D_{R0}} & \text{on } \partial D_{L0}, \\ \displaystyle\int_{\partial D_{R0}} \partial_\nu \phi_v ds = -\int_{\partial D_{L0}} \partial_\nu \phi_h ds & = 2\pi, \\ \phi_v(\mathbf{x}) = O\left(\dfrac{1}{|\mathbf{x}|}\right) & \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{4.13}$$

Then,

$$\phi_v(x, y) = \log |(x, y) + (p_v, 0)| - \log |(x, y) - (p_v, 0)|$$

where

$$p_v = \sqrt{\epsilon} + O(\epsilon).$$

Let β_v be the constant as

$$\beta_v = \frac{u|_{\partial D_{R0}} - u|_{\partial D_{L0}}}{\phi_v|_{\partial D_{R0}} - \phi_v|_{\partial D_{L0}}}$$

and we define a harmonic function u_v as

$$u_v = u - \beta_v \phi_v - (u - \beta \phi_v)|_{\partial D_{R0}}.$$

The solution u is decomposed into $\beta_v \phi_v + (u - \beta_v \phi_v)|_{\partial D_{R0}}$ and u_v as

$$u = (\beta_v \phi_v + (u - \beta_v \phi_v)|_{\partial D_{R0}}) + u_v.$$

Hence, we obtain the desirable decomposition

$$u = \beta_v \nabla \phi_v + \nabla u_v.$$

By direct calculation, there is a constant C regardless of ϵ and δ_v such that

$$\left| \nabla \phi_v - 2 \frac{\sqrt{\epsilon}}{\epsilon + y^2} (1, 0) \right| \le C,$$

and Lemma 4.1 implies $|\beta| \le 3\|H\|_{L^\infty(\Omega_4)}$. If we prove the boundedness of ∇u_v such that

$$\|\nabla u_v\|_{L^\infty(N_v)} \le C\|H\|_{L^\infty(\Omega_4)}$$

for some constant C, then we can obtain the main result (1.9).

The remainder of the proof is to prove the boundedness of ∇u_v. From the definition of β_v, we have $\beta_v \phi_v|_{\partial D_{R0}} - \beta_v \phi_v|_{\partial D_{L0}} = u|_{\partial D_{R0}} - u|_{\partial D_{L0}}$, $u_v|_{\partial D_{R0}} - u_v|_{\partial D_{L0}} = 0$ and $|\beta_v| \le 3\|H\|_{L^\infty(\Omega_4)}$ by Lemma 4.1. Similarly to (4.4), Lemma 4.3 implies that

$$\|u_v\|_{L^\infty\left(\Omega_4 \setminus \overline{D_{R0} \cup D_{L0}}\right)} \le C\|H\|_{L^\infty(\Omega_4)}. \tag{4.14}$$

From definition,

$$u_v = 0 \text{ on } \partial D_{L0} \cup \partial D_{R0}. \tag{4.15}$$

They are the conditions analogous to (4.4) and (4.5). In the same way as the proof of (1.8), we have

$$\|\nabla u_v\|_{L^\infty(N_v)} \le C\|H\|_{L^\infty(\Omega_4)}.$$

We are done. □

5. Proof of Theorem 1.2. The proof is mainly concerned with the fourth equality (1.13) in N_v, since the third equality (1.12) in N_h follows immediately from Theorem 1.1, and the bounds (1.10) and (1.11) are also derived easily from (1.13) and (1.12). Owing to the linearity problem, we consider two cases when $H(x, y) = x$, and when $H(x, y) = y$, separately. Let u_a and u_b be the solutions for $H(x, y) = x$ and $H(x, y) = y$, respectively.

In the first case when $H(x, y) = x$ for $(x, y) \in \mathbb{R}^2$, Theorem 1.1 presents a constant μ_0 satisfying

$$\nabla u_a(x, y) = \mu_0 \frac{\sqrt{\epsilon}}{\epsilon + y^2} (1, 0) + R_{a2}(x, y) \tag{5.1}$$

for $(x, y) \in N_v$, while $\|R_{a2}(x, y)\|_{L^\infty(N_v)}$ is bounded regardless of small $\epsilon > 0$ and $\delta > 0$. Proposition 3.2 provides a positive constant C_1 regardless of ϵ and δ such that

$$\frac{1}{C_1}\sqrt{\epsilon} \le u_a|_{D_{R0}} - u_a|_{D_{L0}} \le C_1\sqrt{\epsilon}.$$

