QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXXVI, NUMBER 2
JUNE 2018, PAGES 361–382
http://dx.doi.org/10.1090/qam/1481
Article electronically published on September 20, 2017

THE LARGE-TIME DEVELOPMENT OF THE SOLUTION TO AN INITIAL-VALUE PROBLEM FOR THE KORTEWEG-DE VRIES EQUATION: IV. TIME DEPENDENT COEFFICIENTS

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Abstract. In this paper, we consider an initial-value problem for the Korteweg-de Vries equation with time dependent coefficients. The normalized variable coefficient Korteweg-de Vries equation considered is given by

$$u_t + \Phi(t)uu_x + \Psi(t)u_{xxx} = 0, \quad -\infty < x < \infty, \quad t > 0,$$

where x and t represent dimensionless distance and time respectively, whilst $\Phi(t)$, $\Psi(t)$ are given functions of $t(>0)$. In particular, we consider the case when the initial data has a discontinuous expansive step, where $u(x,0) = u_+$ for $x \geq 0$ and $u(x,0) = u_-$ for $x < 0$. We focus attention on the case when $\Phi(t) = t^\delta$ (with $\delta > -\frac{2}{3}$) and $\Psi(t) = 1$. The constant states u_+, u_- $(< u_+)$ and δ are problem parameters. The method of matched asymptotic coordinate expansions is used to obtain the large-t asymptotic structure of the solution to this problem, which exhibits the formation of an expansion wave in $x \geq \frac{u_-}{(\delta+1)} t^{(\delta+1)}$ as $t \to \infty$, while the solution is oscillatory in $x < \frac{u_-}{(\delta+1)} t^{(\delta+1)}$ as $t \to \infty$. We conclude with a brief discussion of the structure of the large-t solution of the initial-value problem when the initial data is step-like being continuous with algebraic decay as $|x| \to \infty$, with $u(x,t) \to u_+$ as $x \to \infty$ and $u(x,t) \to u_-(< u_+)$ as $x \to -\infty$.

1. Introduction. In this paper we consider the following initial-value problem for the normalized variable coefficient Korteweg-de Vries equation, namely,

$$u_t + \Phi(t)uu_x + \Psi(t)u_{xxx} = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{1.1}$$

$$u(x,0) = \begin{cases} u_-, & x < 0, \\ u_+, & x \geq 0, \end{cases} \tag{1.2}$$

$$u(x,t) \to \begin{cases} u_-, & x \to -\infty, \\ u_+, & x \to \infty, \end{cases} \quad t \geq 0, \tag{1.3}$$

Received March 27, 2017 and, in revised form, August 2, 2017.
2010 *Mathematics Subject Classification.* Primary 35Q53.

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where u_+ and u_- ($< u_+$) are parameters and the monomial functions $\Phi(t)$ and $\Psi(t)$ are algebraic functions of t. We consider, without loss of generality, the case when

$$\Phi(t) = t^\delta \ (\delta > -1), \quad \Psi(t) = 1, \tag{1.4}$$

and restrict attention in the analysis presented in this paper to $\delta > -\frac{2}{3}$ (excluding the case when $\delta = 0$ which has been considered in [24] and [25]). We note that the initial data is a discontinuous expansive step. In what follows we label initial-value problem (1.1)-(1.3) (with (1.4)) as IVP.

The more general situation of equation (1.1) with $\Phi(t) = t^\alpha$ ($\alpha > -1$) and $\Psi(t) = t^\beta$ ($\beta > -1$) where $\alpha \neq \beta$ can be transformed to equation (1.1) with (1.4) by the change of variables

$$u = (\beta + 1)^{-\delta}\bar{u}, \quad \tau = \int_0^t s^\beta \, \mathrm{d}s,$$

where $\delta = \frac{\alpha-\beta}{\beta+1}$ ($\in (-1, \infty)$). When $\alpha = \beta$ the change of variable $\tau = \int_0^t s^\alpha \, \mathrm{d}s$ transforms (1.1) to the classical Korteweg-de Vries equation. We also note that equation (1.1) (with (1.4)) can be transformed, on writing $u = t^{-\delta}v$, to the generalized Korteweg-de Vries equation

$$v_t - \frac{\delta}{t}v + vv_x + v_{xxx} = 0. \tag{1.5}$$

Equation (1.5) corresponds to the classical Korteweg-de Vries equation when $\delta = 0$, the cylindrical Korteweg-de Vries equation when $\delta = -\frac{1}{2}$ (see for example [27]) and the spherical Korteweg-de Vries equation when $\delta = -1$ (see for example [28]).

The classical Korteweg-de Vries equation ($\delta = 0$) was named after D. J. Korteweg and G. de Vries who derived the equation in 1895 (see [21]). However, the equation had already appeared earlier in the work of Rayleigh [33] and Boussinesq [3]. Although the Korteweg-de Vries was originally derived in the context of shallow water waves as a canonical equation combining both nonlinearity and dispersion it arises in the modelling of many physical phenomenon including for example: ion-acoustic waves [37], the anharmonic lattice [39], waves in the atmosphere and ocean [29] and pressure waves in liquid-gas bubble mixtures [38]. Clearly, the literature relating to the Korteweg-de Vries equation is vast and we make no attempt in this here to summarize it, rather we make reference only to the most salient to this present paper. However, the interested reader is referred to the following excellent reviews and books on the subject (and the extensive lists of references contained therein) [1], [35], [4], [8], [10], [14], [15], [16], [26], [30], [31] and [34].

The variable coefficient Korteweg-de Vries equation (1.1) arises in the modelling of numerous complex physical systems (for example, waves in elastic tubes [5] and water waves moving over a shelf [19] and [6]), but is far less studied in the literature than its constant coefficient counterpart. This is due in part to the fact that methods of solution such as inverse scattering which can be applied to the classical Korteweg-de Vries equation are not applicable in general to equations of the form (1.1). We further note that in the majority of the existing studies on initial-value problems for the variable coefficient Korteweg-de Vries equation the focus has been on soliton propagation (see for example, [11], [17], [19] and [12]). Finally, we draw the attention of the interested reader to a number of interesting studies, given in [9], [20], [32] and [36], that identify exact solutions of the variable coefficient Korteweg-de Vries equation.

It is well known (see [20] for example) that equation (1.1) will pass the Painlevé test if and only if the functions $\Phi(t)$ and $\Psi(t)$ satisfy the condition,

$$\Psi(t) = \Phi(t)\left(c_0 \int_0^t \Phi(s)\mathrm{d}s + c_1\right),$$

where c_0 and c_1 are arbitrary constants with $c_0^2 + c_1^2 \neq 0$. Consideration of (1.4) indicates that this condition can only be satisfied when $\delta = 0$ and equation (1.1) reduces to its integrable constant coefficient counterpart the classical Korteweg-de Vries equation, or when $\delta = -\frac{1}{2}$ and equation (1.1) reduces to the integrable cylindrical Korteweg-de Vries equation which in turn is related to the classical Korteweg-de Vries equation.

It is the purpose of this paper to obtain the detailed asymptotic structure of the initial-value problem IVP as $t \to \infty$, uniformly for $-\infty < x < \infty$. The methodology we employ is based on the method of matched asymptotic coordinate expansions and was developed in the context of reaction-diffusion equations (see for example the monograph [23]). This technique uses matched asymptotic coordinate expansions to transfer information from the initial data (1.2) as $t \to 0$, via asymptotic structures when $t = O(1)$ as $|x| \to \infty$, into the asymptotic structure as $t \to \infty$. The initial-value problem for the classical Korteweg-de Vries equation when the initial data has a discontinuous expansive (compressive) step has recently been considered via this approach in [24] ([25]).

The structure of the paper is as follows: in Section 2 we develop the complete large-t solution, $u(x,t)$, of IVP which exhibits the formation of an expansion wave profile, where

$$u(x,t) \to \begin{cases} u_+, & x > \frac{u_+}{(\delta+1)}t^{(\delta+1)}, \\ (\delta+1)xt^{-(\delta+1)}, & \frac{u_-}{(\delta+1)}t^{(\delta+1)} \leq x \leq \frac{u_+}{(\delta+1)}t^{(\delta+1)}, \end{cases} \tag{1.6}$$

as $t \to \infty$, uniformly in x, while $u(x,t)$ is oscillatory (oscillating about $u = u_-$) for $x < \frac{u_-}{(\delta+1)}t^{(\delta+1)}$ as $t \to \infty$. We conclude in Section 3 with a brief discussion of the structure of the large-t solution of the initial-value problem when the initial data is 'step-like' being continuous with algebraic decay as $|x| \to \infty$, with $u(x,t) \to u_+$ as $x \to \infty$ and $u(x,t) \to u_-(< u_+)$ as $x \to -\infty$. Specifically, we consider

$$u(x,0) = \begin{cases} u_- + \dfrac{A_L}{(-x)^\gamma} + O\Big(E(|x|)\Big) & \text{as} \quad x \to -\infty, \\ u_+ + \dfrac{A_R}{x^\gamma} + O\Big(E(|x|)\Big) & \text{as} \quad x \to \infty, \end{cases} \tag{1.7}$$

where $A_L\,(> 0), A_R\,(< 0)$ and $\gamma\,(> 0)$ are parameters and $E(|x|)$ is linearly exponentially small in x as $|x| \to \infty$. This change in initial data leads to a significant change in the detailed structure of the large-t solution of the initial-value problem (1.1), (1.7), (1.3) and (1.4). These changes in structure illustrate just how sensitive the large-t solution is to changes in the initial data. We note that although the large-t attractor for the solution of this initial-value problem is again an expansion wave the details of all the asymptotic regions that constitute the large-t structure are now modified (see Section 3 for full details). In particular, the oscillations observed in the large-t solution of IVP in $x < \frac{u_-}{(\delta+1)}t^{(\delta+1)}$ are not present up to $O\left(t^{-\gamma(\delta+1)}\right)$ as $t \to \infty$ in the solution to this problem.

Finally, we note that the analysis presented in this paper is formal in nature being based on the method of matched asymptotic coordinate expansions. Further, in the analysis of some of the boundary value problems considered we have had to make conjectures based on the available supporting numerical evidence. It is hoped that the structure of the large-time asymptotic solution of IVP presented in this paper will form the basis for a more rigorous analysis of this interesting initial-value problem in the future.

2. Asymptotic solution as $t \to \infty$. In this section we develop the asymptotic structure of the solution to IVP as $t \to \infty$. We must first begin by examining the asymptotic structure of the solution of IVP as $t \to 0$.

