QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXXVI, NUMBER 2
JUNE 2018, PAGES 383–405
http://dx.doi.org/10.1090/qam/1485
Article electronically published on September 21, 2017

CONVERGENCE OF A MASS-LUMPED FINITE ELEMENT METHOD FOR THE LANDAU-LIFSHITZ EQUATION

By

EUGENIA KIM (*Department of Mathematics, University of California, Berkeley, CA 94720*)

AND

JON WILKENING (*Department of Mathematics and Lawrence Berkeley National Laboratory, University of California, Berkeley, CA 94720*)

Abstract. The dynamics of the magnetic distribution in a ferromagnetic material is governed by the Landau-Lifshitz equation, which is a nonlinear geometric dispersive equation with a nonconvex constraint that requires the magnetization to remain of unit length throughout the domain. In this article, we present a mass-lumped finite element method for the Landau-Lifshitz equation. This method preserves the nonconvex constraint at each node of the finite element mesh, and is energy nonincreasing. We show that the numerical solution of our method for the Landau-Lifshitz equation converges to a weak solution of the Landau-Lifshitz-Gilbert equation using a simple proof technique that cancels out the product of weakly convergent sequences. Numerical tests for both explicit and implicit versions of the method on a unit square with periodic boundary conditions are provided for structured and unstructured meshes.

Micromagnetics is the study of the behavior of ferromagnetic materials at sub-micron length scales, including magnetization reversal and hysteresis effects [22]. The dynamics of the magnetic distribution of a ferromagnetic material occupying a region $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$ is governed by the Landau-Lifshitz (LL) equation [22, 25, 36]. The magnetization

Received August 8, 2017.
2010 *Mathematics Subject Classification.* Primary 65M60, 35Q60; Secondary 78M10.
Key words and phrases. Landau-Lifshitz equation, Landau-Lifshitz-Gilbert equation, micromagnetics, finite element methods, mass-lumped method, convergence, weak solutions.
The first author was supported in part by the U.S. Department of Energy, Office of Science, Office of Workforce Development for Teachers and Scientists, Office of Science Graduate Student Research (SCGSR) program. The SCGSR program is administered by the Oak Ridge Institute for Science and Education for the DOE under contract number DE-AC05-06OR23100.
The second author was supported in part by the U.S. Department of Energy, Office of Science, Applied Scientific Computing Research, under award number DE-AC02-05CH11231.
E-mail address: kim107@math.berkeley.edu
E-mail address: wilkening@berkeley.edu

$m(x,t) : \Omega \times [0,T] \to \mathbb{S}^2 \subset \mathbb{R}^3$ satisfies

$$\begin{cases} \partial_t m = -m \times h - \alpha m \times (m \times h) & \text{in } \Omega, \\ \frac{\partial m}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ m(x,0) = m_0(x), \end{cases} \tag{0.1}$$

where α is a dimensionless damping parameter and h is an effective field given by

$$h(m) := -\frac{\delta \mathcal{E}}{\delta m}(m) = \eta \Delta m - Q(m_2 e_2 + m_3 e_3) + h_s(m) + h_e. \tag{0.2}$$

Here η is the exchange constant, Q is an anisotropy constant, h_s is the stray field, h_e is an external field, and $\frac{\delta \mathcal{E}}{\delta m}$ is the functional derivative of the Landau-Lifshitz energy, defined by

$$\mathcal{E}(m) = \frac{\eta}{2} \int_\Omega |\nabla m|^2 + \frac{Q}{2} \int_\Omega m_2^2 + m_3^2 - \frac{1}{2} \int_\Omega h_s(m) \cdot m - \int_\Omega h_e \cdot m. \tag{0.3}$$

The first term is the exchange energy, which tries to align the magnetization locally; the second term is the anisotropy energy, which tries to orient the magnetization in certain easy direction taken to be e_1; the third term is the stray field energy, which is induced by the magnetization distribution inside the material; and the last term is the external field energy, which tries to align the magnetization with an external field. We denote the lower order terms in (0.2) by

$$\bar{h}(m) := -Q(m_2 e_2 + m_3 e_3) + h_s(m) + h_e. \tag{0.4}$$

When considering mathematical properties such as existence and regularity of the solution, these terms can be considered lower order compared to the exchange term [6]. They also have fewer derivatives than the exchange term, and can be treated as lower order when developing numerical methods.

The stray field h_s depends on m via $h_s = -\nabla \phi$, where the potential ϕ satisfies

$$\begin{gathered} \Delta \phi = \begin{cases} \nabla \cdot m & \text{in } \Omega, \\ 0 & \text{on } \partial\Omega, \end{cases} \\ [\phi]_{\partial\Omega} = 0, \qquad \left[\frac{\partial \phi}{\partial \nu}\right]_{\partial\Omega} = -m \cdot \nu. \end{gathered} \tag{0.5}$$

Here $[v]_{\partial\Omega}(x) = v(x^+) - v(x^-)$ is the jump in v across the boundary $\partial\Omega$ from inside $(-)$ to outside $(+)$; see [25].

There are several equivalent forms of the Landau-Lifshitz (LL) equation. The following is the Landau-Lifshitz-Gilbert (LLG) equation:

$$\partial_t m - \alpha m \times \partial_t m = -(1 + \alpha^2)(m \times h). \tag{0.6}$$

Also, the equation

$$\alpha \partial_t m + m \times \partial_t m = (1+\alpha^2)(h - (h \cdot m)m) \tag{0.7}$$

is equivalent to LL and LLG; see [6]. If only the gyromagnetic term is present in equation (0.1), i.e. if $\partial_t m = -m \times \Delta m$, it is called a Schrödinger map into $\mathbb{S}^2$ [29]. This is a geometric generalization of the linear Schrödinger equation. If only the damping term is present, i.e. if $\partial_t m = -m \times (m \times \Delta m)$, it is called a harmonic map heat flow into $\mathbb{S}^2$ [29].

In 1935, Landau and Lifshitz [37] calculated the structure of the domain walls between antiparallel domains, which started the theory of micromagnetics. The theory was further developed by W. F. Brown, Jr. in [15]. Applications of micromagnetics include magnetic sensor technology, magnetic recording, and magnetic storage devices such as hard drives and magnetic memory (MRAM) [22].

Finite difference methods for micromagnetics can be derived in two different ways [41]. The first is a field-based approach in which the effective field h is discretized directly, and the other is an energy-based approach in which the effective field is derived from the discretized energy. Finite element methods can also be derived in a number of ways. In [44], the Landau-Lifshitz-Gilbert equation (0.6) is used to obtain a discrete system by approximating the magnetization by piecewise linear function on a finite element mesh and then applying time integration in the resulting system of ODEs. In [22], the effective field h is calculated by taking a functional derivative of the discretized energy, where the magnetization in the energy formula is approximated by piecewise linear functions. Extensive work has also been done developing time stepping schemes for micromagnetics [18, 25]. In [31], semi-analytic integration in time was introduced, which is explicit and first order in time and allows stepsize control. In [34, 38], geometric integration methods were applied to the Landau-Lifshitz equation. In [26, 47], a Gauss-Seidel projection method was developed that treats the gyromagnetic term and damping term separately to overcome the difficulty of the stiffness and the nonlinearity of the equation.

However, relatively little work has been done deriving error estimates or establishing rigorous convergence results for weak solutions. In a series of papers [4, 6, 8], Alouges and various co-authors introduced a convergent finite element method based on equation (0.7), an equivalent form of the Landau-Lifshitz equation, that is first order in time. The idea is to use a tangent plane formulation at each timestep, where the velocity vector lies in the finite element space perpendicular to the magnetization at each node. One advantage of this method is that at each step, only a linear system has to be solved, although the Landau-Lifshitz-Gilbert equation is nonlinear. More recently, they developed a formally second order in time scheme [7, 35] that performs better than first order in practice, though not fully at second order. Another finite element scheme was introduced by Bartels and Prohl in [11] based on the Landau-Lifshitz-Gilbert equation, which is an implicit, unconditionally stable method, but involves solving nonlinear equations at each timestep. This method is second order in time; however, there is still a time step constraint, namely that k/h^2 remain bounded, to guarantee the existence of the solutions of the nonlinear systems. Cimrák [19] introduced a finite element method based on the Landau-Lifshitz equation, which is an implicit, unconditionally stable method, but also has nonlinear inner iterations. We note that the Backward Euler method and higher-order diagonally implicit Runge-Kutta (DIRK) methods [30] generally involve solving nonlinear equations at each internal Runge-Kutta stage when applied to nonlinear PDEs.

In this article, we introduce a family of mass-lumped finite element methods for the Landau-Lifshitz equation. The implicit version is similar in computational complexity to the algorithms in [4, 6, 8] in that each timestep involves solving a large sparse linear system. The explicit method is more efficient than the explicit version of [4, 6, 8] as it is completely explicit — it does not even require that a linear system involving a mass matrix be solved as the effective mass matrix is diagonal. The method involves finding the velocity vector in the tangent plane of the magnetization by discretizing the Landau-Lifshitz equation instead of the Landau-Lifshitz-Gilbert equation, as was done in [4, 6–8, 35]. By building a numerical scheme based on the Landau-Lifshitz equation instead of the Landau-Lifshitz-Gilbert equation, we can naturally apply the scheme to limiting cases such as the Schrödinger map or harmonic map heat flow [12, 20, 29, 32, 33]. The main result of the paper is a proof that the numerical solution of our scheme for the Landau-Lifshitz equation converges to a weak solution of the Landau-Lifshitz-Gilbert equation, using a simple technique that cancels out the product of two weakly convergent sequences. Our proof builds on tools developed in [8]. For simplicity, we defer the treatment of the stray field to future work. This term poses computational challenges [1, 2, 13, 16, 21, 23, 27, 39, 40, 43, 46, 48], but does not affect the convergence results since it is a lower order term in comparison to the exchange term; see [8].

