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UNIFORM STABILITY AND MEAN-FIELD LIMIT OF A THERMODYNAMIC CUCKER-SMALE MODEL

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Abstract. We present a uniform-in-time stability and uniform mean-field limit of a thermodynamic Cucker-Smale model with small diffusion velocityf (for short, the SDV-TCS model). The original Cucker-Smale model deals with flocking dynamics of mechanical particles, in which the position and momentum are only macroscopic observables. Thus, the original Cucker-Smale model cannot describe some thermodynamic phenomena resulting from the temperature variations among particles and internal variables not

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taken into account. In [SIAM J. Math. Anal. 50 (2018), pp. 3092–3121] and [Arch. Rational. Mech. Anal. 223 (2017), pp. 1397–1425], a new thermodynamically consistent particle model was proposed from the system of gas mixtures in a rational way. In this paper, we discuss two issues for the SDV-TCS model. First we present a uniform stability of the SDV-TCS model with respect to initial data in the sense that the distance between two solutions is uniformly bounded by that of initial data in a mixed Lebesgue norm. Second, we derive a uniform mean-field limit from the SDV-TCS model to the Vlasov-type kinetic equation for some class of initial data whose empirical measure approximation guarantees exponential flocking in the SDV-TCS model.

1. Introduction. Collective phenomena in many-body systems are ubiquitous in our complex networks, e.g., herding of oxen, flocking of birds, and swarming of fish, etc. So far, most mathematical modeling of such a collective dynamic has been made on the basic assumption of mechanical particles, say point particles ignoring the internal structure and states (see [7, 34–36]). However, in order to provide a realistic modeling for collective behaviors of active particles, we need to introduce a dynamical system for the mechanical observables (position and momentum) and internal variables (temperature, excitation level, etc.). Recently, a new particle model has been introduced in [13, 21] for the flocking description of thermodynamically consistent Cucker-Smale particles. This new thermodynamical Cucker-Smale model (in short the TCS model) was derived based on the natural analogy between the mathematical model of flocking and the homogeneous mixture of fluids [30–32] which takes into account the balance law of energy of the single species particles with *internal energy*. Under the simplifying assumption that all particles have the same specific heat, the internal energy coincides with the *temperature* of each constituent. Like all physical systems in the classical framework, the governing dynamical system should be Galilean invariant, i.e., the equations are the same in any inertial frame. In the present case, as the velocity of the center of mass v is constant, without loss of generality we can choose the reference inertial frame moving with the center of mass. In this frame, $v = 0$ and the velocity of of the ith particle v_i coincide with the diffusion velocity $v_i - v$. Assuming the diffusion velocities are sufficiently small, we obtain the following (SDV-TCS model) system (we refer to [13, 21] for details):

$$
\begin{aligned}
&\frac{dx_i}{dt} = v_i, \quad t > 0, \quad i = 1, \cdots, N, \\
&\frac{dv_i}{dt} = \frac{\kappa_1}{N} \sum_{j=1}^{N} \phi(\|x_j - x_i\|_p) \left(\frac{v_j}{T_j} - \frac{v_i}{T_i} \right), \\
&\frac{dT_i}{dt} = \frac{\kappa_2}{N} \sum_{j=1}^{N} \zeta(\|x_j - x_i\|_p) \left(\frac{1}{T_i} - \frac{1}{T_j} \right), \\
&\sum_{i=1}^{N} x_i = 0, \quad \sum_{i=1}^{N} v_i = 0,
\end{aligned}
\tag{1.1}
$$

subject to initial data:

$$
(x_i, v_i, T_i)(0) = \left(x_i^0, v_i^0, T_i^0 \right). \tag{1.2}
$$

Here x_i, v_i, and T_i are the position, velocity, and temperature of the ith particle, respectively, and $\|\cdot\|_p$ is the ℓ_p-metric defined in (1.5). Communication weight ϕ and ζ are Lipschitz continuous nonnegative functions representing the degree of interactions, respectively,

$$\phi \in \mathrm{Lip}(\mathbb{R}_+;\mathbb{R}_+), \quad (\phi(r_2)-\phi(r_1))(r_2-r_1) \le 0, \quad r_1, r_2 \ge 0,$$
$$\zeta \in \mathrm{Lip}(\mathbb{R}_+;\mathbb{R}_+), \quad (\zeta(r_2)-\zeta(r_1))(r_2-r_1) \le 0, \quad r_1, r_2 \ge 0.$$

Note that for the pure mechanical case with constant and the same temperature, the SDV-TCS model reduces to the C-S model [10]:

$$\begin{aligned} \frac{dx_i}{dt} &= v_i, \quad t>0, \quad i=1,\cdots,N, \\ \frac{dv_i}{dt} &= \frac{\kappa}{N}\sum_{j=1}^{N} \phi(\|x_j-x_i\|_p)(v_j-v_i). \end{aligned} \tag{1.3}$$

In [21], the authors considered the constant communication weight, and this constant communication was further generalized to the metric dependent case in [13].

The model (1.1) takes into account the mutual interactions not only of a "*mechanical*"-type but also of a thermodynamic-type, that is, the "*temperature effect*" due to the presence of different "*internal energy*" compared to the C-S model which has been extensively studied from different perspectives in literature, e.g., asymptotic flocking estimates [4, 6, 10, 18, 20, 22, 23], collision avoiding [8], hierarchical and rooted leadership [24, 25, 33], application to flight navigation [29], noisy effects [1, 9, 11, 19], interaction with fluids [2, 3, 16], local-in-time mean-field limit [20], kinetic and hydrodynamic description [12, 17, 23], and variants of the C-S model [26], etc., (see recent survey papers [5, 7, 27] for details). For a large system with $N \gg 1$, it is not possible to integrate the system (1.3) numerically. Thus, for a large system, we consider the corresponding mean-field kinetic model to approximate system (1.1) with $N \gg 1$ (see [20, 23] for the C-S model). Let $f = f(x, v, T, t)$ be a one-particle probability density function of the TCS ensemble, where x, v, and T denote the position, velocity, and temperature, respectively. Then, using the BBGKY hierarchy [23], the dynamic of f is governed by the Vlasov-McKean-type equation in the extended phase space $(x, v, T) \in \mathbb{R}^{2d} \times \mathbb{R}_+$:

$$\begin{aligned} &\partial_t f + \nabla_x \cdot (vf) + \nabla_v \cdot (\mathcal{F}[f]f) + \partial_T(\mathcal{G}[f]f) = 0, \quad x, v \in \mathbb{R}^d,\ T \in \mathbb{R}_+,\ t>0, \\ &\mathcal{F}[f](x,v,T,t) := -\kappa_1 \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \phi(\|x-y\|_p)\left(\frac{v}{T}-\frac{v_*}{T_*}\right) f(y,v_*,T_*,t)\,dv_*dydT_*, \\ &\mathcal{G}[f](x,v,T,t) := \kappa_2 \int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \zeta(\|x-y\|_p)\left(\frac{1}{T}-\frac{1}{T_*}\right) f(y,v_*,T_*,t)\,dv_*dydT_*. \end{aligned} \tag{1.4}$$

It is well known in the kinetic theory that the macroscopic observables can be effectively calculated as moments of the kinetic density f. In what follows, we introduce a mixed norm and the concept of uniform-in-time stability. For this, we set

$$
\begin{aligned}
&X := (x_1, \cdots, x_N), \quad V := (v_1, \cdots, v_N), \quad T := (T_1, \cdots, T_N), \quad Z := (X, V, T), \\
&||x_i||_p := \left(\sum_{k=1}^{d} |x_i^k|^p \right)^{\frac{1}{p}}, \quad ||X||_{p,q} := \left(\sum_{i=1}^{N} ||x_i||_p^q \right)^{\frac{1}{q}}, \quad ||X||_p := ||X||_{p,p},
\end{aligned} \tag{1.5}
$$

where $1 \leq p, q \leq \infty$ and introduce the concept of uniform-in-time stability for (1.1) in $\ell_{p,q}$-norm with respect to initial data. For two solutions Z and $\bar{Z}$ to (1.1)–(1.2), we define the distance $d_{p,q}(Z, \bar{Z})$ as follows:

$$
d_{p,q}(Z, \bar{Z}) := \left(\sum_{i=1}^{N} ||x_i(t) - \bar{x}_i(t)||_p^q \right)^{\frac{1}{q}} + \left(\sum_{i=1}^{N} ||v_i(t) - \bar{v}_i(t)||_p^q \right)^{\frac{1}{q}} + \left(\sum_{i=1}^{N} |T_i(t) - \bar{T}_i(t)|^q \right)^{\frac{1}{q}}.
$$

Then, it is easy to see that $d_{p,q}$ is a metric on the N-body extended phase space $\mathbb{R}^{Nd} \times \mathbb{R}^{Nd} \times \mathbb{R}^N$.

Definition 1.1. For $p, q \in [1, \infty]$, let Z and $\bar{Z}$ be two solutions to (1.1)–(1.2) with initial data Z^0 and $\bar{Z}^0$, respectively. Then, we say that the uniform stability with respect to initial data holds if there exists a positive constant G independent of t such that

$$
d_{p,q}(Z(t), \bar{Z}(t)) \leq G d_{p,q}(Z^0, \bar{Z}^0), \quad t \geq 0.
$$

In this paper, we are interested in the following questions:

- Question 1 (Uniform-in-time stability): Does the SDV-TCS model (1.1) satisfy a uniform $\ell_{p,q}$-stability in the sense of Definition 1.1? If yes, then provide a sufficient framework that guarantees the uniform stability.
- Question 2 (Uniform-in-time mean-field limit): Is it possible to establish the mean-field limit from the particle SDV-TCS model (1.1) to the kinetic TCS model (1.4) for all time?

The main results of this paper are three-fold. First, we present the improved flocking estimate using the diameter of each variable. Second, we provide a class of initial data which leads to the uniform stability in the sense of Definition 1.1 (see Theorem 3.4). Finally, we provide a sufficient framework in terms of initial measure leading to the uniform-in-time mean-field limit (see Theorem 3.6) and the flocking estimate (see Corollary 3.1). As a byproduct of the mean-field limit, we also establish the unique solvability of the measure-valued solution constructed through the mean-field limit procedure which exhibits a flocking estimate. Recently, in [14] the authors studied a well-posedness of classical solutions to the corresponding hydrodynamic model which can be obtained from (1.4) using the corresponding moment system and monokinetic ansatz as a closure relation for the infinite moment system.

The rest of this paper is organized as follows. In Section 2, we will briefly review basic properties of (1.1) such as conservation laws, monotonicity of entropy, and existence of the trapping set in temperature variables. We also provide a measure-theoretic formulation of the kinetic model (1.4) such as concepts of measure-valued solutions and Wasserstein distance. In Section 3, we first recall the previous flocking estimate from [13] and provide our three results such as the flocking estimates formulated in terms of diameters defined in (1.6), uniform stability of some class of solutions, and unique solvability of measure-valued solutions for some class of initial measures. In Section 4, we provide the proof of Theorem 3.2 on the flocking estimates for (1.1). In Section 5, we provide the proof of Theorem 3.4 on the uniform stability. In Section 6, we present the proof of Theorem 3.6 on the uniform-in-time mean-field limit from (1.1) to (1.4). Section 7 is devoted to a brief summary of this paper. In Appendices A, B, and C, we provide detailed and lengthy proofs for Lemma 5.2 and Lemma 5.3, respectively.

NOTATION. For spatial, velocity, and temperature configurations X, V, and T, we set

$$\mathcal{D}_X := \max_{1\le i,j\le N} \|x_i - x_j\|_p, \quad \mathcal{D}_V := \max_{1\le i,j\le N} \|v_i - v_j\|_p, \quad \mathcal{D}_T := \max_{1\le i,j\le N} |T_i - T_j|. \tag{1.6}$$

2. Preliminaries. In this section, we present a theoretical minimum for later sections. First we study basic a priori estimates for the SDV-TCS model (1.1) such as conserved quantities and monotonicity of an entropy. Second, we will briefly discuss measure-theoretical framework for the mean-field limit.

2.1. *The SDV-TCS model.* In this subsection, we study the conservation laws and monotonicity of the SDV-TCS (1.1) model and existence of the trapping region. We first recall the concept of asymptotic flocking as follows.

DEFINITION 2.1 ([13]). Let $\{(x_i, v_i, T_i)\}$ be a configuration whose temporal evolution is governed by system (1.1). Then, the configuration exhibits an asymptotic flocking if the following conditions hold:

(1) (Asymptotic aggregation): The velocity and temperature variations vanish asymptotically:

$$\lim_{t\to\infty} \mathcal{D}_V(t) = 0, \quad \lim_{t\to\infty} \mathcal{D}_T(t) = 0.$$

(2) (Uniform coherence): The spatial variations stay bounded uniformly in time:

$$\sup_{t\ge 0} \mathcal{D}_X(t) < \infty.$$

In the following proposition, we discuss the conservation laws and monotonicity of entropy.

PROPOSITION 2.1 ([21]). For $\tau \in (0, \infty]$, let (x_i, v_i, T_i) be a solution to system (1.1)–(1.2) in a time interval $[0, \tau)$. Then, the following a priori estimates hold:

(1) The total momentum and temperature are conserved:

$$\frac{d}{dt}\sum_{i=1}^N v_i(t) = 0 \quad \text{and} \quad \frac{d}{dt}\sum_{i=1}^N T_i(t) = 0, \quad t \in [0, \tau).$$

(2) The entropy $\mathcal{S}(t) := \sum_{i=1}^N \ln T_i$ is monotonically increasing:

$$\frac{d\mathcal{S}}{dt} \ge 0.$$

Proof. Since the proof is exactly the same as in Lemma 4.1 and Lemma 4.2 in [21], we omit it here. ☐

Note that the R.H.S. of equation (1.1) is Lipschitz continuous as long as the temperature of each particle T_i is strictly above zero. Hence to guarantee the existence of a solution of (1.1) for whole time interval using standard Cauchy-Lipschitz theory, we must verify that temperature is globally away from zero for all particles. To do this, it is more convenient to work with the coldness $C_i = T_i^{-1}$ satisfying the following dynamics:

$$\begin{aligned} \frac{dC_i}{dt} &= \frac{d}{dt}\frac{1}{T_i} = -\frac{\kappa_2}{NT_i^2}\sum_{j=1}^N \zeta(\|x_j - x_i\|_p)\left(\frac{1}{T_i} - \frac{1}{T_j}\right) \\ &= \frac{\kappa_2}{N}\sum_{j=1}^N \zeta(\|x_j - x_i\|_p) C_i^2 (C_j - C_i). \end{aligned} \tag{2.1}$$

Next, we show that there exists a positively invariant set for system (2.1). For this, we set

$$C_m(t) := \min_{1\le j\le N} C_j(t), \quad C_M(t) := \max_{1\le j\le N} C_j(t), \quad \mathcal{D}_C(t) := C_M(t) - C_m(t), \quad t \ge 0.$$

Note that the extremal indices are also depend on t and $\mathcal{D}_C(t)$ is a Lipschitz continuous function so that it is almost everywhere differentiable.

Lemma 2.2. For $\tau \in (0, \infty]$, let $\{(x_i, v_i, T_i)\}$ be a solution in a time interval $[0, \tau)$ to system (1.1)–(1.2) with initial data $\{(x_i^0, v_i^0, T_i^0)\}$:

$$0 < C_j(0) < \infty, \quad j = 1, \cdots, N.$$

Then, we have

$$0 < C_m(0) \le C_j(t) \le C_M(0) < \infty, \quad t \in [0, \tau). \tag{2.2}$$

Proof. For the desired estimate (2.2), it suffices to show that extremal coldness C_m and C_M satisfy

$$C_m(t) \ge C_m(0), \quad C_M(t) \le C_M(0), \quad t \ge 0.$$

Since the solution of the system (2.1) is analytic, in a finite-time interval $[0, \tau)$ the crossings between C_i and C_j are finite. Therefore, there exists a sequence of times $0 \le t_0 < t_1 < t_2 < \cdots t_l = \tau$ such that the solution is collision free on each sub-time-interval (t_i, t_{i+1}), $i = 0, \cdots, l-1$. For an interval (t_i, t_{i+1}), we assume that C_i is well-ordered as follows:

$$0 < C_1(t) < C_2(t) < \cdots < C_N(t), \quad t \in (t_i, t_{i+1}).$$

Note that $C_m = C_1$ and $C_M = C_N$ satisfy

$$\begin{aligned} \frac{dC_1}{dt} &= \frac{\kappa_2}{N}\sum_{j=1}^N \zeta(\|x_j - x_1\|_p) C_1^2 (C_j - C_1) \ge 0, \quad t \in (t_i, t_{i+1}), \\ \frac{dC_N}{dt} &= \frac{\kappa_2}{N}\sum_{j=1}^N \zeta(\|x_j - x_N\|_p) C_N^2 (C_j - C_N) \le 0, \end{aligned}$$

where we used the positivity of ζ. Then, we have

$$C_M(t) \le C_M(t_i), \quad C_m(t) \ge C_m(t_i), \quad t \in [t_i, t_{i+1}).$$

This also yields the contraction of the diameter $\mathcal{D}(\mathcal{C})$ in the sub-interval $[t_i, t_{i+1})$:

$$\mathcal{D}_C(t) = C_M(t) - C_m(t) \le C_M(t_i) - C_m(t_i) = \mathcal{D}_C(t_i), \quad t \in [t_i, t_{i+1}).$$

By the continuity of $\mathcal{D}_C(\cdot)$, we have

$$\mathcal{D}_C(t) \le \mathcal{D}_C(0), \quad t \ge 0.$$

□

REMARK 2.3. We observe that the result of Lemma 2.2 implies that if initial temperatures satisfy

$$0 < T_m^0 := \min_i T_i^0 \quad \text{and} \quad T_M^0 := \max_i T_i^0 < \infty,$$

then we obtain for $\tau < \infty$,

$$0 < T_m^0 \le T_i(t) \le T_M^0, \quad t \in (0, \tau).$$

2.2. *A measure-theoretic framework.* In this subsection, we briefly present a measure-theoretic framework to be used in Section 6. In [20], the authors investigated the local-in-time mean-field limit from the particle Cucker-Smale model to the kinetic Cucker-Smale equation using the particle method and Wasserstein framework.

First, we review several basics of measure-valued solutions to the kinetic SDV-TCS model (1.4). Let $\mathcal{P}(\mathbb{R}^{2d} \times \mathbb{R}_+)$ be the set of all probability measures on the phase space $\mathbb{R}^{2d} \times \mathbb{R}_+$, which can be understood as normalized nonnegative bounded linear functionals on $\mathcal{C}_0(\mathbb{R}^{2d} \times \mathbb{R}_+)$. For a probability measure $\mu \in \mathcal{P}(\mathbb{R}^{2d} \times \mathbb{R}_+)$, we use a standard duality relation:

$$\langle \mu, f \rangle = \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} f(x, v, T) d\mu(x, v, T), \quad f \in \mathcal{C}_0(\mathbb{R}^{2d} \times \mathbb{R}_+).$$

Next, we recall several definitions to be used in this paper.

DEFINITION 2.4 ([20]). For $\tau \in [0, \infty)$, $\mu_t \in L^\infty([0, T); \mathcal{P}(\mathbb{R}^{2d} \times \mathbb{R}_+))$ is a measure-valued solution to (1.4) with initial data $\mu_0 \in \mathcal{P}(\mathbb{R}^{2d} \times \mathbb{R}_+)$ if the following two conditions hold:

(1) Total mass is normalized and μ is weakly continuous in t:

$$\langle \mu_t, 1 \rangle = 1, \quad \langle \mu_t, f \rangle \text{ is continuous in } t \quad \forall\ f = f(x, v, T) \in \mathcal{C}_0^1(\mathbb{R}^{2d} \times \mathbb{R}_+).$$

(2) μ satisfies equation (1.4) in a weak sense:

$$\langle \mu_t, \varphi(\cdot, \cdot, \cdot, t) \rangle - \langle \mu_0, \varphi(\cdot, \cdot, \cdot, 0) \rangle = \int_0^t \langle \mu_s, \partial_s \varphi + v \cdot \nabla_x \varphi + \mathcal{F} \cdot \nabla_v \varphi + \mathcal{G} \partial_T \varphi \rangle \, ds, \quad (2.3)$$

where φ is a test function in $\mathcal{C}_0(\mathbb{R}^{2d} \times \mathbb{R}_+ \times [0, \tau))$ and $\mathcal{F}$ and $\mathcal{G}$ are given by the relations in (1.4).