2.1. *Asymptotic solution to IVP as* $\boldsymbol{t \to 0}$. Consideration of the initial data (1.2) indicates that the structure of the asymptotic solution to IVP as $t \to 0$ has three asymptotic regions for $x \in (-\infty, \infty)$, namely,

$$\left.\begin{array}{lll} \text{region } \mathbf{I}: & x = o(1), & u(x,t) = O(1) \\ \text{region } \mathbf{II}^+: & x = O(1)\ (>0), & u(x,t) = u_+ - o(1) \\ \text{region } \mathbf{II}^-: & x = O(1)\ (<0), & u(x,t) = u_- + o(1) \end{array}\right\} \quad \text{as} \quad t \to 0. \tag{2.1}$$

Consideration of equation (1.1) for $t \ll 1$ indicates that the small-time solution of IVP follows, after minor modification, that given in [24]. For brevity we omit the details and summarize as follows:

Region **I**. $x = O(t^{\frac{1}{3}})$ as $t \to 0$.
$\eta = xt^{-\frac{1}{3}} = O(1)$ as $t \to 0$, and,

$$u(\eta,t) = \left(\frac{(u_- + 2u_+)}{3} - (u_- - u_+)\int_0^{3^{-\frac{1}{3}}\eta} \mathrm{Ai}(s)\mathrm{d}s\right) + o(1), \tag{2.2}$$

as $t \to 0$ with $\eta = O(1)$, and where Ai(.) is the standard Airy function (see [2]).

Region $\mathbf{II}^+$. $x = O(1)\,(>0)$ as $t \to 0$.

$$u(x,t) = u_+ - e^{-\phi(x,t)}, \tag{2.3}$$

as $t \to 0$ with $x = O(1)(>0)$, and where

$$\phi(x,t) = \begin{cases} \frac{2}{3\sqrt{3}}x^{\frac{3}{2}}t^{-\frac{1}{2}} - \frac{1}{4}\ln t + \frac{3}{4}\ln x - \ln\left(\frac{(u_+-u_-)3^{\frac{1}{4}}}{2\sqrt{\pi}}\right) + O\left(t^{(\delta+\frac{1}{2})}\right), & \delta > -\frac{1}{2}, \\ \frac{2}{3\sqrt{3}}x^{\frac{3}{2}}t^{-\frac{1}{2}} - \frac{1}{4}\ln t + \frac{3}{4}\ln x - \frac{2u_+}{\sqrt{3}}x^{\frac{1}{2}} - \ln\left(\frac{(u_+-u_-)3^{\frac{1}{4}}}{2\sqrt{\pi}}\right) + o(1), & \delta = -\frac{1}{2}, \\ \frac{2}{3\sqrt{3}}x^{\frac{3}{2}}t^{-\frac{1}{2}} - \frac{u_+}{(\delta+1)\sqrt{3}}x^{\frac{1}{2}}t^{(\delta+\frac{1}{2})} - \frac{1}{4}\ln t + \frac{3}{4}\ln x - \ln\left(\frac{(u_+-u_-)3^{\frac{1}{4}}}{2\sqrt{\pi}}\right) + o(1), & \\ & -\frac{2}{3} < \delta < -\frac{1}{2}, \end{cases} \tag{2.4}$$

as $t \to 0$ with $x = O(1)\,(>0)$.

Region $\mathbf{II}^-$. $x = O(1)\,(<0)$ as $t \to 0$.

$$u(x,t) = u_- + \left(e^{\psi_+(x,t)} + e^{\psi_-(x,t)}\right) \tag{2.5}$$

as $t \to 0$ with $x = O(1)\,(<0)$, and where

$$\psi_\pm(x,t) = \begin{cases} \pm i\frac{2}{3\sqrt{3}}(-x)^{\frac{3}{2}}t^{-\frac{1}{2}} + \frac{1}{4}\ln t + \left[\pm i\frac{\pi}{4} - \frac{3}{4}\ln(-x) + \ln\left(\frac{(u_+-u_-)3^{\frac{1}{4}}}{2\sqrt{\pi}}\right)\right] \\ \qquad\qquad + O\left(t^{(\delta+\frac{1}{2})}\right), \quad \delta > -\frac{1}{2}, \\ \pm i\frac{2}{3\sqrt{3}}(-x)^{\frac{3}{2}}t^{-\frac{1}{2}} + \frac{1}{4}\ln t + \Big[\pm i\frac{\pi}{4} \pm i\frac{2u_-}{\sqrt{3}}(-x)^{\frac{1}{2}} - \frac{3}{4}\ln(-x) \\ \qquad\qquad + \ln\left(\frac{(u_+-u_-)3^{\frac{1}{4}}}{2\sqrt{\pi}}\right)\Big] + o\,(1)\,, \quad \delta = -\frac{1}{2}, \\ \pm i\frac{2}{3\sqrt{3}}(-x)^{\frac{3}{2}}t^{-\frac{1}{2}} \pm i\frac{u_-}{(\delta+1)\sqrt{3}}(-x)^{\frac{1}{2}}t^{(\delta+\frac{1}{2})} + \frac{1}{4}\ln t + \Big[\pm i\frac{\pi}{4} - \frac{3}{4}\ln(-x) \\ \qquad\qquad + \ln\left(\frac{(u_+-u_-)3^{\frac{1}{4}}}{2\sqrt{\pi}}\right)\Big] + o\,(1)\,, \quad -\frac{2}{3} < \delta < -\frac{1}{2}, \end{cases} \tag{2.6}$$

as $t \to 0$ with $x = O(1)(<0)$.

The asymptotic structure as $t \to 0$ is now complete, with the expansions in regions $\mathbf{I}$, $\mathbf{II}^+$ and $\mathbf{II}^-$ providing a uniform approximation to the solution of IVP as $t \to 0$.

2.2. *Asymptotic solution to IVP as* $|\boldsymbol{x}| \to \boldsymbol{\infty}$. We now investigate the asymptotic structure of the solution to IVP as $|x| \to \infty$ with $t = O(1)$. We first determine the structure of the solution to IVP as $x \to \infty$ with $t = O(1)$. The form of expansion (2.3) (with (2.4)) of region $\mathbf{II}^+$ for $x \gg 1$ as $t \to 0$ suggests that in this region, which we label as region $\mathbf{III}^+$, we write

$$u(x,t) = u_+ - e^{-\theta(x,t)} \tag{2.7}$$

as $x \to \infty$ with $t = O(1)$, and where

$$\theta(x,t) = \theta_0(t)x^{\frac{3}{2}} + \theta_1(t)x^{\frac{1}{2}} + \theta_2(t)\ln x + \theta_3(t) + o(1) \tag{2.8}$$

as $x \to \infty$ with $t = O(1)$. On substituting (2.7) and (2.8) into equation (1.1) and solving at each order in turn, we find (after matching with (2.3) as $t \to 0^+$) that

$$\begin{aligned} u(x,t) = u_+ - \exp\Bigg(& -\frac{2}{3\sqrt{3}}t^{-\frac{1}{2}}x^{\frac{3}{2}} + \frac{u_+}{(\delta+1)\sqrt{3}}t^{(\delta+\frac{1}{2})}x^{\frac{1}{2}} - \frac{3}{4}\ln x \\ & + \frac{1}{4}\ln t + \ln\left(\frac{(u_+-u_-)3^{\frac{1}{4}}}{2\sqrt{\pi}}\right) + o(1)\Bigg) \end{aligned} \tag{2.9}$$

as $x \to \infty$ with $t = O(1)$. Expansion (2.9) remains uniform for $t \gg 1$ provided that $x \gg \lambda(t)$, but becomes nonuniform when $x = O\left(\lambda(t)\right)$ for $t \gg 1$, where

$$\lambda(t) = \begin{cases} t^{-1}, & -\frac{2}{3} < \delta < 0, \\ t^{-(\delta+1)}, & \delta > 0. \end{cases} \tag{2.10}$$

We next investigate the structure of the solution to IVP as $x \to -\infty$ with $t = O(1)$, which we label as region $\mathbf{III}^-$. The details in this case follow, after minor modification, those given above and we obtain in region $\mathbf{III}^-$ that

$$u(x,t) = u_- + \left(e^{\hat{\psi}_+(x,t)} + e^{\hat{\psi}_-(x,t)}\right) \tag{2.11}$$

as $x \to -\infty$ with $t = O(1)$, and where

$$\begin{aligned}\hat{\psi}_\pm(x,t) = \pm i\frac{2}{3\sqrt{3}}t^{-\frac{1}{2}}(-x)^{\frac{3}{2}} \pm i\frac{u_-}{(\delta+1)\sqrt{3}}t^{(\delta+\frac{1}{2})}(-x)^{\frac{1}{2}} - \frac{3}{4}\ln(-x) + \frac{1}{4}\ln t \\ \pm i\frac{\pi}{4} + \ln\left(\frac{(u_+ - u_-)3^{\frac{1}{4}}}{2\sqrt{\pi}}\right) + o(1)\end{aligned} \tag{2.12}$$

as $x \to -\infty$ with $t = O(1)$. Expansion (2.12) remains uniform for $t \gg 1$ provided that $(-x) \gg \lambda(t)$, but becomes nonuniform when $(-x) = O(\lambda(t))$ as $t \to \infty$, where $\lambda(t)$ is given by (2.10).

2.3. *Asymptotic solution to IVP as* $\boldsymbol{t} \to \infty$. As $t \to \infty$, the asymptotic expansions (2.9) and (2.11) (with (2.12)) of regions $\mathbf{III}^+$ ($x \to \infty$, $t = O(1)$) and $\mathbf{III}^-$ ($x \to -\infty$, $t = O(1)$), respectively, continue to remain uniform provided $|x| \gg \lambda(t)$ as $t \to \infty$. However, as already noted, a nonuniformity develops when $|x| = O(\lambda(t))$ as $t \to \infty$, where $\lambda(t)$ depends on δ and is given by (2.10). Therefore, in what follows we must consider the cases when $-\frac{2}{3} < \delta < 0$ and when $\delta > 0$ separately.

2.3.1. $\boldsymbol{\delta > 0}$. We now investigate the structure of IVP as $t \to \infty$ when $\delta > 0$ and $u_+ > u_-$. Before we begin we note that a schematic representation of the asymptotic structure of IVP as $t \to \infty$ in this case is given in Figure 1. We recall from Section 2.2 that expansions (2.9) and (2.11) (with (2.12) of regions $\mathbf{III}^+$ ($x \to \infty, t = O(1)$) and $\mathbf{III}^-$ ($x \to -\infty, t = O(1)$) respectively, continue to remain uniform provided $|x| \gg t^{(\delta+1)}$ as $t \to \infty$. However, as already noted, a nonuniformity develops when $|x| = O\left(t^{(\delta+1)}\right)$ as $t \to \infty$. We begin by considering the asymptotic structure as $t \to \infty$ moving in from the far field region $\mathbf{III}^+$, when $x \gg t^{(\delta+1)}$ as $t \to \infty$. To proceed we introduce a new region labelled as region $\mathbf{IV}^+$, in which $x = O\left(t^{(\delta+1)}\right)$ as $t \to \infty$.

