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ASYMPTOTICS OF DIFFUSION-LIMITED FAST REACTIONS

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Abstract. We are concerned with the fast-reaction asymptotics $\lambda \to \infty$ for a semilinear coupled diffusion-limited reaction system in contact with infinite reservoirs of reactants. We derive the system of limit equations and prove the uniqueness of its solutions for equal diffusion coefficients. Additionally, we emphasize the structure of the limit free boundary problem. The key tools of our analysis include (uniform with respect to λ) L^1-estimates for both fluxes and products of reaction and a *balanced formulation*, where combinations of the original components which balance the fast reaction are used.

1. Introduction.

We consider a chemical reaction-diffusion system of the form

$$\begin{array}{rcll} u_t - \nabla \cdot a\nabla u &=& -\varphi(\lambda, u, v) - \psi(u, w) & u\,|_{t=0} = u_0, \\ v_t - \nabla \cdot b\nabla v &=& -\varphi(\lambda, u, v) & v\,|_{t=0} = v_0, \\ w_t - \nabla \cdot c\nabla w &=& +\varphi(\lambda, u, v) - \psi(u, w) & w\,|_{t=0} = w_0, \end{array} \tag{1.1}$$

on $\mathcal{Q} = (0, T) \times \Omega$ corresponding to the reaction path

$$A + B \xrightarrow{\varphi} C \qquad A + C \xrightarrow{\psi} P = \text{(product)}, \tag{1.2}$$

where A, B, C are chemical components (reactants) with the concentrations u, v, w. $T > 0$ is a time of physical interest. Here φ gives the rate at which A, B are consumed and C produced by the first reaction, which we will take as very fast, while ψ gives the rate at which A, C are consumed by the second, slower, reaction. A paradigmatic model (the usual mass action kinetics) has

$$\varphi(\lambda, u, v) = \lambda uv, zx \qquad\qquad \psi(u, w) = \mu uw \tag{1.3}$$

for which we scale t so $\mu = 1$.

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The boundary conditions we will impose for the more general problem (1.1) are in the same spirit as for (2.3):

$$\begin{gathered} u = \alpha \quad \text{on } \Gamma^A, \qquad\qquad v = \beta \quad \text{on } \Gamma^B, \\ \text{no-flux conditions for } u, v, w \text{ elsewhere on } \Gamma = \partial\Omega, \end{gathered} \tag{1.4}$$

where Γ^A, Γ^B are nonempty disjoint relatively closed subsets of Γ and α, β are adequately regular functions with $0 \le \alpha, \beta \le \bar{\mathrm{B}}$ and, when needed, extended by zero to the whole of Ω.

Most of our effort in this paper is to justify such an analysis in a somewhat different context in which, rather than considering the fast reaction $A + B \to C$ in isolation, we also consider the evolution of its reaction product C, and permit this to affect A, B. As a model system we specifically couple the fast reaction with another, slower, reaction ψ, adjoining w as the concentration of the intermediate product[1] C and imposing boundary conditions which admit an unbounded supply of reactants.

What is novel for the full (three component) problem we consider here is that one must also track the concentration w of the resultant C of the fast reaction for boundary conditions (1.4) which admit a potentially infinite supply of the fast reactants A, B and which permit the evolution of A, B also to depend on the fast reaction product C. The first part of the analysis, bounding q, is somewhat similar to the two component analysis, but the equation for y still involves w so further analysis is needed, both for compactness and also for the uniqueness. We have here been able to obtain uniqueness also for the more general model by introducing another auxiliary function albeit under the somewhat restrictive assumption that the diffusion coefficients are the same for each component.

By imposing the choice of boundary conditions (2.3), we are considering a nonisolated system. The original motivating example (see [12] and the references cited therein) considered the reaction as taking place in a thin film (hydrodynamic boundary layer for a single bubble in a bubble reactor) so one supply condition came from the oxygen concentration inside the bubble (taken here as essentially constant on the relevant time scale) and the other came from the concentration of the other fast reactant in the bulk fluid of the reactor (outside the film). This scenario is also related to chemical reactions within a permeable membrane with the concentrations maintained on the two sides; see for instance the setup behind the modified film model reported in [9].

For finite λ everything is very smooth (the problem is well-posed, the solution is smooth and λ-bounded), but in the limit $\lambda \to \infty$ the regularity lowers and freely moving interfaces separating the reactants are allowed to occur.

It has long been understood that the two component form — i.e., (1.1), (1.3) with $\psi \equiv 0$ so we ignore C and the w-equation — becomes a free boundary problem in the limit: what one sees is a partition of Ω into A-regions ($u > 0$) and B-regions ($v > 0$) within which the fast reaction cannot occur, although the diffusive flux then carries the reactants to an interface where the (infinitely fast) reaction can occur.

[1]Note that we are not tracking the subsequent product P, which we assume has no further interaction, direct or indirect, with A, B, C.

This analysis of the simple situation with only the single fast reaction goes back at least a century (cf. [19] and note [7]) and has continued[2] in, e.g., [8]. A principal tool of these analyses is the introduction of $y = u - v$ (easiest to work with if $a \equiv b$) since the rapid reaction term φ then drops out yet one can recover $u = y_+ = \max\{y, 0\}$ and $v = -y_- = -\min\{y, 0\}$. The argument for existence of a limit is then by compactness, once one has obtained a λ-independent bound on

$$q(t, s) = \varphi(\lambda, u(t, s), v(t, s)). \tag{1.5}$$

A nice analysis of this two component problem with $d = 1$ was given by Evans [8], treating the uniqueness and regularity of the free boundary $s = \bar{s}(t)$. More recently, several papers, cf. for instance [5, 10, 11, 16], and [18], have considered problems of this nature under various structural assumptions on the system.

We collect the assumptions behind our results and give the concept of the weak formulation in Section 2. In Section 3 we note some preliminary results and conjectures and then in Section 4 we obtain λ-independent estimates in the general context of (1.1) with boundary conditions (1.4). These estimates give suitable compactness to ensure the existence of subsequential limits, whose behavior we then discuss in Sections 5, 6. Those become true limits once one shows uniqueness of solutions to the limit problem in Section 7.

2. Assumptions. Concept of weak formulation. We rely on the following set of assumptions:

(H1) Concerning the choice of the geometry, we may, for example, think of such possibilities as taking Ω to be an annulus in $\mathbb{R}^2$. With somewhat greater concern for regularity, one might take Ω to be a cylinder with bases Γ^A, Γ^B or a more general region with Γ^A, Γ^B somewhat more arbitrary boundary patches or, modifying the setting a bit, a smooth manifold with disconnected boundary. Essentially, we take Ω a $C^{0,1}$-domain and assume that both Γ^A and Γ^B have nonzero measure with $\Gamma^A \cap \Gamma^B = \emptyset$.