By the Mean Value Theorem, there exists a point $(x_a, 0) \in N_v$ such that $-\frac{1}{2}\epsilon < x_a < \frac{1}{2}\epsilon$ and

$$\frac{1}{C_1}\frac{1}{\sqrt{\epsilon}} \le \partial_x u_a(x_a, 0) \le C_1 \frac{1}{\sqrt{\epsilon}}.$$

By (5.1), the coefficient μ_0 is bounded below as

$$\mu_0 \ge \frac{1}{C_1} - \sqrt{\epsilon}\, \|R_{a2}(x, y)\|_{L^\infty(N_v)} \ge \frac{1}{2}\frac{1}{C_1}$$

for small $\epsilon > 0$ due to the boundedness of $\|R_{a2}(x, y)\|_{L^\infty(N_v)}$. Theorem 1.1 provides an upper bound for μ_a so to obtain a constant $C_2 > 0$ satisfying

$$\frac{1}{C_2} \le \mu_0 \le C_2$$

regardless of ϵ and δ. Hence, we have the estimate (1.13) for ∇u_a in N_v with (1.14).

In the second case when $H(x, y) = y$ for $(x, y) \in \mathbb{R}^2$, it follows from Theorem 1.1 that

$$\nabla u_b(x, y) = \mu_b \frac{\sqrt{\epsilon}}{\epsilon + y^2}(1, 0) + R_{b2}(x, y) \text{ for any } (x, y) \in N_v \tag{5.2}$$

and for a proper constant μ_b, and $\|R_{b2}\|_{L^\infty(N_v)}$ is bounded regardless of ϵ and δ. By Proposition 3.4 for $H = y$,

$$\frac{1}{\sqrt{\epsilon}} \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \partial_x u_b(x, 0)dx = \frac{1}{\sqrt{\epsilon}} \left(u_b|_{\partial D_{R0}} - u_b|_{\partial D_{L0}}\right) = 0.$$

Applying (5.2) to here,

$$|\mu_b| \le C_3\sqrt{\epsilon}.$$

Applying the inequality to (5.2), there exists a constant C_4 regardless of ϵ and δ such that

$$\|\nabla u_b\|_{L^\infty(N_v)} \leq C_4.$$

Therefore, in the case when $H(x,y) = ax + by$ in $\mathbb{R}^2$, we have the desirable asymptote as

$$\begin{aligned}\nabla u &= a\nabla u_a + b\nabla u_b \\ &= a\mu_0 \frac{\sqrt{\epsilon}}{\epsilon + y^2}(1,0) + R_2\end{aligned}$$

in N_v and the remainder term R_2 is bounded, since

$$R_2 = aR_{a2} + b\nabla u_b + bR_{b2}. \qquad \square$$

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References

[1] Habib Ammari, Giulio Ciraolo, Hyeonbae Kang, Hyundae Lee, and Kihyun Yun, *Spectral analysis of the Neumann-Poincaré operator and characterization of the stress concentration in anti-plane elasticity*, Arch. Ration. Mech. Anal. **208** (2013), no. 1, 275–304, DOI 10.1007/s00205-012-0590-8. MR3021549

[2] Habib Ammari, Hyeonbae Kang, Hyundae Lee, Jungwook Lee, and Mikyoung Lim, *Optimal estimates for the electric field in two dimensions* (English, with English and French summaries), J. Math. Pures Appl. (9) **88** (2007), no. 4, 307–324, DOI 10.1016/j.matpur.2007.07.005. MR2384571

[3] Habib Ammari, Hyeonbae Kang, and Mikyoung Lim, *Gradient estimates for solutions to the conductivity problem*, Math. Ann. **332** (2005), no. 2, 277–286, DOI 10.1007/s00208-004-0626-y. MR2178063

[4] Ivo Babuška, Börje Andersson, Paul J. Smith, and Klas Levin, *Damage analysis of fiber composites. I. Statistical analysis on fiber scale*, Comput. Methods Appl. Mech. Engrg. **172** (1999), no. 1-4, 27–77, DOI 10.1016/S0045-7825(98)00225-4. MR1685902

[5] Ellen Shiting Bao, Yan Yan Li, and Biao Yin, *Gradient estimates for the perfect conductivity problem*, Arch. Ration. Mech. Anal. **193** (2009), no. 1, 195–226, DOI 10.1007/s00205-008-0159-8. MR2506075