Region $\mathbf{IV}^+$. To examine region $\mathbf{IV}^+$ we introduce the scaled coordinate

$$y = \frac{x}{t^{(\delta+1)}}, \tag{2.13}$$

where $y = O(1)$ as $t \to \infty$ in region $\mathbf{IV}^+$, whilst the form of expansion (2.9) in region $\mathbf{III}^+$, when $t \gg 1$ and $x = O\left(t^{(\delta+1)}\right)$ requires that we expand as

$$u(y,t) = u_+ - e^{-f(y,t)} \tag{2.14}$$

as $t \to \infty$ with $y = O(1)$, and where

$$f(y,t) = f_0(y)\, t^{\frac{1}{2}(3\delta+2)} + f_1(y)\ln t + f_2(y) + o(1) \tag{2.15}$$

as $t \to \infty$ with $y = O(1)$, and where $f_0(y) > 0$. On substituting (2.14) and (2.15) into equation (1.1) (when written in terms of y and t) and solving at each order in turn, we find (after matching with (2.9) as $y \to \infty$) that

$$\begin{aligned} u(y,t) = u_+ - \exp\Bigg(- \frac{2}{3\sqrt{3}}\left(y - \frac{u_+}{(\delta+1)}\right)^{\frac{3}{2}} t^{\frac{1}{2}(3\delta+2)} - \frac{1}{4}(3\delta+2)\ln t \\ -\frac{3}{4}\ln\left(y - \frac{u_+}{(\delta+1)}\right) + \ln\left(\frac{(u_+ - u_-)3^{\frac{1}{4}}}{2\sqrt{\pi}}\right) + o(1)\Bigg) \end{aligned} \tag{2.16}$$

as $t \to \infty$ with $y = O(1)$ $\left(\in \left(\frac{u_+}{(\delta+1)}, \infty\right)\right)$. Expansion (2.16) becomes nonuniform when $y = \frac{u_+}{(\delta+1)} + O\left(t^{-\delta}\right)$ as $t \to \infty$ (that is, when $x = \frac{u_+}{(\delta+1)} t^{(\delta+1)} + O(t)$ as $t \to \infty$). To proceed we introduce a localized region, region $\mathbf{V}^+$.

Region $\mathbf{V}^+$. To investigate region $\mathbf{V}^+$ we introduce the scaled coordinate η via

$$\eta = \left(y - \frac{u_+}{\delta+1}\right) t^{\delta}, \tag{2.17}$$

as $t \to \infty$ with $\eta = O(1)$, and look for an expansion of the form (as suggested by (2.16))

$$u(\eta,t) = u_+ - e^{-\hat{f}(\eta,t)} \tag{2.18}$$

as $t \to \infty$ with $\eta = O(1)$, and where

$$\hat{f}(\eta,t) = \hat{f}_0(\eta)t + \hat{f}_1(\eta)\ln t + \hat{f}_2(\eta) + o(1) \tag{2.19}$$

as $t \to \infty$ with $\eta = O(1)$, and where $\hat{f}_0(\eta) > 0$. On substituting (2.18) and (2.19) into equation (1.1) (when written in terms of η and t) and solving at each order in turn, we find (after matching with (2.16) as $\eta \to \infty$) that

$$u(\eta,t) = u_+ - \exp\left(-\frac{2}{3\sqrt{3}}\eta^{\frac{3}{2}}\, t - \frac{1}{2}\ln t + \hat{H}^+(\eta) + o(1)\right) \tag{2.20}$$

as $t \to \infty$ with $\eta = O(1)\,(>0)$, and where the function $\hat{H}^+(\eta) : (0,\infty) \to \mathbb{R}$ is undetermined, but having

$$\hat{H}^+(\eta) \sim -\frac{3}{4}\ln\eta + \ln\left(\frac{(u_+ - u_-)3^{\frac{1}{4}}}{2\sqrt{\pi}}\right) \quad \text{as} \quad \eta \to \infty.$$

Expansion (2.20) becomes nonuniform when $\eta = O(t^{-\frac{2}{3}})$ as $t \to \infty$ (that is, when $y = \frac{u_+}{\delta+1} + O\left(t^{-\frac{1}{3}(3\delta+2)}\right)$ as $t \to \infty$). Therefore, we must now introduce a second localized region $\mathbf{CR}^+$ (corner region) in which $y = \frac{u_+}{(\delta+1)} + O\left(t^{-\frac{1}{3}(3\delta+2)}\right)$ as $t \to \infty$.

Region $\mathbf{CR}^+$. To examine region $\mathbf{CR}^+$ we write

$$y = \frac{u_+}{(\delta+1)} + \xi\, t^{-\frac{1}{3}(3\delta+2)} \tag{2.21}$$

in region $\mathbf{CR}^+$, with $\xi = O(1)$ as $t \to \infty$. It follows from (2.21) and expansion (2.20) in region $\mathbf{V}^+$, that we should expand as

$$u(\xi,t) = u_+ + F(\xi)\Theta(t) + o\left(\Theta(t)\right) \quad \text{as} \quad t \to \infty \tag{2.22}$$

with $\xi = O(1)$, and the gauge function $\Theta(t) = o(1)$ as $t \to \infty$ is to be determined. On substituting (2.22) into equation (1.1) (when written in terms of ξ and t) we obtain

$$\Theta'(t)F - \frac{\Theta(t)}{t}\frac{\xi}{3}F + t^{(\delta-\frac{1}{3})}\Theta^2(t)FF_\xi + \frac{\Theta(t)}{t}F_{\xi\xi\xi} = 0. \tag{2.23}$$

A nontrivial balance (that retains the most structure in equation (2.23)) requires

$$t^{(\delta-\frac{1}{3})}\Theta^2(t) \sim \frac{\Theta(t)}{t} \quad \text{as} \quad t \to \infty,$$

and so, without loss of generality, we have that

$$\Theta(t) = t^{-\frac{1}{3}(3\delta+2)}. \tag{2.24}$$

We observe that all terms in (2.23) are retained at leading order as $t \to \infty$ and (2.23) becomes

$$F_{\xi\xi\xi} + FF_\xi - \frac{\xi}{3}F_\xi - \frac{1}{3}(3\delta+2)F = 0, \quad -\infty < \xi < \infty. \tag{2.25}$$

We note that equation (2.25) admits the solution

$$F(\xi) = (\delta+1)\xi, \quad -\infty < \xi < \infty.$$

Now matching expansion (2.20) (as $\eta \to 0^+$) with expansion (2.22) (as $\xi \to \infty$) requires first that

$$\hat{H}^+(\eta) \sim \left(\frac{3\delta}{2} + \frac{1}{4}\right)\ln\eta + \ln D \quad \text{as} \quad \eta \to 0^+, \tag{2.26}$$

with $D > 0$ as yet undetermined, after which we require that

$$F(\xi) \sim -D\,\xi^{\left(\frac{3\delta}{2}+\frac{1}{4}\right)}e^{-\frac{2}{3\sqrt{3}}\xi^{\frac{3}{2}}} \quad \text{as} \quad \xi \to \infty. \tag{2.27}$$

Finally for u to remain bounded as $t \to \infty$ when $y = \frac{u_+}{(\delta+1)} - O(1)$ then we require,

$$\xi^{-1}F(\xi) \quad \text{is bounded as} \quad \xi \to -\infty. \tag{2.28}$$

The leading order problem is now complete, and is given by (2.25), (2.27) and (2.28). The boundary value problem (2.25)-(2.28) is both nonlinear and nonautonomous. A numerical study of initial-value problem (2.25) and (2.27) using a shooting method has been carried out in [24] for the case when $\delta = 0$, and this approach can readily be extended to consider the current situation where $\delta > 0$. Following [24] it is straightforward to conjecture, via the shooting method, that for fixed $\delta > 0$ (for the values of δ tested) there exists a value $D = D^*$ such that boundary value problem (2.25)-(2.28) has a unique solution, say $F = F^*(\xi)$, for $-\infty < \xi < \infty$. Moreover, $F^*(\xi)$ is monotone increasing in $-\infty < \xi < \infty$, such that $F^*(\xi) < 0$ for all $-\infty < \xi < \infty$, and

$$F^*(\xi) = (\delta+1)\xi + \alpha(-\xi)^{-\frac{1}{(3\delta+2)}} + O\left((-\xi)^{-\frac{(\delta+1)}{(3\delta+2)}}\exp\left(-\frac{2}{3\sqrt{3}}(3\delta+2)^{\frac{1}{2}}(-\xi)^{\frac{3}{2}}\right)\right) \tag{2.29}$$

as $\xi \to -\infty$, where α is a constant. We note that (2.29) was obtained by developing the boundary condition as $\xi \to -\infty$. We note as in the case when $\delta = 0$ initial-value problem (2.25) and (2.27) admits solutions which blow up at finite-ξ and solutions which are oscillatory in $\xi < 0$. However, the former do not satisfy condition (2.28), while the latter can following the discussion given in [24] be ruled out at this stage. We conclude that $D = D^*$ (where the value of D^* depends on the value of δ), and $F = F^*(\xi)$ for $-\infty < \xi < \infty$.

Finally, when $\delta = 0$ we recall from [25] that on making the substitution

$$F(\xi) = 2\,3^{\frac{1}{3}}\ \left(W_{\hat{\eta}}(\hat{\eta}) - W^2(\hat{\eta})\right), \quad \xi = 3^{\frac{1}{3}}\ \hat{\eta} \tag{2.30}$$

boundary value problem (2.25)-(2.28) becomes

$$W_{\hat{\eta}\hat{\eta}} = \hat{\eta}W + 2W^3, \quad -\infty < \hat{\eta} < \infty, \tag{2.31}$$

$$W(\hat{\eta}) \sim \frac{D}{2.3^{1/4}}\hat{\eta}^{-1/4}\exp\left(-\frac{2}{3}\hat{\eta}^{2/3}\right) \quad \text{as} \quad \hat{\eta} \to \infty, \tag{2.32}$$

$$(-\hat{\eta})^{1/2}W(\hat{\eta}) \quad \text{is bounded as} \quad \hat{\eta} \to -\infty, \tag{2.33}$$

where we recognize equation (2.31) as the second Painlevé equation. Moreover, it has been established in [13], that equation (2.31) has a solution $W = W^*(\hat{\eta})$, $-\infty < \hat{\eta} < \infty$, for which

$$W^*(\hat{\eta}) \sim \begin{cases} \frac{1}{2\sqrt{\pi}}\hat{\eta}^{-\frac{1}{4}}\ \exp\left(-\frac{2}{3}\hat{\eta}^{\frac{3}{2}}\right) & \text{as} \quad \hat{\eta} \to \infty, \\ \left(-\frac{\hat{\eta}}{2}\right)^{\frac{1}{2}} & \text{as} \quad \hat{\eta} \to -\infty. \end{cases} \tag{2.34}$$

A comparison of $(2.34)_1$ with (2.32) establishes that $D^* = \frac{3^{1/4}}{\sqrt{\pi}}$, while a comparison of (2.30) (with $(2.34)_2$) with (2.29) (when $\delta = 0$) gives that $\alpha = -\sqrt{\frac{3}{2}}$. Unfortunately, similar results for $\delta > 0$ are not available.