REMARK 2.5. Note that for a solution $\{(x_i, v_i, T_i)\}$ to (1.1), the empirical measure,

$$\mu_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{x_i} \otimes \delta_{v_i} \otimes \delta_{T_i},$$

is a measure-valued solution in the sense of Definition 2.4. Thus, ODE solution to (1.1) can be understood as a measure-valued solution for the corresponding kinetic equation (1.4). Hence, we can treat the particle and kinetic approximation models in the common framework. Likewise, the classical solution for the kinetic C-S model (1.4) can also be understood as a measure-valued solution.

We now discuss how to measure the distance between the solutions of (1.1) and (1.4) by equipping a metric on the probability measure space $\mathcal{P}(\mathbb{R}^{2d}\times\mathbb{R}_+)$, and the concept of local-in-time mean-field limit. In fact, we can endow Wasserstein-(p,q) distance $W_{p,q}$ in the space $\mathcal{P}(\mathbb{R}^{2d}\times\mathbb{R}_+)$.

Definition 2.6 ([28, 37]).

(1) For $p\in[1,\infty]$ and $q\in[1,\infty)$, let $\mathcal{P}_q(\mathbb{R}^{2d}\times\mathbb{R}_+)$ be a collection of all probability measures with a finite qth moment: for some $z_0\in\mathbb{R}^{2d}\times\mathbb{R}_+$

$$\langle \mu, \|z-z_0\|_p^q\rangle < \infty.$$

Then, Wasserstein-(p,q) distance $W_{p,q}(\mu,\nu)$ is defined as follows. For any $\mu,\nu\in\mathcal{P}_q(\mathbb{R}^{2d}\times\mathbb{R}_+)$,

$$\begin{cases} W_{p,q}(\mu,\nu) := \inf\limits_{\gamma\in\Gamma(\mu,\nu)}\left(\displaystyle\int_{(\mathbb{R}^{2d}\times\mathbb{R}_+)^2}\|z-z^*\|_p^q d\gamma(z,z^*)\right)^{\frac{1}{q}}, \\ W_q(\mu,\nu) := W_{q,q}(\mu,\nu), \end{cases}$$

where $\Gamma(\mu,\nu)$ denotes the collection of all probability measures on $\mathbb{R}^{2d}\times\mathbb{R}_+\times\mathbb{R}^{2d}\times\mathbb{R}_+$ with marginals μ and ν.

(2) If $\lim\limits_{q\to\infty} W_{p,q}$ exists, then we define $W_{p,\infty}$ metric as the limit.

(3) For any $\tau\in(0,\infty]$, the kinetic equation (1.4) is derivable from the particle model (1.1) in $[0,\tau)$, or equivalently the mean-field limit from the particle system (1.1) to the kinetic equation (1.4), which is valid in $[0,\tau)$, if for every solution μ_t of the kinetic equation (1.4) with initial data μ_0, the following condition holds: for some $q\in[1,\infty]$ and $t\in[0,\tau)$,

$$\lim_{N\to+\infty} W_q(\mu_0^N,\mu_0)=0 \quad\Longleftrightarrow\quad \lim_{N\to+\infty} W_q(\mu_t^N,\mu_t)=0,$$

where μ_t^N is a measure-valued solution of the particle system (1.1) with initial data μ_0^N.

For later use, we quote two results on the approximation of a measure by empirical measures and mean-field limit in any finite-time interval without proofs.

Proposition 2.2 ([37]). For any given $p\in[1,\infty)$ and $\mu\in\mathcal{P}(\mathbb{R}^{2d}\times\mathbb{R}_+)$ with compact support, there exists a sequence of empirical measures μ^N such that

$$\mu^N \text{ has the common compact support with } \mu \quad\text{and}\quad \lim_{N\to+\infty} W_p(\mu^N,\mu)=0.$$

Remark 2.7. The construction of the approximation can be done by following the method of Theorem 6.18 in the book [37] by finding a sequence of atomic measures $\sum_{j=1}^N a_j\delta_j$ with real numbers a_j such that $\sum_{j=1}^N a_j = 1$.

3. Presentation of main results. In this section, we first review the previous results and then discuss our main results, namely, "*uniform stability*" and "*uniform mean-field limit*". The proofs for these results will be the main subjects in the following sections.

3.1. *Previous results.* In this subsection, we provide a previous result on the SDV-TCS model in [13].

PROPOSITION 3.1 ([13]). Suppose that for given initial data (x_i^0, v_i^0, T_i^0), there exists a positive constant Γ satisfying the relations:

$$\|X^0\|_{2,2}, \quad \|V^0\|_{2,2} < \Gamma, \quad \phi(2\sqrt{2}\Gamma) > 2\frac{T_M^0}{\kappa_1} + \frac{\phi(0)T_M^0(T_M^0 - T_m^0)}{2(T_m^0)^2}. \tag{3.1}$$

Let (x_i, v_i, T_i) be a global solution to (1.1). Then, the following assertions hold:

(1) A uniform boundedness of position and exponential flocking for velocity occur:

$$\|X\|_{2,2} < 2\Gamma, \quad \|V\|_{2,2} \le \|V_0\|_{2,2}e^{-2t}, \quad t \ge 0.$$

(2) The temperature fluctuation decays to zero exponentially fast:

$$\|T - T_c\|_{2,2} \le \|T^0 - T_c\|_{2,2} \exp\left[-\frac{\kappa_2}{(T_M^0)^2}\zeta(2\sqrt{2}\Gamma)t\right],$$

where $T_c := \frac{1}{N}\sum_{i=1}^N T_i^0$ is an initial mean temperature, which is constant.

REMARK 3.1. Note that the condition (3.1) is independent of the communication weight ζ appearing in the temperature equation (1.1). Thus, once the mechanical variables reach to flocking state, then flocking in temperature follows from the uniform boundedness in spatial diameter.

3.2. *Main results.* In this subsection, we briefly present our main results and discuss the comparison with previous results. Unlike the results in Proposition 3.1 where the sufficient frame (3.1) is characterized in terms of ℓ_2-norm, we use diameters for spatial position, velocity, and temperature in this paper.

3.2.1. *Uniform-in-time stability.* In this part, we present a uniform-in-time stability of the solutions to (1.1) exhibiting a flocking estimate. First of all, let us introduce an admissible set of initial data:

$$\mathcal{S}(C, \mathcal{D}_X^\infty) := \Bigg\{ Z^0 \in \mathbb{R}^{(2d+1)N} \;:\; \mathcal{D}_T(0) \le \frac{(T_m^0)^2\phi(\mathcal{D}_X(0))}{2CT_M^0},$$
$$\mathcal{D}_V(0) \le \frac{\kappa_1}{2T_M^0}\int_{\mathcal{D}_X(0)}^{\mathcal{D}_X^\infty}\phi(s)ds, \quad \frac{8}{3C}\phi(\mathcal{D}_X(0)) \le \phi(\mathcal{D}_X^\infty) < \phi(\mathcal{D}_X(0)) \Bigg\}.$$

Note that the set $\mathcal{S}(C, \mathcal{D}_X^\infty)$ is independent of the communication weight $\zeta = \zeta(s)$ in the temperature equation $(1.1)_3$. Our first result is concerned with flocking estimates in terms of diameters.

THEOREM 3.2 (Improved flocking estimate). Let $Z = (X, V, T)$ be a global solution to (1.1)–(1.2) with initial data $Z^0 \in \mathcal{S}(C, \mathcal{D}_X^\infty)$. Then, asymptotic flocking occurs:

$$\text{(i)} \sup_{0\le t<\infty} \mathcal{D}_X(t) \le \mathcal{D}_X^\infty, \quad \mathcal{D}_T(t) \le \mathcal{D}_T(0)e^{-\frac{\kappa_2}{(T_M^0)^2}\zeta(\mathcal{D}_X^\infty)t} \quad t \ge 0,$$

$$\text{(ii)} \; \mathcal{D}_V(t) \le \mathcal{D}_V(0)\exp\left[-\frac{\kappa_1\phi(\mathcal{D}_X^\infty)}{T_M^0}t + \frac{2\kappa_1(T_M^0)^2\mathcal{D}_T(0)}{\kappa_2(T_m^0)^2\zeta(\mathcal{D}_X^\infty)}\right].$$

REMARK 3.3. We next comment on the comparison between Proposition 3.1 and Theorem 3.2 as follows:

1. In Proposition 3.1, the condition for flocking implies not only the initial temperature difference $T_M^0 - Tm^0$ should be small, but also the maximum temperature T_M^0 itself also should be small with respect to the interaction kernel ϕ. However, the conditions (3.1) require only the initial temperature difference $\mathcal{D}_T(0)$ has constraint with respect to ϕ, which allows us to get a wider class of initial configurations.

2. Note that the framework in Theorem 3.2 is formulated in terms of diameters of spatial and velocity configurations. This new result can be used to establish the existence of measure-valued solutions for the kinetic model (1.4) in the whole time interval.

Next, we present our second result on the uniform stability of solutions with respect to initial data in the set $\mathcal{S}(C, \mathcal{D}_X^\infty)$.

THEOREM 3.4 (Uniform in-time stability). For $p, q \in [1, \infty)$, let Z and $\bar{Z}$ be two global solutions to (1.1)–(1.2) with initial data $Z^0, \bar{Z}^0 \in \mathcal{S}(C, \mathcal{D}_X^\infty)$, respectively, and zero sum conditions:

$$\sum_{i=1}^N v_i(t) = \sum_{i=1}^N \bar{v}_i(t) = 0, \quad t \geq 0.$$

Then, the uniform stability holds in the sense of Definition 1.1, i.e., there exists a positive constant G independent of t such that

$$d_{p,q}(Z(t), \bar{Z}(t)) \leq G d_{p,q}(Z^0, \bar{Z}^0), \quad t \geq 0.$$

REMARK 3.5. For the Cucker-Smale model, the uniform stability has been established in [15] in the class of flocking solutions.

3.2.2. *Uniform-in-time mean-field limit.* In this part, we present our third main result on the uniform mean-field limit of (1.1), which yields the global existence of measure-valued solutions to (1.4).

THEOREM 3.6 (Uniform mean-field limit). Suppose the initial measure $\mu_0 \in \mathcal{P}(\mathbb{R}^{2d} \times \mathbb{R}_+)$ has a compact support, zero mean velocity, and the first two finite velocity moments:

$$\int_{\mathbb{R}^{2d} \times \mathbb{R}_+} v\mu_0(dx, dv, dT) = 0, \quad \int_{\mathbb{R}^{2d} \times \mathbb{R}_+} (1 + |v|^2)\mu_0(dx, dv, dT) \leq m_2. \tag{3.2}$$

Moreover, suppose there are positive constants C and $\mathcal{X}_\infty$ such that:

$$\mathcal{D}_T^{\mu_0} \leq \frac{(T_m^0)^2 \phi(\mathcal{D}_X^{\mu_0})}{2CT_M^0}, \quad \mathcal{D}_V^{\mu_0} \leq \frac{\kappa_1}{2T_M^0} \int_{\mathcal{D}_X^{\mu_0}}^{\mathcal{X}_\infty} \phi(s) ds, \quad \frac{8}{3C} \phi(\mathcal{D}_X^{\mu_0}) \leq \phi(\mathcal{X}_\infty) < \phi(\mathcal{D}_X^{\mu_0}), \tag{3.3}$$

where $\mathcal{D}_X^{\mu_0}, \mathcal{D}_V^{\mu_0}$, and $\mathcal{D}_T^{\mu_0}$ are diameters of restricted support of μ_0 on spatial, velocity, and temperature coordinates, respectively. Then, the following assertions hold. For $p, q \in [1, \infty]$,

(1) There exists a unique measure-valued solution $\mu_t \in L^\infty\left([0, \infty); \mathcal{P}(\mathbb{R}^{2d} \times \mathbb{R}_+)\right)$ to (1.4) with initial data μ_0: μ_t is approximated by empirical measure μ_t^N in Wasserstein-(q, p) distance uniformly in time:

$$\overline{\lim_{N \to +\infty}} \sup_{t \in [0, +\infty)} W_{q,p}(\mu_t^N, \mu_t) = 0.$$

(2) Moreover, if ν_t is the measure-valued solution to (1.4) with initial measure ν_0 with compact support, zero mean velocity and finite moments and the same initial condition on support with μ, then there exists nonnegative constant $\bar{G}$ independent of t such that

$$W_{q,p}(\mu_t,\nu_t) \le \bar{G} W_{q,p}(\mu_0,\nu_0), \quad t \in [0,\infty).$$

As a direct corollary of Theorem 3.6, we obtain the flocking estimate of the measure-valued solution whose existence is guaranteed by the previous theorem.

Corollary 3.1. Let μ_t be a measure-valued solution to (1.1) with initial measure $\mu_0 \in \mathcal{P}(\mathbb{R}^{2d} \times \mathbb{R}_+)$ satisfying the condition (3.2)–(3.3). Then, an asymptotic flocking estimate holds:

$$\left(\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \|v\|_q^p d\mu_t\right)^{\frac{1}{p}} \le d^2 \exp\left[-\frac{\kappa_1\phi(\mathcal{D}_X^\infty)}{T_M^0}t + \frac{2\kappa_1 (T_M^0)^2 \mathcal{D}_T(0)}{\kappa_2 (T_m^0)^2 \zeta(\mathcal{D}_X^\infty)}\right] \left(\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \|v\|_q^p d\mu_0\right)^{\frac{1}{p}}.$$

4. An improved asymptotic flocking estimate. In this section, we present an asymptotic flocking estimate formulated in terms of diameters which yields a finer result compared to [13].

Our asymptotic flocking estimates will be based on the following system of differential inequalities (SDI): for a.e. $t > 0$:

$$\begin{aligned} &\left|\frac{d}{dt}\mathcal{D}_X(t)\right| \le \mathcal{D}_V(t), \quad \frac{d}{dt}\mathcal{D}_T(t) \le -\frac{\kappa_2\zeta(\mathcal{D}_X(t))}{(T_M^0)^2}\mathcal{D}_T(t), \\ &\frac{d}{dt}\mathcal{D}_V(t) \le -\frac{\kappa_1\phi(\mathcal{D}_X)}{T_M^0}\mathcal{D}_V + 2\frac{\kappa_1\mathcal{D}_T}{(T_m^0)^2}\mathcal{D}_V. \end{aligned} \tag{4.1}$$

4.1. *Auxiliary functions.* We first assume $0 \le \phi_{ij} := \phi(|x_i - x_j|) \le 1$ and consider the corresponding function Φ_{ij} as below:

$$\Phi_{ij}(t) := \frac{\phi(|x_i - x_j|)}{N} + \left(1 - \frac{\sum_{j=1}^N \phi(|x_i - x_j|)}{N}\right)\delta_{ij},$$

where δ_{ij} is the Kronecker delta. Then Φ_{ij} is nonnegative and satisfies

$$\Phi_{ij} \ge \frac{\phi_{ij}}{N}, \qquad \sum_{j=1}^N \Phi_{ij} = 1, \qquad \sum_{j=1}^N \Phi_{ij}\left(\frac{v_j}{T_j} - \frac{v_i}{T_i}\right) = \sum_{j=1}^N \frac{\phi(|x_i - x_j|)}{N}\left(\frac{v_j}{T_j} - \frac{v_i}{T_i}\right).$$

Similarly we can define Ψ_{ij} as below:

$$\Psi_{ij}(t) := \frac{\zeta_{ij}}{N} + \left(1 - \frac{\sum_{j=1}^N \zeta_{ij}}{N}\right)\delta_{ij}.$$

Then we have

$$\Psi_{ij} \ge \frac{\zeta_{ij}}{N}, \qquad \sum_{j=1}^N \Psi_{ij} = 1, \qquad \sum_{j=1}^N \Psi_{ij}\left(\frac{1}{T_i} - \frac{1}{T_j}\right) = \sum_{j=1}^N \frac{\zeta_{ij}}{N}\left(\frac{1}{T_i} - \frac{1}{T_j}\right).$$

In the following subsections, we derive differential inequalities (4.1) separately.

4.2. *Derivation of SDI.* In this subsection, we provide differential inequalities (4.1). The first differential inequality $(4.1)_1$ is rather obvious, thus we only provide derivation of the second and third differential inequalities. Let $\{(x_i, v_i, T_i)\}$ be a solution to (1.1).

- (Derivation of $(4.1)_2$): For any i and j, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|T_i - T_j|^2 \\
&\quad = (T_i - T_j)(\dot{T}_i - \dot{T}_j) \\
&\quad = \kappa_2 (T_i - T_j)\left[\sum_{k=1}^N \frac{\zeta_{ik}}{N}\left(\frac{1}{T_i} - \frac{1}{T_k}\right) - \sum_{k=1}^N \frac{\zeta_{jk}}{N}\left(\frac{1}{T_j} - \frac{1}{T_k}\right)\right] \\
&\quad = \kappa_2 (T_i - T_j)\left[\sum_{k=1}^N \Psi_{ik}\left(\frac{1}{T_i} - \frac{1}{T_k}\right) - \sum_{k=1}^N \Psi_{jk}\left(\frac{1}{T_j} - \frac{1}{T_k}\right)\right] \\
&\quad = \kappa_2 (T_i - T_j)\left(\frac{1}{T_i} - \frac{1}{T_j}\right) - (T_i - T_j)\left(\sum_{k=1}^N \frac{\Psi_{ik} - \Psi_{jk}}{T_k}\right) \\
&\quad = \kappa_2 (T_i - T_j)\left(\frac{1}{T_i} - \frac{1}{T_j}\right) \\
&\quad - \kappa_2 (T_i - T_j)\left[\sum_{k=1}^N \frac{\Psi_{ik} - \min\{\Psi_{ik}, \Psi_{jk}\} + \min\{\Psi_{ik}, \Psi_{jk}\} - \Psi_{jk}}{T_k}\right].
\end{aligned} \tag{4.2}$$

We set extremal indices M and m satisfying

$$T_M := \max_{1 \le i \le N} T_i \quad \text{and} \quad T_m := \min_{1 \le i \le N} T_i.$$

We now apply the relation (4.2) for M and m to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|T_M - T_m|^2 \\
&\quad = \kappa_2 (T_M - T_m)\left(\frac{1}{T_M} - \frac{1}{T_m}\right) \\
&\quad - \kappa_2 (T_M - T_m)\left[\sum_{k=1}^N \frac{\Psi_{Mk} - \min\{\Psi_{Mk}, \Psi_{mk}\} + \min\{\Psi_{Mk}, \Psi_{mk}\} - \Psi_{mk}}{T_k}\right] \\
&\quad \le \kappa_2 (T_M - T_m)\left(\frac{1}{T_M} - \frac{1}{T_m}\right) - (T_M - T_m)\left(\sum_{k=1}^N \frac{\Psi_{Mk} - \min\{\Psi_{Mk}, \Psi_{mk}\}}{T_M}\right) \\
&\quad + \kappa_2 (T_M - T_m)\left(\sum_{k=1}^N \frac{\Psi_{mk} - \min\{\Psi_{Mk}, \Psi_{mk}\}}{T_m}\right) \\
&\quad = -\kappa_2 (T_M - T_m)\left(\frac{1}{T_m} - \frac{1}{T_M}\right)\sum_{k=1}^N \min\{\Psi_{Mk}, \Psi_{mk}\}.
\end{aligned} \tag{4.3}$$

Note that

$$\Psi_{ij} \ge \frac{\zeta_{ij}}{N} \ge \frac{\zeta(\mathcal{D}_X)}{N}, \quad T_M \le T_M^0, \quad T_m \le T_M^0. \tag{4.4}$$

Then, we use (4.3) and (4.4) to obtain

$$\frac{1}{2}\frac{d}{dt}\mathcal{D}_T^2 \le -\frac{\kappa_2\zeta(\mathcal{D}_X)}{(T_M^0)^2}\mathcal{D}_T^2, \quad t>0.$$

This yields the desired estimate $(4.1)_2$.