The paper is organized as follows. In section 1, we introduce a finite element mesh and review the weak formulation of the Landau-Lifshitz-Gilbert equation. In section 2, the main algorithm and the main theorem will be introduced. In section 3, we conduct a numerical test for the equation $h = \Delta m$ on the unit square with periodic boundary conditions, where an exact analytical solution is known from [24]. In section 4, the proof of the main theorem will be presented.

1. Weak solutions, meshes and the finite element space.

Let us denote $\Omega_T = \Omega \times (0, T)$. The definition of a weak solution of the Landau-Lifshitz-Gilbert equation is given by

DEFINITION 1.1. Let $m_0(x) \in H^1(\Omega)^3$ with $|m_0(x)| = 1$ *a.e.* Then m is a weak solution of (0.6) if for all $T > 0$,

(i) $m(x,t) \in H^1(\Omega_T)^3$, $|m(x,t)| = 1$ *a.e.*,
(ii) $m(x,0) = m_0(x)$ in the trace sense,
(iii) m satisfies

$$\int_{\Omega_T} \partial_t m \cdot \phi - \alpha \int_{\Omega_T} (m \times \partial_t m) \cdot \phi \tag{1.1}$$
$$= (1+\alpha^2)\eta \sum_{l=1}^{d} \int_{\Omega_T} (m \times \partial_{x_l} m) \cdot \partial_{x_l} \phi - (1+\alpha^2) \sum_{l=1}^{d} \int_{\Omega_T} (m \times \bar{h}(m)) \cdot \phi.$$

for all $\phi \in H^1(\Omega_T)^3$.

(iv) m satisfies an energy inequality

$$C\int_{\Omega_T} |\partial_t m|^2 + \mathcal{E}(m(x,T)) \le \mathcal{E}(m(x,0)). \tag{1.2}$$

for some constant $C > 0$, where the energy $\mathcal{E}(m)$ is defined in equation (0.3).

In [6,11], the value of C in (iv) is taken to be $C = \frac{\alpha}{1+\alpha^2}$. The existence of global weak solution of the Landau-Lifshitz equation in $\Omega \subset \mathbb{R}^3$ into $\mathbb{S}^2$ was proved in [9,28]. The nonuniqueness of weak solution was proved in [9].

Let the domain $\Omega \subset \mathbb{R}^d$ where $d = 2$ or 3 be discretized into triangular or tetrahedral elements $\{\mathcal{T}_h\}_h$ of mesh size at most h with vertices $(x_i)_{i=1}^N$. Let the family of partitions $\mathcal{T} = \{\mathcal{T}_h\}_h$ be admissible, shape regular and uniform [14]. Let $\{\phi_i\}_{1\le i\le N}$ be piecewise linear nodal basis functions for $\mathcal{T}$, such that $\phi_i(x_j) = \delta_{ij}$, where δ_{ij} is a Kronecker delta function. We will consider a vector-valued finite element space F^h defined by $F^h = \{w^h \mid w^h(x) = \sum_{i=1}^N w_i^h\phi_i(x),\ w_i^h \in \mathbb{R}^3\}$. The discrete magnetization m^h is required to belong to the submanifold M^h of F^h defined by $M^h = \{m^h \in F^h \mid m^h(x) = \sum_{i=1}^N m_i^h\phi_i(x),\ |m_i^h| = 1\}$. We define the interpolation operator $I_h : C^0(\Omega,\mathbb{R}^3) \to F^h$ by

$$I_h(m) = \sum_{i=1}^N m(x_i)\phi_i(x). \tag{1.3}$$

We need the following additional conditions for our finite element method: There exist some constants $C_1, C_2, C_3, C_4 > 0$ such that

$$\begin{aligned}
&C_1h^d \le b_i = \int_\Omega \phi_i \le C_2h^d,\\
&|M_{ij}| = \left|\int_\Omega \phi_i\phi_j\right| \le C_3h^d,\\
&|\partial_{x_l}\phi_i| \le \frac{C_4}{h}\\
&\int_\Omega \nabla\phi_i\cdot\nabla\phi_j \le 0, \qquad \text{for } i \ne j,
\end{aligned} \tag{1.4}$$

for all $h > 0$, $i, j = 1,\dots,N$ and $l = 1,\dots,d$.

2. The finite element scheme, the algorithm, and the main theorem.

To illustrate how we obtain Algorithm 1 below, consider the simple case in which only the exchange energy term is present in the effective field, i.e. $h = \eta\triangle m$ from (0.2). Let's first consider the weak form of the Landau-Lifshitz equation with $h = \eta\triangle m$,

$$\begin{aligned}
\int_{\Omega_T} \partial_t m\cdot w = {}& \eta\sum_{l=1}^d\int_{\Omega_T}(m\times\partial_{x_l}m)\cdot\partial_{x_l}w\\
&- \alpha\eta\sum_{l=1}^d\int_{\Omega_T}\partial_{x_l}m\cdot\partial_{x_l}w + \alpha\eta\sum_{l=1}^d\int_{\Omega_T}(\partial_{x_l}m\cdot\partial_{x_l}m)(m\cdot w).
\end{aligned} \tag{2.1}$$

Taking this weak form as a hint, we would like to find $v = \sum_{j=1}^{N} v_j \phi_j \in F^h$ such that

$$\int_\Omega \sum_{j=1}^{N} v_j\phi_j \cdot w_i\phi_i = \eta \sum_{l=1}^{d}\sum_{j=1}^{N}\int_\Omega (m_i \times m_j \partial_{x_l}\phi_j)\cdot \partial_{x_l}\phi_i \, w_i$$
$$- \alpha\eta \sum_{l=1}^{d}\sum_{j=1}^{N}\int_\Omega m_j\partial_{x_l}\phi_j \cdot \partial_{x_l}\phi_i w_i + \alpha\eta \sum_{l=1}^{d}\sum_{j=1}^{N}\int_\Omega (\partial_{x_l}\phi_j m_j \cdot m_i)(m_i \cdot w_i \partial_{x_l}\phi_i) \tag{2.2}$$

for $i = 1, \dots, N$, where $m = \sum_{j=1}^{N} m_j\phi_j(x) \in M^h$, $w \in (C^\infty(\Omega))^3$ and $w_i = I_h(w)(x_i) = w(x_i)$. Then, with w_i as $(1,0,0)$, $(0,1,0)$ or $(0,0,1)$ in equation (2.2), we obtain

$$(\mathbf{M}v)_i = \eta\, m_i \times (\mathbf{A}m)_i + \alpha\eta\, m_i \times (m_i \times (\mathbf{A}m)_i) \tag{2.3}$$

for $i = 1, \dots, N$, where $\mathbf{M} = \begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{bmatrix}$ are $3N \times 3N$ block diagonal matrices with each block M and A a mass or stiffness matrix, i.e. $M_{ij} = \int_\Omega \phi_i\phi_j$, and $A_{ij} = \sum_{l=1}^{d}\int_\Omega \partial_{x_l}\phi_i\partial_{x_l}\phi_j$. Note that $m_i \cdot (\mathbf{M}v)_i = 0$, so approximating v by $\hat{v} = \frac{\mathbf{M}v}{b}$ yields a tangent vector to the constraint manifold M^h, where $b_i = \int_\Omega \phi_i$. The left hand side of (2.3) is then $b_i\hat{v}_i$ which is a mass-lumping approximation. This suggests the following algorithm.

Algorithm 1. For a given time $\bar{T} > 0$, set $J = [\frac{\bar{T}}{k}]$.

(1) Set an initial discrete magnetization m^0 at the nodes of the finite element mesh described in section 1 above.

(2) For $j = 0, \dots, J$,

a. compute a velocity vector $\hat{v}_i^j$ at each node by

$$\hat{v}_i^j = \frac{(\mathbf{M}v^j)_i}{b_i} = \frac{\eta\, m_i^j \times (\mathbf{A}m + \theta k\mathbf{A}\hat{v})_i^j + \alpha\eta\, m_i^j \times (m_i^j \times (\mathbf{A}m + \theta k \mathbf{A}\hat{v})_i^j)}{b_i}$$
$$- \frac{m_i^j \times (\mathbf{M}\bar{h}(m) + \theta k\mathbf{M}\bar{h}(\hat{v}))_i^j + \alpha m_i^j \times (m_i^j \times (\mathbf{M}\bar{h}(m) + \theta k \mathbf{M}\bar{h}(\hat{v}))_i^j)}{b_i} \tag{2.4}$$

for $\theta \in [0,1]$ and for $i = 1, \dots, N$.

b. Compute $m_i^{j+1} = \frac{m_i^j + k\hat{v}_i^j}{|m_i^j + k\hat{v}_i^j|}$ for $i = 1, \dots, N$.

We define the time-interpolated magnetization and velocity as in [8]:

Definition 2.1. For $(x,t) \in \Omega \times [jk, (j+1)k) \subset \Omega \times [0,T)$, where $T = Jk$, define

$$m^{h,k}(x,t) = m^j(x),$$
$$\bar{m}^{h,k}(x,t) = \frac{t - jk}{k} m^{j+1}(x) + \frac{(j+1)k - t}{k} m^j(x),$$
$$\hat{v}^{h,k}(x,t) = \hat{v}^j(x),$$
$$v^{h,k}(x,t) = v^j(x).$$

The main theorem in this article is the following theorem, which is proved in section 4.