Now as $\xi \to -\infty$, we move out of the corner region, region $\mathbf{CR}^+$, into region **EW** (expansion wave region), where $y = O(1)$ $\left(\in \left(-\infty, \frac{u_+}{(\delta+1)}\right)\right)$ as $t \to \infty$.

Region **EW**. It follows, via (2.22), (2.24) and (2.29) that in region **EW** we have that $u(y,t) = O(1)$ as $t \to \infty$. Hence we expand as

$$u(y,t) = G_0(y) + G_1(y)t^{-(\delta+1)} + o\left(t^{-(\delta+1)}\right) \tag{2.35}$$

as $t \to \infty$ with $y = O(1)$ $\left(\in \left(-\infty, \frac{u_+}{(\delta+1)}\right)\right)$. On substitution of (2.35) into equation (1.1) (when written in terms of y and t) we obtain the leading order problem as

$$G_0'\Big(G_0 - (\delta+1)y\Big) = 0, \quad -\infty < y < \frac{u_+}{(\delta+1)}, \tag{2.36}$$

$$G_0(y) \sim (\delta+1)y \quad \text{as} \quad y \to \left(\frac{u_+}{(\delta+1)}\right)^-, \tag{2.37}$$

with the final condition being the matching condition with region $\mathbf{CR}^+$. The solution of (2.36), (2.37) is readily obtained as

$$G_0(y) = (\delta+1)y, \quad -\infty < y < \frac{u_+}{(\delta+1)}. \tag{2.38}$$

The function $G_1(y)$ remains undetermined, being a remnant of the global evolution when $t = O(1)$. However, matching to region $\mathbf{CR}^+$ requires that

$$G_1(y) \sim \alpha\left(\frac{u_+}{(\delta+1)} - y\right)^{-\frac{1}{(3\delta+2)}} \quad \text{as} \quad y \to \left(\frac{u_+}{(\delta+1)}\right)^-.$$

The solution in region **EW** is therefore given by

$$u(y,t) = (\delta+1)y + O\left(t^{-(\delta+1)}\right) \tag{2.39}$$

as $t \to \infty$ with $y = O(1)$ $\left(\in \left(-\infty, \frac{u_+}{(\delta+1)}\right)\right)$. We will establish later in this analysis that $y \in \left(\frac{u_-}{(\delta+1)}, \frac{u_+}{(\delta+1)}\right)$ as $t \to \infty$ in region **EW**.

We next develop the asymptotic structure of $u(y,t)$ as $t \to \infty$, moving in from region **III**$^-$ (when $(-y) \gg 1$) to $y = O(1)$ as $t \to \infty$. To proceed we introduce a new region, labelled as region **IV**$^-$.

Region **IV**$^-$. It follows from expansion (2.11) (with (2.12)) that in region **IV**$^-$ we should expand as

$$u(y,t) = u_- + \left(e^{g^+(y,t)} + e^{g^-(y,t)}\right) \tag{2.40}$$

as $t \to \infty$, and where

$$g^{\pm}(y,t) = \pm i\, g_0(y) t^{\frac{1}{2}(3\delta+2)} + g_1(y)\ln t + g_2(y) \pm i\, g_3(y) + o(1) \tag{2.41}$$

as $t \to \infty$ with $y = O(1)(< \frac{u_+}{(\delta+1)})$. On substituting (2.40) and (2.41) into equation (1.1) (when written in terms of y and t) and solving at each order in turn, we find (after matching with (2.11), (2.12) as $y \to \infty$) that

$$\begin{aligned} u(y,t) = u_- &+ \exp\left(i\frac{2}{3\sqrt{3}}\left(\frac{u_-}{(\delta+1)} - y\right)^{\frac{3}{2}} t^{\frac{1}{2}(3\delta+2)} - \frac{1}{4}(3\delta+2)\ln t \right. \\ &\left. -\frac{3}{4}\ln\left(\frac{u_-}{(\delta+1)} - y\right) + i\frac{\pi}{4} + \ln\left(\frac{(u_+ - u_-)3^{\frac{1}{4}}}{2\sqrt{\pi}}\right) + o(1)\right) \\ &+ \exp\left(-i\frac{2}{3\sqrt{3}}\left(\frac{u_-}{(\delta+1)} - y\right)^{\frac{3}{2}} t^{\frac{1}{2}(3\delta+2)} - \frac{1}{4}(3\delta+2)\ln t \right. \\ &\left. -\frac{3}{4}\ln\left(\frac{u_-}{(\delta+1)} - y\right) - i\frac{\pi}{4} + \ln\left(\frac{(u_+ - u_-)3^{\frac{1}{4}}}{2\sqrt{\pi}}\right) + o(1)\right) \end{aligned} \tag{2.42}$$

as $t \to \infty$ with $y = O(1)$ $\left(\in \left(-\infty, \frac{u_-}{(\delta+1)}\right)\right)$. The large-$t$ solution of IVP is therefore oscillatory in $y < \frac{u_-}{(\delta+1)}$, with the oscillatory envelope being of $O\left(t^{-\frac{1}{4}(3\delta+2)}\right)$ as $t \to \infty$. Expansion (2.42) becomes nonuniform when $y = \frac{u_-}{(\delta+1)} + O\left(t^{-\delta}\right)$ as $t \to \infty$ (that is, when $x = \frac{u_-}{(\delta+1)} t^{(\delta+1)} + O(t)$ as $t \to \infty$). To proceed we introduce a localized region, region **V**$^-$.

Region **V**$^-$. To investigate region **V**$^-$ we introduce the scaled coordinate $\bar{\eta}$ via

$$\bar{\eta} = \left(y - \frac{u_-}{\delta+1}\right) t^{\delta}, \tag{2.43}$$

as $t \to \infty$ with $\bar{\eta} = O(1)$, and look for an expansion of the form (as suggested by (2.42))

$$u(\bar{\eta},t) = u_- + \left(e^{\hat{g}^+(\bar{\eta},t)} + e^{\hat{g}^-(\bar{\eta},t)}\right) \tag{2.44}$$

as $t \to \infty$ with $\bar{\eta} = O(1)$, and where

$$\hat{g}^{\pm}(\bar{\eta}, t) = \pm\, i\, \hat{g}_0(\bar{\eta})t + \hat{g}_1(\bar{\eta}) \ln t + \hat{g}_2(\bar{\eta}) \pm\, i\, \hat{g}_4(\bar{\eta}) + o(1) \tag{2.45}$$

as $t \to \infty$ with $\bar{\eta} = O(1)$. On substituting (2.44) and (2.45) into equation (1.1) (when written in terms of $\bar{\eta}$ and t) and solving at each order in turn, we find (after matching with (2.42) as $\bar{\eta} \to -\infty$) that

$$\begin{aligned} u(\bar{\eta}, t) = u_- &+ \exp\left(i\, \frac{2}{3\sqrt{3}}\, (-\bar{\eta})^{\frac{3}{2}}\, t - \frac{1}{2} \ln t + H_0(\bar{\eta}) + i\, H_1(\bar{\eta}) + o(1) \right) \\ &+ \exp\left(-\, i\, \frac{2}{3\sqrt{3}}\, (-\bar{\eta})^{\frac{3}{2}}\, t - \frac{1}{2} \ln t + H_0(\bar{\eta}) - i\, H_1(\bar{\eta}) + o(1) \right) \end{aligned} \tag{2.46}$$

as $t \to \infty$ with $\bar{\eta} = O(1)\,(< 0)$, and where the functions $H_0(\bar{\eta})$ and $H_1(\bar{\eta})$ remain undetermined, but matching with region IV$^-$ requires that

$$H_1(\bar{\eta}) \sim -\frac{3}{4} \ln(-\bar{\eta}) + \ln\left(\frac{(u_+ - u_-)3^{\frac{1}{4}}}{2\sqrt{\pi}} \right), \quad H_0(\bar{\eta}) \sim \frac{\pi}{4} \quad \text{as} \quad \bar{\eta} \to -\infty.$$

It is instructive in what follows to rewrite expansion (2.46) in terms of the cosine function to give

$$u(\bar{\eta}, t) = u_- + \frac{\hat{H}_1(\bar{\eta})}{t^{\frac{1}{2}}} \cos\left(\frac{2}{3\sqrt{3}}\, (-\bar{\eta})^{\frac{3}{2}}\, t + H_0(\bar{\eta}) \right) + o\left(\frac{1}{t^{\frac{1}{2}}} \right) \tag{2.47}$$

as $t \to \infty$ with $\bar{\eta} = O(1)\,(< 0)$, and where $\hat{H}_1(\bar{\eta}) = 2 \exp\left(H_1(\bar{\eta})\right)$. We note that the oscillatory envelope of the large-t solution of IVP in region $\mathbf{V}^-$ of $O\left(t^{-\frac{1}{2}}\right)$ as $t \to \infty$. Expansion (2.47) becomes nonuniform when $\bar{\eta} = O(t^{-\frac{2}{3}})$ as $t \to \infty$ (that is, when $y = \frac{u_-}{(\delta+1)} + O\left(t^{-\frac{1}{3}(3\delta+2)}\right)$ as $t \to \infty$). Therefore, we must now introduce a second localized region $\mathbf{CR}^-$ (corner region) in which $y = \frac{u_-}{(\delta+1)} + O\left(t^{-\frac{1}{3}(3\delta+2)}\right)$ as $t \to \infty$.

Region $\mathbf{CR}^-$. Thus we write

$$y = \frac{u_-}{(\delta + 1)} + \xi\, t^{-\frac{1}{3}(3\delta+2)} \tag{2.48}$$

in region $\mathbf{CR}^-$, with $\xi = O(1)$ as $t \to \infty$. It follows from expansion (2.47) and (2.48) that in region $\mathbf{CR}^-$ we should expand as

$$u(\xi, t) = u_- + F(\xi)t^{-\frac{1}{3}(3\delta+2)} + o\left(t^{-\frac{1}{3}(3\delta+2)}\right) \tag{2.49}$$

as $t \to \infty$ with $\xi = O(1)$. On substitution of expansion (2.49) into equation (1.1) (when written in terms of ξ and t) we obtain at leading order

$$F_{\xi\xi\xi} + FF_{\xi} - \frac{\xi}{3}F_{\xi} - \frac{1}{3}\,(3\delta + 2)\,F = 0, \quad -\infty < \xi < \infty, \tag{2.50}$$

which we recognize as equation (2.25) of region $\mathbf{CR}^+$. We recall that equation (2.50) admits the solution

$$F(\xi) = (\delta + 1)\xi, \quad -\infty < \xi < \infty.$$

Now matching expansion (2.47) (as $\bar{\eta} \to 0^-$) with expansion (2.49) (as $\xi \to -\infty$) requires first that

$$\hat{H}_1(\bar{\eta}) \sim \beta_0(-\bar{\eta})^{\left(\frac{3}{2}\delta + \frac{1}{4}\right)}, \quad H_0(\bar{\eta}) \sim \beta_1 \quad \text{as} \quad \bar{\eta} \to 0^-,$$

with the constants $\beta_0\,(> 0)$ and β_1 undetermined at this stage, after which matching requires that

$$F(\xi) \sim \beta_0(-\xi)^{\left(\frac{3}{2}\delta + \frac{1}{4}\right)} \cos\left(\frac{2}{3\sqrt{3}}(-\xi)^{\frac{3}{2}} + \beta_1\right) \quad \text{as} \quad \xi \to -\infty. \tag{2.51}$$