(H2) The diffusion coefficients satisfy $a, b, c \in L^\infty(\Omega)$ and there exist constants $\underline{A}, \underline{B}, \underline{C} \in (0, \infty)$ such that $a > \underline{A}$, $b > \underline{B}$, and $c > \underline{C}$.

(H3) We consider

$$\begin{gathered} \varphi = 0 \text{ if either } u = 0 \text{ or } v = 0, \\ \varphi(\lambda, u, v) \nearrow \infty \text{ as } \lambda \to \infty \quad \text{ if both } u, v > 0, \end{gathered} \tag{2.1}$$

making the first reaction in (1.2) very "fast" for large λ: essentially A, B react instantaneously if both would be simultaneously present. For ψ we assume

$$\begin{gathered} \psi(u, \cdot),\ \psi(\cdot,\ w) \text{ are nondecreasing with } \psi(0, \cdot) = \psi(\cdot, 0) = 0, \\ |\psi(u, w) - \psi(\hat{u}, \hat{w})| \le L|u - \hat{u}|(1 + |w|) + L|w - \hat{w}|, \end{gathered} \tag{2.2}$$

where in the Lipschitz condition we have taken advantage of the fact that we will have a pointwise bound $0 \le u \le \bar{\mathrm{B}}$. Note that the stoichiometric example (1.3)

[2]with variations, especially noting that this two component system also models competitive population models where the fast reaction has no associated product. More slow reactions can be included, it is however essential that the fast reaction is irreversible to get the separation into A- and B-regions.

satisfies both (2.1) and (2.2). Essentially, we will be working with $\varphi, \psi \geq 0$ for nonnegative concentrations (setting $\varphi, \psi = 0$ if these might be negative).

(H4) The Dirichlet data satisfies $\alpha \in L^\infty_+(\Gamma^A)$ and $\beta \in L^\infty_+(\Gamma^B)$.

(H5) The initial concentrations satisfy $(u_0, v_0, w_0) \in \left[L^\infty_+(\Omega)\right]^3$.

The assumptions (H1)–(H5) perfectly fit the reaction-diffusion scenario we have in mind. Note however that our focus does not lie in finding the optimal regularity setting. We are only interested in the asymptotic behaviour of the problem and its solution as λ goes to infinity. Regarding (H2): for obtaining the uniform in λ L^1-bound from Theorem 4.1 one needs some regularity on the diffusion coefficients of the reactant species if they are nonconstant; for instance, take $a, b \in C^1(\bar{\Omega})$. As (H3) is concerned, we note for comparison that the system treated in [21] and [12] is

$$\begin{array}{rclcc} & \underline{\text{on} \quad (0,1)} & & \underline{\text{at } s=0} & \underline{\text{at } s=1} \\ -au_{ss} & = & -\lambda uv - uw & u = \alpha & u_s = 0 \\ -bv_{ss} & = & -\lambda uv & v_s = 0 & v = \beta \\ -cw_{ss} & = & +\lambda uv - uw & w_s = 0 & w_s = 0 \end{array} \tag{2.3}$$

with a, b, c, α, β positive constants modeling reaction and diffusion inside a permeable membrane separating two reservoirs with constant reactant level.

Note that (H1)-(H5) hold throughout the manuscript. The only notable exception is the characterization of the limit free boundary in Section 6. It is also worth noting that the uniqueness argument from Section 7 holds for equal diffusion coefficients (a sub-case of (H2)).

We denote the space of test functions corresponding to the unknowns u and v as follows:

$$V_u := \{\xi \in H^1(\Omega) \text{ such that } \xi\|_{\Gamma^A} = 0\},$$
$$V_v := \{\eta \in H^1(\Omega) \text{ such that } \eta\|_{\Gamma^B} = 0\}.$$

We note here the weak formulation of (1.1)–(1.4).

Definition 2.1 (Weak formulation). The triplet

$$(u - \alpha, v - \beta, w) \in [L^2((0,T), H^1(\Omega)) \cap H^1((0,T); L^2(\Omega))]^3$$

is called weak solution to (1.1)–(1.4) if and only if for all $(\xi, \eta, \zeta) \in V_u \times V_w \times H^1(\Omega)$ the following identities hold:

$$\begin{aligned} \langle \xi, u_t \rangle + \langle \nabla \xi, a \nabla u \rangle &= \langle \xi, u_\nu \rangle_{\partial\Omega} - \langle \xi, \varphi \rangle - \langle \xi, \psi \rangle, \\ \langle \eta, v_t \rangle + \langle \nabla \eta, b \nabla v \rangle &= \langle \eta, v_\nu \rangle_{\partial\Omega} - \langle \eta, \varphi \rangle, \\ \langle \zeta, w_t \rangle + \langle \nabla \zeta, c \nabla w \rangle &= \langle \zeta, \varphi \rangle - \langle \zeta, \psi \rangle, \end{aligned} \tag{2.4}$$

where u_ν, v_ν, w_ν are the fluxes $\mathbf{n} \cdot a \nabla u$, etc., and, as usual, the dualities $\langle \cdot, \cdot \rangle$ pivot on the L^2 inner products.

3. Preliminary results.

Remark 3.1. Our intuitive picture of the consequence of the first reaction in (1.2) being very fast is that we should expect a resulting partition into A- and B-regions, i.e., regions with $u > 0$, $v \equiv 0$ or with $u \equiv 0$, $v > 0$.

Later we will show quite generally that

$$u,\ v \geq 0 \text{ but nowhere are both } u > 0 \text{ and } v > 0. \tag{3.1}$$

However it is informative to see this in the context of (1.3) since in that case we can employ a standard formal singular perturbation analysis to observe the resulting transient when (3.1) would not hold at the initial time t_0. Setting $\varepsilon = 1/\lambda \to 0$ and $\tau = \lambda(t - t_0)$, the first equations of (2.3) become $\left[\frac{du}{d\tau} - \varepsilon a u_{ss} = -uv - \varepsilon uw, \quad \frac{dv}{d\tau} - \varepsilon b v_{ss} = -uv\right]$ and formally setting[3] $\varepsilon = 0$ in this we then get the reduced system

$$\frac{du}{d\tau} = -uv = \frac{dv}{d\tau} \qquad \text{so}$$

$$\frac{du}{d\tau} = -uv = -u(u - c) \qquad \text{with } c = c(s) = u(s, t_0) - v(s, t_0)$$

which can be solved explicitly. Within a layer of duration $\mathcal{O}(1/\lambda)$, we have a rapid transient for which, as $\tau \to \infty$, one has

$$\begin{array}{llll} u \to [u_0 - v_0],\ v \to 0 & (A\text{-region}) & \text{if} & u_0 > v_0 \geq 0, \\ v \to [v_0 - u_0],\ u \to 0 & (B\text{-region}) & \text{if} & v_0 > u_0 \geq 0, \\ \quad u, v \to 0 & & \text{if} & u_0 = v_0 \geq 0, \end{array} \tag{3.2}$$

with an exponential decay rate when $u_0 \neq v_0$ and $1/\tau$ when $u_0 = v_0 > 0$. Thus, at each s we obtain 'adjusted' initial data $u, v\,|_{t=t_0+;\,\tau=\infty}$ such that (3.1) holds. In view of this, our subsequent analysis will always take the 'initial data' as subsequent to any such transient so satisfying (3.1) as well as $0 \leq u_0, v_0 \leq \bar{\mathrm{B}}$.