[6] Ellen Shiting Bao, Yan Yan Li, and Biao Yin, *Gradient estimates for the perfect and insulated conductivity problems with multiple inclusions*, Comm. Partial Differential Equations **35** (2010), no. 11, 1982–2006, DOI 10.1080/03605300903564000. MR2754076

[7] Eric Bonnetier and Michael Vogelius, *An elliptic regularity result for a composite medium with "touching" fibers of circular cross-section*, SIAM J. Math. Anal. **31** (2000), no. 3, 651–677, DOI 10.1137/S0036141098333980. MR1745481

[8] B. Budiansky, G. F. Carrier, *High shear stresses in stiff fiber composites*, J. Appl. Mech. **51** (1984), 733–735.

[9] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983. MR737190

[10] Hyeonbae Kang, Hyundae Lee, and KiHyun Yun, *Optimal estimates and asymptotics for the stress concentration between closely located stiff inclusions*, Math. Ann. **363** (2015), no. 3-4, 1281–1306, DOI 10.1007/s00208-015-1203-2. MR3412359

[11] Hyeonbae Kang, Mikyoung Lim, and KiHyun Yun, *Asymptotics and computation of the solution to the conductivity equation in the presence of adjacent inclusions with extreme conductivities*, J. Math. Pures Appl. (9) **99** (2013), no. 2, 234–249, DOI 10.1016/j.matpur.2012.06.013. MR3007847

[12] Hyeonbae Kang, Mikyoung Lim, and KiHyun Yun, *Characterization of the electric field concentration between two adjacent spherical perfect conductors*, SIAM J. Appl. Math. **74** (2014), no. 1, 125–146, DOI 10.1137/130922434. MR3162415

[13] J. Lekner, *Analytical expression for the electric field enhancement between two closely-spaced conducting spheres,* J. Electrostatics **68** (2010), 299–304.

[14] J. Lekner, *Near approach of two conducting spheres: enhancement of external electric field,* J. Electrostatics **69** (2011), 559-563.

[15] John Lekner, *Electrostatics of two charged conducting spheres*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **468** (2012), no. 2145, 2829–2848, DOI 10.1098/rspa.2012.0133. MR2960928

[16] Haigang Li, Yanyan Li, Ellen Shiting Bao, and Biao Yin, *Derivative estimates of solutions of elliptic systems in narrow regions*, Quart. Appl. Math. **72** (2014), no. 3, 589–596, DOI 10.1090/S0033-569X-2014-01339-0. MR3237564

[17] Yanyan Li and Louis Nirenberg, *Estimates for elliptic systems from composite material*, Comm. Pure Appl. Math. **56** (2003), no. 7, 892–925, DOI 10.1002/cpa.10079. Dedicated to the memory of Jürgen K. Moser. MR1990481

[18] Yan Yan Li and Michael Vogelius, *Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients*, Arch. Ration. Mech. Anal. **153** (2000), no. 2, 91–151, DOI 10.1007/s002050000082. MR1770682

[19] Mikyoung Lim and Kihyun Yun, *Blow-up of electric fields between closely spaced spherical perfect conductors*, Comm. Partial Differential Equations **34** (2009), no. 10-12, 1287–1315, DOI 10.1080/03605300903079579. MR2581974

[20] Mikyoung Lim and KiHyun Yun, *Strong influence of a small fiber on shear stress in fiber-reinforced composites*, J. Differential Equations **250** (2011), no. 5, 2402–2439, DOI 10.1016/j.jde.2010.12.024. MR2756070

[21] M. Lim, S. Yu, *Asymptotic analysis for superfocusing of the electric field in between two nearly touching metallic spheres*, preprint, arXiv:1504.04691

[22] Kihyun Yun, *Estimates for electric fields blown up between closely adjacent conductors with arbitrary shape*, SIAM J. Appl. Math. **67** (2007), no. 3, 714–730, DOI 10.1137/060648817. MR2300307

[23] KiHyun Yun, *Optimal bound on high stresses occurring between stiff fibers with arbitrary shaped cross-sections*, J. Math. Anal. Appl. **350** (2009), no. 1, 306–312, DOI 10.1016/j.jmaa.2008.09.057. MR2476915

[24] KiHyun Yun, *An optimal estimate for electric fields on the shortest line segment between two spherical insulators in three dimensions*, J. Differential Equations **261** (2016), no. 1, 148–188, DOI 10.1016/j.jde.2016.03.005. MR3487255