• (Derivation of $(4.1)_3$): For any i and j, we use a similar argument as in (4.2) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_i - v_j\|^2 \\
&\quad = (v_i - v_j)\cdot(\dot v_i - \dot v_j) \\
&\quad = \kappa_1 (v_i - v_j)\cdot \left[\sum_{k=1}^N \frac{\phi_{ik}}{N}\left(\frac{v_k}{T_k} - \frac{v_i}{T_i}\right) - \sum_{k=1}^N \frac{\phi_{jk}}{N}\left(\frac{v_k}{T_k} - \frac{v_j}{T_j}\right)\right] \\
&\quad = \kappa_1 (v_i - v_j)\cdot \left[\sum_{k=1}^N \Phi_{ik}\left(\frac{v_k}{T_k} - \frac{v_i}{T_i}\right) - \sum_{k=1}^N \Phi_{jk}\left(\frac{v_k}{T_k} - \frac{v_j}{T_j}\right)\right] \\
&\quad = \kappa_1 (v_i - v_j)\cdot \left(\frac{v_j}{T_j} - \frac{v_i}{T_i}\right) - \kappa_1 (v_i - v_j)\cdot\left(\sum_{k=1}^N \frac{(\Phi_{ik}-\Phi_{jk})v_k}{T_k}\right) \\
&\quad = \kappa_1 (v_i - v_j)\cdot \left(\frac{v_j}{T_j} - \frac{v_i}{T_i}\right) \\
&\quad - \kappa_1 (v_i - v_j)\cdot\left[\sum_{k=1}^N \left(\frac{\Phi_{ik} - \min\{\Phi_{ik},\Phi_{jk}\} + \min\{\Phi_{ik},\Phi_{jk}\} - \Phi_{jk}}{T_k}\right) v_k\right].
\end{aligned} \tag{4.5}$$

On the other hand, for a given time t, we choose indices i_t and j_t satisfying the relation:

$$\mathcal{D}_V(t) = \|v_{i_t}(t) - v_{j_t}(t)\|.$$

Then, we claim: for each $k = 1, 2, \cdots, N$,

$$(v_{i_t} - v_{j_t})\cdot v_{j_t} \le (v_{i_t} - v_{j_t})\cdot v_k \le (v_{i_t} - v_{j_t})\cdot v_{i_t}. \tag{4.6}$$

⋄ (The derivation of the first inequality in (4.6)): By definition of $\mathcal{D}_V$, we have

$$(v_{i_t} - v_{j_t})\cdot(v_{i_t} - v_k) \le \mathcal{D}_V^2 = \|v_{i_t} - v_{j_t}\|^2.$$

This yields the desired first inequality in (4.6):

$$(v_{i_t} - v_{j_t})\cdot v_{j_t} \le (v_{i_t} - v_{j_t})\cdot v_k.$$

⋄ (The derivation of the second inequality in (4.6)): Note that

$$(v_{i_t} - v_{j_t})\cdot(v_k - v_{j_t}) \le \mathcal{D}_V^2 = \|v_{i_t} - v_{j_t}\|^2.$$

This again yields

$$(v_{i_t} - v_{j_t})\cdot v_{i_t} \ge (v_{i_t} - v_{j_t})\cdot v_k.$$

Now, we combine (4.5) and (4.6) to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_{i_t}-v_{j_t}\|^2 \\
&\quad \le \kappa_1(v_{i_t}-v_{j_t})\cdot\left(\frac{v_{j_t}}{T_{j_t}}-\frac{v_{i_t}}{T_{i_t}}\right)-\kappa_1(v_{i_t}-v_{j_t})\cdot v_{j_t}\left(\sum_{k=1}^N\frac{\Phi_{i_tk}-\min\{\Phi_{i_tk},\Phi_{j_tk}\}}{T_k}\right)\\
&\quad + \kappa_1(v_{i_t}-v_{j_t})\cdot v_{i_t}\left(\sum_{k=1}^N\frac{\Phi_{j_tk}-\min\{\Phi_{i_tk},\Phi_{j_tk}\}}{T_k}\right)\\
&\quad = \kappa_1(v_{i_t}-v_{j_t})\cdot v_{j_t}\left(\frac{1}{T_{j_t}}-\sum_{k=1}^N\frac{\Phi_{i_tk}}{T_k}\right)+\kappa_1(v_{i_t}-v_{j_t})\cdot v_{i_t}\left(\sum_{k=1}^N\frac{\Phi_{j_tk}}{T_k}-\frac{1}{T_{i_t}}\right)\\
&\quad -\kappa_1\sum_{k=1}^N\frac{\min\{\Phi_{i_tk},\Phi_{j_tk}\}}{T_k}(v_{i_t}-v_{j_t})^2\\
&\quad = \kappa_1(v_{i_t}-v_{j_t})\cdot v_{j_t}\left(\sum_{k=1}^N\Phi_{i_tk}\left(\frac{1}{T_{j_t}}-\frac{1}{T_k}\right)\right)\\
&\quad + \kappa_1(v_{i_t}-v_{j_t})\cdot v_{i_t}\left(\sum_{k=1}^N\Phi_{j_tk}\left(\frac{1}{T_k}-\frac{1}{T_{i_t}}\right)\right)-\kappa_1\sum_{k=1}^N\frac{\min\{\Phi_{i_tk},\Phi_{j_tk}\}}{T_k}(v_{i_t}-v_{j_t})^2\\
&\quad \le -\kappa_1\frac{\phi(\mathcal{D}_X)}{T_M^0}\mathcal{D}_V^2+2\kappa_1\mathcal{D}_V^2\frac{\mathcal{D}_T}{(T_m^0)^2}.
\end{aligned}
\tag{4.7}
$$

The last inequality follows from the estimation

$$
\|v_{j_t}\| = \|v_{j_t}-v_c\| = \frac{1}{N}\left\|\sum_{k=1}^N(v_{j_t}-v_k)\right\| \le \mathcal{D}_V, \quad T_M \le T_M^0, \quad T_m \ge T_m^0,
$$

where we use the fact that mean velocity v_c is equal to zero.

Note that $\mathcal{D}_V$ is Lipschitz continuous with respect to t and thus, it is differentiable a.e. Now, we consider the instant t at which $\mathcal{D}_v$ is differentiable. In this case, we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{D}_V^2 &= \lim_{\Delta t\to 0}\frac{\mathcal{D}_V^2(t)-\mathcal{D}_V^2(t-\Delta t)}{\Delta t}\\
&= \lim_{\Delta t\to 0}\frac{\|v_{i_t}(t)-v_{j_t}(t)\|^2-\mathcal{D}_V^2(t-\Delta t)}{\Delta t}\\
&\le \lim_{\Delta t\to 0}\frac{\|v_{i_t}(t)-v_{j_t}(t)\|^2-\|v_{i_t}(t-\Delta t)-v_{j_t}(t-\Delta t)\|^2}{\Delta t}\\
&= \frac{d}{dt}\|v_{i_t}(t)-v_{j_t}(t)\|^2.
\end{aligned}
\tag{4.8}
$$

Thus, we combine (4.7) and (4.8) to get

$$
\frac{d}{dt}\mathcal{D}_V \le -\frac{\kappa_1\phi(\mathcal{D}_X)}{T_M^0}\mathcal{D}_V+2\kappa_1\mathcal{D}_V\frac{\mathcal{D}_T}{(T_m^0)^2}.
$$

4.3. *Proof of Theorem* 3.2. In this subsection, we present the proof of our first result. For this, we will provide decay and uniform boundedness estimates for a system of differential inequalities:

LEMMA 4.1. Let $\mathcal{X}$, $\mathcal{V}$, and $\mathcal{T}$ be Lipschitz continuous functions satisfying the following system of differential inequalities:

$$\left|\frac{d}{dt}\mathcal{X}\right| \le \mathcal{V}, \quad \frac{d}{dt}\mathcal{T} \le -c\zeta(\mathcal{X})\mathcal{T}, \quad \frac{d}{dt}\mathcal{V} \le -a\phi(\mathcal{X})\mathcal{V} + b\mathcal{T}\mathcal{V}, \tag{4.9}$$

where a, b, and c are positive constants. Suppose that initial data $(\mathcal{X}^0, \mathcal{V}^0, \mathcal{T}^0)$ satisfies the following relations for some positive constant $\mathcal{X}^\infty$:

$$b\mathcal{T}^0 \le \frac{a\phi(\mathcal{X}^0)}{4}, \quad \mathcal{V}^0 \le \frac{a}{2}\int_{\mathcal{X}^0}^{\mathcal{X}^\infty} \phi(s)ds, \quad \frac{2}{3}\phi(\mathcal{X}^0) \le \phi(\mathcal{X}^\infty) < \phi(\mathcal{X}^0). \tag{4.10}$$

Then, we have

$$\text{(i)} \sup_{0\le t<\infty} \mathcal{X}(t) \le \mathcal{X}^\infty,$$

$$\text{(ii)}\ \mathcal{V}(t) \le \mathcal{V}^0 e^{-a\phi(\mathcal{X}^\infty)t + \frac{b\mathcal{T}^0}{c\zeta(\mathcal{X}^\infty)}}, \quad \mathcal{T}(t) \le \mathcal{T}^0 e^{-c\zeta(\mathcal{X}^\infty)t}, \quad t \ge 0.$$

Proof. We use the standard bootstrapping argument. For this, we define a set $\mathcal{A}$:

$$\mathcal{A} := \left\{ t \in (0,\infty] : \ b\mathcal{T}(s) < \frac{a\phi(\mathcal{X}(s))}{2} \text{ for } 0 \le s \le t \right\}.$$

Below, we will verify the following assertions:

$$\mathcal{A} \ne \emptyset \quad \text{and} \quad \sup \mathcal{A} = \infty.$$

- Step A (Nonemptyness of $\mathcal{A}$): Since

$$b\mathcal{T}^0 \le \frac{a\phi(\mathcal{X}^0)}{4} < \frac{a\phi(\mathcal{X}^0)}{2},$$

the continuity of the solution implies that there exists $\tau \in (0,\infty)$ such that

$$b\mathcal{T}(s) < \frac{a\phi(\mathcal{X}(s))}{2}, \quad s \in [0,\tau), \quad \text{i.e.,} \quad \tau \in \mathcal{A}.$$

Thus, we have $\tau^* := \sup \mathcal{A} \in (0,\infty]$.

- Step B ($\tau^* = \infty$): Suppose that τ^* is finite. In this case, we have

$$b\mathcal{T}(\tau^*) = \frac{a\phi(\mathcal{X}(\tau^*))}{2}. \tag{4.11}$$

On the other hand, it follows from $(4.9)_2$ that we have

$$\frac{d\mathcal{V}}{dt} \le -\frac{a\phi(\mathcal{X})}{2}\mathcal{V} \le -\frac{a\phi(\mathcal{X})}{2}\frac{d\mathcal{X}}{dt} = -\frac{a}{2}\frac{d}{dt}\int_0^{\mathcal{X}} \phi(s)ds, \quad 0 \le t \le \tau^*.$$

We integrate the above relation from 0 to τ^* to obtain

$$\mathcal{V}(0) - \mathcal{V}(\tau^*) \ge \frac{a}{2}\int_{\mathcal{X}(0)}^{\mathcal{X}(\tau^*)} \phi(s)ds.$$

This yields

$$\frac{a}{2}\int_{\mathcal{X}^0}^{\mathcal{X}^\infty}\phi(s)ds \geq \frac{a}{2}\int_{\mathcal{X}^0}^{\mathcal{X}(\tau^*)}\phi(s)ds \quad \Longleftrightarrow \quad \int_{\mathcal{X}(\tau^*)}^{\mathcal{X}^\infty}\phi(s)ds \geq 0,$$

which immediately implies the boundedness of $\mathcal{X}(\tau^*)$:

$$\mathcal{X}(\tau^*) \leq \mathcal{X}^\infty.$$

Now, we use (4.10) and monotonic decreasing property of ζ to obtain

$$b\mathcal{T}(\tau^*) \leq b\mathcal{T}(0) \leq \frac{a\phi(\mathcal{X}(0))}{4} \leq \frac{3a\phi(\mathcal{X}^\infty)}{8} \leq \frac{3a\phi(\mathcal{X}(\tau^*))}{8} < \frac{a\phi(\mathcal{X}(\tau^*))}{2},$$

which contradicts (4.11). Hence we conclude $\tau^* = \infty$.

• Step C (Flocking estimate): Once we have a uniform boundedness of $\mathcal{X}$, the other decay estimates can be obtained as follows. First, it follows from $(4.9)_1$ that

$$\frac{d}{dt}\mathcal{T} \leq -c\zeta(\mathcal{X})\mathcal{T} \leq -c\zeta(\mathcal{X}^\infty)\mathcal{T}.$$

This yields the desired exponential decay estimate of $\mathcal{T}$:

$$\mathcal{T}(t) \leq \mathcal{T}^0 e^{-c\zeta(\mathcal{X}^\infty)t}, \quad t \geq 0. \tag{4.12}$$

Next, we use the uniform boundedness of $\mathcal{X}$, (4.12), and $(4.9)_2$ to obtain a Grönwall's inequality for $\mathcal{V}$:

$$\frac{d\mathcal{V}}{dt} \leq -a\phi(\mathcal{X})\mathcal{V} + b\mathcal{T}\mathcal{V} \leq \left(-a\phi(\mathcal{X}^\infty) + b\mathcal{T}^0 e^{-c\zeta(\mathcal{X}^\infty)t}\right)\mathcal{V}.$$

This yields an exponential decay estimate of $\mathcal{V}$:

$$\begin{aligned}\mathcal{V}(t) &\leq \mathcal{V}^0 \exp\left(-a\phi(\mathcal{X}^\infty)t + b\mathcal{T}^0\int_0^t e^{-c\zeta(\mathcal{X}^\infty)s}ds\right)\\ &\leq \mathcal{V}^0 e^{\frac{b\mathcal{T}(0)}{c\zeta(\mathcal{X}^\infty)}} e^{-a\phi(\mathcal{X}^\infty)t}, \quad t \geq 0.\end{aligned}$$

□

As a direct application of Lemma 4.1, we obtain the proof of Theorem 3.2 as follows. We first take $C = 4$ in Theorem 3.2 and then choose a, b, and c in (4.9) as follows:

$$a = \frac{\kappa_1}{T_M^0}, \quad b = \frac{2\kappa_1}{(T_m^0)^2}, \quad c = \frac{\kappa_2}{(T_M^0)^2}, \quad \mathcal{T} = \mathcal{D}_T, \quad \mathcal{V} = \mathcal{D}_V, \quad \mathcal{X} = \mathcal{D}_X.$$

Then, the results of (4.1) imply the desired flocking estimate.

5. Uniform-in-time stability estimate. In this section, we will present the uniform stability of system (1.1) with respect to initial data.

5.1. *Uniform stability.* In this subsection, we will derive the uniform stability estimate based on a Grönwall-type inequality. More precisely, we have the following lemma.

LEMMA 5.1. Suppose that three nonnegative Lipschitz continuous functions $\mathcal{X}$, $\mathcal{V}$, and $\mathcal{T}$ satisfy the coupled differential inequalities: for some positive constants α, β, and γ and for a.e. $t > 0$,

$$\left|\frac{d\mathcal{X}}{dt}\right| \le \mathcal{V}, \quad \frac{d\mathcal{V}}{dt} \le -\alpha\mathcal{V} + \gamma e^{-\alpha t}\mathcal{X} + \gamma e^{-\alpha t}\mathcal{T}, \quad \frac{d\mathcal{T}}{dt} \le \gamma e^{-\alpha t}\mathcal{T} + \gamma e^{-\alpha t}\mathcal{X}, \tag{5.1}$$

subject to initial data:

$$(\mathcal{X}, \mathcal{V}, \mathcal{T})(0) = (\mathcal{X}^0, \mathcal{V}^0, \mathcal{T}^0).$$

Then, there exists a positive constant M such that

(i) $\sup_{0\le t<\infty} \mathcal{X}(t) \le M(\mathcal{X}^0 + \mathcal{V}^0 + \mathcal{T}^0)$,

(ii) $\mathcal{V}(t) \le M(\mathcal{X}^0 + \mathcal{V}^0 + \mathcal{T}^0)e^{-\frac{\alpha t}{2}}, \quad \mathcal{T}(t) \le M(\mathcal{X}^0 + \mathcal{V}^0 + \mathcal{T}^0), \quad t \ge 0.$

Proof. It follows from $(5.1)_1$ that we have

$$\mathcal{X}(t) \le \mathcal{X}^0 + \int_0^t \mathcal{V}(s)ds. \tag{5.2}$$

We use a Grönwall-type inequality in $(5.1)_3$ to find

$$\frac{d\mathcal{T}}{dt} \le \gamma e^{-\alpha t}\mathcal{T} + \gamma e^{-\alpha t}\mathcal{X},$$

which yields

$$\begin{aligned} \mathcal{T}(t) &\le \left(\mathcal{T}(0) + \int_0^t \gamma e^{-\alpha s}\mathcal{X}(s)ds\right)\exp\left(\int_0^t \gamma e^{-\alpha s}ds\right) \\ &\le \left(\mathcal{T}(0) + \int_0^t \gamma e^{-\alpha s}\mathcal{X}(s)ds\right)e^{\frac{\gamma}{\alpha}}. \end{aligned} \tag{5.3}$$

Note that once we have a desired exponential decay estimate for $\mathcal{V}$, the relations (5.2) and (5.3) imply the desired estimates. We again substitute (5.2) and (5.3) into the differential inequality for $\mathcal{V}$ in (5.1) to obtain

$$\begin{aligned} \frac{d\mathcal{V}}{dt} &\le -\alpha\mathcal{V} + \gamma e^{-\alpha t}\mathcal{X} + \gamma e^{-\alpha t}\mathcal{T} \\ &\le -\alpha\mathcal{V} + \gamma e^{-\alpha t}\left(\mathcal{X}^0 + \int_0^t \mathcal{V}(s)ds\right) + \gamma e^{-\alpha t}e^{\frac{\gamma}{\alpha}}\left(\mathcal{T}^0 + \int_0^t \gamma e^{-\alpha s}\mathcal{X}(s)ds\right) \\ &\le \gamma e^{-\alpha t}\left(\mathcal{X}^0 + \int_0^t \mathcal{V}(s)ds\right) + \gamma e^{-\alpha t}e^{\frac{\gamma}{\alpha}}\left(\mathcal{T}^0 + \int_0^t \gamma e^{-\alpha s}\mathcal{X}(s)ds\right). \end{aligned} \tag{5.4}$$

We use a Grönwall-type inequality in the above relation (5.4) to obtain

$$\mathcal{V}(t) \le \mathcal{V}^0 e^{-\alpha t} + \gamma e^{-\alpha t}\int_0^t \left(\mathcal{X}^0 + \int_0^s \mathcal{V}(\tau)d\tau\right)ds + \gamma e^{-\alpha t}e^{\frac{\gamma}{\alpha}}\int_0^t \left(\mathcal{T}^0 + \int_0^s \gamma\mathcal{X}(\tau)d\tau\right)ds. \tag{5.5}$$

We set

$$\mathcal{M}_V(t) := \max_{0\le\tau\le t} \mathcal{V}(\tau), \quad t \ge 0.$$

Then, it follows from (5.2) that we have

$$\mathcal{X}(t) \le \mathcal{X}^0 + \int_0^t \mathcal{V}(s)ds \le \mathcal{X}^0 + t\mathcal{M}_V(t). \tag{5.6}$$

Since $\mathcal{M}_V(t)$ is an increasing function of t, (5.3) and (5.6) imply

$$\mathcal{T}(t) \le e^{\frac{\gamma}{\alpha}} \left[\mathcal{T}^0 + \frac{\gamma \mathcal{X}^0}{\alpha} + \frac{\gamma \mathcal{M}_V(t)}{\alpha^2} \right]. \tag{5.7}$$