REMARK 3.2. Clearly, we must have $u, v, w \geq 0$ for these to represent concentrations and we will complement that by showing that upper bounds on u, v can also be obtained by quite similar arguments.

LEMMA 3.1. Let u, v, w satisfy (2.4) with (1.4). Assume a, b, c, α, β and the initial data are nonnegative and that $\varphi, \psi = 0$ where any argument is negative. Then u, v, w are nonnegative on $\mathcal{Q} = (0, T) \times \Omega$.

Proof. First we take $\xi = u_- = \min\{u, 0\} \leq 0$ in (2.4) and note (cf., e.g., [24]) that:
where $\xi \neq 0$ one has $\varphi, \psi = 0$ and (a.e.) $\xi_t = u_t$ so $\xi u_t = (\frac{1}{2}\xi^2)_t$,
where $\nabla\xi \neq 0$ one has (a.e.) $\nabla\xi = \nabla u$ so $\langle \nabla\xi, a\nabla u\rangle \geq 0$,
on Γ^A one has $u = \alpha \geq 0$ so $\xi = 0$; on $\partial\Omega\backslash\Gamma^A$ one has $u_\nu = 0$.
Using this in (2.4) and integrating, one gets $\frac{1}{2}\|\xi(t)\|^2 \leq \frac{1}{2}\|\xi(0)\|^2 = 0$ as one has $\xi = 0$ at $t = 0$. Thus, $\xi \equiv 0$ so $u \geq 0$. Showing $v, w \geq 0$ uses essentially the same argument. □

LEMMA 3.2. Let $(u, v, w)^\lambda$ satisfy (2.4), (1.4). Assume the boundary data α, β and the initial data are also bounded above independently of λ, i.e., $\alpha, \beta \leq \bar{\mathrm{B}}$ on Γ^A, Γ^B and $u_0, v_0 \leq \bar{\mathrm{B}}$ at $t = 0$. Then $u^\lambda, v^\lambda \leq \bar{\mathrm{B}}$ on $\mathcal{Q}$.

Proof. We now take $\xi = (u - \bar{\mathrm{B}})_+ = \max\{u - \bar{\mathrm{B}}, 0\} \geq 0$ and, as in the proof of Lemma 3.1, note that $\xi u_t = \xi\,\xi_t$ and $\nabla\xi \cdot \nabla u = |\nabla\xi|^2$ with $\xi\varphi$ and $\xi\psi$ nonnegative.

[3]This neglects the diffusive transport and the second reaction as negligible on this fast time scale; for diffusive transport this just requires enough regularity to ensure that $u_{ss}, v_{ss} = o(\lambda)$. Note that omitting spatial transport decouples the PDEs so we have independent ODEs at each s.

Further, the boundary term vanishes and $\xi = 0$ at $t = 0$. We then have $\xi \equiv 0$ so $u - \bar{\mathrm{B}} \leq 0$ on $\mathcal{Q}$. The argument for $v \leq \bar{\mathrm{B}}$ is essentially the same. $\square$

4. Compactness and convergence. Our primary goal in this section is to obtain Lemma 4.2, showing that u^λ, v^λ, w^λ all lie in a compact subset of $L^1(\mathcal{Q})$ so we have subsequential convergence as $\lambda \to \infty$. We then characterize such limit solutions to some extent, but defer to Section 7 a proof of uniqueness to make this true convergence.

4.1. *λ-independent estimates.* To this end, however, our major task is to get a λ-independent estimate for $q = \varphi(\lambda, u, v)$ as in (1.5). Somewhat counterintuitively, this estimate is entirely independent of the specifications of the functions φ, ψ although the argument is tailored to the form of the boundary conditions (1.4).

THEOREM 4.1. Assume $(u, v, w)^\lambda$ satisfies (1.1), (1.4) with data as in Lemmas 3.1, 3.2. Then there is a uniform $L^1(\mathcal{Q})$ bound B', independent of λ, for q, i.e.,

$$\|q\|_1 = \int_{\mathcal{Q}} q \ \leq \mathrm{B}' \qquad \text{for } \ 0 \leq q = q^\lambda = \varphi(\lambda, u^\lambda, v^\lambda).$$

Proof. It is convenient to introduce a function $\vartheta \in C^2(\overline{\Omega})$ (independent of λ) such that

$$0 \leq \vartheta \leq 1 \text{ with } \left\{ \begin{array}{l} \vartheta \equiv 0 \text{ in a neighborhood of } \Gamma^A, \\ \vartheta \equiv 1 \text{ in a neighborhood of } \Gamma^B. \end{array} \right. \tag{4.1}$$

This is always possible for disjoint closed sets Γ^A, Γ^B. We begin by using $\xi = \vartheta$ in (2.4) to get

$$\begin{aligned} \int_\Omega \vartheta q \ &= -\left(\int_\Omega \vartheta u\right)_t - \int_\Omega \nabla\vartheta \cdot a\nabla u - \int_\Omega \vartheta\psi + \int_{\partial\Omega} \vartheta u_\nu \\ &\leq -\left(\int_\Omega \vartheta u\right)_t + \int_\Omega (\nabla \cdot a\nabla\vartheta) u - \int_{\partial\Omega} \vartheta_\nu u \end{aligned}$$

on noting that $\vartheta\psi \geq 0$ and that $\vartheta u_\nu \equiv 0$ on $\partial\Omega$ since ϑ vanishes on Γ^A and elsewhere $u_\nu = 0$. Now we note that our assumptions that a is smooth and $\vartheta \in C^2$ imply a bound on $\nabla \cdot a\nabla\vartheta$ on Ω and on $\vartheta_\nu = \mathbf{n} \cdot a\nabla\vartheta$ on $\partial\Omega$ whence, as $u \leq \bar{\mathrm{B}}$ by Lemma 3.2, the last two terms are bounded. Integrating over $(0, T)$ gives an integral over $\mathcal{Q}$, and we have bounded $\int_{\mathcal{Q}} \vartheta q$ with an estimate independent of λ.