THEOREM 2.2. Let $m_0 \in H^1(\Omega, \mathbb{S}^2)$ and suppose $m_0^h \to m_0$ in $H^1(\Omega)$ as $h \to 0$. Let $\theta \in [0,1]$, and for $0 \leq \theta < \frac{1}{2}$, assume that $\frac{k}{h^2} \leq C_0$, for some $C_0 > 0$. If the triangulation $\mathcal{T} = \{\mathcal{T}_h\}_h$ satisfies condition (1.4), then the sequence $\{m^{h,k}\}$, constructed by Algorithm 1 and Definition 2.1, has a subsequence that converges weakly to a weak solution of the Landau-Lifshitz equation.

3. Numerical results. Before proving Theorem 2.2, we demonstrate the effectiveness of the scheme on a test problem. We conduct a numerical experiment for the Landau-Lifshitz equation (0.1) with effective field involving only the exchange energy term, with $h = \Delta m$ in equation (0.2), on the unit square with periodic boundary conditions. This corresponds to setting $\eta = 1$ and $\bar{h} = 0$ in equation (2.4) in Algorithm 1. For the convergence study, we used an explicit method ($\theta = 0$) and an implicit method ($\theta = 0.5$) on a structured and unstructured mesh. The unstructured mesh with point and line sources, which is an arbitrary mesh, was generated using DistMesh [42], with an example shown in Figure 1.

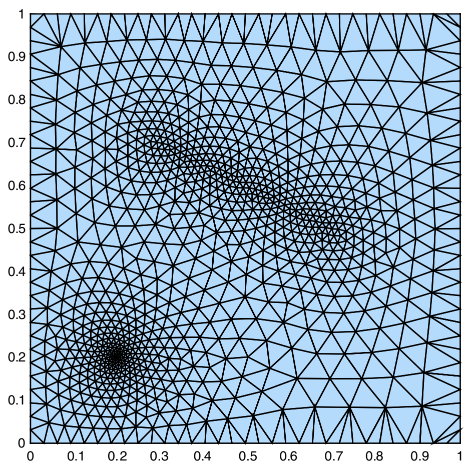
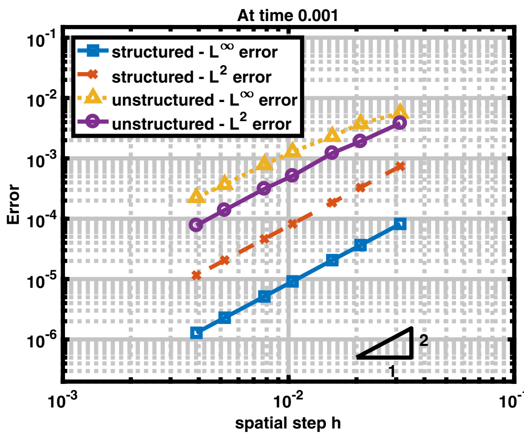


FIG. 1. Unstructured mesh with point and line sources, with $h = 1/32$.

The L^∞ and L^2 errors were measured relative to an exact solution for the Landau-Lifshitz equation with $h = \Delta m$ from [24], namely

$$\begin{aligned} m^x(x_1, x_2, t) &= \frac{1}{d(t)} \sin\beta \cos(k(x_1 + x_2) + g(t)), \\ m^y(x_1, x_2, t) &= \frac{1}{d(t)} \sin\beta \sin(k(x_1 + x_2) + g(t)), \\ m^z(x_1, x_2, t) &= \frac{1}{d(t)} e^{2k^2\alpha t} \cos\beta. \end{aligned} \tag{3.1}$$

Here $\beta = \frac{\pi}{24}$, $k = 2\pi$, $d(t) = \sqrt{\sin^2\beta + e^{4k^2\alpha t}\cos^2\beta}$ and $g(t) = \frac{1}{\alpha}\log(\frac{d(t)+e^{2k^2\alpha t}\cos\beta}{1+\cos\beta})$. The numerical results are summarized in Tables 1 and 2. Figure 2 shows the convergence rate of the methods, which is first order in the time step k and second order in the mesh size h.



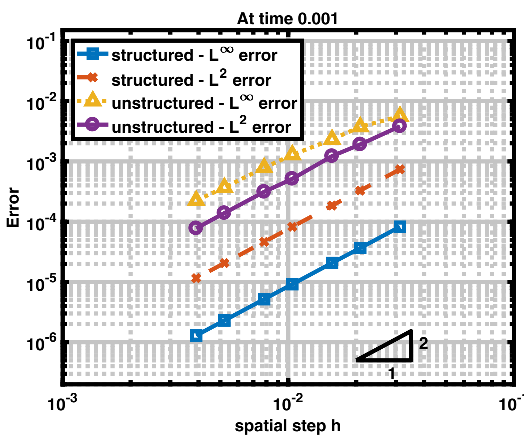


FIG. 2. Convergence plot, Top: Explicit method, Bottom: Implicit method.

TABLE 1. **Explicit method ($\theta = 0$)** : L^∞ and L^2 error and convergence rates on a structured and unstructured mesh with spatial step h, time step $k = 8 \cdot 10^{-5} h^2$ and time 0.001.

	Structured mesh				Unstructured mesh			
$\frac{1}{h}$	$\|\|m - m^h\|\|_{L^\infty}$	rate	$\|\|m - m^h\|\|_{L^2}$	rate	$\|\|m - m^h\|\|_{L^\infty}$	rate	$\|\|m - m^h\|\|_{L^2}$	rate
32	8.22e-05	2.00	7.40e-04	2.00	5.61e-03	1.28	3.83e-03	1.65
64	2.06e-05	2.00	1.85e-04	2.00	2.32e-03	1.56	1.22e-03	1.97
128	5.15e-06	2.00	4.63e-05	2.00	7.87e-04	1.81	3.13e-04	2.01
256	1.29e-06		1.16e-05		2.25e-04		7.77e-05	

TABLE 2. **Implicit method** ($\theta = \frac{1}{2}$) : L^∞ and L^2 error and convergence rates on a structured and unstructured mesh, with spatial step h, time step $k = 0.00256h^2$ and time 0.001.

	Structured mesh				Unstructured mesh			
$\frac{1}{h}$	$\|\|m - m^h\|\|_{L^\infty}$	rate	$\|\|m - m^h\|\|_{L^2}$	rate	$\|\|m - m^h\|\|_{L^\infty}$	rate	$\|\|m - m^h\|\|_{L^2}$	rate
32	8.26e-05	2.00	7.40e-04	2.00	5.61e-03	1.28	3.83e-03	1.65
64	2.07e-05	2.00	1.85e-04	2.00	2.32e-03	1.56	1.22e-03	1.97
128	5.17e-06	2.00	4.63e-05	2.00	7.87e-04	1.81	3.13e-04	2.01
256	1.29e-06		1.16e-05		2.25e-04		7.77e-05	

3.1. *Going beyond first order in time.* In this section, we propose a method which is second order in time, by replacing the nonlinear projection step 2 (b) in Algorithm 1 by a linear projection step, and test the convergence order. In Algorithm 1, step 2 (a) can be viewed as the predictor step and 2 (b) as the corrector step. The corrector step was used to conserve the length of the magnetization at each node. By replacing this nonlinear projection by a linear projection step, it not only preserves the length of the magnetization, but also makes the method higher order. Moreover, it has a similar complexity as the nonlinear projection step in that you only need to solve a 3×3 matrix equation for each node. We defer a rigorous analysis to future work and present here the modified algorithm and some convergence test results.

ALGORITHM 2. For a given time $\bar{T} > 0$, set $J = [\frac{\bar{T}}{k}]$.

(1) Set an initial discrete magnetization m^0 at the nodes of the finite element mesh described in section 1 above.

(2) For $j = 0, \dots, J$,

a. compute an intermediate magnetization vector m_i^* at each node by

$$\frac{m_i^* - m_i^j}{k} = \hat{v}_i^j = \frac{(\mathbf{M}v^j)_i}{b_i}$$
$$= \frac{\eta\, m_i^j \times (\mathbf{A}m + \theta k \mathbf{A}\hat{v})_i^j + \alpha\eta\, m_i^j \times (m_i^j \times (\mathbf{A}m + \theta k\mathbf{A}\hat{v})_i)^j}{b_i}$$
$$- \frac{m_i^j \times (\mathbf{M}\bar{h}(m) + \theta k \mathbf{M}\bar{h}(\hat{v}))_i^j + \alpha m_i^j \times (m_i^j \times (\mathbf{M}\bar{h}(m) + \theta k\mathbf{M}\bar{h}(\hat{v}))_i)^j}{b_i}$$

for $\theta \in [0,1]$ and for $i = 1, \dots, N$.

b. Compute m_i^{j+1} for $i = 1, \dots, N$.

$$\frac{m_i^{j+1} - m_i^j}{k} =$$
$$= \frac{\eta\, \frac{m_i^{j+1} + m_i^j}{2} \times (\mathbf{A}m_i^{j+1/2}) + \alpha\eta\, \frac{m_i^{j+1} + m_i^j}{2} \times (m_i^{j+1/2} \times (\mathbf{A}m^{j+1/2})}{b_i}$$
$$- \frac{\frac{m_i^{j+1} + m_i^j}{2} \times \mathbf{M}\bar{h}(m^{j+1/2})_i + \alpha \frac{m_i^{j+1} + m_i^j}{2} \times (m_i^{j+1/2} \times (\mathbf{M}\bar{h}(m_i^{j+1/2}))}{b_i}$$

where $m^{j+1/2} = \frac{m^j + m^*}{2}$ for $i = 1, \dots, N$.