Now we will try to estimate $\mathcal{M}_V(t)$ by using (5.5):

$$\begin{aligned}
\mathcal{M}_V(t) &\le \mathcal{V}^0 + \gamma e^{-\alpha t} \int_0^t \left(\mathcal{X}^0 + s\mathcal{M}_V(s) \right) ds + \gamma e^{-\alpha t} e^{\frac{\gamma}{\alpha}} \left[\mathcal{T}^0 + \frac{\gamma \mathcal{X}^0}{\alpha} + \frac{\gamma \mathcal{M}_V(t)}{\alpha^2} \right] \\
&\le \mathcal{V}^0 + \gamma \mathcal{X}^0 t e^{-\alpha t} + \gamma e^{-\alpha t} \mathcal{M}_V(t) \int_0^t s ds + \gamma e^{-\alpha t} e^{\frac{\gamma}{\alpha}} \left[\mathcal{T}^0 + \frac{\gamma \mathcal{X}^0}{\alpha} + \frac{\gamma \mathcal{M}_V(t)}{\alpha^2} \right] \\
&\le M_0(\mathcal{V}^0 + \mathcal{X}^0 + \mathcal{T}^0) + \left[\frac{\gamma}{2} e^{-\alpha t} t^2 + \frac{\gamma^2}{\alpha^2} e^{\frac{\gamma}{\alpha}} e^{-\alpha t} \right] \mathcal{M}_V(s).
\end{aligned}$$

Hence, when t is large enough such that $\left[\frac{\gamma}{2} e^{-\alpha t} t^2 + \frac{\gamma^2}{\alpha^2} e^{\frac{\gamma}{\alpha}} e^{-\alpha t} \right] < \frac{1}{2}$, we have

$$\mathcal{M}_V(t) \le 2M_0 \left(\mathcal{V}^0 + \mathcal{X}^0 + \mathcal{T}^0 \right).$$

Since $\mathcal{M}_V$ is an increasing function, we have a positive constant M such that

$$\mathcal{M}_V(t) \le M \left(\mathcal{V}^0 + \mathcal{X}^0 + \mathcal{T}^0 \right). \tag{5.8}$$

• Step A (Decay estimate for $\mathcal{V}$): Now, we return to the estimation for $\mathcal{V}$:

$$\begin{aligned}
\mathcal{V}(t) &\le e^{-\alpha t} \mathcal{V}^0 + \gamma e^{-\alpha t} \int_0^t \left(\mathcal{X}^0 + \int_0^s \mathcal{V}(\tau) d\tau \right) ds + \gamma e^{\frac{\gamma}{\alpha}} \int_0^t e^{-\alpha t} \left(\mathcal{T}^0 + \int_0^s \gamma \mathcal{X}(\tau) d\tau \right) ds \\
&\le e^{-\alpha t} \mathcal{V}^0 + \gamma e^{-\alpha t} \int_0^t \mathcal{X}(0) + M(\mathcal{X}^0 + \mathcal{V}^0 + \mathcal{T}^0) s ds + \gamma e^{\frac{\gamma}{\alpha}} \mathcal{T}^0 t e^{-\alpha t} \\
&+ \gamma^2 e^{\frac{\gamma}{\alpha}} e^{-\alpha t} \int_0^t \int_0^s (\mathcal{X}^0 + \tau M_V(\tau)) d\tau ds \\
&\le e^{-\alpha t} \mathcal{V}^0 + \gamma e^{-\alpha t} \int_0^t \mathcal{X}(0) + M(\mathcal{X}^0 + \mathcal{V}^0 + \mathcal{T}^0) s ds + \gamma e^{\frac{\gamma}{\alpha}} \mathcal{T}^0 t e^{-\alpha t} \\
&+ \gamma^2 e^{\frac{\gamma}{\alpha}} e^{-\alpha t} \left[\frac{\mathcal{X}^0 t^2}{2} + \frac{t^3 M \left(\mathcal{V}^0 + \mathcal{X}^0 + \mathcal{T}^0 \right)}{6} \right] \\
&\le e^{-\alpha t} \mathcal{V}^0 + \gamma e^{-\alpha t} \left[(\mathcal{X}^0 + e^{\frac{\gamma}{\alpha}} \mathcal{T}^0) t + \frac{M t^2}{2} (\mathcal{X}^0 + \mathcal{V}^0 + \mathcal{T}^0) \right] \\
&+ \gamma^2 e^{\frac{\gamma}{\alpha}} e^{-\alpha t} \left[\frac{\mathcal{X}^0 t^2}{2} + \frac{t^3 M \left(\mathcal{V}^0 + \mathcal{X}^0 + \mathcal{T}^0 \right)}{6} \right].
\end{aligned}$$

By sacrificing some decay rate, we take the constant M_i such that

$$M_1 \ge \gamma t e^{-\frac{\alpha t}{2}}, \quad M_2 \ge \frac{1}{2} M \gamma t^2 e^{-\frac{\alpha t}{2}}, \quad M_3 \ge \frac{1}{6} M \gamma^2 t^3 e^{-\frac{\alpha t}{2}}, \quad t \ge 0.$$

Then, we have

$$\mathcal{V}(t) \le G_v(\mathcal{X}^0 + \mathcal{V}^0 + \mathcal{T}^0)e^{-\frac{\alpha t}{2}}$$

for large constant G_v.

• Step B (Uniform bound estimate for $\mathcal{X}$): By direct calculation, we have

$$\mathcal{X}(t) \le \mathcal{X}^0 + \int_0^t \mathcal{V}(s)ds \le \mathcal{X}^0 + \frac{2G_v}{\alpha}(\mathcal{X}^0 + \mathcal{V}^0 + \mathcal{T}^0).$$

• Step C (Uniform bound estimate for $\mathcal{T}$): It follows from (5.1) and the uniform boundedness of $\mathcal{X}$ that we have the desired uniform boundedness of $\mathcal{T}$,

$$\mathcal{T}(t) \le G_T(\mathcal{X}^0 + \mathcal{V}^0 + \mathcal{T}^0), \quad t \ge 0.$$

□

5.2. *Derivation of SDI for mechanical variables.* In this subsection, let us recall that the center of mass is on the origin of the reference frame:

$$\sum_{i=1}^N x_i = 0, \quad \sum_{i=1}^N v_i = 0, \quad t \ge 0. \tag{5.9}$$

We also assume the global boundedness of the temperature such as

$$T_m^0 \le T_i(t),\ \bar{T}_i(t) \le T_M^0 \quad \forall\ t,\ 1 \le i \le N.$$

We next define Lyapunov functional $\|X - \bar{X}\|_p$, $\|V - \bar{V}\|_p$, and $\|T - \bar{T}\|_p$: for $p \in [1, \infty)$,

$$\begin{cases} \|X - \bar{X}\|_p^p := \dfrac{1}{N}\displaystyle\sum_{i=1}^N \|x_i - \bar{x}_i\|_p^p, \\ \|V - \bar{V}\|_p^p := \dfrac{1}{N}\displaystyle\sum_{i=1}^N \|v_i - \bar{v}_i\|_p^p, \\ \|T - \bar{T}\|_p^p := \dfrac{1}{N}\displaystyle\sum_{i=1}^n |T_i - \bar{T}_i|^p. \end{cases}$$

Below, we present a series of lemmas without proofs. The proofs will be presented in the appendices.

LEMMA 5.2 (ℓ_1-estimates). For $p = 1$, let $\{(x_i, v_i, T_i)\}_{i=1}^N$ and $\{(\bar{x}_i, \bar{v}_i, \bar{T}_i)\}_{i=1}^N$ be the smooth solutions to (1.1) with zero sum conditions (5.9). Then, the following estimates hold:

$$\begin{cases} \dfrac{d}{dt}\|X - \bar{X}\|_1 \le \|V - \bar{V}\|_1, \quad \text{a.e. } t > 0, \\ \dfrac{d}{dt}\|V - \bar{V}\|_1 \le \dfrac{4\kappa_1 \mathcal{D}_V \|\phi\|_{Lip}}{T_m^0}\|X - \bar{X}\|_1 + \dfrac{2\kappa_1\phi(0)}{(T_m^0)^2}\|T - \bar{T}\|_1\mathcal{D}_V - \dfrac{\kappa_1\phi(\mathcal{D}_{\bar{X}})}{T_M^0}\|V - \bar{V}\|_1, \\ \dfrac{d}{dt}\|T - \bar{T}\|_1 \le \dfrac{2\kappa_2\|\zeta\|_{Lip}\mathcal{D}_T}{(T_m^0)^2}\|X - \bar{X}\|_1. \end{cases}$$

Proof. We postpone its proof to Appendix A. □

LEMMA 5.3 (ℓ_p-estimates). For $p \in (1,\infty]$, let $\{(x_i,v_i,T_i)\}_{i=1}^N$ and $\{(\bar{x}_i,\bar{v}_i,\bar{T}_i)\}_{i=1}^N$ be the smooth solutions to (1.1) with zero sum conditions (5.9). Then, the following estimates hold:

$$\begin{cases} \dfrac{d}{dt}\|X-\bar{X}\|_p \le \|V-\bar{V}\|_p, \quad \text{a.e. } t>0, \\ \dfrac{d}{dt}\|V-\bar{V}\|_p \le \dfrac{4\kappa_1\mathcal{D}_V\|\phi\|_{Lip}}{T_m^0}\|X-\bar{X}\|_p + \dfrac{2\kappa_1\phi(0)}{(T_m^0)^2}\|T-\bar{T}\|_p\mathcal{D}_V \\ \qquad -\left(\dfrac{2\kappa_1\phi(\mathcal{D}_{\bar{X}})}{T_M^0} - \dfrac{\kappa_1\mathcal{D}_T\phi(0)}{(T_m^0)^2}\right)\|V-\bar{V}\|_p, \\ \dfrac{d}{dt}\|T-\bar{T}\|_p \le \dfrac{2\kappa_2\|\zeta\|_{Lip}\mathcal{D}_T}{(T_m^0)^2}\|X-\bar{X}\|_p \\ \qquad +\kappa_2\zeta(0)\left(\dfrac{\mathcal{D}_T\mathcal{D}_{\bar{T}} + T_m^0\mathcal{D}_T + T_m^0\mathcal{D}_{\bar{T}}}{(T_m^0)^4}\right)\|T-\bar{T}\|_p. \end{cases} \tag{5.10}$$

Proof. We postpone the proof to Appendix B. □

5.3. *Proof of Theorem* 3.4. We are now ready to prove Theorem 3.4. For this, we consider two cases:

$$\text{Either } p=q, \quad \text{or} \quad p\neq q.$$

• Case A ($q=p$): It follows from (5.10) that we have the system of dissipative differential inequalities needed to use in the proof of Lemma 5.1. By Theorem 3.2, we know that the $\mathcal{D}_V$ is exponentially decaying. By using Lemma 5.1, there exists a positive constant G independent of time t such that

$$\begin{aligned} \|X(t)-\bar{X}(t)\|_p &+ \|V(t)-\bar{V}(t)\|_p + \|T(t)-\bar{T}(t)\|_p \\ &\le G\left(\|X^0-\bar{X}^0\|_p + \|V^0-\bar{V}^0\|_p + \|T^0-\bar{T}^0\|_p\right), \quad t\ge 0. \end{aligned}$$

• Case B ($q\neq p$): Note that in a finite dimensional normed space, all ℓ_p norm are equivalent to the ℓ_q norm if $p,q\in[1,\infty)$. For $x=(x^1,\cdots,x^d)$, we have

$$\|x\|_p = \left(\sum_{k=1}^d |x^k|^p\right)^{\frac{1}{p}} \le d^{\frac{1}{p}}\|x\|_\infty \le d^{\frac{1}{p}}\|x\|_1, \qquad \left(\frac{1}{d}\sum_{k=1}^d |x^k|\right)^q \le \frac{1}{d}\sum_{k=1}^d |x^k|^q.$$

This yields

$$\|x\|_1 \le d^{1-\frac{1}{q}}\|x\|_q.$$

Thus, we have

$$\|x\|_p \le d^{1-\frac{1}{q}+\frac{1}{p}}\|x\|_q$$

which means that $\|x\|_p$ and $\|x\|_q$ are equivalent. For the case of $p=\infty$ or $q=\infty$, equivalence comes directly from the estimation $\|x\|_\infty \le \|x\|_1 \le d\|x\|_\infty$. Moreover, the sum

$$\left(\sum_{i=1}^N ||x_i(t)-\tilde{x}_i(t)||_q^q\right)^{\frac{1}{q}} + \left(\sum_{i=1}^N ||v_i(t)-\tilde{v}_i(t)||_q^q\right)^{\frac{1}{q}} + \left(\sum_{i=1}^N |T_i(t)-\tilde{T}_i(t)|^q\right)^{\frac{1}{q}}$$

is equivalent to

$$\left(\sum_{i=1}^{N}||x_i(t)-\tilde{x}_i(t)||_p^q\right)^{\frac{1}{q}}+\left(\sum_{i=1}^{N}||v_i(t)-\tilde{v}_i(t)||_p^q\right)^{\frac{1}{q}}+\left(\sum_{i=1}^{N}|T_i(t)-\tilde{T}_i(t)|^q\right)^{\frac{1}{q}}.$$

If we define the norm

$$\begin{aligned}&W(X,V,T,\bar{X},\bar{V},\bar{T})_p^q\\&\quad:=\left(\sum_{i=1}^{N}||x_i(t)-\tilde{x}_i(t)||_p^q\right)^{\frac{1}{q}}+\left(\sum_{i=1}^{N}||v_i(t)-\tilde{v}_i(t)||_p^q\right)^{\frac{1}{q}}+\left(\sum_{i=1}^{N}|T_i(t)-\tilde{T}_i(t)|^q\right)^{\frac{1}{q}}.\end{aligned}$$

By the equivalence of the norm, we have

$$W_p^q(t)\le C_1W_q^q(t)\le C_1MW_q^q(0)\le C_1MC_2W_p^q(0).$$

Since the constants C_1, M, C_2 are independent of $X, V, T, \bar{X}, \bar{V}, \bar{T}$ we find our G_0

REMARK 5.4. Note that the local-in-time stability is quite standard in many other models. However, in this special kind of dissipative flocking model, the velocity and temperature fluctuations exponentially decay to zero. Therefore, the velocity and temperature differences between two solutions also exponentially decay to zero, which is integrable on the whole time interval. Thanks to this integrability, we have a uniform-in-time stability in the flocking model.

6. Uniform mean-field limit. In this section, we revisit the mean-field limit from the particle system (1.1) to the kinetic equation (1.4). Thanks to the new uniform $\ell_{q,p}$-stability estimate in the previous section, we can now establish the uniform mean-field limit valid in the whole time interval.

6.1. *Proof of Theorem* 3.6. In this subsection, we are now ready to present the uniform-in-time mean-field limit results generalizing the earlier result in [20] as a direct application of the uniform stability result in the previous section.

6.1.1. *Global existence of measure-valued solutions.* Suppose that initial probability measure $\mu_0 \in \mathcal{P}(\mathbb{R}^{2d}\times\mathbb{R}_+)$ has a compact support. Then, we claim that there exists a measure-valued solution $\mu_t \in L^\infty\big([0,\infty);\mathcal{P}(\mathbb{R}^{2d}\times\mathbb{R}_+)\big)$ to (1.4) with initial data μ_0:

(1) μ_t is approximated by μ_t^N in Wasserstein-(q,p) distance uniformly in time:

$$\overline{\lim_{N\to+\infty}}\sup_{t\in[0,+\infty)}W_{q,p}(\mu_t^N,\mu_t)=0.$$

(2) μ_t is unique in the class of a measure-valued solution with initial data μ_0 and has uniformly compact support.

For the simplicity of presentation, we start with W_p and we split its proof into several steps. Let μ_0 be a measure in $\mathcal{P}(\mathbb{R}^{2d}\times\mathbb{R}_+)$.

• Step A (Extraction of Cauchy approximation for μ_0 in W_p): Let μ_0^N be an approximation of μ_0 satisfying

$$\lim_{N\to+\infty}W_p(\mu_0^N,\mu_0)=0. \tag{6.1}$$

The existence of such approximation is guaranteed by Proposition 2.2. Then, owing to (6.1), for any $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that

$$W_p(\mu_0^n, \mu_0^m) < \varepsilon \quad \text{for } n, m > N(\varepsilon).$$

• Step B (Calculation of $W_p(\mu_0^n, \mu_0^m)$): Since the empirical measures μ_0^n and μ_0^m obtained in Step A are both concentrated at finite points, we denote them by

$$\mu_0^n := \frac{1}{n}\sum_{i=1}^n \delta_{(x_i^0, v_i^0, T_i^0)}, \quad \mu_0^m := \frac{1}{m}\sum_{j=1}^m \delta_{(\bar{x}_j^0, \bar{v}_j^0, \bar{T}_j^0)}.$$

Then, we can find an optimal plan (a_{ij}) whose entries are nonnegative real numbers satisfying

$$W_p^p(\mu_0^n, \mu_0^m) = \frac{1}{nm}\sum_{i=1}^n\sum_{j=1}^m a_{ij}\left(\|x_i^0 - \bar{x}_j^0\|_p^p + \|v_i^0 - \bar{v}_j^0\|_p^p + |T_i^0 - \bar{T}_j^0|^p\right), \tag{6.2}$$

where a_{ij} satisfies the constraints:

$$\sum_{i=1}^n a_{ij} = n, \qquad \sum_{j=1}^m a_{ij} = m.$$

• Step C (Approximation of $W_p(\mu_0^n, \mu_0^m)$): To associate (6.2) with $\ell_{p,p}$-distance between Z_0 and $\bar{Z}_0$, we approximate (6.2) with rational coefficients r_{ij} instead of real ones a_{ij} with some small error. More precisely, we find proper rational numbers r_{ij} such that they have the same denominator D_{mn} and

$$|r_{ij} - a_{ij}| \le \frac{\varepsilon^p}{d_x(0)^p + d_v(0)^p + d_\tau(0)^p}, \qquad \sum_{i=1}^n r_{ij} = n, \qquad \sum_{j=1}^m r_{ij} = m, \tag{6.3}$$

where

$$d_x(0) := \max_{1\le i,j\le N}\|x_i^0 - \bar{x}_j^0\|_p, \quad d_v(0) := \max_{1\le i,j\le N}\|v_i^0 - \bar{v}_j^0\|_p \quad \text{and} \quad d_\tau(0) := \max_{1\le i,j\le N}|T_i^0 - \bar{T}_j^0|.$$

Then, it follows from (6.2) and (6.3) that we have

$$\begin{aligned}
&\left|W_p^p(\mu_0^n, \mu_0^m) - \frac{1}{nm}\sum_{i=1}^n\sum_{j=1}^m r_{ij}\left(\|x_i^0 - \bar{x}_j^0\|_p^p + \|v_i^0 - \bar{v}_j^0\|_p^p + |T_i^0 - \bar{T}_j^0|^p\right)\right| \\
&\quad \le \frac{1}{nm}\sum_{i=1}^n\sum_{j=1}^m |a_{ij} - r_{ij}|\|z_i^0 - \bar{z}_j^0\|_p^p \\
&\quad \le (d_x(0)^p + d_v(0)^p + d_\tau(0)^p)\frac{1}{nm}\sum_{i=1}^n\sum_{j=1}^m |a_{ij} - r_{ij}| \\
&\quad \le \varepsilon^p,
\end{aligned} \tag{6.4}$$

where we used (6.3) and the fact that (x_i^0, v_i^0, T_i^0) and $(\bar{x}_i^0, \bar{v}_i^0, \bar{T}_i^0)$ are included in a common compact set and

$$\|z_i^0 - \bar{z}_j^0\|_p^p \le d_x(0)^p + d_v(0)^p + d_\tau(0)^p.$$

We set

$$r_{ij} := \frac{N_{ij}}{D_{mn}}, \qquad N_{ij} \in \mathbb{Z}_+, \qquad M_{mn} := D_{mn} mn,$$

to rewrite

$$\begin{aligned}
&\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} r_{ij}\left(\|x_i^0 - \bar{x}_j^0\|_p^p + \|v_i^0 - \bar{v}_j^0\|_p^p + |T_i^0 - \bar{T}_j^0|^p\right) \\
&\quad = \frac{1}{M_{mn}}\sum_{i=1}^{n}\sum_{j=1}^{m} N_{ij}\left(\|x_i^0 - \bar{x}_j^0\|_p^p + \|v_i^0 - \bar{v}_j^0\|_p^p + |T_i^0 - \bar{T}_j^0|^p\right) \\
&\quad = \frac{1}{M_{mn}}\sum_{k=1}^{M_{mn}} \|z_k^0 - \bar{z}_k^0\|_p^p.
\end{aligned} \tag{6.5}$$