Next we use $\eta = (1 - \vartheta)$ for the v-equation in (2.4) in an essentially similar fashion, e.g., now noting that $(1 - \vartheta)v_\nu \equiv 0$ on $\partial\Omega$ since $1 - \vartheta$ vanishes near Γ^B and elsewhere $v_\nu = 0$. The argument bounding $\int_{\mathcal{Q}} (1 - \vartheta) q$ is then much the same. Adding these gives the desired $L^1(\mathcal{Q})$ estimate for $q = q^\lambda$. $\square$

We may remark that the reaction product C is irrelevant to this estimation, even if we were to permit φ and so q^λ to depend on w, except to be able to assert the nonnegativity of ψ or, at least, to bound $-\psi$ from above. On the other hand, the argument here is very much dependent on the nature of the boundary conditions (1.4) under consideration.

4.2. *Compactness.* A first compactness result is an immediate corollary of Theorem 4.1, noting that the space $L^1(\mathcal{Q})$ is isometrically embedded in the dual space $\mathcal{M}$, so Alaoglu's Theorem applies to the norm-bounded set $S^q = \{q^\lambda : \lambda > 0\}$.

COROLLARY 4.2. For every sequence (λ_k) there is a subsequence $(\lambda_{k(j)})$ for which $q^{\lambda_{k(j)}} \overset{*}{\rightharpoonup} \bar{q}$ (weak-* convergence in $\mathcal{M} = [C(\overline{\mathcal{Q}})]^*$). □

We continue here by introducing three linear operators

$$\mathcal{L}_A,\, \mathcal{L}_B,\, \mathcal{L}_C : \quad L^1(\mathcal{Q}) \longrightarrow L^1(\mathcal{Q}) : \quad f \mapsto \omega$$

defined, respectively, by

$$\begin{aligned}
&\begin{cases} \omega_t - \nabla\cdot a\nabla\omega = f \\ \text{with } \ \omega = 0 \text{ at } t=0 \end{cases} \text{and} \begin{cases} \omega = 0 & \text{on } \Gamma^A \\ \omega_\nu = 0 & \text{elsewhere on } \partial\Omega, \end{cases}\\
&\begin{cases} \omega_t - \nabla\cdot b\nabla\omega = f \\ \text{with } \ \omega = 0 \text{ at } t=0 \end{cases} \text{and} \begin{cases} \omega = 0 & \text{on } \Gamma^B \\ \omega_\nu = 0 & \text{elsewhere on } \partial\Omega, \end{cases}\\
&\begin{cases} \omega_t - \nabla\cdot c\nabla\omega = f \\ \text{with } \ \omega = 0 \text{ at } t=0 \end{cases} \text{and} \quad \omega_\nu = 0 \text{ on } \partial\Omega.
\end{aligned} \tag{4.2}$$

LEMMA 4.1. Assume $\Omega, \Gamma^A, \Gamma^B$ are as above and a, b, c are sufficiently smooth. Then each of $\mathcal{L}_A$, $\mathcal{L}_B$, $\mathcal{L}_C$ is a compact linear operator on $L^1(\Omega)$.

Proof. We refer the reader to [3, 4] for the proof; see also [2, 22]. □

For the solutions $(u,\, v,\, w)^\lambda$ of (1.1) with the specified boundary conditions (1.4) we define the sets

$$\begin{aligned}
&S^u = \{u^\lambda\}, \quad S^v = \{v^\lambda\}, \quad S^w = \{w^\lambda\}, \qquad \text{and}\\
&S^q = \{q^\lambda = \varphi(\lambda, u^\lambda, v^\lambda)\}, \quad S^\psi = \{\psi^\lambda = \psi(u^\lambda, w^\lambda)\}.
\end{aligned}$$

We will also, slightly extending the definitions used above for the initial data, now let u_0, v_0, w_0 be the solutions on $\mathcal{Q}$ of

$$\begin{aligned}
u_t - \nabla\cdot a\nabla u &= 0, & u\,|_{t=0} &= u_0,\\
v_t - \nabla\cdot b\nabla v &= 0, & v\,|_{t=0} &= v_0,\\
w_t - \nabla\cdot c\nabla w &= 0, & w\,|_{t=0} &= w_0,
\end{aligned} \tag{4.3}$$

with (1.4) so we have

$$\begin{aligned}
u^\lambda &= u_0 + \mathcal{L}_A f_A, & f_A &= f_A^\lambda = -q^\lambda - \psi^\lambda,\\
v^\lambda &= v_0 + \mathcal{L}_B f_B, & f_B &= f_B^\lambda = -q^\lambda,\\
w^\lambda &= w_0 + \mathcal{L}_C f_C, & f_C &= f_C^\lambda = +q^\lambda - \psi^\lambda.
\end{aligned}$$

Note that Lemma 3.2 and Theorem 4.1 show that the sets S^u, S^v, S^q are each bounded in $L^1(\mathcal{Q})$; we will show as part of the proof of Lemma 4.2 below that the sets S^w, S^ψ are also bounded.

LEMMA 4.2. The sets S^u, S^v, S^w are precompact in $L^1(\mathcal{Q})$. Thus, for any sequence $\lambda_k \to \infty$ there is a subsequence $\lambda = \lambda_{k(j)}$ for which $u^\lambda \to \bar{u}$, $v^\lambda \to \bar{v}$, $w^\lambda \to \bar{w}$ in $L^1(\mathcal{Q})$ and pointwise a.e. while $q^\lambda \overset{*}{\rightharpoonup} \bar{q}$.

Proof. Let $\hat{w} = \hat{w}^\lambda$ be the solution of $\hat{w}_t - \nabla\cdot c\nabla\hat{w} = q$, noting that S^q is bounded by Theorem 4.1. Since $f_C = q - \psi \le q$, one then has $0 \le w^\lambda = w_0 + \mathcal{L}f_C \le \hat{w} = w_0 + \mathcal{L}_C q$ (using an argument much like those in Lemmas 3.1 and 3.2). Thus,

$$\|w^\lambda\|_1 \le \|\hat{w}\|_1 \le \|w_0\|_1 + \|\mathcal{L}_C\|\,\|q^\lambda\|_1$$

which bounds S^w and, with (2.2), bounds S^ψ. This then bounds $\{f_A^\lambda\} \subset (-S^q - S^\psi)$ so S^u, now viewed as $(\{u_0\} + \mathcal{L}_A\{f_A^\lambda\})$, is precompact in $L^1(\mathcal{Q})$ by Lemma 4.1. Similarly, $S^v \subset \{v_0\} + \mathcal{L}_B(-S^q)$ and $S^w \subset (\{w_0\} + \mathcal{L}_C\{f_C^\lambda\})$ are precompact in $L^1(\mathcal{Q})$. The subsequential weak convergence is then immediate by Alaoglu's Theorem. $\square$