As before, we conduct a numerical test for the Landau-Lifshitz equation (0.1) with effective field involving only the exchange energy term, with $h = \Delta m$ in equation (0.2), on the unit square with periodic boundary conditions, to compare the two algorithms. For the convergence study, we used an implicit method ($\theta = 0.5$) on a structured and unstructured mesh. One of the unstructured meshes was shown in Figure 1. The L^2 errors were measured relative to an analytical solution (3.1) for the Landau-Lifshitz equation with $h = \Delta m$. The numerical results are summarized in Table 3. Figure 3 shows the convergence rates of the methods, which shows first order in k for Algorithm 1 and second order convergence for Algorithm 2.

TABLE 3. **Implicit method** ($\theta = \frac{1}{2}$) : L^2 error and convergence rates on a structured and unstructured mesh, with spatial step h, time step $k = 0.04h$ and time 0.01.

	Structured mesh				Unstructured mesh			
$\frac{1}{h}$	Alg. 1	rate	Alg. 2	rate	Alg. 1	rate	Alg. 2	rate
32	5.14e-04	1.32	1.87e-04	2.01	2.10e-03	1.61	1.91e-03	1.72
64	2.06e-04	1.18	4.66e-05	2.00	6.87e-04	1.75	5.81e-04	2.01
128	9.12e-05	1.10	1.16e-05	2.00	2.04e-04	1.57	1.44e-04	2.03
256	4.27e-05		2.91e-06		6.84e-05		3.53e-05	

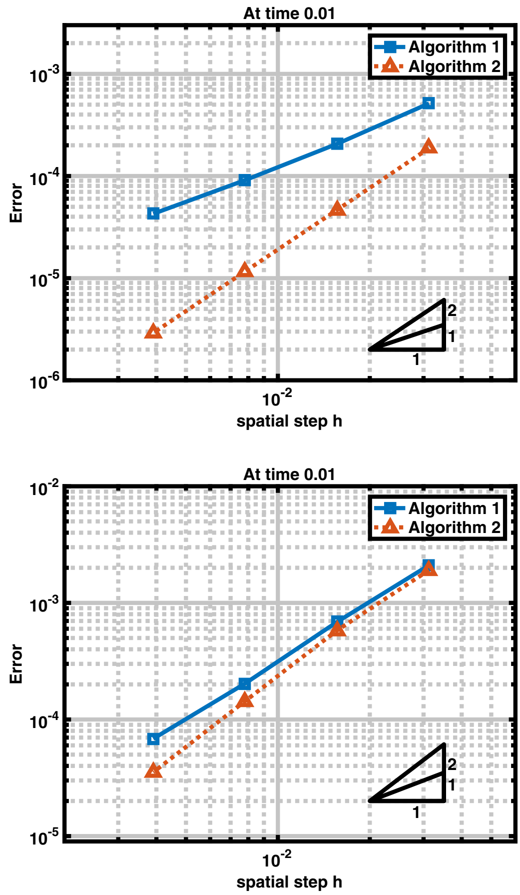


FIG. 3. Convergence plot, Top: Structured mesh, Bottom: Unstructured mesh.

4. Proof of Theorem 2.2. In this section, we present the proof of the theorem, which states that the sequence $\{m^{h,k}\}$, constructed by Algorithm 1 and Definition 2.1, has a subsequence that converges weakly to a weak solution m of the Landau-Lifshitz-Gilbert equation under some conditions. That is, we show that the limit m satisfies Definition 1.1. In section 4.1, we derive a discretization of the weak form of the Landau-Lifshitz-Gilbert equation satisfied by the $\{m^{h,k}\}$, namely (4.3). In section 4.2, we derive energy estimates to show that the sequences $m^{h,k}$, $\bar{m}^{h,k}$ and $\hat{v}^{h,k}$ converge to m in a certain sense made precise in section 4.3. In section 4.4, we show that each term of the discretization of the weak form converges to the appropriate limit, so that the limit m satisfies the weak form of the Landau-Lifshitz-Gilbert equation. In section 4.5, we show that the limit m satisfies the energy inequality (1.2) in Definition 1.1 (iv). Finally, in section 4.6, we establish that the magnitude of m is 1 a.e. in Ω_T.

4.1. *Equations that $m^{h,k}$ and $v^{h,k}$ satisfy.* In this section, we derive a discretization of the weak form of the Landau-Lifshitz-Gilbert equation. This form is easier to use for the proof of Theorem 2.2, since it does not involve the product of the weakly convergent sequences. In general, a product of weakly convergent sequences is not weakly convergent. It is convergent only in some certain cases, such as when the sequences satisfy the hypothesis of the div-curl lemma [17, 45].

The generalized version of equation (2.2) including all the terms in the effective field h (0.2) and with $0 \le \theta \le 1$ is

$$\begin{aligned} \int_\Omega v^{h,k} \cdot w^h = {} & \eta \sum_{l,i} \int_\Omega (m_i^{h,k} \times \partial_{x_l}(m^{h,k} + \theta k \hat{v}^{h,k})) \cdot \partial_{x_l}\phi_i \, w_i^h \\ & - \alpha\eta \sum_{l,i} \int_\Omega \partial_{x_l}(m^{h,k} + \theta k \hat{v}^{h,k}) \cdot \partial_{x_l}\phi_i w_i^h \\ & + \alpha\eta \sum_{l,i} \int_\Omega (\partial_{x_l}(m^{h,k} + \theta k \hat{v}^{h,k}) \cdot m_i^{h,k})(m_i^{h,k} \cdot w_i^h)\partial_{x_l}\phi_i \\ & - \sum_i \int_\Omega (m_i^{h,k} \times \bar{h}(m^{h,k} + \theta k \hat{v}^{h,k})) \cdot \phi_i w_i^h \\ & + \alpha \sum_i \int_\Omega \bar{h}(m^{h,k} + \theta k \hat{v}^{h,k}) \cdot \phi_i w_i^h \\ & - \alpha \sum_i \int_\Omega (\bar{h}(m^{h,k} + \theta k \hat{v}^{h,k}) \cdot m_i)(m_i \cdot w_i^h \phi_i). \end{aligned} \tag{4.1}$$

In fact, by taking w_i^h as $(1,0,0), (0,1,0)$ and $(0,0,1)$ in (4.1), we get (2.4) in Algorithm 1. Setting $w^h = \sum_{j=1}^N (m_j^{h,k} \times u_j^h)\phi_j$ in (4.1), we have

$$\begin{aligned} -\sum_i \int_\Omega (m_i^{h,k} \times v^{h,k}) \cdot u_i^h \phi_i = {} & \eta \sum_l \int_\Omega \partial_{x_l}(m^{h,k} + \theta k \hat{v}^{h,k}) \cdot \partial_{x_l} u^h \\ & - \eta \sum_{l,i} \int_\Omega (\partial_{x_l}(m^{h,k} + \theta k \hat{v}^{h,k}) \cdot m_i^{h,k}) \, (m_i^{h,k} \cdot u_i^h)\partial_{x_l}\phi_i \\ & + \alpha\eta \sum_{l,i} \int_\Omega (m_i^{h,k} \times \partial_{x_l}(m^{h,k} + \theta k \hat{v}^{h,k})) \cdot \partial_{x_l}\phi_i u_i^h \\ & - \sum_i \int_\Omega (\bar{h}(m^{h,k} + \theta k \hat{v}^{h,k})) \cdot \phi_i u_i^h \\ & + \sum_i \int_\Omega (\bar{h}(m^{h,k} + \theta k \hat{v}^{h,k}) \cdot m_i^{h,k}) \, (m_i^{h,k} \cdot u_i^h)\phi_i \\ & - \alpha \sum_i \int_\Omega (m_i^{h,k} \times \bar{h}(m^{h,k} + \theta k \hat{v}^{h,k})) \cdot u_i^h \phi_i. \end{aligned} \tag{4.2}$$

Equations (4.1) and (4.2) have terms that contain the product of weakly convergent sequences, namely the third term of the right hand side of (4.1), and the second term of the right hand side of (4.2), $\alpha\eta \sum_{l,i} \int_\Omega (\partial_{x_l}(m^{h,k} + \theta k \hat{v}^{h,k}) \cdot m_i^{h,k})(m_i^{h,k} \cdot w_i^h)\partial_{x_l}\phi_i$. By adding α times equation (4.2) to equation (4.1), we eliminate the terms that contain the product of weakly convergent sequences:

$$\int_\Omega \left[v^{h,k} \cdot w^h - \alpha \sum_i (m_i^{h,k} \times v^{h,k}) \cdot w_i^h \phi_i\right]$$
$$= (1+\alpha^2)\Bigg[\eta \sum_{l,i} \int_\Omega \left[(m_i^{h,k} \times \partial_{x_l} m^{h,k}) \cdot \partial_{x_l}\phi_i w_i^h + \theta k (m_i^{h,k} \times \partial_{x_l} \hat{v}^{h,k}) \cdot \partial_{x_l}\phi_i w_i^h\right]$$
$$- \sum_i \int_\Omega \left[(m_i^{h,k} \times \bar{h}(m^{h,k})) \cdot w_i^h \phi_i + \theta k (m_i^{h,k} \times \bar{h}(\hat{v}^{h,k})) \cdot w_i^h \phi_i\right]\Bigg]. \tag{4.3}$$

This is a similar procedure to subtracting α times the following equation:

$$m \times \partial_t m = -m \times (m \times h) + \alpha m \times h \tag{4.4}$$

from the Landau-Lifshitz equation (0.1) to get the Landau-Lifshitz-Gilbert equation (0.6). Here, equation (4.4) is obtained by taking $m\times$ the Landau-Lifshitz equation (0.1).