We combine (6.4) and (6.5) to obtain

$$\left| W_p^p(\mu_0^n, \mu_0^m) - \frac{1}{M_{mn}}\sum_{k=1}^{M_{mn}} \|z_k^0 - \bar{z}_k^0\|_p^p \right| \le \varepsilon^p. \tag{6.6}$$

• Step D (Lifting the information at 0 to $t > 0$): We use the uniform ℓ_p-stability to show that $\{\mu_t^n\}$ is Cauchy. Note that $\sum_{i,j} N_{ij} = M_{mn}$, and z_k^0 and $\bar{z}_k^0$ are $(2d+1)$-vectors chosen from $\{(x_i^0, v_i^0, T_i^0) | 1 \le i \le n\}$ and $\{(\bar{x}_j^0, \bar{v}_j^0, \bar{T}_j^0) | 1 \le j \le m\}$, respectively. On the other hand, we consider the term

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{m} \frac{N_{ij}}{M_{mn}}\left(\|x_i^0 - \bar{x}_j^0\|_p^p + \|v_i^0 - \bar{v}_j^0\|_p^p + |T_i^0 - \bar{T}_j^0|^p\right), \\
&\sum_{i=1}^{n} \frac{N_{ij}}{D_{mn}} = n, \qquad \sum_{j=1}^{m} \frac{N_{ij}}{D_{mn}} = m.
\end{aligned}$$

It actually corresponds to a plan between μ_0^n and μ_0^m, which we denote by $\gamma(\mu_0^n, \mu_0^m)$,

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}\times\mathbb{R}_+\times\mathbb{R}^{2d}\times\mathbb{R}_+} \|z - \bar{z}\|_p^p \gamma(\mu_0^n, \mu_0^m) \\
&\quad = \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{N_{ij}}{M_{mn}}\left(\|x_i^0 - \bar{x}_j^0\|_p^p + \|v_i^0 - \bar{v}_j^0\|_p^p + |T_i^0 - \bar{T}_j^0|^p\right) \\
&\quad = \frac{1}{M_{mn}}\sum_{k=1}^{M_{mn}} \|z_k^0 - \bar{z}_k^0\|_p^p.
\end{aligned}$$

Then, we have

$$\begin{aligned}
W_p^p(\mu_t^n,\mu_t^m) &\le \int_{\mathbb{R}^{2d}\times\mathbb{R}_+\times\mathbb{R}^{2d}\times\mathbb{R}_+} \|z-\bar z\|_p^p \gamma(\mu_t^n,\mu_t^m)\\
&= \frac{1}{M_{mn}}\sum_{k=1}^{M_{mn}} \|z_k(t)-\bar z_k(t)\|_p^p = \frac{1}{M_{mn}}\sum_{k=1}^{M_{mn}} \left(\|x_k-\bar x_k\|_p^p + \|v_k-\bar v_k\|_p^p + |T_k-\bar T_k|^p\right)\\
&\le \frac{1}{M_{mn}}\left[\left(\sum_{k=1}^{M_{mn}}\|x_k(t)-\bar x_k(t)\|_p^p\right)^{\frac1p} + \left(\sum_{k=1}^{M_{mn}}\|v_k(t)-\bar v_k(t)\|_p^p\right)^{\frac1p}\right.\\
&\qquad\qquad \left. + \left(\sum_{k=1}^{M_{mn}}|T_k(t)-\bar T_k(t)|^p\right)^{\frac1p}\right]^p\\
&\le \frac{G^p}{M_{mn}}\left[\left(\sum_{k=1}^{M_{mn}}\|x_k^0(t)-\bar x_k^0(t)\|_p^p\right)^{\frac1p} + \left(\sum_{k=1}^{M_{mn}}\|v_k^0(t)-\bar v_k^0(t)\|_p^p\right)^{\frac1p}\right.\\
&\qquad\qquad \left. + \left(\sum_{k=1}^{M_{mn}}|T_k^0(t)-\bar T_k^0(t)|^p\right)^{\frac1p}\right]^p\\
&\le \frac{3^{p-1}G^p}{M_{mn}}\sum_{k=1}^{M_{mn}}\left(\|x_k^0-\bar x_k^0\|_p^p + \|v_k^0-\bar v_k^0\|_p^p + |T_k^0-\bar T_k^0|^p\right)\\
&\le \frac{3^{p-1}G^p}{M_{mn}}\sum_{k=1}^{M_{mn}}\|z_k^0-\bar z_k^0\|_p^p \le 3^{p-1}G^p\left(W_p^p(\mu_0^n,\mu_0^m)+\varepsilon^p\right)\\
&\le 2\cdot 3^{p-1}G^p\varepsilon^p < 3^pG^p\varepsilon^p. \qquad (6.7)
\end{aligned}$$

Now for any $\varepsilon > 0$, we can find a positive integer L such that for any $n, m > L$, we have

$$W_p(\mu_t^n,\mu_t^m) < 3G\varepsilon.$$

This shows that the sequence $\{\mu_t^n\}_{n\ge1}$ is a Cauchy sequence in W_p-metric, thus we can find a limit measure μ_t. We next apply similar arguments as in [20] and show that the limit μ_t is the unique measure-valued solution of the kinetic equation (1.4) with initial data μ_0. Moreover, because of our estimates (6.7), we can conclude that for any ε, there exists a positive constant L, such that

$$\sup_{t\in[0,+\infty)} W_p(\mu_t^n,\mu_t) < 6G\varepsilon \quad \text{for } n > L.$$

This yields

$$\overline{\lim_{N\to+\infty}}\sup_{t\in[0,+\infty)} W_p(\mu_t^N,\mu_t) = 0.$$

The uniform compact support of μ_t follows this uniform convergence.

REMARK 6.1. 1. For $p, q \in [1,\infty]$, it is well known that ℓ_p and ℓ_q norms in $\mathbb{R}^d$ are equivalent. Using this relation, we obtain a uniform estimate in Wasserstein-(q,p) metric:

$$\begin{aligned}
W_{q,p}^p(\mu_t^n, \mu_t^m) &\le \int_{\mathbb{R}^{2d}\times\mathbb{R}_+\times\mathbb{R}^{2d}\times\mathbb{R}_+} \|z-\bar{z}\|_q^p \gamma(\mu_t^n,\mu_t^m) \\
&= \frac{1}{N_{mn}}\sum_{k=1}^{N_{mn}} \|z_k(t)-\bar{z}_k(t)\|_q^p \\
&\le 3^{p-1}G^p d^{p-1+\frac{p}{q}} \frac{1}{N_{mn}}\sum_{k=1}^{N_{mn}} \|z_{k0}-\bar{z}_{k0}\|_p^p \\
&\le 3^{p-1}G^p d^{p-1+\frac{p}{q}} (W_p^p(\mu_0^n,\mu_0^m)+\varepsilon^p) \\
&\le 2\cdot 3^{p-1}G^p d^{p-1+\frac{p}{q}}\varepsilon^p,
\end{aligned} \tag{6.8}$$

which indicates the uniform limit in $W_{q,p}$-metric.

2. It is clear that μ_t^N is a measure-valued solution in the sense of (2.3) with initial data μ_0^N. From the weak convergence from μ_t^N to μ_t, we can take the limit of (2.3) to prove that μ_t is a measure-valued solution with initial data μ_0.

6.1.2. *Uniform stability.* In this part, we study a uniform stability of measure-valued solutions whose existences and asymptotic behaviors are guaranteed in Section 6.1. For measures μ_0 and ν_0 in $\mathcal{P}(\mathbb{R}^{2d}\times\mathbb{R}_+)$ satisfying conditions in Theorem 3.6, let μ_t and ν_t be measure-valued solutions to (1.4). Then, it follows from (6.1) that for any $\varepsilon \ll 1$, there exists $N_0(\varepsilon)\in\mathbb{N}$ such that

$$W_p(\mu_t,\mu_t^n) < \frac{\varepsilon}{2}, \qquad W_p(\nu_t^n,\nu_t) < \frac{\varepsilon}{2}, \qquad n \ge N_0(\varepsilon).$$

Then, we use the above estimates and (6.7) to obtain

$$\begin{aligned}
W_p^p(\mu_t,\nu_t) &\le (W_p(\mu_t,\mu_t^n)+W_p(\mu_t^n,\nu_t^n)+W_p(\nu_t^n,\nu_t))^p \\
&\le (\varepsilon + W_p(\mu_t^n,\nu_t^n))^p \\
&\le 2^{p-1}\left(\varepsilon^p + W_p^p(\mu_t^n,\nu_t^n)\right) \\
&\le 2^{p-1}\left(2\varepsilon^p + G^p W_p^p(\mu_0^n,\nu_0^n)\right).
\end{aligned}$$

Letting $n\to\infty$, we have

$$W_p^p(\mu_t,\nu_t) \le 2^p\varepsilon^p + 2^{p-1}G^pW_p^p(\mu_0,\nu_0).$$

Since ε was arbitrary, we have the uniform W_p-stability:

$$W_p(\mu_t,\nu_t) \le 2^{\frac{p-1}{p}} G W_p(\mu_0,\nu_0), \quad t\ge 0.$$

Then, with the same argument as before, we have,

$$W_{q,p}(\mu_t,\nu_t) \le 2^{\frac{p-1}{p}} d^2 G W_{q,p}(\mu_0,\nu_0), \quad t\ge 0.$$

6.2. *Proofs of Corollary* 3.1. It follows from Theorem 3.2 that we have

$$\left(\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \|v\|_q^p d\mu_t^N\right)^{\frac{1}{p}} \le d^2 e^{-K\psi(2x_\infty^\infty)t}\left(\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \|v\|_q^p d\mu_0^N\right)^{\frac{1}{p}}. \tag{6.9}$$

Due to the common compact support of μ_t^N, we can view $\|v\|_q^p$ as a test function. Then, it follows that

$$\lim_{N\to 0} W_{q,p}(\mu_t^N, \mu_t) = 0,$$

which implies the weak convergence of μ_t^N to μ_t. Thus, by taking limiting on (6.9), we have

$$\left(\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \|v\|_q^p d\mu_t \right)^{\frac{1}{p}} \le d^2 e^{-K\psi(2x_\infty^\infty)t} \left(\int_{\mathbb{R}^{2d}\times\mathbb{R}_+} \|v\|_q^p d\mu_0 \right)^{\frac{1}{p}}.$$

7. Conclusion. In this paper, we provided three main results. First, we provided a new improved flocking estimate in terms of diameters of spatial, velocity, and temperature configurations. Second, we studied the uniform $\ell_{q,p}$-stability for the SDV-TCS model, which means that the suitable distance between two solutions is uniformly bounded by that of initial data in the class of suitable solutions exhibiting asymptotic flocking dynamics. Third, we provided a unique solvability and asymptotic behavior of measure-valued solutions for the corresponding mean-field kinetic equation as a direct application of uniform stability. Although we dealt with the simplified TCS model with small diffusion velocities, the same analysis can also be applied to the full TCS model with much heavier technical calculations.

Appendix A. Proof of Lemma 5.2. In this appendix, we present a proof of Lemma 5.2 which is rather lengthy. Therefore we split its proof into two parts.

A.1. *Spatial and velocity variations in ℓ_1-norm.* It is easy to see that the differences $x_i^k - \bar{x}_i^k$ and $v_i^k - \bar{v}_i^k$ satisfy

$$\begin{aligned}
&\frac{d}{dt}(x_i^k - \bar{x}_i^k) = v_i^k - \bar{v}_i^k, \quad t > 0, \quad 1 \le i \le N, \quad 1 \le k \le d, \\
&\frac{d}{dt}(v_i^k - \bar{v}_i^k) = \frac{\kappa_1}{N} \sum_{j=1}^N (\phi(\|x_j - x_i\|_1) - \phi(\|\bar{x}_j - \bar{x}_i\|_1)) \left(\frac{v_j^k}{T_j} - \frac{v_i^k}{T_i} \right) \\
&\qquad + \frac{\kappa_1}{N} \sum_{j=1}^N \phi(\|\bar{x}_j - \bar{x}_i\|_1) \left[\frac{v_j^k}{T_j} - \frac{v_j^k}{\bar{T}_j} - \left(\frac{v_i^k}{T_i} - \frac{v_i^k}{\bar{T}_i} \right) \right] \\
&\qquad + \frac{\kappa_1}{N} \sum_{j=1}^N \phi(\|\bar{x}_j - \bar{x}_i\|_1) \left(\frac{v_j^k - \bar{v}_j^k}{\bar{T}_j} - \frac{v_i^k - \bar{v}_i^k}{\bar{T}_i} \right).
\end{aligned} \tag{A.1}$$

We multiply

$$\sigma_x^{i,k} := \mathrm{sgn}(x_i^k - \bar{x}_i^k) \qquad \text{and} \qquad \sigma_v^{i,k} := \mathrm{sgn}(v_i^k - \bar{v}_i^k)$$

to the first and second equation of (A.1) to obtain

$$
\begin{aligned}
&\left|\frac{d}{dt}|x_i^k - \bar{x}_i^k|\right| \le |v_i^k - \bar{v}_i^k|, \quad t > 0,\\
&\frac{d}{dt}|v_i^k - \bar{v}_i^k| = \frac{\kappa_1}{N}\sum_{j=1}^N (\phi(\|x_j - x_i\|_1) - \phi(\|\bar{x}_j - \bar{x}_i\|_1))\left(\frac{v_j^k}{T_j} - \frac{v_i^k}{T_i}\right)\sigma_v^{i,k}\\
&\qquad + \frac{\kappa_1}{N}\sum_{j=1}^N \phi(\|\bar{x}_j - \bar{x}_i\|_1)\left[\frac{v_j^k}{T_j} - \frac{v_j^k}{\bar{T}_j} - \left(\frac{v_i^k}{T_i} - \frac{v_i^k}{\bar{T}_i}\right)\right]\sigma_v^{i,k}\\
&\qquad + \frac{\kappa_1}{N}\sum_{j=1}^N \phi(\|\bar{x}_j - \bar{x}_i\|_1)\left(\frac{v_j^k - \bar{v}_j^k}{\bar{T}_j} - \frac{v_i^k - \bar{v}_i^k}{\bar{T}_i}\right)\sigma_v^{i,k}\\
&\qquad \le \frac{\kappa_1}{N}\sum_{j=1}^N |\phi(\|x_j - x_i\|_1) - \phi(\|\bar{x}_j - \bar{x}_i\|_1)|\left|\frac{v_j^k}{T_j} - \frac{v_i^k}{T_i}\right|\\
&\qquad + \frac{\kappa_1}{N}\sum_{j=1}^N \phi(\|\bar{x}_j - \bar{x}_i\|_1)\left|\frac{v_j^k}{T_j} - \frac{v_j^k}{\bar{T}_j} - \left(\frac{v_i^k}{T_i} - \frac{v_i^k}{\bar{T}_i}\right)\right|\\
&\qquad + \frac{\kappa_1}{N}\sum_{j=1}^N \phi(\|\bar{x}_j - \bar{x}_i\|_1)\left(\frac{v_j^k - \bar{v}_j^k}{\bar{T}_j} - \frac{v_i^k - \bar{v}_i^k}{\bar{T}_i}\right)\sigma_v^{i,k}.
\end{aligned}
\tag{A.2}
$$

We next take a sum of (A.2) with respect to i and k to obtain

$$
\begin{aligned}
&\frac{d}{dt}\sum_{i,k}|x_i^k - \bar{x}_i^k| \le \sum_{i,k}|v_i^k - \bar{v}_i^k|, \quad t > 0,\\
&\frac{d}{dt}\sum_{i,k}|v_i^k - \bar{v}_i^k| \le \frac{\kappa_1}{N}\sum_{i,j}^N |\phi(\|x_j - x_i\|_1) - \phi(\|\bar{x}_j - \bar{x}_i\|_1)|\sum_k\left|\frac{v_j^k}{T_j} - \frac{v_i^k}{T_i}\right|\\
&\qquad + \frac{\kappa_1}{N}\sum_{i,j}^N \phi(\|\bar{x}_j - \bar{x}_i\|_1)\sum_k\left|\frac{v_j^k}{T_j} - \frac{v_j^k}{\bar{T}_j} - \left(\frac{v_i^k}{T_i} - \frac{v_i^k}{\bar{T}_i}\right)\right|\\
&\qquad + \frac{\kappa_1}{N}\sum_{i,j}^N \phi(\|\bar{x}_j - \bar{x}_i\|_1)\sum_k\left(\frac{v_j^k - \bar{v}_j^k}{\bar{T}_j} - \frac{v_i^k - \bar{v}_i^k}{\bar{T}_i}\right)\sigma_v^{i,k}\\
&\qquad =: \mathcal{I}_{11} + \mathcal{I}_{12} + \mathcal{I}_{13}.
\end{aligned}
\tag{A.3}
$$

We next estimate the terms $\mathcal{I}_{1i},\ i = 1, 2, 3$ separately.

- Case A.1 (Estimate of $\mathcal{I}_{11}$): We use Lipschitz continuity of $\phi(\cdot)$ to obtain

$$
\begin{aligned}
&|\phi(\|x_j - x_i\|_1) - \phi(\|\bar{x}_j - \bar{x}_i\|_1)|\\
&\qquad \le \|\phi\|_{Lip}|\|x_j - x_i\|_1 - \|\bar{x}_j - \bar{x}_i\|_1| \le \|\phi\|_{Lip}(\|x_j - \bar{x}_j\|_1 + \|x_i - \bar{x}_i\|_1).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\mathcal{I}_{11} &= \frac{\kappa_1}{N}\sum_{i,j}|\phi(\|x_j-x_i\|_1)-\phi(\|\bar{x}_j-\bar{x}_i\|_1)|\sum_k\left|\frac{v_j^k}{T_j}-\frac{v_i^k}{T_i}\right| \\
&\le \frac{\kappa_1}{N}\sum_{i,j}|\phi(\|x_j-x_i\|_1)-\phi(\|\bar{x}_j-\bar{x}_i\|_1)|\sum_k\left(\left|\frac{v_j^k}{T_j}\right|+\left|\frac{v_i^k}{T_i}\right|\right) \\
&\le \frac{\kappa_1}{N}\sum_{i,j}|\phi(\|x_j-x_i\|_1)-\phi(\|\bar{x}_j-\bar{x}_i\|_1)|\sum_k\left(\left|\frac{v_j^k}{T_m}\right|+\left|\frac{v_i^k}{T_m}\right|\right) \\
&\le \frac{\kappa_1}{N}\sum_{i,j}\|\phi\|_{Lip}(\|x_j-\bar{x}_j\|_1+\|x_i-\bar{x}_i\|_1)\sum_k\left(\left|\frac{v_j^k}{T_m}\right|+\left|\frac{v_i^k}{T_m}\right|\right).
\end{aligned}
\tag{A.4}
$$

On the other hand, we use $v_c = \frac{1}{N}\sum_{i=1}^N v_i = 0$ to see

$$
\sum_k |v_i^k| = \sum_k |v_i^k - v_c^k| \le \sum_k\sum_j \frac{1}{N}|v_i^k - v_j^k| \le \max_{i,j}\|v_i - v_j\|_1 = \mathcal{D}_V. \tag{A.5}
$$

Then, we use (A.5) to obtain

$$
\begin{aligned}
\mathcal{I}_{11} &\le \frac{\kappa_1}{N}\sum_{i,j}\|\phi\|_{Lip}(\|x_j-\bar{x}_j\|_1+\|x_i-\bar{x}_i\|_1)\sum_k\left(\left|\frac{v_j^k}{T_m}\right|+\left|\frac{v_i^k}{T_m}\right|\right) \\
&\le \frac{\kappa_1}{N}\sum_{i,j}\|\phi\|_{Lip}(\|x_j-\bar{x}_j\|_1+\|x_i-\bar{x}_i\|_1)\left(\frac{2\mathcal{D}_V}{T_m}\right) \\
&\le \frac{4\kappa_1\mathcal{D}_V\|\phi\|_{Lip}}{T_m}\sum_j(\|x_j-\bar{x}_j\|_1) \\
&\le \frac{4N\kappa_1\mathcal{D}_V\|\phi\|_{Lip}}{T_m^0}\|X-\bar{X}\|_1,
\end{aligned}
\tag{A.6}
$$

where we use the relation $T_m \ge T_m^0$ in the last inequality.