The matching condition between expansion (2.49) (as $\xi \to \infty$) and expansion (2.39) (as $y \to \left(\frac{u_-}{(\delta+1)}\right)^-$) of region $\mathbf{EW}$ then requires that

$$F(\xi) \sim (\delta + 1)\xi \quad \text{as} \quad \xi \to \infty. \tag{2.52}$$

The leading order problem in region $\mathbf{CR}^-$ is now complete, being the nonlinear nonautonomous boundary value problem (2.50), (2.51) and (2.52). The boundary condition (2.52) can be developed to give

$$F(\xi) = (\delta + 1)\xi + \xi^{-\frac{1}{(3\delta+2)}}\left(\bar{\alpha} + \bar{R}\cos\left(\frac{2}{3}\sqrt{\delta + \frac{2}{3}}\,\xi^{\frac{3}{2}} + \beta_2\right)\right) + o\left(\xi^{-\frac{1}{(3\delta+2)}}\right) \tag{2.53}$$

as $\xi \to \infty$, with β_2, $\bar{\alpha}$ and $\bar{R}$ constants to be determined. However, consideration of (2.49) (with (2.53)) as we move into region $\mathbf{EW}$ reveals that matching with expansion (2.39) of region $\mathbf{EW}$ requires that $\bar{R} = 0$, and

$$G_1(y) \sim \bar{\alpha}\left(y - \frac{u_-}{(\delta + 1)}\right)^{-\frac{1}{(3\delta+2)}} \quad \text{as} \quad y \to \left(\frac{u_-}{(\delta + 1)}\right)^+,$$

giving that

$$F(\xi) = (\delta + 1)\xi + \bar{\alpha}\xi^{-\frac{1}{(3\delta+2)}} + o\left(\xi^{-\frac{1}{(3\delta+2)}}\right) \tag{2.54}$$

as $\xi \to \infty$. The boundary value problem (2.50), (2.51) and (2.54) is both nonlinear and nonautonomous. A numerical study of initial-value problem (2.50) and (2.54) using a shooting method reveals that a unique solution exists, which is oscillatory in $\xi < 0$, being of the form (2.51) for $(-\xi) \gg 1$, for each $\bar{\alpha} \geq \alpha^+$ (where $\alpha^+ > 0$ is a constant which depends on $\delta > 0$). However, when $\bar{\alpha} < \alpha^+$ (excluding $\bar{\alpha} = 0$) the solution to initial-value problem (2.50) and (2.54) blows up at finite ξ, whilst when $\bar{\alpha} = 0$ the solution to initial-value problem (2.50) and (2.54) is $F(\xi) = (\delta + 1)\xi$. We conclude, by making the conjecture (based on the available supporting numerical evidence), that boundary value problem (2.50), (2.51) and (2.54) has a unique solution for each $\bar{\alpha} \geq \alpha^+$, but no solution for $\bar{\alpha} < \alpha^+$. Moreover, for a specified $\bar{\alpha} \geq \alpha^+$, then β_0 and β_1 are fixed uniquely. The parameter $\bar{\alpha}$ remains undetermined in this analysis. Finally, we note that $\alpha^+ \to 0$ as $\delta \to 0^+$.

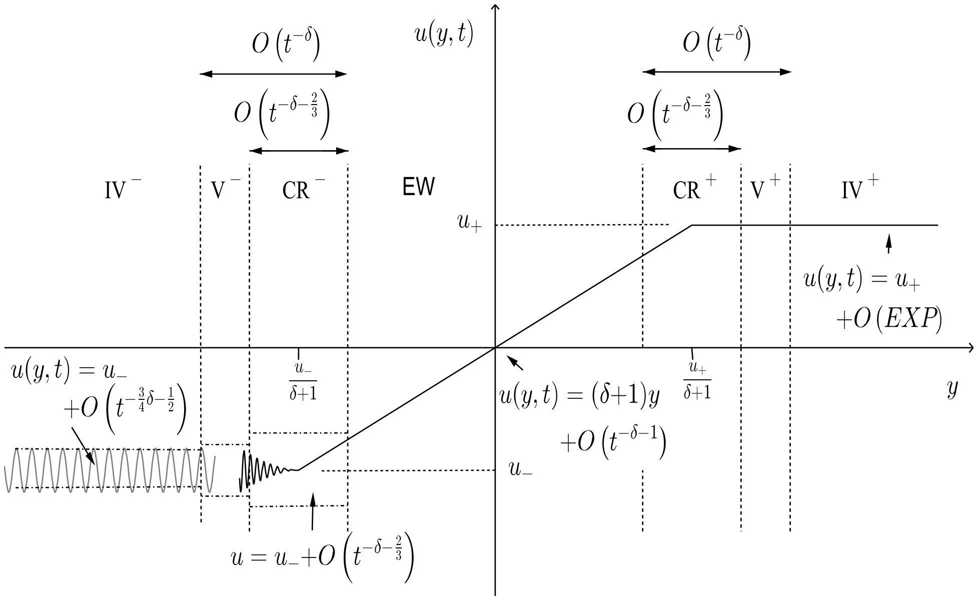

FIG. 1. A schematic representation of the asymptotic structure of $u(y,t)$ in the (y,u) plane as $t \to \infty$ for IVP when $\delta > 0$ and $u_+ > u_-$. Here (EXP) denotes terms exponentially in t as $t \to \infty$ and $y = xt^{-(\delta+1)}$. We note that the solution to IVP is oscillatory in regions IV$^-$ and V$^-$, and we recall that $u = u_- + O\left(t^{-\frac{1}{2}}\right)$ as $t \to \infty$ in region V$^-$ while $u = u_+ - O\left(t^{-\delta-\frac{2}{3}}\right)$ as $t \to \infty$ in region CR$^+$.

The asymptotic structure to the solution of IVP as $t \to \infty$ when $\delta > 0$ and $u_+ > u_-$ is now complete. A uniform approximation has been given through regions $\mathbf{IV}^{\pm}$, $\mathbf{V}^{\pm}$, $\mathbf{CR}^{\pm}$ and **EW**. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Figure 1. The large-t attractor for the solution of IVP when $\delta > 0$ and $u_+ > u_-$ is the expansion wave which allows for the adjustment of the solution from u_+ to u_-.

2.3.2. $\boldsymbol{-\frac{2}{3} < \delta < 0}$. We now investigate the structure of IVP as $t \to \infty$ when $-\frac{2}{3} < \delta < 0$ and $u_+ > u_-$. We recall from Section 2.2 that expansions (2.9) and (2.11) (with (2.12) of regions $\mathbf{III}^+$ $(x \to \infty, t = O(1))$ and $\mathbf{III}^-$ $(x \to -\infty, t = O(1))$ respectively, continue to remain uniform provided $|x| \gg t$ as $t \to \infty$. However, as already noted, a nonuniformity develops when $|x| = O\,(t)$ as $t \to \infty$. We begin by considering the asymptotic structure as $t \to \infty$ moving in from region $\mathbf{III}^+$, when $x \gg t$ as $t \to \infty$. To proceed we introduce a new region labelled as region **R**, in which $x = O\,(t)$ as $t \to \infty$. To examine region **R** we introduce the scaled coordinate

$$\zeta = \frac{x}{t}, \tag{2.55}$$

where $\zeta = O(1)$ as $t \to \infty$ in region **R**. Following Section 2.3.1 we readily find that in region **R** we have

$$u(\zeta,t) = u_+ - \begin{cases} \exp\left(-\frac{2}{3\sqrt{3}}\zeta^{\frac{3}{2}}t + \frac{u_+}{\sqrt{3}(\delta+1)}y^{\frac{1}{2}}t^{(\delta+1)} - \frac{1}{2}\ln t + H_2(\zeta) + o(1)\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\frac{1}{2} \le \delta < 0, \\ \exp\Bigg(- \frac{2}{3\sqrt{3}}\zeta^{\frac{3}{2}}t + \left[\frac{u_+}{\sqrt{3}(\delta+1)}y^{\frac{1}{2}}t^{(\delta+1)} + O\left(t^{(2\delta+1)}\right)\right] - \frac{1}{2}\ln t \\ \qquad +H_2(\zeta) + o(1)\Bigg), \qquad\qquad\qquad -\frac{2}{3} < \delta < -\frac{1}{2}, \end{cases} \tag{2.56}$$

as $t \to \infty$ with $\zeta = O(1)\,(> 0)$, and where the function $H_2(\zeta) : (0,\infty) \to \mathbb{R}$ remains undetermined being a remnant of the global evolution when $t = O(1)$, but having

$$H_2(\zeta) \sim -\frac{3}{4}\ln\zeta + \ln\left(\frac{(u_+ - u_-)3^{\frac{1}{4}}}{2\sqrt{\pi}}\right) \quad \text{as} \quad \zeta \to \infty.$$

Further, we will require that

$$H_2(\zeta) \sim -\frac{3}{4}\ln\zeta + \mathcal{C}_0 \quad \text{as} \quad \zeta \to 0^+,$$

where $\mathcal{C}_0\,(> 0)$ is a constant. Expansion (2.56) becomes nonuniform when $\zeta = O\left(t^{\delta}\right)$ as $t \to \infty$ (that is, when $x = O\left(t^{(\delta+1)}\right)$ as $t \to \infty$).

We next develop the asymptotic structure of $u\zeta, t)$ as $t \to \infty$, moving in from region **III**$^-$ (when $(-\zeta) \gg 1$) to $\zeta = O(1)$ as $t \to \infty$. To proceed we introduce a new region, labelled **L**. Following Section 2.3.1 we obtain in region **L** that

$$u(\zeta,t) = u_- + \left(e^{k^+(\zeta,t)} + e^{k^-(\zeta,t)}\right) \tag{2.57}$$

as $t \to \infty$ with $\zeta = O(1)\,(< 0)$, and where

$$k^{\pm}(\zeta,t) = \begin{cases} \pm i\,\frac{2}{3\sqrt{3}}(-\zeta)^{\frac{3}{2}}t \pm\, i\,\frac{u_-}{\sqrt{3}(\delta+1)}(-\zeta)^{\frac{1}{2}}t^{(\delta+1)} - \frac{1}{2}\ln t + H_3(\zeta) \pm\, i\,H_4(\zeta) + o(1), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\frac{1}{2} \le \delta < 0, \\ \pm i\,\frac{2}{3\sqrt{3}}(-\zeta)^{\frac{3}{2}}t \pm\, i\,\left[\frac{u_-}{\sqrt{3}(\delta+1)}(-\zeta)^{\frac{1}{2}}t^{(\delta+1)} + O\left(t^{(2\delta+1)}\right)\right] - \frac{1}{2}\ln t + H_3(\zeta) \\ \qquad \pm\, i\,H_4(\zeta) + o(1), \qquad\qquad\qquad -\frac{3}{2} < \delta < -\frac{1}{2}, \end{cases} \tag{2.58}$$

as $t \to \infty$, with $\zeta = O(1)\,(< 0)$, and where the functions $H_3(\zeta)$ and $H_4(\zeta)$ remain undetermined, but matching to the far field requires that

$$H_3(\zeta) \sim -\frac{3}{4}\ln(-\zeta) + \ln\left(\frac{(u_+ - u_-)3^{\frac{1}{4}}}{2\sqrt{\pi}}\right), \quad H_4(\zeta) \sim \frac{\pi}{4} \quad \text{as} \quad \zeta \to -\infty.$$

Expansion (2.58) becomes nonuniform when $(-\zeta) = O(t^{\delta})$ as $t \to \infty$ (that is, when $(-x) = O\left(t^{(\delta+1)}\right)$ as $t \to \infty$).