4.2. *Energy inequality.* In this section, we derive the energy inequalities we will need to prove Theorem 2.2, namely (4.17) for $0 \le \theta < \frac{1}{2}$ and (4.18) for $\frac{1}{2} \le \theta \le 1$. We will use Theorem 1 from [8], which states that the exchange energy is decreased after renormalization. This result goes back to [5, 10]:

THEOREM 4.1. For the P^1 approximation in $\Omega \subset \mathbb{R}^2$, if

$$\int_\Omega \nabla\phi_i \cdot \nabla\phi_j \le 0, \qquad \text{for } i \ne j, \tag{4.5}$$

then for all $w = \sum_{i=1}^N w_i\phi_i \in F^h$ such that $|w_i| \ge 1$ for $i = 1, \ldots, N$, we have

$$\int_\Omega \left|\nabla I_h(\frac{w}{|w|})\right|^2 \le \int_\Omega |\nabla w|^2 . \tag{4.6}$$

In 3D, we have (4.6), if an additional condition that all dihedral angles of the tetrahedra of the mesh are smaller than $\frac{\pi}{2}$ is satisfied, along with (4.5). Also, we will use inequality (14) of [8],

$$\left\|\bar{h}(m)\right\|_{L^2} \le C_5 \left\|m\right\|_{L^2} + C_5 \tag{4.7}$$

and equation (25) from [7],

$$\left\|h_s(m)\right\|_{L^2} \le C_5 \left\|m\right\|_{L^2} \tag{4.8}$$

where C_5 are positive constants, depending only on Ω. Furthermore, we will use an inequality (20) of [8] in the proof, which states there exists $C_6 > 0$ such that for all $1 \le p < \infty$ and all $\phi_h \in F^h$, we have

$$\frac{1}{C_6} \left\|\phi_h\right\|_{L^p}^p \le h^d \sum_{i=1}^N |\phi_h(x_i)|^p \le C_6 \left\|\phi_h\right\|_{L^p}^p . \tag{4.9}$$

Moreover, we will assume that there exists $C_7 > 0$ such that

$$\int_\Omega |\nabla v^h|^2 \le \frac{C_7}{h^2}\int_\Omega |v^h|^2 \tag{4.10}$$

for all $v^h \in F^h$.

Taking $w^h = \sum_{j=1}^N (m_j^{h,k} \times u_j^h)\phi_j$ in (4.3), and setting $u^h = \hat{v}^{h,k}$, we have

$$\begin{aligned} -\alpha\sum_i\int_\Omega v_i^{h,k}\cdot\hat{v}_i\phi_i = (1+\alpha^2)\Big[\eta\sum_{l,i}\int_\Omega &\left[(\partial_{x_l} m^{h,k}\cdot\partial_{x_l}\phi_i\hat{v}_i) + \theta k(\partial_{x_l}\hat{v}^{h,k}\cdot\partial_{x_l}\phi_i\hat{v}_i)\right] \\ &-\sum_i\int_\Omega\left[(\bar{h}(m^{h,k})\cdot\hat{v}_i)\phi_i + \theta k(\bar{h}(\hat{v}^{h,k})\cdot\hat{v}_i)\phi_i\right]\Big] \end{aligned} \tag{4.11}$$

where we have used the fact $m_i^{h,k}\cdot\hat{v}_i^{h,k} = 0$ for $i = 1,\dots,N$. This equation can be written as

$$(\nabla m, \nabla\hat{v}) = -\theta k\,\|\nabla\hat{v}\|_{L^2}^2 - \frac{\alpha}{1+\alpha^2}\frac{1}{\eta}\sum_i\frac{|(\mathbf{M}v)_j|^2}{b_j} + \frac{1}{\eta}(\bar{h}(m),v) + \frac{\theta k}{\eta}(\bar{h}(\hat{v}),v). \tag{4.12}$$

We now derive an energy estimate. We have

$$\begin{aligned} \frac{1}{2}\left\|\nabla m^{j+1}\right\|_{L^2}^2 &\le \frac{1}{2}\left\|\nabla m^j + k\nabla\hat{v}^j\right\|_{L^2}^2 = \frac{1}{2}\left\|\nabla m^j\right\|_{L^2}^2 + k(\nabla m^j,\nabla\hat{v}^j) + \frac{1}{2}k^2\left\|\nabla\hat{v}^j\right\|_{L^2}^2 \\ &\le \frac{1}{2}\left\|\nabla m^j\right\|_{L^2}^2 - k(\frac{\alpha}{1+\alpha^2})\frac{1}{\eta}\sum_i\frac{|(\mathbf{M}v)_i^j|^2}{b_i^2}b_i + \frac{1}{2}k^2\left\|\nabla\hat{v}^j\right\|_{L^2}^2 - \theta k^2\left\|\nabla\hat{v}^j\right\|_{L^2}^2 \\ &\qquad + \frac{k}{\eta}(\bar{h}(m^j),\hat{v}^j) + \theta\frac{k^2}{\eta}(\bar{h}(\hat{v}^j),\hat{v}^j) \\ &\le \frac{1}{2}\left\|\nabla m^j\right\|_{L^2}^2 - k(\frac{\alpha}{1+\alpha^2})\frac{1}{\eta}\frac{C_1}{C_6}\left\|\hat{v}^j\right\|_{L^2}^2 - (\theta-\frac{1}{2})k^2\left\|\nabla\hat{v}^j\right\|_{L^2}^2 + \frac{k}{\eta}(\bar{h}(m^j),\hat{v}^j) \\ &\qquad + \theta\frac{k^2}{\eta}(\bar{h}_e,\hat{v}^j) \end{aligned} \tag{4.13}$$

where the first inequality is obtained by Theorem 4.1, the second inequality by equation (4.12), and the last inequality by the fact $(h_s(\hat{v}^j),\hat{v}^j) < 0$. We have the estimate for the last two terms of the above inequality:

$$\left|(\bar{h}(m^j)+\theta k\bar{h}_e,\hat{v}^j)\right| \le \left\|\bar{h}(m^j)+\theta k\bar{h}_e\right\|_{L^2}\left\|\hat{v}^j\right\|_{L^2} \le C_8\left\|\hat{v}^j\right\|_{L^2} \le \epsilon\left\|\hat{v}^j\right\|_{L^2}^2 + \frac{1}{4\epsilon}C_8^2 \tag{4.14}$$

for some $C_8 > 0$, where the second inequality is obtained by equation (4.7) and the last inequality by Young's inequality with $\epsilon = \frac{1}{2}\frac{\alpha}{1+\alpha^2}\frac{C_1}{C_6}$. Summing the inequality (4.13) from $j = 0,\dots,J-1$ and using (4.14), we get

$$\frac{1}{2}\left\|\nabla m^J\right\|_{L^2}^2 + k(\frac{1}{2\eta}(\frac{\alpha}{1+\alpha^2})\frac{C_1}{C_6} - C_7(\frac{1}{2}-\theta)\frac{k}{h^2})\sum_{j=0}^{J-1}\left\|\hat{v}^j\right\|_{L^2}^2 \le \frac{1}{2}\left\|\nabla m^0\right\|_{L^2}^2 + C_9 T \tag{4.15}$$

with $\frac{k}{h^2} \leq C_0 < \frac{1}{2}\frac{\alpha}{1+\alpha^2}\frac{C_1}{C_6}\frac{1}{C_7\eta}$, for $0 \leq \theta < \frac{1}{2}$, and

$$\begin{aligned}\frac{1}{2}\left\|\nabla m^J\right\|_{L^2}^2 + k(\frac{1}{2\eta}\frac{\alpha}{1+\alpha^2})\frac{C_1}{C_6}\sum_{j=0}^{J-1}\left\|\hat{v}^j\right\|_{L^2}^2 \\ + (\theta - \frac{1}{2})k^2\sum_{j=0}^{J-1}\left\|\nabla\hat{v}^j\right\|_{L^2}^2 \leq \frac{1}{2}\left\|\nabla m^0\right\|_{L^2}^2 + C_9 T\end{aligned} \tag{4.16}$$

for $\frac{1}{2} \leq \theta \leq 1$, and for some $C_9 > 0$.

In summary, we have the energy inequalities

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla m^{h,k}(\cdot,T)|^2 + (\frac{1}{2\eta}(\frac{\alpha}{1+\alpha^2})\frac{C_1}{C_6} - C_7C_0)\int_{\Omega_T}|\hat{v}^{h,k}|^2 \\ \leq \frac{1}{2}\int_\Omega |\nabla m^{h,k}(\cdot,0)|^2 + C_9 T\end{aligned} \tag{4.17}$$

with $C_0 < \frac{1}{2}\frac{\alpha}{1+\alpha^2}\frac{C_1}{C_6}\frac{1}{C_7\eta}$, for $0 \leq \theta < \frac{1}{2}$ and

$$\begin{aligned}\frac{1}{2}\int_\Omega \left|\nabla m^{h,k}(\cdot,T)\right|^2 + (\frac{1}{2\eta}\frac{\alpha}{1+\alpha^2})\frac{C_1}{C_6}\int_{\Omega_T}\left|\hat{v}^{h,k}\right|^2 \\ + (\theta - \frac{1}{2})k\int_{\Omega_T}\left|\nabla\hat{v}^{h,k}\right|^2 \leq \frac{1}{2}\int_\Omega\left|\nabla m^{h,k}(\cdot,0)\right|^2 + C_9 T.\end{aligned} \tag{4.18}$$

for $\frac{1}{2} \leq \theta \leq 1$.

4.3. *Weak convergence of $m^{h,k}$, $\bar{m}^{h,k}$ and $\hat{v}^{h,k}$.* In this section, we show the weak convergence of $\bar{m}^{h,k}$ and $\hat{v}^{h,k}$ and strong convergence of $m^{h,k}$ in some sense, based on the energy estimates (4.17) and (4.18). We follow similar arguments from section 6 of [7].