5. The limit problem.

THEOREM 5.1. The subsequential limits given by Lemma 4.2 satisfy, in an appropriate sense, the limit problem

$$\begin{array}{rclll} \bar{u}_t - \nabla\cdot a\nabla\bar{u} & = & -\bar{q} - \psi(\bar{u},\bar{w}), & & \bar{u}\,|_{t=0} = u_0, \\ \bar{v}_t - \nabla\cdot b\nabla\bar{v} & = & -\bar{q}, & & v\,|_{t=0} = v_0, \\ \bar{w}_t - \nabla\cdot c\nabla\bar{w} & = & +\bar{q} - \psi(\bar{u},\bar{w}), & & \bar{w}\,|_{t=0} = w_0, \end{array} \tag{5.1}$$

with (1.4).

Proof. Since the initial and boundary conditions are independent of λ, we focus attention on the equations in the interior of $\mathcal{Q}$. To this end, we consider C^∞ test functions ξ, η, ζ with support in the interior. Then we have no boundary terms on integrating the time derivatives by parts and applying the Divergence Theorem twice in (1.1) so we get, somewhat as in (2.4),

$$\begin{array}{rl} -\langle \xi_t, u\rangle - \langle \nabla\cdot a\nabla\xi, u\rangle & = -\langle \xi, q\rangle - \langle \xi, \psi\rangle, \\ -\langle \eta_t, v\rangle - \langle \nabla\cdot b\nabla\eta, v\rangle & = -\langle \eta, q\rangle, \\ -\langle \zeta_t, w\rangle - \langle \nabla\cdot c\nabla\zeta, w\rangle & = +\langle \zeta, q\rangle - \langle \zeta, \psi\rangle, \end{array} \tag{5.2}$$

with $u = u^\lambda$, etc. Of course $\langle \xi, q\rangle \to \langle \xi, \bar{q}\rangle$ by the definition of $q \overset{*}{\rightharpoonup} \bar{q}$ and similarly for $\langle \eta, q\rangle$ and $\langle \zeta, q\rangle$. We then note that Krasnosel'skĭi's Theorem on the continuity of Nemytsky operators (see e.g. [13]) ensures that $\psi^\lambda = \psi(u^\lambda, w^\lambda) \to \psi(\bar{u},\bar{w})$ as $(u,v)^\lambda \to (\bar{u},\bar{v})$. Since the functions ξ_t, η_t, ζ_t and $\nabla\cdot a\nabla\xi$, $\nabla\cdot b\nabla\eta$, $\nabla\cdot c\nabla\zeta$ are smooth by hypothesis, it is easy to see that each term here converges and, at least in this weak sense, the limits satisfy (5.1). $\square$

As anticipated, one consequence of this is the partition of $\mathcal{Q}$ into A-regions ($\bar{u} > 0$) and B-regions ($\bar{v} > 0$) with an interfacial set where $\bar{u} = \bar{v} = 0$.

THEOREM 5.2. Any subsequential limit solution $(\bar{u},\bar{v})$ as in Lemma 4.2 must satisfy (3.1), i.e., pointwise a.e. on $\mathcal{Q}$ one has $\bar{u} = 0$ or $\bar{v} = 0$.

Proof. If the conclusion were false, we would have existence of some set $E \subset \mathcal{Q}$ of positive measure with $\bar{u},\bar{v} \geq 2\varepsilon > 0$ on E. Since we have convergence $u^\lambda \to \bar{u}$, $v^\lambda \to \bar{v}$ pointwise a.e., we would have existence of some λ_* and some set $E' \subset E$ of positive measure with $u^\lambda, v^\lambda \geq \varepsilon$ on E' for all $\lambda \geq \lambda_*$ so, by (2.1),

$$\int_{\mathcal{Q}} q^\lambda \geq \int_{E'} \varphi(\lambda, u^\lambda, v^\lambda) \geq \int_{E'} \varphi(\lambda, \varepsilon, \varepsilon) \to \infty$$

as $\lambda \to \infty$, which would contradict Theorem 4.1. $\square$

We next introduce $y = \bar{u} - \bar{v}$; this trick goes back at least a century [19]; see also [7]. From (3.1) we immediately see that $y = \bar{u}$ (with $\bar{v} = 0$) when $y > 0$ and that $y = -\bar{v}$

(with $\bar{u} = 0$) when $y < 0$ so we recover $\bar{u} = y_+$ and $\bar{v} = -y_-$. This auxiliary function y satisfies the equation[4]

$$y_t - \nabla \cdot D(y)\nabla y = -\psi(y_+, \bar{w}), \qquad D(y) = \begin{cases} a & \text{if } y > 0, \\ b & \text{if } y < 0, \end{cases} \tag{5.3}$$

together with the boundary condition that

$$\begin{gathered} y = \alpha \;\text{ on } \Gamma^A, \qquad\qquad y = -\beta \;\text{ on } \Gamma^B, \\ y_\nu = \mathbf{n} \cdot D(y)\nabla y = 0 \quad \text{elsewhere on } \partial\Omega. \end{gathered} \tag{5.4}$$

It will then be convenient to define the 'y-flux' as

$$F = -D(y)\nabla y = \begin{cases} -a\nabla y = -a\nabla u & \text{where } y > 0, \\ -b\nabla y = b\nabla v & \text{where } y < 0, \end{cases} \tag{5.5}$$

so $y_t = \nabla F - \psi$. Note that F in (5.5) is the diffusive flux of the reactant A when $0 < y = u$ and is the (reversed) diffusive flux of B when $0 > y = -v$. There is no jump in F across a separating interface since the fast reaction consumes equal amounts of A and B and ψ is a function.

6. The 1-dimensional case. In the 1-dimensional case it is possible to provide a description in greater detail. Within this section, then, we will assume $\Omega = (0, 1)$ with $\Gamma^A = \{0\}$, $\Gamma^B = \{1\}$.

THEOREM 6.1. The solution $y = \bar{u} - \bar{v}$ of (5.3) is locally Hölder continuous in $\mathcal{Q}$. The A- and B-regions of Theorem 5.2 must be open subsets of $\mathcal{Q}$ and on these $\bar{q} \equiv 0$ and the flux F is locally Hölder continuous.