Therefore, to complete the asymptotic structure in this case it remains to investigate the region when $|\zeta| = O(t^{\delta})$ as $t \to \infty$. To investigate this region we introduce the scaled coordinate

$$y = \frac{x}{t^{(\delta+1)}} \tag{2.59}$$

as $t \to \infty$ with $y = O(1)$. The asymptotic structure of the solution to IVP as $t \to \infty$ follows, after some minor modification, that given through regions $\mathbf{IV}^{\pm}$, $\mathbf{V}^{\pm}$, $\mathbf{CR}^{\pm}$ and **EW** in Section 2.3.1 and is summarized here for brevity.

Region $\mathbf{IV}^{+}$. $x = O\left(t^{(\delta+1)}\right)$ as $t \to \infty$
$y = \frac{x}{t^{(\delta+1)}} = O(1)\ \left(\in \left(\frac{u_+}{(\delta+1)}, \infty\right)\right)$ as $t \to \infty$, and

$$u(y,t) = u_+ - \exp\Bigg(-\frac{2}{3\sqrt{3}}\left(y - \frac{u_+}{(\delta+1)}\right)^{\frac{3}{2}} t^{\frac{1}{2}(3\delta+2)} - \frac{1}{4}(3\delta+2)\ln t \\ -\frac{3}{4}\ln\left(y - \frac{u_+}{(\delta+1)}\right) + \ln \mathcal{C}_0 + o(1)\Bigg)$$

as $t \to \infty$ with $y = O(1)\ \left(\in \left(\frac{u_+}{(\delta+1)}, \infty\right)\right)$.

Region $\mathbf{V}^{+}$. $x = \frac{u_+}{(\delta+1)} t^{(\delta+1)} + O(t)$ as $t \to \infty$
$\eta = \left(y - \frac{u_+}{(\delta+1)}\right) t^{\delta} = O(1)\,(>0)$ as $t \to \infty$, and

$$u(\eta,t) = u_+ - \exp\left(-\frac{2}{3\sqrt{3}}\eta^{\frac{3}{2}}\, t - \frac{1}{2}\ln t + \hat{H}^{+}(\eta) + o(1)\right)$$

as $t \to \infty$ with $\eta = O(1)\,(>0)$, and where the function $\hat{H}^{+}(\eta) : (0,\infty) \to \mathbb{R}$ is undetermined, but having

$$\hat{H}^{+}(\eta) \sim \begin{cases} -\frac{3}{4}\ln\eta + \ln\mathcal{C}_0 & \text{as} \quad \eta \to \infty, \\ \left(\frac{3}{2}\delta + \frac{1}{4}\right)\ln\eta + \ln D^* & \text{as} \quad \eta \to 0^+, \end{cases}$$

where $D^*\,(>0)$ is a constant.

Region $\mathbf{CR}^{+}$. $x = \frac{u_+}{(\delta+1)} + O\left(t^{\frac{1}{3}}\right)$ as $t \to \infty$
$\xi = \eta t^{\frac{2}{3}} = O(1)$ as $t \to \infty$, and

$$u(\xi,t) = u_+ + F^*(\xi) t^{-\frac{1}{3}(3\delta+2)} + o\left(t^{-\frac{1}{3}(3\delta+2)}\right)$$

as $t \to \infty$ with $\xi = O(1)$, and where $F^*(\xi)$ is the solution to boundary value problem (2.25), (2.27) and (2.28) when $D = D^*$. Also

$$F^*(\xi) \sim \begin{cases} -D^*\xi^{\left(\frac{3}{2}\delta+\frac{1}{4}\right)} e^{-\frac{2}{3\sqrt{3}}\xi^{\frac{3}{2}}} & \text{as} \quad \xi \to \infty, \\ (\delta+1)\xi + \alpha(-\xi)^{-\frac{1}{(3\delta+2)}} & \text{as} \quad \xi \to -\infty, \end{cases} \tag{2.60}$$

with D^* and α discussed earlier. A numerical study of initial value problem (2.25) and (2.27) using a shooting method reveals that there exists a value $D = D^* > 0$ such that boundary condition (2.28) is satisfied for each $D \in (0, D^*]$, whilst for each $D \in (D^*, \infty)$ the solution blows up at finite-ξ. In particular, the solution of (2.25), (2.27) is oscillatory in $\xi < 0$ when $0 < D < D^*$, while $F(\xi) \sim (\delta+1)\xi$ as $\xi \to -\infty$ when $D = D^*$. Confirming that a unique solution, $F^*(\xi)$, to boundary value problem (2.25), (2.27) and (2.28) satisfying condition $(2.60)_2$ exists when $D = D^*$.

Region **EW**. $x = O\left(t^{(\delta+1)}\right)$ as $t \to \infty$
$y = \frac{x}{t^{(\delta+1)}} = O(1) \left(\in \left(\frac{u_-}{(\delta+1)}, \frac{u_+}{(\delta+1)}\right)\right)$ as $t \to \infty$, and

$$u(y,t) = (\delta+1)y + O\left(t^{-(\delta+1)}\right)$$

as $t \to \infty$ with $y = O(1) \left(\in \left(\frac{u_-}{(\delta+1)}, \frac{u_+}{(\delta+1)}\right)\right)$.

Region **CR**$^-$. $x = \frac{u_-}{(\delta+1)} + O\left(t^{\frac{1}{3}}\right)$ as $t \to \infty$
$\xi = \bar{\eta} t^{\frac{2}{3}} = O(1)$ as $t \to \infty$, and

$$u(\xi,t) = u_+ + F(\xi) t^{-\frac{1}{3}(3\delta+2)} + o\left(t^{-\frac{1}{3}(3\delta+2)}\right)$$

as $t \to \infty$ with $\xi = O(1)$, and where $F(\xi)$ is the solution to boundary value problem (2.50), (2.51) and (2.54), and has

$$F(\xi) \sim \begin{cases} (\delta+1)\xi + \bar{\alpha}\xi^{-\frac{1}{(3\delta+2)}} & \text{as} \quad \xi \to \infty, \\ \beta_0 (-\xi)^{\left(\frac{3}{2}\delta+\frac{1}{4}\right)} \cos\left(\frac{2}{3\sqrt{3}}(-\xi)^{\frac{3}{2}} + \beta_1\right) & \text{as} \quad \xi \to -\infty, \end{cases} \tag{2.61}$$

with β_0 and β_1 as discussed earlier. A numerical study of initial value problem (2.50) and (2.54) using a shooting method confirms that a unique solution, $F(\xi)$, to boundary value problem (2.50), (2.51) and (2.54) exists for each $\bar{\alpha} > 0$, but no solution exists for $\bar{\alpha} \leq 0$. Although we have not been able to determine $\bar{\alpha}$ in this analysis, a specified $\bar{\alpha}$ fixes β_0 and β_1 uniquely.

Region **V**$^-$. $x = \frac{u_-}{(\delta+1)} + O(t)$ as $t \to \infty$
$\bar{\eta} = \left(y - \frac{u_-}{(\delta+1)}\right) t^{\delta} = O(1) (< 0)$ as $t \to \infty$, and

$$u(\bar{\eta},t) = u_- + \frac{\hat{H}_1(\bar{\eta})}{t^{\frac{1}{2}}} \cos\left(\frac{2}{3\sqrt{3}}(-\bar{\eta})^{\frac{3}{2}} t + H_0(\bar{\eta})\right) + o\left(\frac{1}{t^{\frac{1}{2}}}\right) \tag{2.62}$$

as $t \to \infty$ with $\bar{\eta} = O(1)\,(< 0)$, and where the functions $\hat{H}_1(\bar{\eta})$ and $H_0(\bar{\eta})$ are undetermined, having

$$\hat{H}_1(\bar{\eta}) \sim \beta_0 (-\bar{\eta})^{\left(\frac{3}{2}\delta+\frac{1}{4}\right)}, \quad H_0(\bar{\eta}) \sim \beta_1 \quad \text{as} \quad \bar{\eta} \to 0^-.$$

Region **IV**$^-$. $x = O\left(t^{(\delta+1)}\right)$ as $t \to \infty$
$y = \frac{x}{t^{(\delta+1)}} = O(1) \left(\in \left(-\infty, \frac{u_-}{(\delta+1)}\right)\right)$ as $t \to \infty$, and

$$u(y,t) = u_- + \left(e^{\hat{k}^+(y,t)} + e^{\hat{k}^-(y,t)}\right) \tag{2.63}$$

as $t \to \infty$, and where

$$\hat{k}^{\pm}(y,t) = \pm i \frac{2}{3\sqrt{3}} \left(\frac{u_-}{(\delta+1)} - y\right)^{\frac{3}{2}} t^{\frac{1}{2}(3\delta+2)} - \frac{1}{2}\ln t + H_5(y) \pm i\, H_6(y) + o(1)$$

as $t \to \infty$ with $y = O(1)$ $\left(\in \left(-\infty, \frac{u_-}{(\delta+1)}\right)\right)$, and where the functions $H_5(y)$ and $H_6(y)$ remain undetermined.

The undetermined functions in expansions (2.58), (2.63), (2.62) of regions $\mathbf{L}$, $\mathbf{IV}^-$, $\mathbf{V}^-$ respectively, are remnants from the evolution when $t = O(1)$. Although, they remain undetermined they must allow for matching between regions $\mathbf{L}$, $\mathbf{IV}^-$ and $\mathbf{V}^-$.

The asymptotic structure to the solution of IVP as $t \to \infty$ when $-\frac{2}{3} < \delta < 0$ and $u_+ > u_-$ is now complete. A uniform approximation has been given through regions $\mathbf{L}$, $\mathbf{IV}^\pm$, $\mathbf{V}^\pm$, $\mathbf{CR}^\pm$, $\mathbf{EW}$ and $\mathbf{R}$. The large-t attractor for the solution of IVP when $-\frac{3}{2} < \delta < 0$ and $u_+ > u_-$ is the expansion wave which allows for the adjustment of the solution from u_+ to u_-.