Since we have

$$\left|\frac{m_i^{j+1} - m_i^j}{k}\right| \leq \left|\hat{v}_i^j\right| \tag{4.19}$$

for $i = 1, \ldots, N$ and $j = 0, \ldots, J-1$, we have

$$\left\|\partial_t \bar{m}^{h,k}\right\|_{L^2(\Omega)} = \left\|\frac{m^{j+1} - m^j}{k}\right\|_{L^2(\Omega)} \leq C_6\left\|\hat{v}^{h,k}\right\|_{L^2(\Omega)}. \tag{4.20}$$

Thus, we have

$$\left\|\partial_t \bar{m}^{h,k}\right\|_{L^2(\Omega_T)} = \left\|\frac{m^{j+1} - m^j}{k}\right\|_{L^2(\Omega_T)} \leq C_6\left\|\hat{v}^{h,k}\right\|_{L^2(\Omega_T)} \tag{4.21}$$

which is bounded by the energy inequalities, (4.17) for $0 \leq \theta < \frac{1}{2}$ and (4.18) for $\frac{1}{2} \leq \theta \leq 1$. Hence, $\bar{m}^{h,k}$ is bounded in $H^1(\Omega_T)$ and $\hat{v}^{h,k}$ is bounded in $L^2(\Omega_T)$ by (4.21) and by the energy inequalities, (4.17) for $0 \leq \theta < \frac{1}{2}$ and (4.18) for $\frac{1}{2} \leq \theta \leq 1$. Thus, by passing to subsequences, there exist $m \in H^1(\Omega_T)$ and $\hat{v} \in L^2(\Omega_T)$ such that

$$\begin{aligned}&\bar{m}^{h,k} \to m \text{ weakly in } H^1(\Omega_T), \\ &\bar{m}^{h,k} \to m \text{ strongly in } L^2(\Omega_T), \\ &\hat{v}^{h,k} \to \hat{v} \text{ weakly in } L^2(\Omega_T).\end{aligned} \tag{4.22}$$

Moreover, we have

$$\left|m_i^{j+1} - m_i^j - k\hat{v}_i^j\right| = \left|\frac{m_i^j + k\hat{v}_i^j}{|m_i^j + k\hat{v}_i^j|} - m_i^j - k\hat{v}_i^j\right| = \left|1 - |m_i^j + k\hat{v}_i^j|\right| \leq \frac{1}{2}k^2\left|\hat{v}_i^j\right|^2, \quad (4.23)$$

since $|m_i^j + k\hat{v}_i^j| = \sqrt{1 + k^2|\hat{v}_i^j|^2} \leq 1 + \frac{1}{2}k^2|\hat{v}_i^j|^2$, for $i = 1, \ldots, N$ and $j = 0, \ldots, J-1$. Thus,

$$\left\|\partial_t \bar{m}^{h,k} - \hat{v}^{h,k}\right\|_{L^1(\Omega_T)} \leq \frac{1}{2}kC_2C_6\left\|\hat{v}^{h,k}\right\|^2_{L^2(\Omega_T)} \quad (4.24)$$

which converges to 0 as $h, k \to 0$, so

$$\partial_t m = \hat{v}. \quad (4.25)$$

Furthermore, since

$$\left\|m^{h,k} - \bar{m}^{h,k}\right\|_{L^2(\Omega_T)} = \left\|(t - jk)\frac{m^{j+1} - m^j}{k}\right\|_{L^2(\Omega_T)} \leq k\left\|\partial_t \bar{m}^{h,k}\right\|_{L^2(\Omega_T)} \quad (4.26)$$

and the right hand side goes to 0 as $h, k \to 0$, we have

$$m^{h,k} \to m \text{ strongly in } L^2(\Omega_T). \quad (4.27)$$

In summary, we have shown that there exist a subsequence of $\{\bar{m}^{h,k}\}$ that converges weakly in $H^1(\Omega \times (0,T))$, a subsequence of $\{\bar{v}^{h,k}\}$ that converges weakly in $L^2(\Omega \times (0,T))$, and a subsequence of $\{m^{h,k}\}$ converges strongly in $L^2(\Omega_T)$ based on the energy estimates (4.17) and (4.18). However, in our numerical tests in section 3, it was not necessary to take subsequences and the method was in fact second order in space and first order in time. Thus, there is still a gap in what we are able to prove and the practical performance of the algorithm in cases where the weak solution is unique and sufficiently smooth.

4.4. *The proof that the limit m actually satisfies Landau-Lifshitz-Gilbert equation.* In this section, we show that each term of equation (4.3) converges to the appropriate limit, so that the limit m of the sequences $\{\bar{m}^{h,k}\}$ and $\{m^{h,k}\}$ satisfies the weak form of the Landau-Lifshitz-Gilbert equation (1.1) in Definition 1.1.

LEMMA 4.2. Let the sequences $\{m^{h,k}\}$, $\{\bar{m}^{h,k}\}$, $\{\hat{v}^{h,k}\}$, and $\{v^{h,k}\}$ be defined by Definition 2.1. Also, let $m \in H^1(\Omega_T)$ be the limit as in (4.22) and (4.27). Moreover, let's assume $w \in (C^\infty(\Omega_T))^3 \cap (H^1(\Omega_T))^3$, and $w^h = I_h(w) \in F_h$ as in equation (1.3). Then we have

$$\lim_{h,k\to 0}\int_{\Omega_T} v^{h,k} \cdot w^h = \lim_{h,k\to 0}\int_0^T \sum_{j=1}^N \hat{v}_j^{h,k} \cdot w_j^h \int_\Omega \phi_j = \int_{\Omega_T} \partial_t m \cdot w. \quad (4.28)$$

Proof. The difference between the last two terms is bounded by

$$\left|\int_{\Omega_T} I_h(\hat{v}^{h,k} \cdot w^h) - \hat{v}^{h,k} \cdot w^h\right| + \left|\int_{\Omega_T} \hat{v}^{h,k} \cdot w^h - \partial_t m \cdot w\right|. \quad (4.29)$$

The first term of (4.29) has the following estimate. For each element L, we have $\hat{v}^{h,k}(\cdot,t)\cdot w^h(\cdot,t)\in C^\infty(L)$ and

$$\begin{aligned}&\left\|I_h(\hat{v}^{h,k}\cdot w^h)-\hat{v}^{h,k}\cdot w^h\right\|^2_{L^2(L)}\le C_{10}h^4\left\|\Delta(\hat{v}^{h,k}\cdot w^h)\right\|^2_{L^2(L)}\\&\le C_{10}h^4(\left\|\Delta\hat{v}^{h,k}\cdot w^h\right\|^2_{L^2(L)}+\left\|\nabla\hat{v}^{h,k}\cdot\nabla w^h\right\|^2_{L^2(L)}+\left\|\hat{v}^{h,k}\cdot\Delta w^h\right\|^2_{L^2(L)})\\&\le C_{10}h^4((\left\|\nabla\hat{v}^{h,k}\cdot\nabla w^h\right\|^2_{L^2(L)})\end{aligned}\tag{4.30}$$

for some $C_{10}>0$, where the first inequality is obtained by the Bramble-Hilbert lemma [14], and in the last inequality we have used $\Delta\hat{v}^{h,k}=0$ and $\Delta w^h=0$ in L, since $\hat{v}^{h,k}$ and w^h are the sum of piecewise linear functions. We have the estimate

$$\begin{aligned}\left\|\nabla\hat{v}^{h,k}\right\|^2_{L^2(\Omega)}&\le\sum_L\int_L|\sum_i(\hat{v}^{h,k})_i\nabla\phi_i|^2\le\frac{C_{11}}{h^2}\sum_L|\sum_{i\in I_L}(\hat{v}^{h,k})_i|^2|L|\\&\le C_{12}h^{d-2}\sum_{i=1}^N|(\hat{v}^{h,k})_i|^2\le\frac{C_{13}}{h^2}\left\|\hat{v}^{h,k}\right\|^2_{L^2(\Omega)}\end{aligned}\tag{4.31}$$

for some constants $C_{11},C_{12},C_{13}>0$ and I_L is the index of nodes of L, where the second inequality is obtained by (4.10), and the last inequality by (4.9). Hence,

$$\left\|I_h(\hat{v}^{h,k}\cdot w^h)-\hat{v}^{h,k}\cdot w^h\right\|^2_{L^2(\Omega_T)}\le C_{10}h^4\left\|\nabla\hat{v}^{h,k}\cdot\nabla w^h\right\|^2_{L^2(\Omega_T)}\le C_{14}h^2\left\|\hat{v}^{h,k}\right\|_{L^2(\Omega_T)}\tag{4.32}$$

for some constant $C_{14}>0$. Therefore, the first term of (4.29) goes to 0 as $h,k\to 0$. Moreover, the second term of (4.29) goes to 0 by the weak convergence of $\hat{v}^{h,k}$ to $\partial_t m$ which are equations (4.22) and (4.25) . □

LEMMA 4.3. Under the same assumptions of Lemma 4.2, we have

$$\begin{aligned}\lim_{h,k\to 0}\int_{\Omega_T}\sum_i(m_i^{h,k}\times v^{h,k})\cdot w_i^h\phi_i&=\lim_{h,k\to 0}\int_{\Omega_T}\sum_i(m_i^{h,k}\times\hat{v}_i^{h,k})\cdot w_i^h\phi_i\\&=\int_{\Omega_T}(m\times\partial_t m)\cdot w.\end{aligned}\tag{4.33}$$