• Case A.2 (Estimate of $\mathcal{I}_{12}$): We use the boundedness of temperatures and ϕ

$$
\begin{aligned}
\mathcal{I}_{12} &= \frac{\kappa_1}{N}\sum_{i,j}^{n}\phi(\|\bar{x}_j-\bar{x}_i\|_1)\sum_k\left|\frac{v_j^k}{T_j}-\frac{v_j^k}{\bar{T}_j}-\left(\frac{v_i^k}{T_i}-\frac{v_i^k}{\bar{T}_i}\right)\right| \\
&\le \frac{\kappa_1}{N}\sum_{i,j}^{n}\phi(0)\sum_k\left(\left|\frac{v_j^k}{T_j}-\frac{v_j^k}{\bar{T}_j}\right|+\left|\frac{v_i^k}{T_i}-\frac{v_i^k}{\bar{T}_i}\right|\right) \\
&= 2\kappa_1\sum_{i}^{N}\phi(0)\sum_k\left|v_i^k\left(\frac{1}{T_i}-\frac{1}{\bar{T}_i}\right)\right| \\
&\le 2\kappa_1\sum_{i}^{N}\phi(0)\sum_k\left|v_i^k\left(\frac{|T_i-\bar{T}_i|}{T_m^2}\right)\right| \\
&\le \frac{2\kappa_1\phi(0)}{T_m^2}\sum_{i}^{N}|T_i-\bar{T}_i|\sum_k|v_i^k| \\
&\le \frac{2\kappa_1\phi(0)}{T_m^2}\sum_{i}^{N}|T_i-\bar{T}_i|\mathcal{D}_V \\
&= \frac{2N\kappa_1\phi(0)}{T_m^2}\|T-\bar{T}\|_1\mathcal{D}_V \le \frac{2N\kappa_1\phi(0)}{(T_m^0)^2}\|T-\bar{T}\|_1\mathcal{D}_V.
\end{aligned}
\tag{A.7}
$$

• Case A.3 (Estimate of $\mathcal{I}_{13}$): We use a standard trick of switching $i\leftrightarrow j$ to find

$$
\begin{aligned}
\mathcal{I}_{13} &= \frac{\kappa_1}{N}\sum_{i,j}^{n}\phi(\|\bar{x}_j-\bar{x}_i\|_1)\sum_k\left(\frac{v_j^k-\bar{v}_j^k}{\bar{T}_j}-\frac{v_i^k-\bar{v}_i^k}{\bar{T}_i}\right)\sigma_v^{i,k} \\
&= -\frac{\kappa_1}{N}\sum_{i,j}^{n}\phi(\|\bar{x}_j-\bar{x}_i\|_1)\sum_k\left(\frac{v_j^k-\bar{v}_j^k}{\bar{T}_j}-\frac{v_i^k-\bar{v}_i^k}{\bar{T}_i}\right)\sigma_v^{j,k} \\
&= -\frac{\kappa_1}{2N}\sum_{i,j}^{N}\phi(\|\bar{x}_j-\bar{x}_i\|_1)\sum_k\left(\frac{v_j^k-\bar{v}_j^k}{\bar{T}_j}-\frac{v_i^k-\bar{v}_i^k}{\bar{T}_i}\right)(\sigma_v^{j,k}-\sigma_v^{i,k}) \\
&\le -\frac{\kappa_1}{2N}\phi(\mathcal{D}_{\bar{X}})\sum_{i,j,k}\left(\frac{v_j^k-\bar{v}_j^k}{\bar{T}_j}-\frac{v_i^k-\bar{v}_i^k}{\bar{T}_i}\right)(\sigma_v^{j,k}-\sigma_v^{i,k}) \\
&\le -\frac{\kappa_1}{2N}\phi(\mathcal{D}_{\bar{X}})\sum_{i,j,k}\left(\frac{v_j^k-\bar{v}_j^k}{\bar{T}_M}-\frac{v_i^k-\bar{v}_i^k}{\bar{T}_M}\right)(\sigma_v^{j,k}-\sigma_v^{i,k}) \\
&= -\frac{\kappa_1}{2N\bar{T}_M}\phi(\mathcal{D}_{\bar{X}})\sum_{i,j,k}\big((v_j^k-\bar{v}_j^k)-(v_i^k-\bar{v}_i^k)\big)\sigma_v^{j,k} \\
&\quad+\frac{\kappa_1}{2N\bar{T}_M}\phi(\mathcal{D}_{\bar{X}})\sum_{i,j,k}\big((v_j^k-\bar{v}_j^k)-(v_i^k-\bar{v}_i^k)\big)\sigma_v^{i,k} \\
&= -\frac{\kappa_1\phi(\mathcal{D}_{\bar{X}})}{\bar{T}_M}\sum_{i,k}|v_i^k-\bar{v}_i^k| = -\frac{\kappa_1\phi(\mathcal{D}_{\bar{X}})}{\bar{T}_M}\sum_i\|v_i-\bar{v}_i\|_1 \\
&= -\frac{\kappa_1N\phi(\mathcal{D}_{\bar{X}})}{\bar{T}_M}\|V-\bar{V}\|_1 \le -\frac{\kappa_1N\phi(\mathcal{D}_{\bar{X}})}{\bar{T}_M^0}\|V-\bar{V}\|_1,
\end{aligned}
\tag{A.8}
$$

where we used the fact that

$$
\begin{aligned}
&(\sigma_v^{j,k} - \sigma_v^{i,k}) \left[\frac{(v_j^k - \bar{v}_j^k)}{\bar{T}_j} - \frac{(v_i^k - \bar{v}_i^k)}{\bar{T}_i} \right] \\
&\quad = \left(1 - \frac{\sigma_v^{i,k}}{\sigma_v^{j,k}}\right) \frac{|v_j^k - \bar{v}_j^k|}{\bar{T}_j} + \left(1 - \frac{\sigma_v^{j,k}}{\sigma_v^{i,k}}\right) \frac{|v_i^k - \bar{v}_i^k|}{\bar{T}_i} \\
&\quad \geq \left(1 - \frac{\sigma_v^{i,k}}{\sigma_v^{j,k}}\right) \frac{|v_j^k - \bar{v}_j^k|}{\bar{T}_M} + \left(1 - \frac{\sigma_v^{j,k}}{\sigma_v^{i,k}}\right) \frac{|v_i^k - \bar{v}_i^k|}{\bar{T}_M} \geq 0, \\
&\sum_{i,j,k} \sigma_v^{j,k} \left[(v_j^k - \bar{v}_j^k) - (v_i^k - \bar{v}_i^k) \right] = N \sum_{j,k} |v_j^k - \bar{v}_j^k|, \\
&\sum_{i,j,k} \sigma_v^{i,k} \left[(v_j^k - \bar{v}_j^k) - (v_i^k - \bar{v}_i^k) \right] = -N \sum_{i,k} |v_i^k - \bar{v}_i^k|.
\end{aligned}
$$

Finally, in (A.3) we combine estimates (A.6), (A.7), and (A.4) to obtain

$$
\begin{aligned}
&\frac{d}{dt}\|X - \bar{X}\|_1 \leq \|V - \bar{V}\|_1, \\
&\frac{d}{dt}\|V - \bar{V}\|_1 \leq \frac{4\kappa_1 \mathcal{D}_V \|\phi\|_{Lip}}{T_m^0}\|X - \bar{X}\|_1 + \frac{2\kappa_1\phi(0)}{(T_m^0)^2}\|T - \bar{T}\|_1 \mathcal{D}_V - \frac{\kappa_1 \phi(\mathcal{D}_{\bar{X}})}{T_M^0}\|V - \bar{V}\|_1.
\end{aligned}
$$

A.2. *Temperature variations in ℓ_1.* We next return to the second part of the proof of Lemma 5.2. It follows from (1.1) that we have

$$
\begin{aligned}
\frac{d}{dt}(T_i - \bar{T}_i) &= \frac{\kappa_2}{N}\sum_{j=1}^N \left[\zeta(\|x_i - x_j\|_1)\left(\frac{1}{T_i} - \frac{1}{T_j}\right) - \zeta(\|\bar{x}_i - \bar{x}_j\|_1)\left(\frac{1}{\bar{T}_i} - \frac{1}{\bar{T}_j}\right) \right] \\
&= \frac{\kappa_2}{N}\sum_{j=1}^N \left[\zeta(\|x_i - x_j\|_1) - \zeta(\|\bar{x}_i - \bar{x}_j\|_1) \right] \left(\frac{1}{T_i} - \frac{1}{T_j}\right) \\
&\quad + \frac{\kappa_2}{N}\sum_{j=1}^N \zeta(\|\bar{x}_i - \bar{x}_j\|_1) \left[\left(\frac{1}{T_i} - \frac{1}{\bar{T}_i}\right) - \left(\frac{1}{T_j} - \frac{1}{\bar{T}_j}\right) \right].
\end{aligned} \tag{A.9}
$$

We multiply $\sigma_T^i := \mathrm{sgn}(T_i - \bar{T}_i)$ to (A.9) to obtain

$$
\begin{aligned}
\sum_{i=1} \frac{d}{dt}|T_i - \bar{T}_i| &= \frac{\kappa_2}{N}\sum_{i,j=1}^N \left(\zeta(\|x - x\|_1) - \zeta(\|\bar{x}_i - \bar{x}_j\|_1) \right) \left(\frac{1}{T_i} - \frac{1}{T_j}\right)\sigma_T^i \\
&\quad + \frac{\kappa_2}{N}\sum_{i,j=1}^N \zeta(\|\bar{x}_i - \bar{x}_j\|_1) \left[\left(\frac{1}{T_i} - \frac{1}{\bar{T}_i}\right) - \left(\frac{1}{T_j} - \frac{1}{\bar{T}_j}\right) \right]\sigma_T^i \\
&\leq \frac{\kappa_2}{N}\sum_{i,j=1}^N \left| \zeta(\|x - x\|_1) - \zeta(\|\bar{x}_i - \bar{x}_j\|_1) \right| \left|\frac{1}{T_i} - \frac{1}{T_j}\right| \\
&\quad + \frac{\kappa_2}{N}\sum_{i,j=1}^N \zeta(\|\bar{x}_i - \bar{x}_j\|_1) \left[\left(\frac{1}{T_i} - \frac{1}{\bar{T}_i}\right) - \left(\frac{1}{T_j} - \frac{1}{\bar{T}_j}\right) \right]\sigma_T^i \\
&=: \mathcal{I}_{21} + \mathcal{I}_{22}.
\end{aligned} \tag{A.10}
$$

- Case B.1 (Estimate of $\mathcal{I}_{21}$): We use Lipschitz continuity of $\zeta(\cdot)$ to obtain

$$
\begin{aligned}
&|\zeta(\|x_j - x_i\|_1) - \zeta(\|\bar{x}_j - \bar{x}_i\|_1)| \\
&\qquad \le \|\zeta\|_{Lip} |\|x_j - x_i\|_1 - \|\bar{x}_j - \bar{x}_i\|_1| \le \|\zeta\|_{Lip}(\|x_j - \bar{x}_j\|_1 + \|x_i - \bar{x}_i\|_1).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\mathcal{I}_{21} &= \frac{\kappa_2}{N} \sum_{i,j=1}^{N} |\zeta(\|x_i - x_j\|_1) - \zeta(\|\bar{x}_i - \bar{x}_j\|_1)| \left| \frac{1}{T_i} - \frac{1}{T_j} \right| \\
&\le \frac{\kappa_2}{N} \sum_{i,j=1}^{N} |\zeta(\|x_i - x_j\|_1) - \zeta(\|\bar{x}_i - \bar{x}_j\|_1)| \frac{|T_i - T_j|}{T_m^2} \\
&\le \frac{\kappa_2}{N} \sum_{i,j=1}^{N} \|\zeta\|_{Lip}(\|x_j - \bar{x}_j\|_1 + \|x_i - \bar{x}_i\|_1) \frac{|T_i - T_j|}{T_m^2} \\
&\le 2\kappa_2 \|\zeta\|_{Lip} \frac{\mathcal{D}_T}{T_m^2} \sum_{i=1}^{N} \|x_i - \bar{x}_i\|_1 = 2N\kappa_2 \|\zeta\|_{Lip} \frac{\mathcal{D}_T}{T_m^2} \|X - \bar{X}\|_1 \\
&\le 2N\kappa_2 \|\zeta\|_{Lip} \frac{\mathcal{D}_T}{(T_m^0)^2} \|X - \bar{X}\|_1.
\end{aligned} \tag{A.11}
$$

- Case B.2 (Estimate of $\mathcal{I}_{22}$): We use a standard trick of switching $i \leftrightarrow j$ to have

$$
\begin{aligned}
\mathcal{I}_{22} &= \frac{\kappa_2}{N} \sum_{i,j=1}^{N} \zeta(\|x_i - \bar{x}_j\|_1) \left[\left(\frac{1}{T_i} - \frac{1}{\bar{T}_i} \right) - \left(\frac{1}{T_j} - \frac{1}{\bar{T}_j} \right) \right] \sigma_T^i \\
&= \frac{\kappa_2}{2N} \sum_{i,j=1}^{N} \zeta(\|\bar{x}_i - \bar{x}_j\|_1) \left[\left(\frac{1}{T_i} - \frac{1}{\bar{T}_i} \right) - \left(\frac{1}{T_j} - \frac{1}{\bar{T}_j} \right) \right] (\sigma_T^i - \sigma_T^j) \\
&= -\frac{\kappa_2}{2N} \sum_{i,j=1}^{N} \zeta(\|\bar{x}_i - \bar{x}_j\|_1) \left(\frac{T_i - \bar{T}_i}{T_i \bar{T}_i} - \frac{T_j - \bar{T}_j}{T_j \bar{T}_j} \right) (\sigma_T^i - \sigma_T^j).
\end{aligned} \tag{A.12}
$$

We again use the following relations:

$$
\begin{aligned}
&(\sigma_T^j - \sigma_T^i) \left[\frac{(T_j - \bar{T}_j)}{T_j \bar{T}_j} - \frac{(T_i - \bar{T}_i)}{T_i \bar{T}_i} \right] \\
&\qquad = \left(1 - \frac{\sigma_T^i}{\sigma_T^j} \right) \frac{|T_j - \bar{T}_j|}{T_j \bar{T}_j} + \left(1 - \frac{\sigma_T^j}{\sigma_T^i} \right) \frac{|T_i - \bar{T}_i|}{T_i \bar{T}_i} \\
&\qquad \ge \left(1 - \frac{\sigma_T^i}{\sigma_T^j} \right) \frac{|T_j - \bar{T}_j|}{T_M^2} + \left(1 - \frac{\sigma_T^j}{\sigma_T^i} \right) \frac{|T_i - \bar{T}_i|}{T_M^2} \ge 0, \\
&\sum_{i,j} \sigma_T^j \left[(T_j - \bar{T}_j) - (T_i - \bar{T}_i) \right] = N \sum_j |T_j - \bar{T}_j| - \sum_j \sigma_T^j \sum_i (T_i - \bar{T}_i), \\
&\sum_{i,j} \sigma_T^i \left[(T_j - \bar{T}_j) - (T_i - \bar{T}_i) \right] = -N \sum_i |T_i - \bar{T}_i| + \sum_j \sigma_T^j \sum_i (T_i - \bar{T}_i),
\end{aligned}
$$

to continue the remaining computations in (A.12):

$$\begin{aligned}
\mathcal{I}_{22} &= -\frac{\kappa_2}{2N}\sum_{i,j=1}^N \zeta(\|\bar{x}_i-\bar{x}_j\|_1)\left(\frac{T_i-\bar{T}_i}{T_i\bar{T}_i}-\frac{T_j-\bar{T}_j}{T_j\bar{T}_j}\right)(\sigma_T^i-\sigma_T^j)\\
&\le -\frac{\kappa_2}{2N}\sum_{i,j=1}^N \zeta(\|\bar{x}_i-\bar{x}_j\|_1)\left(\left(1-\frac{\sigma_T^i}{\sigma_T^j}\right)\frac{|T_j-\bar{T}_j|}{T_M^2}+\left(1-\frac{\sigma_T^j}{\sigma_T^i}\right)\frac{|T_i-\bar{T}_i|}{T_M^2}\right)\\
&= -\frac{\kappa_2}{2NT_M^2}\sum_{i,j=1}^N \zeta(\|\bar{x}_i-\bar{x}_j\|_1)\left(\left(1-\frac{\sigma_T^i}{\sigma_T^j}\right)|T_j-\bar{T}_j|+\left(1-\frac{\sigma_T^j}{\sigma_T^i}\right)|T_i-\bar{T}_i|\right)\\
&\le -\frac{\kappa_2}{2NT_M^2}\sum_{i,j=1}^N \zeta(\mathcal{D}_{\bar{X}})\left(\left(1-\frac{\sigma_T^i}{\sigma_T^j}\right)|T_j-\bar{T}_j|+\left(1-\frac{\sigma_T^j}{\sigma_T^i}\right)|T_i-\bar{T}_i|\right)\\
&\le -\frac{\kappa_2\zeta(\mathcal{D}_{\bar{X}})}{2NT_M^2}\sum_{i,j=1}^N \big((T_i-\bar{T}_i)-(T_j-\bar{T}_j)\big)(\sigma_T^i-\sigma_T^j)\\
&= -\frac{\kappa_2\zeta(\mathcal{D}_{\bar{X}})}{T_M^2}\left(\sum_{i=1}^N|T_i-\bar{T}_i|-\frac{\sum_i\sigma_T^i}{N}\sum_j(T_j-\bar{T}_j)\right)\le 0. \qquad \text{(A.13)}
\end{aligned}$$

In (A.10), we combine estimates (A.11) and (A.13) to obtain the desired differential inequality:

$$\frac{d}{dt}\|T-\bar{T}\|_1 \le 2\kappa_2\|\zeta\|_{Lip}\frac{\mathcal{D}_T}{(T_m^0)^2}\|X-\bar{X}\|_1.$$

Appendix B. Proof of Lemma 5.3. In this appendix, we will derive a system of differential inequalities in ℓ_p norm. Since the proof is extremely long and tedious, we also split its proof into two subsections.