2.3.3. *Summary.* In this section we have obtained, via the method of matched asymptotic coordinate expansions, the uniform asymptotic structure of the large-t solution to the initial-value problem IVP for $-\frac{2}{3} < \delta < 0$ and $\delta > 0$ when $u_+ > u_-$ (the case when $\delta = 0$ having been considered in [24]). In each case the large-t structure was obtained by careful consideration of the asymptotic structures as $t \to 0$ $(-\infty < x < \infty)$ and as $|x| \to \infty$ $(t \geq O(1))$. In both cases the solution, $u(x,t)$, to IVP exhibits the formation of an expansion wave profile in $y > \frac{u_-}{(\delta+1)}$ (where we recall that $y = xt^{-(\delta+1)}$), with

$$u(yt^{(\delta+1)}, t) \to \begin{cases} u_+, & y > \frac{u_+}{(\delta+1)}, \\ (\delta+1)y, & \frac{u_-}{(\delta+1)} \leq y \leq \frac{u_+}{(\delta+1)}, \end{cases}$$

as $t \to \infty$, uniformly in y, while the solution is oscillatory (oscillating about $u = u_-$) for $y < \frac{u_-}{(\delta+1)}$, with the oscillatory envelope for $(-y) \gg 1$ being of order $O\left(t^{-\frac{1}{4}(3\delta+2)}\right)$ $\left[O\left(t^{-\frac{1}{2}}\right)\right]$ as $t \to \infty$ when $\delta > 0$ $\left[-\frac{2}{3} < \delta < 0\right]$, respectively. The rate of convergence to the expansion wave in region $\mathbf{EW}$ is of $O\left(t^{-(\delta+1)}\right)$ as $t \to \infty$. Finally, we conclude by noting that regions $\mathrm{V}^\pm$ present in the large-t solution of IVP are not present in the large-t solution given in [24] for the case $\delta = 0$.

3. Discussion. We conclude by giving a brief overview of the structure of the large-t solution of IVP when the initial data is continuously differentiable and has algebraic decay as $|x| \to \infty$ (step-like initial data), rather than the discontinuous expansive step considered above. Specifically, we consider

$$u(x,0) = \begin{cases} u_- + \dfrac{A_L}{(-x)^\gamma} + O\Big(E(|x|)\Big) & \text{as} \quad x \to -\infty, \\ u_+ + \dfrac{A_R}{x^\gamma} + O\Big(E(|x|)\Big) & \text{as} \quad x \to \infty, \end{cases} \tag{3.1}$$

where $A_L\,(> 0), A_R\,(< 0)$ and $\gamma\,(> 0)$ are parameters and $E(|x|)$ is linearly exponentially small in x as $|x| \to \infty$. In what follows we refer to initial-value problem (1.1), (3.1) and (1.3) as IVP2. The structure of asymptotic solution of IVP2 as $t \to 0$ $(-\infty < x < \infty)$ as $|x| \to \infty$ $(t = O(1))$ follows, after minor modification, that given in [22] ([7]) and is omitted here for brevity. We now review the structure of the asymptotic solution of IVP2 as $t \to \infty$, and focus attention initially on the situation when $0 < \gamma < 3\delta + 2$. The large-t structure of the solution of IVP2 when $\delta > -\frac{2}{3}$ (taken for direct comparison with the

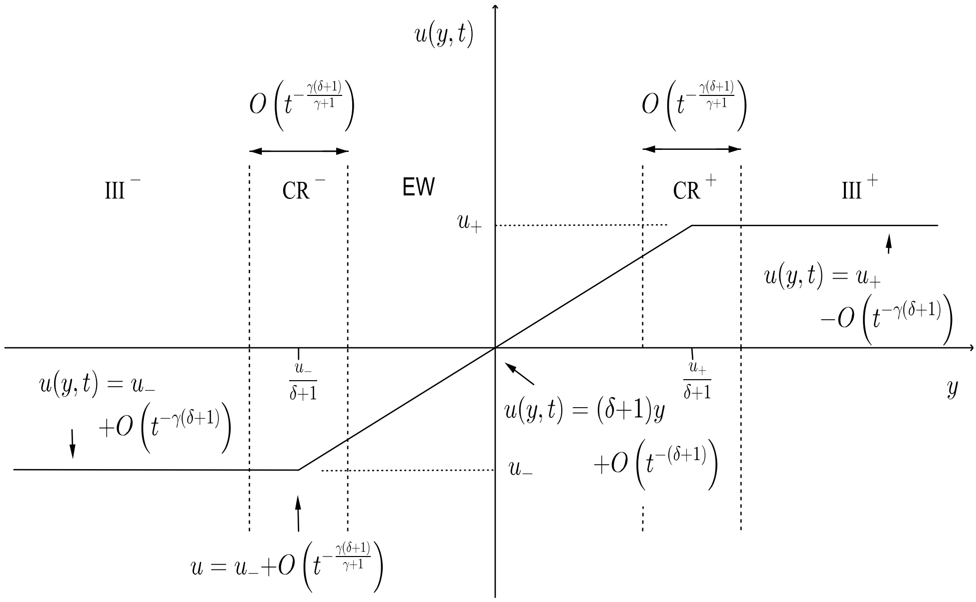

FIG. 2. A schematic representation of the asymptotic structure of $u(y,t)$ in the (y,u) plane as $t \to \infty$ for IVP2 when $\gamma < 3\delta + 2$. We recall that $y = \frac{x}{t^{(\delta+1)}}$ and $u = u_+ - O\left(t^{-\frac{\gamma(\delta+1)}{\gamma+1}}\right)$ as $t \to \infty$ in region CR$^+$.

analysis presented in Section 2), $0 < \gamma < 3\delta + 2$ and $u_+ > u_-$ consists of five asymptotic regions, which in terms of the coordinate y (where $y = xt^{-(\delta+1)}$), are displayed in Figure 2. These five regions are namely, as $t \to \infty$

Region $\mathbf{III}^+$. $x = O\left(t^{(\delta+1)}\right)$ as $t \to \infty$
$y = \frac{x}{t^{(\delta+1)}} = O(1)$ $\left(\in \left(\frac{u_+}{(\delta+1)}, \infty\right)\right)$ as $t \to \infty$, and

$$u(y,t) = u_+ + A_R \left(y - \frac{u_+}{(\delta+1)}\right)^{-\gamma} t^{-\gamma(\delta+1)} + o\left(t^{-\gamma(\delta+1)}\right) \tag{3.2}$$

as $t \to \infty$ with $y = O(1)$ $\left(\in \left(\frac{u_+}{(\delta+1)}, \infty\right)\right)$. Expansion (3.2) becomes nouniform when $y = \frac{u_+}{(\delta+1)} + o(1)$ as $t \to \infty$, and further examination in the case when $0 < \gamma < (3\delta+2)$ reveals that expansion (3.2) becomes nonuniform when $y = \frac{u_+}{(\delta+1)} + O\left(t^{-\frac{\gamma(\delta+1)}{\gamma+1}}\right)$ as $t \to \infty$.

Region $\mathbf{CR}^+$. $x = \frac{u_+}{(\delta+1)} t^{(\delta+1)} + O\left(t^{\frac{(\delta+1)}{(\gamma+1)}}\right)$ as $t \to \infty$
$\xi = \left(y - \frac{u_+}{(\delta+1)}\right) t^{\frac{\gamma(\delta+1)}{\gamma+1}} = O(1)$ as $t \to \infty$, and

$$u(\xi,t) = u_+ + F(\xi) t^{-\frac{\gamma(\delta+1)}{\gamma+1}} + o\left(t^{-\frac{\gamma(\delta+1)}{\gamma+1}}\right) \tag{3.3}$$

as $t \to \infty$ with $\xi = O(1)$, and where $F(\xi)$ satisfies the nonlinear differential equation

$$FF_\xi - \frac{(\delta+1)}{(\gamma+1)}\xi F_\xi - \gamma\frac{(\delta+1)}{(\gamma+1)}F = 0, \quad -\infty < \xi < \infty. \tag{3.4}$$

The matching condition with region $\mathbf{III}^+$ requires

$$F(\xi) \sim A_R \xi^{-\gamma} \quad \text{as} \quad \xi \to \infty. \tag{3.5}$$

We note that equation (3.4) admits the exact solution

$$F(\xi) = \xi, \quad -\infty < \xi < \infty. \tag{3.6}$$

In general, equation (3.4) is of homogeneous type, and admits a quadrature, after which the solution of (3.4) with (3.5) is given implicitly by

$$\xi = \frac{F(\xi)}{(\delta+1)} + \left(\frac{A_R}{F(\xi)}\right)^{\frac{1}{\gamma}}, \quad -\infty < \xi < \infty. \tag{3.7}$$

It follows from (3.7) that (on recalling that $A_R < 0$),

$$F(\xi) < 0 \quad \text{for all} \quad -\infty < \xi < \infty, \tag{3.8}$$

$$F(\xi) \quad \text{is strictly monotone increasing, with } -\infty < \xi < \infty, \tag{3.9}$$

$$F(\xi) \sim A_R\xi^{-\gamma} + \frac{A_R^2\gamma}{(\delta+1)}\xi^{-(2\gamma+1)} \quad \text{as} \quad \xi \to \infty, \tag{3.10}$$

$$F(\xi) \sim (\delta+1)\xi - (-A_R)^{\frac{1}{\gamma}}(\delta+1)^{1-\frac{1}{\gamma}}(-\xi)^{-\frac{1}{\gamma}} \quad \text{as} \quad \xi \to -\infty. \tag{3.11}$$

Region **EW**. $x = O\left(t^{(\delta+1)}\right)$ as $t \to \infty$
$y = \frac{x}{t^{(\delta+1)}} = O(1)\ \left(\in \left(\frac{u_-}{(\delta+1)}, \frac{u_+}{(\delta+1)}\right)\right)$ as $t \to \infty$, and

$$u(y,t) = (\delta+1)y + O\left(t^{-(\delta+1)}\right) \tag{3.12}$$

as $t \to \infty$ with $y = \frac{x}{t^{(\delta+1)}} = O(1)\ \left(\in \left(\frac{u_-}{(\delta+1)}, \frac{u_+}{(\delta+1)}\right)\right)$.

Region $\mathbf{CR}^-$. $x = \frac{u_-}{(\delta+1)}t^{(\delta+1)} + O\left(t^{\frac{(\delta+1)}{(\gamma+1)}}\right)$ as $t \to \infty$
$\xi = \left(y - \frac{u_-}{(\delta+1)}\right)t^{\frac{\gamma(\delta+1)}{\gamma+1}} = O(1)$ as $t \to \infty$, and

$$u(\xi,t) = u_- + \hat{F}(\xi)t^{-\frac{\gamma(\delta+1)}{\gamma+1}} + o\left(t^{-\frac{\gamma(\delta+1)}{\gamma+1}}\right) \tag{3.13}$$

as $t \to \infty$ with $\xi = O(1)$. The details are identical to those of region CR^+, and are not repeated here. In fact. making the replacement of A_R by $(-A_L)$, we obtain for $\hat{F}(\xi)$, $-\infty < \xi < \infty$, that $\hat{F}(\xi) = -F(-\xi)$ when $0 < \gamma < 3\delta + 2$.