Proof. The difference between the last two terms is bounded by

$$\begin{aligned}&\left|\int_{\Omega_T}I_h((m^{h,k})^a(\hat{v}^{h,k})^b(w^h)^c)-(m^{h,k})^a(\hat{v}^{h,k})^b(w^h)^c\right|\\&+\left|\int_{\Omega_T}(m^{h,k})^a(\hat{v}^{h,k})^b(w^h)^c-m^a(\partial_t m)^b w^c\right|\end{aligned}\tag{4.34}$$

for some $a,b,c\in\{1,2,3\}$. The first term of (4.34), has the following estimate. For each element L, we have $(m^{h,k})^a(\hat{v}^{h,k})^b(w^h)^c\in C^\infty(L)$ and

$$\begin{aligned}&\left\|I_h((m^{h,k})^a(\hat{v}^{h,k})^b(w^h)^c)-(m^{h,k})^a(\hat{v}^{h,k})^b(w^h)^c\right\|_{L^1(L)}\\&\le C_{15}h^2(\left\|\Delta((m^{h,k})^a(\hat{v}^{h,k})^b(w^h)^c)\right\|_{L^1(L)})\\&\le C_{15}h^2(\left\|\nabla(m^{h,k})^a\nabla(\hat{v}^{h,k})^b(w^h)^c\right\|_{L^1(L)}+\left\|\nabla(m^{h,k})^a(\hat{v}^{h,k})^b\nabla(w^h)^c\right\|_{L^1(L)}\\&\qquad+\left\|(m^{h,k})^a\nabla(\hat{v}^{h,k})^b\nabla(w^h)^c\right\|_{L^1(L)})\end{aligned}\tag{4.35}$$

for some constant $C_{15} > 0$, where the first inequality is obtained by Bramble-Hilbert lemma, and in the last inequality we have used $\Delta\hat{m}^{h,k} = 0$, $\Delta\hat{v}^{h,k} = 0$ and $\Delta w^h = 0$ in L, since $m^{h,k}$ $\hat{v}^{h,k}$ and w^h are the sum of piecewise linear functions. Hence, we have the estimate

$$\begin{aligned} &\left\| I_h((m^{h,k})^a(\hat{v}^{h,k})^b(w^h)^c) - (m^{h,k})^a(\hat{v}^{h,k})^b(w^h)^c \right\|_{L^1(\Omega_T)} \\ &\le C_{16} h \left\| (\hat{v}^{h,k})^b \right\|_{L^2(\Omega_T)} (\left\| \nabla (m^{h,k})^a \right\|_{L^2(\Omega_T)} + h \left\| \nabla (m^{h,k})^a \right\|_{L^2(\Omega_T)} \\ &+ \left\| (m^{h,k})^a \right\|_{L^2(\Omega_T)}) \end{aligned} \tag{4.36}$$

for some constant $C_{16} > 0$, where we have used Hölder's inequality for all the terms and used (4.31) for the first and the third terms. Therefore, the first term of (4.34) goes to 0 as $h, k \to 0$. Moreover, the second term of (4.34) goes to 0 by the weak convergence of $(\hat{v}^{h,k})^b$ to $(\partial_t m)^b$ established in (4.22) and (4.25), and strong convergence of $(m^{h,k})^a$ to m^a. ☐

Lemma 4.4. Under the same assumptions of Lemma 4.2, we have

$$\lim_{h,k\to 0} \sum_{l,i} \int_{\Omega_T} (m_i^{h,k} \times \partial_{x_l} m^{h,k}) \cdot w_i^h \, \partial_{x_l}\phi_i = \sum_l \int_{\Omega_T} (m \times \partial_{x_l} m) \cdot \partial_{x_l} w. \tag{4.37}$$

Proof. The difference between the last two terms is bounded by

$$\begin{aligned} &\left| \int_{\Omega_T} (\partial_{x_l} m^{h,k})^b \partial_{x_l} I_h((m^{h,k})^c(w^h)^a) - \int_{\Omega_T} (\partial_{x_l} m^{h,k})^b((m^{h,k})^c(\partial_{x_l} w^h)^a) \right| \\ &\qquad + \left| \int_{\Omega_T} (m^{h,k})^c(\partial_{x_l} m^{h,k})^b(\partial_{x_l} w^h)^a - \int_{\Omega_T} m^c(\partial_{x_l} m)^b(\partial_{x_l} w)^a) \right|, \end{aligned} \tag{4.38}$$

for some $a, b, c \in \{1, 2, 3\}$. The first term is bounded by

$$\left\| (\partial_{x_l} m^{h,k})^b \right\|_{L^2(\Omega_T)} \left\| \partial_{x_l} I_h((m^{h,k})^c(w^h)^a) - \partial_{x_l}((m^{h,k})^c(w^h)^a) \right\|_{L^2(\Omega_T)}. \tag{4.39}$$

For each element L, we have $m^{h,k}(\cdot, t)w(\cdot, t) \in C^\infty(L)$, and we have the estimate,

$$\left\| \partial_{x_l} I_h((m^{h,k})^c(w^h)^a) - \partial_{x_l}((m^{h,k})^c(w^h)^a) \right\|_{L^2(L)}^2 \le C_{17} h^2 |(m^{h,k})^c(w^h)^a|_{H^2(L)}^2 \tag{4.40}$$

for some constant $C_{17} > 0$, by the Bramble-Hilbert lemma. Moreover, we have the estimate,

$$\begin{aligned} |(m^{h,k})^c(w^h)^a|_{H^2(L)}^2 = \int_L |\Delta((m^{h,k})^c(w^h)^a)|^2 &\le C_{18} \int_L |\nabla (m^{h,k})^c|^2 |\nabla (w^h)^a|^2 \\ &\le C_{19} \left\| (m^{h,k})^c \right\|_{H^1(L)}^2 \end{aligned} \tag{4.41}$$

for some constants $C_{18}, C_{19} > 0$, since $\Delta m^{h,k} = 0$ and $\Delta w^h = 0$ in L, since $m^{h,k}$ and w^h are the sum of piecewise linear functions. We get the estimate

$$\left\| \partial_{x_l} I_h((m^{h,k})^c(w^h)^a) - \partial_{x_l}((m^{h,k})^c(w^h)^a) \right\|_{L^2(\Omega_T)}^2 \le C_{17} C_{19} h^2 \left\| (m^{h,k})^c \right\|_{H^1(\Omega_T)}^2. \tag{4.42}$$

Therefore, we may conclude that the first term of (4.38) goes to 0 as $h, k \to 0$. Moreover, the second term of (4.38) goes to 0 by the weak convergence of $(\partial_{x_l} m^{h,k})^b$ to $(\partial_{x_l} m)^b$ and strong convergence of $(m^{h,k})^c$ to m^c, which gives (4.22) and (4.27). ☐

LEMMA 4.5. Under the same assumptions of Lemma 4.2, we have

$$\lim_{h,k\to 0}\left|k\sum_i\int_{\Omega_T}(m_i^{h,k}\times\partial_{x_l}\hat{v}^{h,k})^a(\partial_{x_l}w_i^h)^a\right| = 0 \tag{4.43}$$

for $0\le\theta\le 1$.

Proof. An upper bound for the sequence above is

$$\sqrt{k}\left\|\sqrt{k}\,\partial_{x_l}(\hat{v}^{h,k})^c\right\|_{L^2(\Omega_T)}\left\|\nabla(I_h(m^{h,k})^b(w^h)^a)\right\|_{L^2(\Omega_T)}. \tag{4.44}$$

for some $a,b,c\in\{1,2,3\}$. The term $\left\|\sqrt{k}\,\partial_{x_l}(\hat{v}^{h,k})^c\right\|_{L^2(\Omega_T)}$ in (4.44) is uniformly bounded, since $\left\|\sqrt{k}\partial_{x_l}(\hat{v}^{h,k})^c\right\|_{L^2(\Omega)}\le C_7\frac{\sqrt{k}}{h}\left\|(\hat{v}^{h,k})^c\right\|_{L^2(\Omega)}$ is uniformly bounded by (4.17) for $0\le\theta<\frac{1}{2}$, which is obtained by (4.10), and $\left\|\sqrt{k}\partial_{x_l}(\hat{v}^{h,k})^c\right\|_{L^2(\Omega)}$ is uniformly bounded by equation (4.18) for $\frac{1}{2}\le\theta\le 1$. For each element L, we have $m^{h,k}(\cdot,t)w(\cdot,t)\in C^\infty(L)$, so

$$\left\|\nabla I_h((m^{h,k})^b(w^h)^a)-\nabla((m^{h,k})^b(w^h)^a)\right\|_{L^2(L)}^2\le C_{20}h^2(\left\|(\nabla(m^{h,k})^b)\right\|_{L^2(L)}^2), \tag{4.45}$$

for some constant $C_{20}>0$, by the Bramble-Hilbert lemma, and using $\Delta m^{h,k}=0$ and $\Delta w^h=0$ in L, since $m^{h,k}$ and w^h are the sum of piecewise linear functions. Thus, we have

$$\begin{aligned}&\left\|\nabla(I_h(m^{h,k})^b(w^h)^a)\right\|_{L^2(\Omega_T)}^2\\&\qquad\le\left\|\nabla(m^{h,k})^b\right\|_{L^2(\Omega_T)}^2+C_{20}h^2(\left\|(\nabla(m^{h,k})^b)\right\|_{L^2(\Omega_T)}^2),\end{aligned} \tag{4.46}$$

which is uniformly bounded. Hence, (4.44) goes to 0 as $h,k\to 0$. ☐

LEMMA 4.6. Under the same assumptions of Lemma 4.2, we have

$$\lim_{h,k\to 0}\sum_i\int_{\Omega_T}(m_i^{h,k}\times\bar{h}(m^{h,k}))\cdot w_i^h\phi_i=\int_{\Omega_T}(m\times\bar{h}(m))\cdot w. \tag{4.47}$$