Proof. We begin with an estimate for $\bar{w}$. From (5.1) we have $\bar{w} = w_0 + \mathcal{L}_C(\bar{q} - \psi)$ and, since $\psi \geq 0$ so a Weak Maximum Principle argument shows $\mathcal{L}_C(\bar{q} - \psi) \leq \mathcal{L}\bar{q} = \omega$, we have $\bar{w} \in L^1([0, T] \to L^\infty(\Omega))$ if $\omega \in L^1([0, T] \to L^\infty(\Omega))$. Letting $S(t)$ be the semigroup corresponding to $\mathcal{L}_C$, we have $\omega(t) = \int_0^t S(t - s)\bar{q}(s, \cdot)\, ds$. From,[5] e.g., [1, Theorem 4.4], we have $\|S(t)\|_{1\to\infty} \leq Ct^{-1/2}$ where $\|\cdot\|_{1\to\infty}$ denotes the operator norm from $L^1(\Omega)$ to $L^\infty(\Omega)$. Thus,

$$\rho(t) = \|w_0\|_\infty + \|\omega(t)\|_\infty \leq C + \int_0^t C(t - s)^{-1/2}\|\bar{q}(s)\|_1\, ds.$$

We recognize this integral as convolution of the functions $t^{-1/2}$ and $\|\bar{q}(t)\|_1$ so, as $t^{-1/2} \in L^p$ for $1 \leq p < 2$, we have a bound[6] on ω in $L^p([0, T] \to L^\infty(\Omega))$ and so the desired bound on $\bar{w}$.

From Lemma 3.2 and (2.2) it follows that the forcing term $-\psi$ in (5.3) will also be in $L^p([0, T] \to L^\infty(\Omega))$ so, by the Nash-Moser estimates (see, e.g., [14, Theorem III:10.1])

[4]Note that the diffusion coefficient D will be discontinuous at the zero set of y where we cross between A- and B-regions. This is the interfacial region where the reaction is concentrated and the diffusion coefficient has not been defined here. However, we have $\nabla y = 0$ a.e. on the interfacial set where y is constant ($y \equiv 0$) so the value of $D(0)$ is irrelevant.

[5]We could also get the result from the Gaussian estimate for the fundamental solution, which is $\mathcal{O}(t^{-1/2})$ for $d = 1$.

[6]The function ρ will, in general, depend on ω, i.e., on $\bar{q} \in L^1(\mathcal{Q})$. However, the norm of $\rho \in L^p(0, T)$ is bounded with dependence only on the $L^1(\mathcal{Q})$-bound we have obtained for $\bar{q}$.

we have Hölder continuity of y. At any point where $\bar{u} > 0$ we have $y > 0$ so also $y = \bar{u} > 0$ in a neighborhood whence A-regions are open. Similarly, the B-regions are also open in $\mathcal{Q}$.

On an A-region where $\bar{u} > 0$ we have $\bar{v} \equiv 0$ and so $\bar{v}_t = 0$ and $(b\bar{v}_s)_s = 0$. Thus, comparing with the $\bar{v}$-equation of (5.1) on this region, we must have $\bar{q} \equiv 0$. Similarly, on a B-region we have $\bar{u} \equiv 0$ so $\bar{u}_t = 0$, $(a\bar{u}_s)_s = 0$, and $\psi(\bar{u}, \bar{w}) = 0$ whence again $\bar{q} \equiv 0$. Applying [14, Theorem III:11.1] locally (restricted to the region, noting that the diffusion coefficient is smooth there) gives the asserted local Hölder continuity of y_s and so of the A-flux $f = -a\bar{u}_s$ or of the B flux $f = b\bar{v}_s$ as appropriate. □

Suppose we set $A(t) = \{s : y(t,s) > 0\}$, $B(t) = \{s : y(t,s) < 0\}$. These form a set of open intervals which represent (except near the endpoints $s = 0, 1$ where the reactants A, B are supplied) isolated pockets of reactants, separated by interfacial points[7] at which the reaction can occur (delta functions for the Borel measure $\bar{q}$). If $\bar{u} > 0$ on (s_1, s_2) and $\bar{v} > 0$ on (s_2, s_3), necessarily with $\bar{u} = 0 = \bar{v}$ at the intermediate point s_2 by the continuity of y

At such an interface, separating an A-region from a B-region the normal $\mathbf{n} = \pm 1$ crossing outward from the A-region must have y decreasing so $\mathbf{n} \cdot F = |F| > 0$, i.e., a positive outward A-flux; similarly (now with $\mathbf{n}$ reversed) one has a positive outward B-flux from the B-region. Thus, both the isolated A-pocket (s_1, s_2) and also the B-pocket (s_2, s_3) are continually depleted and the same would be true if the interface would have the B-pocket to the left. We expect – and see computationally – that with no compensating source term such an isolated pocket must eventually vanish by coalescence of the endpoints. This provides a third time scale for the problem: one has the fast initial time scale as in Remark 3.1, the asymptotically long time scale of approach to steady state,[8] and this intermediate scale on which all but one of the separating reaction points[9] disappear in pairs by coalescence.

We note a computational example[10] showing in Figure 1 the evolution of the interfaces $y = 0$. For this example, the initial data was consistent ($uv \equiv 0$) with a pair of isolated pockets of A, B so initially three interfaces. In Figure 1 we then see the left- and rightmost interfaces moving toward the center as these pockets shrink with the pocket of B here

[7] The location of these interfacial points depends on t, of course, but we do not treat here the regularity of this dependence, referring the reader to [8] and [20] for some comparable discussion. Here we will simply assume adequate regularity for our purposes.

[8] We note that this has not yet been proved, even for the particular example of (2.3) for which existence of a unique steady state was shown in [21] but convergence to it as $t \to \infty$ has been observed computationally.

[9] Note that we would expect very small pockets to disappear rapidly so, after the initial $\mathcal{O}(1/\lambda)$ transient period (somewhat extended from (3.2) to permit the relevance of diffusion, but still negligible for large λ), there should be at $t = 0+$ only a finite number, necessarily odd, of isolated reaction points, each constituting a free boundary.

[10] This computation, taken from [23], was done with $\lambda = 10^9$, but the picture is essentially independent of λ, even for fairly moderate values. If one were to look at q^λ, the simulations for this and other large values of λ show profiles (for fixed t) spatially of the same form as given theoretically by the singular perturbation analysis of [12] for steady state: scaling as $\lambda^{1/3}$ in height and as $\lambda^{-1/3}$ in width, so converging in $[C(0,1)]^*$ to a delta function as $\lambda \to \infty$. This description applies only to isolated interfaces and, of course, cannot apply to the behaviour near the moment t_* when the pocket of B vanishes with its left and right boundaries coming together.