B.1. *Spatial and velocity variations in ℓ_p norm.* We multiply $p|x_i^k-\bar{x}_i^k|^{p-1}$ to the first equation in (A.2), sum it over i and k, and then use Hölder's inequality to obtain

$$\frac{d}{dt}\sum_{i,k}|x_i^k-\bar{x}_i^k|^p \le p\sum_{i,k}|x_i^k-\bar{x}_i^k|^{p-1}|v_i^k-\bar{v}_i^k| \le p\left(\sum_{i,k}|x_i^k-\bar{x}_i^k|^p\right)^{1-\frac{1}{p}}\left(\sum_{i,k}|v_i^k-\bar{v}_i^k|^p\right)^{\frac{1}{p}}.$$

This yields the first desired differential inequality in Lemma 5.3:

$$\frac{d}{dt}\|X-\bar{X}\|_p \le \|V-\bar{V}\|_p.$$

We next note that $|v_i^k - \bar{v}_i^k|^2$ satisfies

$$\begin{aligned}\frac{d}{dt}|v_i^k - \bar{v}_i^k|^2 &= \frac{2\kappa_1}{N}\sum_{j=1}^N \left(\phi(\|x_j - x_i\|_p) - \phi(\|\bar{x}_j - \bar{x}_i\|_p)\right)\left(\frac{v_j^k}{T_j} - \frac{v_i^k}{T_i}\right)(v_i^k - \bar{v}_i^k)\\ &\quad + \frac{2\kappa_1}{N}\sum_{j=1}^N \phi(\|\bar{x}_j - \bar{x}_i\|_p)\left[\frac{v_j^k}{T_j} - \frac{v_j^k}{\bar{T}_j} - \left(\frac{v_i^k}{T_i} - \frac{v_i^k}{\bar{T}_i}\right)\right](v_i^k - \bar{v}_i^k)\\ &\quad + \frac{2\kappa_1}{N}\sum_{j=1}^N \phi(\|\bar{x}_j - \bar{x}_i\|_p)\left(\frac{v_j^k - \bar{v}_j^k}{\bar{T}_j} - \frac{v_i^k - \bar{v}_i^k}{\bar{T}_i}\right)(v_i^k - \bar{v}_i^k).\end{aligned} \tag{B.1}$$

Then, we use (B.1) to obtain

$$\begin{aligned}&\frac{d}{dt}|v_i^k - \bar{v}_i^k|^p\\ &= \frac{d}{dt}\left(|v_i^k - \bar{v}_i^k|^2\right)^{\frac{p}{2}}\\ &= \frac{p}{2}|v_i^k - \bar{v}_i^k|^{p-2}\frac{d}{dt}|v_i^k - \bar{v}_i^k|^2\\ &= p|v_i^k - \bar{v}_i^k|^{p-2}\left[\frac{\kappa_1}{N}\sum_{j=1}^N \left(\phi(\|x_j - x_i\|_p) - \phi(\|\bar{x}_j - \bar{x}_i\|_p)\right)\left(\frac{v_j^k}{T_j} - \frac{v_i^k}{T_i}\right)(v_i^k - \bar{v}_i^k)\right.\\ &+ \frac{\kappa_1}{N}\sum_{j=1}^N \phi(\|\bar{x}_j - \bar{x}_i\|_p)\left[\frac{v_j^k}{T_j} - \frac{v_j^k}{\bar{T}_j} - \left(\frac{v_i^k}{T_i} - \frac{v_i^k}{\bar{T}_i}\right)\right](v_i^k - \bar{v}_i^k)\\ &+ \left.\frac{\kappa_1}{N}\sum_{j=1}^N \phi(\|\bar{x}_j - \bar{x}_i\|_p)\left(\frac{v_j^k - \bar{v}_j^k}{\bar{T}_j} - \frac{v_i^k - \bar{v}_i^k}{\bar{T}_i}\right)(v_i^k - \bar{v}_i^k)\right].\end{aligned} \tag{B.2}$$

Now, we sum up (B.2) for all i and k to obtain the following relation:

$$\begin{aligned}&\sum_{i,k}\frac{d}{dt}|v_i^k - \bar{v}_i^k|^p\\ &= \frac{p\kappa_1}{N}\sum_{i,j,k}\left(\phi(\|x_j - x_i\|_p) - \phi(\|\bar{x}_j - \bar{x}_i\|_p)\right)\left(\frac{v_j^k}{T_j} - \frac{v_i^k}{T_i}\right)(v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2}\\ &+ \frac{p\kappa_1}{N}\sum_{i,j,k}\phi(\|\bar{x}_j - \bar{x}_i\|_p)\left[\frac{v_j^k}{T_j} - \frac{v_j^k}{\bar{T}_j} - \left(\frac{v_i^k}{T_i} - \frac{v_i^k}{\bar{T}_i}\right)\right](v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2}\\ &+ \frac{p\kappa_1}{N}\sum_{i,j,k}\phi(\|\bar{x}_j - \bar{x}_i\|_p)\left(\frac{v_j^k - \bar{v}_j^k}{\bar{T}_j} - \frac{v_i^k - \bar{v}_i^k}{\bar{T}_i}\right)(v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2}\\ &=: \mathcal{I}_{31} + \mathcal{I}_{32} + \mathcal{I}_{33}.\end{aligned} \tag{B.3}$$

• Case C.1 (Estimate of $\mathcal{I}_{31}$): We use Lipschitz continuity of $\phi(\cdot)$ and Hölder's inequality to obtain

$$
\begin{aligned}
\mathcal{I}_{31} &= \frac{p\kappa_1}{N}\sum_{i,j,k}\left(\phi(\|x_j-x_i\|_p)-\phi(\|\bar{x}_j-\bar{x}_i\|_p)\right)\left(\frac{v_j^k}{T_j}-\frac{v_i^k}{T_i}\right)(v_i^k-\bar{v}_i^k)|v_i^k-\bar{v}_i^k|^{p-2}\\
&\le \frac{p\kappa_1}{N}\sum_{i,j}^{N}\left|\|\phi\|_{Lip}(\|x_j-x_i\|_p-\|\bar{x}_j-\bar{x}_i\|_p)\right|\sum_k\left(\left|\frac{v_j^k}{T_j}\right|+\left|\frac{v_i^k}{T_i}\right|\right)|v_i^k-\bar{v}_i^k|^{p-1}\\
&\le \frac{p\kappa_1}{N}\sum_{i,j}\left|\|\phi\|_{Lip}(\|x_j-\bar{x}_j\|_p+\|x_i-\bar{x}_i\|_p)\right|\sum_k\left(\left|\frac{v_j^k}{T_m}\right|+\left|\frac{v_i^k}{T_m}\right|\right)|v_i^k-\bar{v}_i^k|^{p-1}\\
&\le \frac{p\kappa_1}{N}\frac{\|\phi\|_{Lip}}{T_m}\sum_{i,j}\left|(\|x_j-\bar{x}_j\|_p+\|x_i-\bar{x}_i\|_p)\right|\sum_k(|v_j^k|+|v_i^k|)|v_i^k-\bar{v}_i^k|^{p-1}\\
&\le \frac{p\kappa_1}{N}\frac{\|\phi\|_{Lip}}{T_m}\sum_{i,j}\left|(\|x_j-\bar{x}_j\|_p+\|x_i-\bar{x}_i\|_p)\right|\|v_i-\bar{v}_i\|_p^{p-1}\left(\sum_k(|v_j^k|+|v_i^k|)^p\right)^{1/p}\\
&\le \frac{p\kappa_1}{N}\frac{\|\phi\|_{Lip}}{T_m}\sum_{i,j}\left|(\|x_j-\bar{x}_j\|_p+\|x_i-\bar{x}_i\|_p)\right|\|v_i-\bar{v}_i\|_p^{p-1}\\
&\times\left[\left(\sum_k|v_j^k|^p\right)^{1/p}+\left(\sum_k|v_i^k|^p\right)^{1/p}\right]\qquad(\because\text{ Minkowski's inequality})\\
&\le 2p\kappa_1 N\frac{\|\phi\|_{Lip}}{T_m}\|X-\bar{X}\|_p\|V-\bar{V}\|_p^{p-1}(2\max_i\|v_i\|_p)\\
&\le 4p\kappa_1 N\mathcal{D}_V\frac{\|\phi\|_{Lip}}{T_m}\|X-\bar{X}\|_p\|V-\bar{V}\|_p^{p-1}\\
&\le 4p\kappa_1 N\mathcal{D}_V\frac{\|\phi\|_{Lip}}{T_m^0}\|X-\bar{X}\|_p\|V-\bar{V}\|_p^{p-1},
\end{aligned}\tag{B.4}
$$

where we used the following estimation in the second and third inequalities from below:

$$
\|x_i-\bar{x}_i\|_p\le\|X-\bar{X}\|_p,\quad \|v_i-\bar{v}_i\|_p\le\|V-\bar{V}\|_p
$$

and

$$
\|v_i\|_p=\left\|\frac{1}{N}\sum_j(v_i-v_j)\right\|_p\le\max_{i,j}\|v_i-v_j\|_p\le\mathcal{D}_V.
$$

• Case C.2 (Estimate of $\mathcal{I}_{32}$): By direct calculation, we have

$$\begin{aligned}
\mathcal{I}_{32} &= \frac{p\kappa_1}{N}\sum_{i,j,k}\phi(\|\bar{x}_j-\bar{x}_i\|_p)\left[\frac{v_j^k}{T_j}-\frac{v_j^k}{\bar{T}_j}-\left(\frac{v_i^k}{T_i}-\frac{v_i^k}{\bar{T}_i}\right)\right](v_i^k-\bar{v}_i^k)|v_i^k-\bar{v}_i^k|^{p-2}\\
&\le \frac{p\kappa_1}{N}\sum_{i,j,k}\phi(0)\left(\left|\frac{v_j^k}{T_j}-\frac{v_j^k}{\bar{T}_j}\right|+\left|\frac{v_i^k}{T_i}-\frac{v_i^k}{\bar{T}_i}\right|\right)|v_i^k-\bar{v}_i^k|^{p-1}\\
&\le \frac{p\kappa_1}{NT_m^2}\sum_{i,j,k}\phi(0)\left(|v_j^k||T_j-\bar{T}_j|+|v_i^k||T_i-\bar{T}_i|\right)|v_i^k-\bar{v}_i^k|^{p-1} \qquad \text{(B.5)}\\
&= \frac{p\kappa_1}{NT_m^2}\sum_{i,j}\phi(0)|T_i-\bar{T}_i|\sum_k|v_i^k-\bar{v}_i^k|^{p-1}|v_i^k|\\
&\quad+\frac{p\kappa_1}{NT_m^2}\sum_{i,j}\phi(0)|T_j-\bar{T}_j|\sum_k|v_i^k-\bar{v}_i^k|^{p-1}|v_j^k|.
\end{aligned}$$

We use Hölder's inequality on the $\sum_k|v_i^k-\bar{v}_i^k|^{p-1}|v_j^k|$ in the last term of (B.5) to obtain

$$\begin{aligned}
\mathcal{I}_{32} &\le \frac{p\kappa_1}{NT_m^2}\sum_{i,j}\phi(0)|T_i-\bar{T}_i|\|v_i-\bar{v}_i\|_p^{p-1}\|v_i\|_p+\frac{p\kappa_1}{NT_m^2}\sum_{i,j}\phi(0)|T_j-\bar{T}_j|\|v_i-\bar{v}_i\|_p^{p-1}\|v_j\|_p\\
&\le \frac{p\kappa_1}{NT_m^2}\sum_{i,j}\phi(0)|T_i-\bar{T}_i|\|v_i-\bar{v}_i\|_p^{p-1}\mathcal{D}_V+\frac{p\kappa_1}{NT_m^2}\sum_{i,j}\phi(0)|T_j-\bar{T}_j|\|v_i-\bar{v}_i\|_p^{p-1}\mathcal{D}_V\\
&\le \frac{pN\kappa_1\phi(0)}{T_m^2}\|T-\bar{T}\|_p\|V-\bar{V}\|_p^{p-1}\mathcal{D}_v+\frac{pN\kappa_1\phi(0)}{T_m^2}\|T-\bar{T}\|_p\|V-\bar{V}\|_p^{p-1}\mathcal{D}_V\\
&= \frac{2pN\kappa_1\phi(0)}{T_m^2}\|T-\bar{T}\|_p\|V-\bar{V}\|_p^{p-1}\mathcal{D}_V\\
&\le \frac{2pN\kappa_1\phi(0)}{(T_m^0)^2}\|T-\bar{T}\|_p\|V-\bar{V}\|_p^{p-1}\mathcal{D}_V. \qquad \text{(B.6)}
\end{aligned}$$

• Case C.3 (Estimate of $\mathcal{I}_{33}$): Using a trick to change $i\leftrightarrow j$ we can have the following estimate:

$$\begin{aligned}
\mathcal{I}_{33} &= \frac{p\kappa_1}{N}\sum_{i,j,k}\phi(\|\bar{x}_j-\bar{x}_i\|_p)\left(\frac{v_j^k-\bar{v}_j^k}{\bar{T}_j}-\frac{v_i^k-\bar{v}_i^k}{\bar{T}_i}\right)(v_i^k-\bar{v}_i^k)|v_i^k-\bar{v}_i^k|^{p-2}\\
&= \frac{p\kappa_1}{2N}\sum_{i,j,k}\phi(\|\bar{x}_j-\bar{x}_i\|_p)\left(\frac{v_j^k-\bar{v}_j^k}{\bar{T}_j}-\frac{v_i^k-\bar{v}_i^k}{\bar{T}_i}\right) \qquad \text{(B.7)}\\
&\quad\times\left((v_i^k-\bar{v}_i^k)|v_i^k-\bar{v}_i^k|^{p-2}-(v_j^k-\bar{v}_j^k)|v_j^k-\bar{v}_j^k|^{p-2}\right).
\end{aligned}$$

Here we need to be careful, since the linearity does not hold. Thus, we cannot assume that

$$\left(\frac{v_j^k-\bar{v}_j^k}{\bar{T}_j}-\frac{v_i^k-\bar{v}_i^k}{\bar{T}_i}\right)\left((v_i^k-\bar{v}_i^k)|v_i^k-\bar{v}_i^k|^{p-2}-(v_j^k-\bar{v}_j^k)|v_j^k-\bar{v}_j^k|^{p-2}\right)<0,$$

even though it holds when $T_j=T_i$.

Note that the above inequality holds when $(|v_j^k - \bar{v}_j^k| - |v_i^k - \bar{v}_i^k|)(\bar{T}_j - \bar{T}_i) \le 0$. This is due to the following calculation:

$$\begin{aligned}
&\left(\frac{v_j^k - \bar{v}_j^k}{\bar{T}_j} - \frac{v_i^k - \bar{v}_i^k}{\bar{T}_i}\right)\left((v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2} - (v_j^k - \bar{v}_j^k)|v_j^k - \bar{v}_j^k|^{p-2}\right)\\
&\quad = -\frac{|v_i^k - \bar{v}_i^k|^p}{\bar{T}_i} - \frac{|v_j^k - \bar{v}_j^k|^p}{\bar{T}_j}\\
&\quad + \frac{|v_i^k - \bar{v}_i^k|^{p-2}}{\bar{T}_j}(v_j^k - \bar{v}_j^k)(v_i^k - \bar{v}_i^k) + \frac{|v_j^k - \bar{v}_j^k|^{p-2}}{\bar{T}_i}(v_j^k - \bar{v}_j^k)(v_i^k - \bar{v}_i^k)\\
&\quad \le -\frac{|v_i^k - \bar{v}_i^k|^p}{\bar{T}_i} - \frac{|v_j^k - \bar{v}_j^k|^p}{\bar{T}_j}\\
&\quad + \frac{|v_i^k - \bar{v}_i^k|^{p-2}}{\bar{T}_j}|v_j^k - \bar{v}_j^k||v_i^k - \bar{v}_i^k| + \frac{|v_j^k - \bar{v}_j^k|^{p-2}}{\bar{T}_i}|v_j^k - \bar{v}_j^k||v_i^k - \bar{v}_i^k|\\
&\quad = \left(\frac{|v_j^k - \bar{v}_j^k|}{\bar{T}_j} - \frac{|v_i^k - \bar{v}_i^k|}{\bar{T}_i}\right)\left(|v_i^k - \bar{v}_i^k|^{p-1} - |v_j^k - \bar{v}_j^k|^{p-1}\right).
\end{aligned}$$

Thus, without loss of generality, if we assume $|v_i^k - \bar{v}_i^k| \ge |v_j^k - \bar{v}_j^k|$ and if $\bar{T}_i \le \bar{T}_j$, we can see that

$$\begin{aligned}
&\left(\frac{v_j^k - \bar{v}_j^k}{\bar{T}_j} - \frac{v_i^k - \bar{v}_i^k}{\bar{T}_i}\right)\left((v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2} - (v_j^k - \bar{v}_j^k)|v_j^k - \bar{v}_j^k|^{p-2}\right)\\
&\qquad \le -\left(\frac{|v_i^k - \bar{v}_i^k|}{\bar{T}_i} - \frac{|v_j^k - \bar{v}_j^k|}{\bar{T}_j}\right)\left(|v_i^k - \bar{v}_i^k|^{p-1} - |v_j^k - \bar{v}_j^k|^{p-1}\right) \le 0.
\end{aligned}$$

We will use this fact to gain an appropriate estimate for the functional. Now we define a set

$$S_1 := \Big\{(i,j,k) : (|v_j^k - \bar{v}_j^k| - |v_i^k - \bar{v}_i^k|)(\bar{T}_j - \bar{T}_i) \le 0 \text{ at time } t\Big\}.$$

Then, we can separate $\mathcal{I}_{33}$ into two peices:

$$\begin{aligned}
\mathcal{I}_{33} &= \frac{p\kappa_1}{N}\sum_{i,j,k}\phi(\|\bar{x}_j - \bar{x}_i\|_p)\left(\frac{v_j^k - \bar{v}_j^k}{\bar{T}_j} - \frac{v_i^k - \bar{v}_i^k}{\bar{T}_i}\right)(v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2}\\
&= \frac{p\kappa_1}{N}\sum_{(i,j,k)\in S_1}\phi(\|\bar{x}_j - \bar{x}_i\|_p)\left(\frac{v_j^k - \bar{v}_j^k}{\bar{T}_j} - \frac{v_i^k - \bar{v}_i^k}{\bar{T}_i}\right)(v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2}\\
&+ \frac{p\kappa_1}{N}\sum_{(i,j,k)\notin S_1}\phi(\|\bar{x}_j - \bar{x}_i\|_p)\left(\frac{v_j^k - \bar{v}_j^k}{\bar{T}_j} - \frac{v_i^k - \bar{v}_i^k}{\bar{T}_i}\right)(v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2}\\
&= \mathcal{I}_{331} + \mathcal{I}_{332}.
\end{aligned} \tag{B.8}$$

$\diamond$ Case C.3.1 (Estimate of $\mathcal{I}_{331}$): Estimation of term $\mathcal{I}_{331}$ is very easy since its term's negativity is satisfied if $(i,j,k) \in S$ and then $(j,i,k) \in S$ by the condition. So we are able to use the switching trick in S_1 as follows:

$$\begin{aligned}
\mathcal{I}_{331} &= \frac{p\kappa_1}{N}\sum_{i,j,k\in S_1}\phi(\|\bar{x}_j-\bar{x}_i\|_p)\left(\frac{v_j^k-\bar{v}_j^k}{\bar{T}_j}-\frac{v_i^k-\bar{v}_i^k}{\bar{T}_i}\right)(v_i^k-\bar{v}_i^k)|v_i^k-\bar{v}_i^k|^{p-2}\\
&= \frac{p\kappa_1}{2N}\sum_{i,j,k\in S_1}\phi(\|\bar{x}_j-\bar{x}_i\|_p)\left(\frac{v_j^k-\bar{v}_j^k}{\bar{T}_j}-\frac{v_i^k-\bar{v}_i^k}{\bar{T}_i}\right)\\
&\times\left((v_i^k-\bar{v}_i^k)|v_i^k-\bar{v}_i^k|^{p-2}-(v_j^k-\bar{v}_j^k)|v_j^k-\bar{v}_j^k|^{p-2}\right)\\
&\le \frac{p\kappa_1}{2N}\sum_{i,j,k\in S_1}\phi(\mathcal{D}_{\bar{X}})\left(\frac{v_j^k-\bar{v}_j^k}{\bar{T}_j}-\frac{v_i^k-\bar{v}_i^k}{\bar{T}_i}\right)\\
&\times\left((v_i^k-\bar{v}_i^k)|v_i^k-\bar{v}_i^k|^{p-2}-(v_j^k-\bar{v}_j^k)|v_j^k-\bar{v}_j^k|^{p-2}\right).
\end{aligned}\tag{B.9}$$

Now we consider the case

$$|v_i^k-\bar{v}_i^k|>|v_j^k-\bar{v}_j^k| \quad \text{and} \quad \bar{T}_i\le\bar{T}_j.$$

In this case, we have

$$\begin{aligned}
&\left(\frac{v_j^k-\bar{v}_j^k}{\bar{T}_j}-\frac{v_i^k-\bar{v}_i^k}{\bar{T}_i}\right)((v_i^k-\bar{v}_i^k)|v_i^k-\bar{v}_i^k|^{p-2}-(v_j^k-\bar{v}_j^k)|v_j^k-\bar{v}_j^k|^{p-2})\\
&\quad\le\left(\frac{v_j^k-\bar{v}_j^k}{\bar{T}_j}-\frac{v_i^k-\bar{v}_i^k}{\bar{T}_j}\right)((v_i^k-\bar{v}_i^k)|v_i^k-\bar{v}_i^k|^{p-2}-(v_j^k-\bar{v}_j^k)|v_j^k-\bar{v}_j^k|^{p-2})\\
&\quad=\frac{1}{\bar{T}_j}\left(v_j^k-\bar{v}_j^k-(v_i^k-\bar{v}_i^k)\right)((v_i^k-\bar{v}_i^k)|v_i^k-\bar{v}_i^k|^{p-2}-(v_j^k-\bar{v}_j^k)|v_j^k-\bar{v}_j^k|^{p-2})\\
&\quad\le\frac{1}{T_M}\left(v_j^k-\bar{v}_j^k-(v_i^k-\bar{v}_i^k)\right)((v_i^k-\bar{v}_i^k)|v_i^k-\bar{v}_i^k|^{p-2}-(v_j^k-\bar{v}_j^k)|v_j^k-\bar{v}_j^k|^{p-2}).
\end{aligned}\tag{B.10}$$