Proof. An upper bound for the difference between the sequence and the limit is given by

$$\begin{aligned}&\left|\int_{\Omega_T}(\bar{h}(m^{h,k}))^aI_h((m^{h,k})^b(w^h)^c)-\int_{\Omega_T}(\bar{h}(m^{h,k}))^a(m^{h,k})^b(w^h)^c)\right|\\&\qquad+\left|\int_{\Omega_T}(\bar{h}(m^{h,k}))^a(m^{h,k})^b(w^h)^c-\int_{\Omega_T}(\bar{h}(m))^am^bw^c)\right|\end{aligned} \tag{4.48}$$

for some $a,b,c\in\{1,2,3\}$. The first term of (4.48) is bounded by

$$\left\|\bar{h}(m^{h,k})^a\right\|_{L^2(\Omega_T)}\left\|I_h((m^{h,k})^b(w^h)^c)-(m^{h,k})^b(w^h)^c\right\|_{L^2(\Omega_T)}. \tag{4.49}$$

For each element L, we have $m^{h,k}(\cdot,t)w(\cdot,t)\in C^\infty(L)$, and we get the estimate,

$$\left\|I_h((m^{h,k})^bw^c)-((m^{h,k})^bw^c)\right\|_{L^2(L)}^2\le C_{21}h^4|(m^{h,k})^bw^c|_{H^2(L)}^2 \tag{4.50}$$

for some constant $C_{21} > 0$, by the Bramble-Hilbert lemma. Moreover,

$$\begin{aligned} |(m^{h,k})^b w^c|^2_{H^2(L)} &\leq C_{21} \int_L |\nabla(m^{h,k})^b|^2 |\nabla(w^h)^c|^2 + |(m^{h,k})^b|^2 |\Delta(w^h)^c|^2 \\ &\leq C_{22} \left\|(m^{h,k})^b\right\|^2_{H^1(L)} \end{aligned} \tag{4.51}$$

for some constant $C_{22} > 0$, and using the fact $\Delta m^{h,k} = \Delta w^h = 0$ in L, since $m^{h,k}$ and w^h are the sum of piecewise linear functions. We get the estimate

$$\left\|I_h((m^{h,k})^b w^c) - ((m^{h,k})^b w^c)\right\|^2_{L^2(\Omega_T)} \leq C_{23} h^4 \left\|(m^{h,k})^b\right\|^2_{H^1(\Omega_T)} \tag{4.52}$$

for some constant $C_{23} > 0$. Thus, the first term of (4.48) goes to 0 as $h, k \to 0$, and the second term of (4.48) converges to 0 as $h, k \to 0$, because of the strong convergence of $(\bar{h}(m^{h,k}))^a$ and $(m^{h,k})^b$. □

Lemma 4.7. Under the same assumptions of Lemma 4.2, we have

$$\lim_{h,k\to 0} \left| k \sum_i \int_{\Omega_T} (m_i^{h,k} \times \bar{h}(\hat{v}^{h,k})) \cdot w_i^h \phi_i \right| = 0. \tag{4.53}$$

Proof. An upper bound for the sequence above is

$$k \left\|\bar{h}(\hat{v}^{h,k}))\right\|_{L^2(\Omega_T)} \left\|w^h\right\|_{L^2(\Omega_T)}. \tag{4.54}$$

Since, $\left\|\bar{h}(\hat{v}^{h,k}))\right\|_{L^2(\Omega_T)} \leq (C_5 \left\|\hat{v}^{h,k}\right\|_{L^2(\Omega_T)} + C_5)$ by (4.7), the term $\left\|\bar{h}(\hat{v}^{h,k}))\right\|_{L^2(\Omega_T)}$ in (4.54) is uniformly bounded. Therefore, (4.54) goes to 0 as $h, k \to 0$. □

4.5. *Energy of m.* Recall the definition of the energy $\mathcal{E}(m)$ in (0.3). We follow the same arguments in section 6 of [7]. We have an energy estimate of $m^{h,k}$ as

$$\begin{aligned} \mathcal{E}(m^{j+1}) - \mathcal{E}(m^j) \leq &- k(\frac{\alpha}{1+\alpha^2})\frac{C_1}{C_6}||\hat{v}^j||^2_{L^2} - (\theta - \frac{1}{2})k^2\eta||\nabla\hat{v}^j||^2_{L^2} + k(\bar{h}(m^j), \hat{v}^j) \\ &+ \theta k^2(\bar{h}_e, \hat{v}^j) - \frac{1}{2}\int_\Omega (\bar{h}(m^{j+1}) + \bar{h}(m^j)) \cdot (m^{j+1} - m^j) \end{aligned} \tag{4.55}$$

by (4.13) from section 4.2. For $0 \leq \theta < \frac{1}{2}$, the second term on the right has an upper bound

$$(\theta - \frac{1}{2})k^2\eta \left\|\nabla\hat{v}^j\right\|^2_{L^2(\Omega)} \leq k^2\eta \left\|\nabla\hat{v}^j\right\|^2_{L^2(\Omega)} \leq C_7 k\eta\frac{k}{h^2}\left\|\hat{v}^j\right\|^2_{L^2(\Omega)} \leq C_7 C_0 \eta k \left\|\hat{v}^j\right\|^2_{L^2(\Omega)} \tag{4.56}$$

and by choosing $C_0 \leq \frac{1}{2}\frac{\alpha}{1+\alpha^2}\frac{C_1}{C_6}\frac{1}{C_7\eta}$, this term and the first term on the right hand side of (4.55) can be combined to be less than equal to

$$-\frac{k}{2}(\frac{\alpha}{1+\alpha^2})\frac{C_1}{C_6}||\hat{v}^j||^2_{L^2(\Omega)}. \tag{4.57}$$

The second term on the right of equation (4.55) can be disregarded for $\frac{1}{2} \leq \theta \leq 1$. We will derive the upper bound for the rest of the terms of the right hand side of (4.55).

The third and the last terms on the right can be combined to be written as

$$\left| k(\bar{h}(m^j), \hat{v}^j) - \frac{1}{2}\int_\Omega (\bar{h}(m^{j+1}) + \bar{h}(m^j)) \cdot (m^{j+1} - m^j) \right| \tag{4.58}$$

and has an upper bound

$$\left|\int_\Omega \bar{h}(m^j)\cdot(m^{j+1}-m^j-k\hat{v}^j)\right| + \left|\frac{1}{2}\int_\Omega (\bar{h}(m^{j+1})-\bar{h}(m^j))\cdot(m^{j+1}-m^j)\right|. \tag{4.59}$$

The first term of (4.59) is bounded by

$$C_{24}k^2\left(\left\|\hat{v}^j\right\|_{L^2(\Omega)}\left\|\hat{v}^j\right\|_{L^4(\Omega)}\right) \le C_{24}\frac{k^2}{2}\left(\left\|\hat{v}^j\right\|^2_{L^2(\Omega)}+\left\|\hat{v}^j\right\|^2_{L^4(\Omega)}\right) \tag{4.60}$$

for some constant $C_{24}>0$, by (4.23), and (4.9). The second term of (4.59) is bounded by $C_{25}k^2\left\|\hat{v}^j\right\|^2_{L^2(\Omega)}$ for some constant $C_{25}>0$, by (4.19) and (4.9).

The fourth term on the right has the upper bound $|\theta k^2(\bar{h}_e,\hat{v}^j)| \le C_{26}k^2\left\|\hat{v}^j\right\|_{L^2(\Omega)}$ for some constant $C_{26}>0$. Then (4.55) has an upper bound

$$\begin{aligned}\mathcal{E}(m^{j+1})-\mathcal{E}(m^j)+\frac{k}{2}\left(\frac{\alpha}{1+\alpha^2}\right)\frac{C_1}{C_6}\|\hat{v}^j\|^2_{L^2(\Omega)} \le& C_{27}k^2\left(\left\|\hat{v}^j\right\|^2_{L^4(\Omega)}+\left\|\hat{v}^j\right\|^2_{L^2(\Omega)}\right)\\ \le& C_{28}k^2\left(\left\|\nabla\hat{v}^j\right\|^2_{L^2(\Omega)}+\left\|\hat{v}^j\right\|^2_{L^2(\Omega)}\right)\end{aligned} \tag{4.61}$$

for some constants $C_{27}, C_{28}>0$, by using Sobolev embedding theorem [3], $\left\|\hat{v}^j\right\|_{L^4(\Omega)} \le C_{29}\left\|\nabla\hat{v}^j\right\|_{L^2(\Omega)}$ for some constant $C_{29}>0$. Summing from $j=0,\ldots,J-1$, we get

$$\begin{aligned}&\mathcal{E}(m^J)-\mathcal{E}(m^0)+\frac{1}{2}\left(\frac{\alpha}{1+\alpha^2}\right)\frac{C_1}{C_6}\int_{\Omega_T}|\hat{v}^{h,k}|^2\\ &\qquad\le C_{28}\,k\left(\left\|\nabla\hat{v}^{h,k}\right\|^2_{L^2(\Omega_T)}+\left\|\hat{v}^{h,k}\right\|^2_{L^2(\Omega_T)}\right).\end{aligned} \tag{4.62}$$

Therefore, taking $h,k\to 0$, we get the energy inequality (1.2).

4.6. *Magnitude of* m. By the same argument in [8], we have $|m(x,t)|=1$ *a.e.* for $(x,t)\in\Omega_T$ (See equation (28) and (29) on page 1347 of [8]).

5. Conclusion. We have presented a mass-lumped finite element method for the Landau-Lifshitz equation. We showed that the numerical solution of our method has a subsequence that converges weakly to a weak solution of the Landau-Lifshitz-Gilbert equation. Numerical tests show that the method is second order accurate in space and first order accurate in time when the underlying solution is smooth. A second order in time variant was also presented and tested numerically, but not analyzed rigorously in the present work.

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