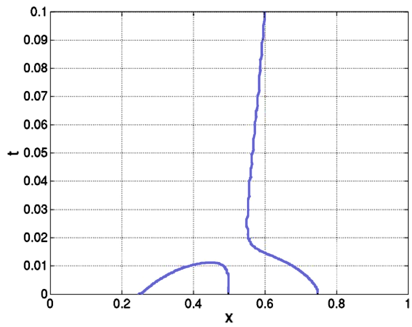

FIG. 1. Evolution in t of the interface.

consumed first, after which the single remaining interfacial point moves more slowly towards its steady state value.

For simplicity of exposition we now restrict our description to the situation with only a single interface at $s = \bar{s}(t)$, noting that interior interfaces behave in the same way. We assume we already know that $\bar{s}(\cdot)$ is fairly smooth. We exploit now the localization of the infinitely fast reaction on the free boundary (similar calculations have been done e.g. in [11, 17]).

THEOREM 6.2. Under the given assumptions, the measure-valued function

$$\bar{q} = \kappa(t)\delta(s - \bar{s}(t)) \qquad \text{with } \kappa(t) = |F(t, \bar{s}(t))| \tag{6.1}$$

is bounded in the space of Radon measures $\mathcal{M}$ on $[0, T]$. Further, $\bar{w} \in C(\mathcal{Q})$.

Proof. Under the assumption of a single interface at $s = \bar{s}(t)$ one has no isolated pockets so $y = u > 0$ for $0 \leq s < \bar{s}(t)$ and $-y = v > 0$ for $\bar{s}(t) < s \leq 1$. We have already noted that the outward A, B-fluxes at $\bar{s}(t)$ each have magnitude $|F|$ and provide the source for the point reaction $\kappa\delta(s - \bar{s})$, giving (6.1).

Now consider the B-region $\mathcal{Q}^B = \{s > \bar{s}(t), 0 < t < T\}$. This is disjoint from the support of $\bar{q}$ so we have the smooth equation $v_t = (bv_s)_s$ on this region with the Dirichlet boundary conditions $v \equiv 0$ at $s = \bar{s}$ on the left and $v \equiv \beta$ at $s = 1$ on the right. Although $\mathcal{Q}^B$ is not of the usual cylindrical form $(0, T) \times \Omega$, we may appeal to [15, Theorem 12.10] to see that the gradient v_s — and so the y-flux F — is Hölder continuous on $\mathcal{Q}^B$, including continuity up to the boundary $\bar{s}$.

In particular, this means that $t \mapsto F(t, \bar{s}(t))$ is continuous, hence bounded on the compact interval $[0, T]$ so $\kappa = |F|$ and $\bar{q}$ are pointwise well-defined, continuous and bounded. With $\|\bar{q}\|_1$ bounded, the same estimate for $\rho(t) = \|\bar{w}(t, \cdot)\|_\infty$ obtained in the proof of Theorem 6.1 now bounds[11] $\bar{w}$ in $L^\infty(\mathcal{Q})$. We then have $\bar{w} \in C(\mathcal{Q})$ either by using again the Nash-Moser estimates or by noting that $\bar{w}$ is a uniform limit of continuous functions w^λ. The continuity of $\bar{w}$ extends up to the boundary of $\mathcal{Q}$. $\square$

[11]With this uniform bound on $\bar{w}$, the Lipschitz condition (2.2) simplifies to

$$|\psi(u, w) - \psi(\hat{u}, \hat{w})| \leq L(|u - \hat{u}| + |w - \hat{w}|). \tag{6.2}$$

7. Uniqueness of the limit. Returning to the general setting, we note that true convergence

$$\bar{u} = \lim_{\lambda\to\infty} u^\lambda, \qquad \bar{v} = \lim_{\lambda\to\infty} v^\lambda, \qquad \bar{w} = \lim_{\lambda\to\infty} w^\lambda \qquad \text{in } L^1(\mathcal{Q}),$$

has not yet been proved since Lemma 4.2 only gives convergent subsequences. To provide it we must complement that by an argument showing that (5.1) has a unique solution, fixing $\bar{q}, \bar{u}, \bar{v}, \bar{w}$. We will do this here when the diffusion coefficients are equal ($a = b = c$) and the dimension $d < 6$. We further assume that $\partial\Omega$ is sufficiently regular that we may take the function ϑ of (4.1) to satisfy

$$\vartheta_\nu = \mathbf{n} \cdot a\nabla\vartheta \equiv 0 \qquad \text{on } \partial\Omega \tag{7.1}$$

and that the data α, β are sufficiently regular, meaning here that $\|\nabla y_0\|_{\mathcal{Q}} < \infty$ (as well as $\|y_0\|_\infty \le \bar{\mathrm{B}}$) where y_0 is the solution of $y_t - \nabla \cdot a\nabla y \equiv 0$ with (5.4).

We consider $y = \bar{u} - \bar{v}$ as above and, to handle the reaction product $\bar{w}$, will reformulate the system (1.1) by introducing another auxiliary function[12]

$$z = \bar{w} + \vartheta\bar{u} + (1-\vartheta)\bar{v} = \bar{w} + \bar{v} + \vartheta y. \tag{7.2}$$

Using the product rule to evaluate $\nabla\cdot a\nabla(\vartheta\bar{u})$ and $\nabla\cdot a\nabla((1-\vartheta)\bar{v})$, we obtain the system

$$\begin{aligned} y_t - \nabla\cdot a\nabla y \quad &= -\psi, \\ z_t - \nabla\cdot a\nabla z \quad &= \Upsilon y - (1+\vartheta)\psi, \end{aligned} \tag{7.3}$$

where[13]

$$\Upsilon : H^1(\Omega) \to L^2(\Omega) : y \mapsto (2a\nabla\vartheta)\cdot\nabla y + (\nabla\cdot a\nabla\vartheta)y. \tag{7.4}$$

Supplementing (5.4), a bit of manipulation in each part of $\partial\Omega$ shows that we have the boundary condition $z_\nu \equiv 0$ for the z-equation of (7.3).

As noted earlier, we can recover $\bar{u} = y_+$ and $\bar{v} = -y_-$ from y and now can also recover $\bar{w} = (z + y_- - \vartheta y)$ from y, z. Thus, if we show y, z are uniquely determined, then we have shown uniqueness for $\bar{u}, \bar{v}, \bar{w}$ and so also for $\bar{q}$.

Our first step is to estimate z.

Lemma 7.1. Under the assumptions above, $\int_0^T \|z(t,\cdot)\|_\infty$ is bounded.