Region $\mathbf{III}^-$. $x = O\left(t^{(\delta+1)}\right)$ as $t \to \infty$
$y = \frac{x}{t^{(\delta+1)}} = O(1)\ \left(\in \left(-\infty, \frac{u_-}{(\delta+1)}\right)\right)$ as $t \to \infty$, and

$$u(y,t) = u_- + A_L\left(\frac{u_-}{(\delta+1)} - y\right)^{-\gamma} t^{-\gamma(\delta+1)} + o\left(t^{-\gamma(\delta+1)}\right) \tag{3.14}$$

as $t \to \infty$ with $y = O(1)$ $\left(\in \left(-\infty, \frac{u_-}{(\delta+1)}\right)\right)$. When $0 < \gamma < (3\delta + 2)$ expansion (3.14) becomes nonuniform when $y = \frac{u_-}{(\delta+1)} + O\left(t^{-\frac{\gamma(\delta+1)}{\gamma+1}}\right)$ as $t \to \infty$.

The asymptotic structure to the solution of IVP2 as $t \to \infty$ when $\delta > -\frac{2}{3}$, $0 < \gamma < 3\delta + 2$ and $u_+ > u_-$ is now complete. A uniform approximation has been given through regions $\mathbf{III}^{\pm}$, $\mathbf{CR}^{\pm}$ and **EW**. The large-t attractor for the solution of IVP2 is the expansion wave which allows for the adjustment of the solution from u_+ to u_-.

Although the large-t solutions of IVP and IVP2 both exhibit the formation of an expansion wave structure we note that there are significant differences in the structure of the solution between the two problems. First, we observe immediately that the oscillations observed in the large-t solution of IVP in $y < \frac{u_-}{(\delta+1)}$ are not present up to $O\left(t^{-\gamma(\delta+1)}\right)$ as $t \to \infty$ in the solution to IVP2 (these oscillations being generated by the discontinuous initial data in IVP). Secondly, the corner regions $\mathbf{CR}^{\pm}$ are of thickness $O\left(t^{-\frac{1}{3}(3\delta+2)}\right)$ as $t \to \infty$ in the large-t solution of IVP, whereas they are thicker being of $O\left(t^{-\frac{\gamma(\delta+1)}{\gamma+1}}\right)$ as $t \to \infty$ in the large-t solution of IVP2 (when $0 < \gamma < 3\delta + 2$). The associated boundary value problems in these corner regions are related and accommodate the required change in structure in the large-t solution of the initial-value problem under consideration.

References

[1] M. J. Ablowitz and H. Segur, *Asymptotic solutions of the Korteweg-deVries equation*, Studies in Appl. Math. **57** (1976/77), no. 1, 13–44. MR0481656

[2] *Handbook of mathematical functions, with formulas, graphs, and mathematical tables*, Edited by Milton Abramowitz and Irene A. Stegun. Third printing, with corrections. National Bureau of Standards Applied Mathematics Series, vol. 55, Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1965. MR0177136

[3] J. Boussinesq, *Théorie de l'intumescence liquid appelée onde solitaire ou de translation, se propageant dans un canal rectangulaire*, Comptes Rendus Acad. Sci. 72 (1871) 755-759.

[4] John P. Boyd, *Weakly nonlocal solitary waves and beyond-all-orders asymptotics*, Mathematics and its Applications, vol. 442, Kluwer Academic Publishers, Dordrecht, 1998. Generalized solitons and hyperasymptotic perturbation theory. MR1636975

[5] Radu C. Cascaval, *Variable coefficient KdV equations and waves in elastic tubes*, Evolution equations, Lecture Notes in Pure and Appl. Math., vol. 234, Dekker, New York, 2003, pp. 57–69. MR2073735

[6] G. A. El and R. H. J. Grimshaw, *Generation of undular bores in the shelves of slowly-varying solitary waves*, Chaos **12** (2002), no. 4, 1015–1026, DOI 10.1063/1.1507381. MR1946775

[7] F. B. Ali Sulaiman, J. A. Leach, and D. J. Needham, *The large-time solution of Burgers' equation with time-dependent coefficients. II. Algebraic coefficients*, Stud. Appl. Math. **137** (2016), no. 3, 273–305, DOI 10.1111/sapm.12130. MR3564301

[8] A. P. Fordy, *Soliton theory: a brief synopsis*, Soliton theory: a survey of results, Nonlinear Sci. Theory Appl., Manchester Univ. Press, Manchester, 1990, pp. 3–22. MR1090584

[9] Zun Tao Fu, Shi Da Liu, Shi Kuo Liu, and Qiang Zhao, *New exact solutions to KdV equations with variable coefficients or forcing* (Chinese, with English and Chinese summaries), Appl. Math. Mech. **25** (2004), no. 1, 67–73, DOI 10.1007/BF02437295; English transl., Appl. Math. Mech. (English Ed.) **25** (2004), no. 1, 73–79. MR2041342

[10] Katrin Grunert and Gerald Teschl, *Long-time asymptotics for the Korteweg-de Vries equation via nonlinear steepest descent*, Math. Phys. Anal. Geom. **12** (2009), no. 3, 287–324, DOI 10.1007/s11040-009-9062-2. MR2525595

[11] R. Grimshaw, *Slowly varying solitary waves. I. Korteweg - de Vries equation*, Proc. Roy. Soc. London Ser. A **368** (1979), no. 1734, 359–375, DOI 10.1098/rspa.1979.0135. MR551468

[12] Roger Grimshaw, Efim Pelinovsky, and Tatjana Talipova, *Solitary wave transformation in a medium with sign-variable quadratic nonlinearity and cubic nonlinearity*, Phys. D **132** (1999), no. 1-2, 40–62, DOI 10.1016/S0167-2789(99)00045-7. MR1705698

[13] S. P. Hastings and J. B. McLeod, *A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vries equation*, Arch. Rational Mech. Anal. **73** (1980), no. 1, 31–51, DOI 10.1007/BF00283254. MR555581

[14] Karl R. Helfrich and W. Kendall Melville, *Long nonlinear internal waves*, Annual review of fluid mechanics. Vol. 38, Annu. Rev. Fluid Mech., vol. 38, Annual Reviews, Palo Alto, CA, 2006, pp. 395–425, DOI 10.1146/annurev.fluid.38.050304.092129. MR2206980

[15] E.M. de Jager, *On the origin of the Korteweg-de Vries equation*, ArXiv: math.HO/0602661 (2006)

[16] A. Jeffrey and T. Kakutani, *Weak nonlinear dispersive waves: A discussion centered around the Korteweg-de Vries equation*, SIAM Rev. **14** (1972), 582–643, DOI 10.1137/1014101. MR0334675

[17] R. S. Johnson, *On the development of a solitary wave moving over an uneven bottom*, Proc. Cambridge Philos. Soc. **73** (1973), 183–203. MR0312817

[18] R. S. Johnson, *A modern introduction to the mathematical theory of water waves*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1997. MR1629555

[19] R.S. Johnson, *Some numerical solutions of a variable-coefficient Korteweg-de Vries equation (with applications to solitary wave development on a shelf)*, J. Fluid Mech. 54 (1972) 81-91.

[20] Nalini Joshi, *Painlevé property of general variable-coefficient versions of the Korteweg-de Vries and nonlinear Schrödinger equations*, Phys. Lett. A **125** (1987), no. 9, 456–460, DOI 10.1016/0375-9601(87)90184-8. MR917430

[21] D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Philos. Mag. (5) **39** (1895), no. 240, 422–443. MR3363408

[22] J. A. Leach, *The large-time solution of Burgers' equation with time-dependent coefficients. I. The coefficients are exponential functions*, Stud. Appl. Math. **136** (2016), no. 2, 163–188, DOI 10.1111/sapm.12098. MR3457798

[23] J. A. Leach and D. J. Needham, *Matched asymptotic expansions in reaction-diffusion theory*, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2004. MR2013330

[24] J. A. Leach and D. J. Needham, *The large-time development of the solution to an initial-value problem for the Korteweg-de Vries equation. I. Initial data has a discontinuous expansive step*, Nonlinearity **21** (2008), no. 10, 2391–2408, DOI 10.1088/0951-7715/21/10/010. MR2439485

[25] J. A. Leach and D. J. Needham, *The large-time development of the solution to an initial-value problem for the Korteweg–de Vries equation. II. Initial data has a discontinuous compressive step*, Mathematika **60** (2014), no. 2, 391–414, DOI 10.1112/S0025579313000284. MR3229496

[26] W. Lick, *Nonlinear wave propergation in fluids*, Annu. Rev. Fluid Mech. 2 (1970), 113–134.

[27] S. Maxon and J. Viecelli, *Cylindrical solitons*, Phys. Fluids 17 (1974), 1614-1616.

[28] S. Maxon and J. Viecelli, *Spherical solitons*, Phys. Rev. Lett. 32 (1974), 4-6.

[29] C.C. Mei. *The Applied Dynamics of Ocean Surface Waves*. World Scientific, Singapore (1989)

[30] John W. Miles, *Solitary waves*, Annual review of fluid mechanics, Vol. 12, Annual Reviews, Palo Alto, Calif., 1980, pp. 11–43. MR565388

[31] Robert M. Miura, *The Korteweg-de Vries equation: a survey of results*, SIAM Rev. **18** (1976), no. 3, 412–459, DOI 10.1137/1018076. MR0404890

[32] N. Nirmala, M. J. Vedan, and B. V. Baby, *Auto-Bäcklund transformation, Lax pairs, and Painlevé property of a variable coefficient Korteweg-de Vries equation. I*, J. Math. Phys. **27** (1986), no. 11, 2640–2643, DOI 10.1063/1.527282. MR861323

[33] Lord Rayleigh, *On waves*, Philos. Mag. 1 (1876), 257–279.

[34] Alwyn Scott (ed.), *Encyclopedia of nonlinear science*, Routledge, New York, 2005. MR2265413

[35] H. Segur and M. J. Ablowitz, *Asymptotic solutions of non-linear evolution equations and a Painleve transcendent*, Physica D (1981), 165–184.

[36] M. Vlieg-Hulstman and W. D. Halford, *Exact solutions to KdV equations with variable coefficients and/or nonuniformities*, Comput. Math. Appl. **29** (1995), no. 1, 39–47, DOI 10.1016/0898-1221(94)00205-Y. MR1308136

[37] H. Washimi and T. Taniuti, *Propergation of ion-accoustic solitary waves of small amplitude*, Phys. Rev. Lett. 17 (1966), 996–998.

[38] L. van Wijngaarden, *On the equation of motion for mixtures of liquid and gas bubbles*, J. Fluid Mech. 33 (1968), 465-474.

[39] N. J. Zabusky, *Solitons and energy transport in nonlinear latticies*, Comput. Phys. Commun. 5 (1973), 1–10.