Proof. We use $\eta = y - y_0$ as test function in the weak form of the difference between the y-equation in (7.3) and the y_0-equation, noting that the initial and boundary terms vanish. Then

$$\|\eta(t)\|^2 + 2\underline{a}\|\nabla\eta\|^2_{\mathcal{Q}(t)} \le -\langle\eta,\psi\rangle_{\mathcal{Q}(t)} \le C\left(1 + \|z\|_{\mathcal{Q}(t)}\right)$$

where we have used (2.2) and the bound $|\eta| \le \bar{\mathrm{B}}$ to bound ψ in terms of $\bar{w}$ and so of z. Applying the Gronwall Inequality then gives a bound $\|\nabla\eta\|^2_{\mathcal{Q}(t)} \le C(1 + \|z\|^2_{\mathcal{Q}(t)})$. We note that $\|\Upsilon y\| \le \|\Upsilon\eta\| + \|\Upsilon y_0\|$ and that $\|\Upsilon\eta\| \le C\|\nabla\eta\|$ since η vanishes on $\Gamma^A \cup \Gamma^B$ so $\|\eta\|_{H^1(\Omega)} \le C\|\nabla\eta\|$. Thus, $\|\Upsilon y\| \le C(1 + \|z\|^2_{\mathcal{Q}(t)})$ in view of the assumed boundedness

[12]A somewhat different auxiliary function $z = w + v$ was introduced in [20]. The resulting system could be made self-contained, but that required an awkward coupling in the boundary conditions.

[13]Up to this point we could get a similar construction without the assumption of equal diffusion coefficients. In that case, however, the operator Υ would no longer be linear and the corresponding uniqueness argument then fails.

of $\|\Upsilon y_0\|_{\mathcal{Q}}$. Using z as test function in the weak form of the z-equation in (7.3), noting the homogeneous boundary condition, we get

$$\|z(t)\|^2 + 2\underline{a}\|\nabla z\|^2_{\mathcal{Q}(t)} \leq \|z(0)\|^2 + 2\langle z, \Upsilon y - (1+\vartheta)\psi\rangle \leq C(1+\|z\|^2)$$

and applying the Gronwall Inequality bounds $\sup_t \|z(t)\|$ on $[0,T]$ as desired – and then also gives bounds on $\|\nabla y\|_{\mathcal{Q}}$ and on $\|\nabla z\|_{\mathcal{Q}}$. In particular, we see that $h = \Upsilon y - (1+\vartheta)\psi$ is bounded in $L^2(\mathcal{Q})$.

Now consider the contraction semigroup $S(\cdot)$ on $L^2(\Omega)$ generated by $G : z \mapsto -\nabla\cdot a\nabla z$ with homogeneous Neumann conditions so

$$z(t) = z_0(t) + \int_0^t S(t-s)\, h(s)\, ds.$$

We then have

$$\|z(t) - z_0(t)\|_{H^{2\sigma}(\Omega)} \leq C\|G^\sigma(z-z_0)\| \leq C\int_0^t \|G^\sigma S(t-s)\|\|h(s)\|\, ds.$$

Since $S(\cdot)$ is an analytic semigroup, $\|G^\sigma S(\tau)\| \leq C\tau^{-\sigma}$ which is in L^p for $p < 1/\sigma$. By Young's inequality the convolution with the L^2 function h is in L^1 provided $\sigma + \frac{1}{2} - 1 < 1$ so $\sigma < 3/2$. On the other hand, by the Sobolev Embedding Theorem, one has $H^{2\sigma}(\Omega) \hookrightarrow C(\overline{\Omega})$ for $2\sigma > d/2$, i.e., we have the desired embedding into $L^\infty(\Omega)$ for $2\sigma > d/2$, which is possible for $\sigma < 3/2$ when $d = 1, \dots, 5$. $\square$

We remark that this improves the $L^1(\mathcal{Q})$ bound on $\bar{w}$ from Lemma 4.2 since, in this setting, it also bounds $\sup_t \|\bar{w}(t)\|$, $\|\nabla\bar{w}\|_{\mathcal{Q}}$ and, since $|y| \leq \bar{\mathrm{B}}$, we have bounded $\|\bar{w}\|_{L^1([0,T]\to L^\infty(\Omega))}$.

THEOREM 7.1. Under the assumptions of this section ($a = b = c$, etc.) the solution of (5.1) is unique so, in this context, the subsequential convergence of Lemma 4.2 is genuine convergence as $\lambda \to \infty$.

Proof. We assume (7.3) might potentially have distinct solutions (y,z) and $(\hat{y},\hat{z})$ corresponding to distinct limit solutions as in Theorem 5.1 and then set $Y = y - \hat{y}$, $Z = z - \hat{z}$ and also

$$\Psi = \psi - \hat{\psi} = \psi(y_+, z - \vartheta y_+) - \psi(\hat{y}_+, \hat{z} - \hat{y}_+).$$

We now get, from (7.3), the system

$$\begin{aligned} Y_t - \nabla\cdot a\nabla Y &= -\Psi, \\ Z_t - \nabla\cdot a\nabla Z &= \Upsilon Y - (1+\vartheta)\Psi, \end{aligned} \tag{7.5}$$

where we have used the important fact that Υ is linear. We now use (2.2) to see that, pointwise, $|\Psi| \leq C|Y|(1 + |z - \vartheta y_+|) + C|Z|$ so $\|\Psi\| \leq \rho(t)\|Y\| + \|Z\|$ with ρ bounding $C(1 + \|z(t)\|_\infty + \bar{\mathrm{B}})$. By Lemma 7.1 we can take ρ bounded in $L^1(0,T)$.

Much as in the proof of Lemma 7.1, we now take Y as test function for the Y-equation in (7.5), noting that the initial and boundary data vanish since these are the same for y and for $\hat{y}$. We get

$$\|Y(t)\|^2 + 2\underline{a}\|\nabla Y\|^2_{\mathcal{Q}(t)} \leq \int_0^T \|Y\|\|\Psi\| \leq \int_0^T \rho\left[\|Y\|^2 + \|Z\|^2\right].$$

Similarly, taking Z as test function for the Z-equation in (7.5) we get, noting that $\|\Upsilon Y\| \leq C\|\nabla Y\|$ on Ω, the estimate

$$\|Z(t)\|^2 + 2\underline{a}\int_0^t \|\nabla Z\|^2 \leq \int_0^T \left[C\|\nabla Y\|\|Z\| + C\|Z\|^2 + \rho\|Y\|\|Z\|\right].$$

Adding and using $C\|\nabla Y\|\|Z\| \leq \underline{a}\|\nabla Y\|^2 + C'\|Z\|^2$, so we can cancel the $\|\nabla Y\|$ term, we get

$$\|Y\|^2 + \|Z\|^2 \leq \int_0^t C\rho(t)\left[\|Y\|^2 + \|Z\|^2\right].$$

It follows from Gronwall's inequality that $\|Y\|^2 + \|Z\|^2 \leq 0\exp[\int C\rho]$ with ρ integrable so we have uniqueness. □

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