Next, we consider the case

$$|v_i^k-\bar{v}_i^k|<|v_j^k-\bar{v}_j^k| \quad \text{and} \quad \bar{T}_i\ge\bar{T}_j.$$

This is symmetric to the former case. Thus the same estimate holds. Finally, in (B.9), we use (B.10) to see

$$\begin{aligned}
\mathcal{I}_{331} &\le \frac{p\kappa_1}{2N}\sum_{i,j,k\in S_1}\phi(\mathcal{D}_{\bar{X}})\left(\frac{v_j^k-\bar{v}_j^k}{\bar{T}_j}-\frac{v_i^k-\bar{v}_i^k}{\bar{T}_i}\right)\\
&\times\left((v_i^k-\bar{v}_i^k)|v_i^k-\bar{v}_i^k|^{p-2}-(v_j^k-\bar{v}_j^k)|v_j^k-\bar{v}_j^k|^{p-2}\right)\\
&\le \frac{p\kappa_1}{2NT_M}\sum_{i,j,k\in S_1}\phi(\mathcal{D}_{\bar{X}})\left(v_j^k-\bar{v}_j^k-(v_i^k-\bar{v}_i^k)\right)\\
&\times\left((v_i^k-\bar{v}_i^k)|v_i^k-\bar{v}_i^k|^{p-2}-(v_j^k-\bar{v}_j^k)|v_j^k-\bar{v}_j^k|^{p-2}\right).
\end{aligned}\tag{B.11}$$

$\diamond$ Case C.3.2 (Estimate of $\mathcal{I}_{331}$): For $\mathcal{I}_{332}$, if $(i,j,k) \notin S_1$, then $(j,i,k) \notin S_1$. Then, we can use the switching trick in $\mathcal{I}_{332}$. We first split $\mathcal{I}_{332}$ into two parts:

$$\begin{aligned}
\mathcal{I}_{332} &= \frac{p\kappa_1}{N} \sum_{i,j,k \notin S_1} \phi(\|\bar{x}_j - \bar{x}_i\|_p) \left(\frac{v_j^k - \bar{v}_j^k}{\bar{T}_j} - \frac{v_i^k - \bar{v}_i^k}{\bar{T}_i} \right) (v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2} \\
&= \frac{p\kappa_1}{N} \sum_{i,j,k \notin S_1} \phi(\|\bar{x}_j - \bar{x}_i\|_p) \left(\frac{v_j^k - \bar{v}_j^k}{\bar{T}_j} - \frac{v_i^k - \bar{v}_i^k}{\bar{T}_j} \right) (v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2} \\
&+ \frac{p\kappa_1}{N} \sum_{i,j,k \notin S_1} \phi(\|\bar{x}_j - \bar{x}_i\|_p) \left(\frac{v_i^k - \bar{v}_i^k}{\bar{T}_j} - \frac{v_i^k - \bar{v}_i^k}{\bar{T}_i} \right) (v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2} \\
&= \frac{p\kappa_1}{N} \sum_{i,j,k \notin S_1} \phi(\|\bar{x}_j - \bar{x}_i\|_p) \left((v_j^k - \bar{v}_j^k) - (v_i^k - \bar{v}_i^k) \right) \frac{1}{\bar{T}_j} (v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2} \\
&+ \frac{p\kappa_1}{N} \sum_{i,j,k \notin S_1} \phi(\|\bar{x}_j - \bar{x}_i\|_p) \left(\frac{v_i^k - \bar{v}_i^k}{\bar{T}_j} - \frac{v_i^k - \bar{v}_i^k}{\bar{T}_i} \right) (v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2}.
\end{aligned} \tag{B.12}$$

If $(i,j,k) \notin S_1$, then $(j,i,k) \notin S_1$. Thus, we can switch i and j to yield

$$\begin{aligned}
\mathcal{I}_{332} &= \frac{p\kappa_1}{N} \sum_{i,j,k \notin S_1} \phi(\|\bar{x}_j - \bar{x}_i\|_p) \left((v_j^k - \bar{v}_j^k) - (v_i^k - \bar{v}_i^k) \right) \frac{1}{\bar{T}_j} (v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2} \\
&+ \frac{p\kappa_1}{N} \sum_{i,j,k \notin S_1} \phi(\|\bar{x}_j - \bar{x}_i\|_p) \left(\frac{v_i^k - \bar{v}_i^k}{\bar{T}_j} - \frac{v_i^k - \bar{v}_i^k}{\bar{T}_i} \right) (v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2} \\
&= \frac{p\kappa_1}{2N} \sum_{i,j,k \notin S_1} \phi(\|\bar{x}_j - \bar{x}_i\|_p) \left((v_j^k - \bar{v}_j^k) - (v_i^k - \bar{v}_i^k) \right) \\
&\times \left(\frac{1}{\bar{T}_j} (v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2} - \frac{1}{\bar{T}_i} (v_j^k - \bar{v}_j^k)|v_j^k - \bar{v}_j^k|^{p-2} \right) \\
&+ \frac{p\kappa_1}{N} \sum_{i,j,k \notin S_1} \phi(\|\bar{x}_j - \bar{x}_i\|_p) \left(\frac{1}{\bar{T}_j} - \frac{1}{\bar{T}_i} \right) |v_i^k - \bar{v}_i^k|^p.
\end{aligned} \tag{B.13}$$

It follows from the simple calculations that if $(i,j,k) \notin S$, then the term

$$\left((v_j^k - \bar{v}_j^k) - (v_i^k - \bar{v}_i^k) \right) \left(\frac{1}{\bar{T}_j} (v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2} - \frac{1}{\bar{T}_i} (v_j^k - \bar{v}_j^k)|v_j^k - \bar{v}_j^k|^{p-2} \right) < 0.$$

By the same argument as in $\mathcal{I}_{331}$, we have

$$
\begin{aligned}
\mathcal{I}_{332} &= \frac{p\kappa_1}{2N} \sum_{i,j,k\notin S} \phi(\|\bar{x}_j - \bar{x}_i\|_p) \left((v_j^k - \bar{v}_j^k) - (v_i^k - \bar{v}_i^k)\right) \\
&\quad \times \left(\frac{1}{\bar{T}_j}(v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2} - \frac{1}{\bar{T}_i}(v_j^k - \bar{v}_j^k)|v_j^k - \bar{v}_j^k|^{p-2}\right) \\
&\quad + \frac{p\kappa_1}{N} \sum_{i,j,k\notin S} \phi(\|\bar{x}_j - \bar{x}_i\|_p) \left(\frac{1}{\bar{T}_j} - \frac{1}{\bar{T}_i}\right) |v_i^k - \bar{v}_i^k|^p \\
&\le \frac{p\kappa_1}{2NT_M} \sum_{i,j,k\notin S} \phi(\mathcal{D}_{\bar{X}}) \left((v_j^k - \bar{v}_j^k) - (v_i^k - \bar{v}_i^k)\right) \\
&\quad \times \left((v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2} - (v_j^k - \bar{v}_j^k)|v_j^k - \bar{v}_j^k|^{p-2}\right) \\
&\quad + \frac{p\kappa_1\mathcal{D}_T}{NT_m^2} \sum_{i,j,k\notin S} \phi(0)|v_i^k - \bar{v}_i^k|^p.
\end{aligned} \tag{B.14}
$$

Since the first part of the estimate of $\mathcal{I}_{332}$ and the estimate of $\mathcal{I}_{331}$ coincide, we have

$$
\begin{aligned}
\mathcal{I}_{33} &= \mathcal{I}_{331} + \mathcal{I}_{332} \\
&\le \frac{p\kappa_1}{2NT_M} \sum_{i,j,k\in S} \phi(\mathcal{D}_{\bar{X}}) \left(v_j^k - \bar{v}_j^k - (v_i^k - \bar{v}_i^k)\right) \\
&\quad \times \left((v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2} - (v_j^k - \bar{v}_j^k)|v_j^k - \bar{v}_j^k|^{p-2}\right) \\
&\quad + \frac{p\kappa_1}{2NT_M} \sum_{i,j,k\notin S} \phi(\mathcal{D}_{\bar{X}}) \left((v_j^k - \bar{v}_j^k) - (v_i^k - \bar{v}_i^k)\right) \\
&\quad \times \left((v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2} - (v_j^k - \bar{v}_j^k)|v_j^k - \bar{v}_j^k|^{p-2}\right) \\
&\quad + \frac{p\kappa_1\mathcal{D}_T}{NT_m^2} \sum_{i,j,k\notin S} \phi(0)|v_i^k - \bar{v}_i^k|^p \\
&= \frac{p\kappa_1}{2NT_M} \sum_{i,j,k} \phi(\mathcal{D}_{\bar{X}}) \left(v_j^k - \bar{v}_j^k - (v_i^k - \bar{v}_i^k)\right) \\
&\quad \times \left((v_i^k - \bar{v}_i^k)|v_i^k - \bar{v}_i^k|^{p-2} - (v_j^k - \bar{v}_j^k)|v_j^k - \bar{v}_j^k|^{p-2}\right) \\
&\quad + \frac{p\kappa_1\mathcal{D}_T}{NT_m^2} \sum_{i,j,k\notin S} \phi(0)|v_i^k - \bar{v}_i^k|^p \\
&\le -\frac{2p\kappa_1 N}{T_M}\phi(\mathcal{D}_{\bar{X}})\|V - \bar{V}\|_p^p + \frac{p\kappa_1\mathcal{D}_T N}{T_m^2}\phi(0)\|V - \bar{V}\|_p^p \\
&\le -\frac{2p\kappa_1 N}{T_M^0}\phi(\mathcal{D}_{\bar{X}})\|V - \bar{V}\|_p^p + \frac{p\kappa_1\mathcal{D}_T N}{(T_m^0)^2}\phi(0)\|V - \bar{V}\|_p^p,
\end{aligned} \tag{B.15}
$$

where we used the zero mean velocity condition.

In (B.3), we combine all estimates (B.4), (B.6), and (B.15) to obtain

$$\begin{aligned} N\frac{d}{dt}\|V-\bar V\|_p^p &\le 4p\kappa_1 N\mathcal{D}_V\frac{\|\phi\|_{Lip}}{T_m^0}\|X-\bar X\|_p\|V-\bar V\|_p^{p-1} \\ &\quad + \frac{2pN\kappa_1\phi(0)}{(T_m^0)^2}\|T-\bar T\|_p\|V-\bar V\|_p^{p-1}\mathcal{D}_V \\ &\quad - \frac{2p\kappa_1 N}{T_M^0}\phi(\mathcal{D}_{\bar X})\|V-\bar V\|_p^p + \frac{p\kappa_1\mathcal{D}_T N}{(T_m^0)^2}\phi(0)\|V-\bar V\|_p^p. \end{aligned}$$

This yields

$$\begin{aligned} \frac{d}{dt}\|V-\bar V\|_p &\le 4\kappa_1\mathcal{D}_V\frac{\|\phi\|_{Lip}}{T_m^0}\|X-\bar X\|_p + \frac{2\kappa_1\phi(0)}{(T_m^0)^2}\|T-\bar T\|_p\mathcal{D}_V \\ &\quad - \left[\frac{2\kappa_1}{T_M^0}\phi(\mathcal{D}_{\bar X}) - \frac{\kappa_1\mathcal{D}_T}{(T_m^0)^2}\phi(0)\right]\|V-\bar V\|_p. \end{aligned}$$

B.2. *Temperature variations in ℓ_p norm.* It follows from (1.1) that we have

$$\begin{aligned} \frac{d}{dt}(T_i-\bar T_i) &= \frac{\kappa_2}{N}\sum_{j=1}^N\left[\zeta(\|x_i-x_j\|_p)\left(\frac{1}{T_i}-\frac{1}{T_j}\right) - \zeta(\|\bar x_i-\bar x_j\|_p)\left(\frac{1}{\bar T_i}-\frac{1}{\bar T_j}\right)\right] \\ &= \frac{\kappa_2}{N}\sum_{j=1}^N\left(\zeta(\|x_i-x_j\|_p)-\zeta(\|\bar x_i-\bar x_j\|_p)\right)\left(\frac{1}{T_i}-\frac{1}{T_j}\right) \\ &\quad + \frac{\kappa_2}{N}\sum_{j=1}^N\zeta(\|\bar x_i-\bar x_j\|_p)\left[\left(\frac{1}{T_i}-\frac{1}{\bar T_i}\right)-\left(\frac{1}{T_j}-\frac{1}{\bar T_j}\right)\right]. \end{aligned} \tag{B.16}$$

We multiply the $2(T_i-\bar T_i)$ in both sides of (B.16) to obtain

$$\begin{aligned} \frac{d}{dt}|T_i-\bar T_i|^2 &= \frac{2\kappa_2}{N}\sum_{j=1}^N\left(\zeta(\|x_i-x_j\|_p)-\zeta(\|\bar x_i-\bar x_j\|_p)\right)\left(\frac{1}{T_i}-\frac{1}{T_j}\right)(T_i-\bar T_i) \\ &\quad + \frac{2\kappa_2}{N}\sum_{j=1}^N\zeta(\|\bar x_i-\bar x_j\|_p)\left[\left(\frac{1}{T_i}-\frac{1}{\bar T_i}\right)-\left(\frac{1}{T_j}-\frac{1}{\bar T_j}\right)\right](T_i-\bar T_i). \end{aligned} \tag{B.17}$$

Next, we use (B.17) to obtain

$$\begin{aligned} &\frac{d}{dt}|T_i-\bar T_i|^p \\ &\quad = \frac{d}{dt}\left(|T_i-\bar T_i|^2\right)^{\frac{p}{2}} \\ &\quad = \frac{p}{2}|T_i-\bar T_i|^{p-2}\frac{d}{dt}|T_i-\bar T_i|^2 \\ &\quad = \frac{p\kappa_2}{N}\sum_{j=1}^N\left(\zeta(\|x_i-x_j\|_p)-\zeta(\|\bar x_i-\bar x_j\|_p)\right)\left(\frac{1}{T_i}-\frac{1}{T_j}\right)(T_i-\bar T_i)|T_i-\bar T_i|^{p-2} \\ &\quad + \frac{p\kappa_2}{N}\sum_{j=1}^N\zeta(\|\bar x_i-\bar x_j\|_p)\left[\left(\frac{1}{T_i}-\frac{1}{\bar T_i}\right)-\left(\frac{1}{T_j}-\frac{1}{\bar T_j}\right)\right](T_i-\bar T_i)|T_i-\bar T_i|^{p-2}. \end{aligned} \tag{B.18}$$

We sum (B.18) over all $i = 1, \cdots N$ to get

$$\begin{aligned}
N\frac{d}{dt}\|T-\bar{T}\|_p^p &= \sum_i \frac{d}{dt}|T_i-\bar{T}_i|^p \\
&= \frac{p\kappa_2}{N}\sum_{i,j=1}^N \left(\zeta(\|x_i-x_j\|_p)-\zeta(\|\bar{x}_i-\bar{x}_j\|_p)\right)\left(\frac{1}{T_i}-\frac{1}{T_j}\right)(T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2} \\
&+ \frac{p\kappa_2}{N}\sum_{i,j=1}^N \zeta(\|\bar{x}_i-\bar{x}_j\|_p)\left[\left(\frac{1}{T_i}-\frac{1}{\bar{T}_i}\right)-\left(\frac{1}{T_j}-\frac{1}{\bar{T}_j}\right)\right](T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2} \\
&=: \mathcal{I}_{41}+\mathcal{I}_{42}.
\end{aligned} \tag{B.19}$$

• Case D.1 (Estimate of $\mathcal{I}_{41}$): We again use the Lipschitz continuity of $\zeta(\cdot)$:

$$\begin{aligned}
&|\zeta(\|x_j-x_i\|_p)-\zeta(\|\bar{x}_j-\bar{x}_i\|_p)| \\
&\quad \le \|\zeta\|_{Lip}\,|\|x_j-x_i\|_p-\|\bar{x}_j-\bar{x}_i\|_p| \le \|\zeta\|_{Lip}(\|x_j-\bar{x}_j\|_p+\|x_i-\bar{x}_i\|_p)
\end{aligned}$$

to obtain

$$\begin{aligned}
\mathcal{I}_{41} &= \frac{p\kappa_2}{N}\sum_{i,j=1}^N \left(\zeta(\|x_i-x_j\|_p)-\zeta(\|\bar{x}_i-\bar{x}_j\|_p)\right)\left(\frac{1}{T_i}-\frac{1}{T_j}\right)(T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2} \\
&\le \frac{p\kappa_2}{N}\sum_{i,j=1}^N |\zeta(\|x_i-x_j\|_p)-\zeta(\|\bar{x}_i-\bar{x}_j\|_p)|\left|\frac{1}{T_i}-\frac{1}{T_j}\right||T_i-\bar{T}_i|^{p-1} \\
&\le \frac{p\kappa_2}{N}\sum_{i,j=1}^N |\zeta(\|x_i-x_j\|_p)-\zeta(\|\bar{x}_i-\bar{x}_j\|_p)|\,\frac{|T_i-T_j|}{T_m^2}|T_i-\bar{T}_i|^{p-1} \\
&\le \frac{p\kappa_2\mathcal{D}_T}{NT_m^2}\sum_{i,j=1}^N \|\zeta\|_{Lip}(\|x_j-\bar{x}_j\|_p+\|x_i-\bar{x}_i\|_p)|T_i-\bar{T}_i|^{p-1} \\
&\le 2p\kappa_2\|\zeta\|_{Lip}N\frac{\mathcal{D}_T}{T_m^2}\|X-\bar{X}\|_p\|T-\bar{T}\|_p^{p-1}.
\end{aligned} \tag{B.20}$$

• Case D.2 (Estimate of $\mathcal{I}_{42}$): In this case, we use a standard trick of switching $i \leftrightarrow j$, but we need to be careful because the equation

$$\begin{aligned}
&\left[\left(\frac{1}{T_i}-\frac{1}{\bar{T}_i}\right)-\left(\frac{1}{T_j}-\frac{1}{\bar{T}_j}\right)\right]\left((T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}-(T_j-\bar{T}_j)|T_j-\bar{T}_j|^{p-2}\right) \\
&\quad = -\left(\frac{T_i-\bar{T}_i}{T_i\bar{T}_i}-\frac{T_j-\bar{T}_j}{T_j\bar{T}_j}\right)\left((T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}-(T_j-\bar{T}_j)|T_j-\bar{T}_j|^{p-2}\right)
\end{aligned} \tag{B.21}$$

can be positive. Thus, as before, we define the set

$$S_2 := \left\{(i,j) : (|T_i-\bar{T}_i|-|T_j-\bar{T}_j|)(T_i\bar{T}_i-T_j\bar{T}_j) < 0\right\}$$

so that for $(i,j) \in S_2$, we can guarantee that (B.21) is negative. Again, we split $\mathcal{I}_{42}$ into two parts:

$$\begin{aligned}
\mathcal{I}_{42} &= \frac{p\kappa_2}{N}\sum_{i,j=1}^{N}\zeta(\|\bar{x}_i-\bar{x}_j\|_p)\left[\left(\frac{1}{T_i}-\frac{1}{\bar{T}_i}\right)-\left(\frac{1}{T_j}-\frac{1}{\bar{T}_j}\right)\right](T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}\\
&= \frac{p\kappa_2}{N}\sum_{i,j\in S_2}\zeta(\|\bar{x}_i-\bar{x}_j\|_p)\left[\left(\frac{1}{T_i}-\frac{1}{\bar{T}_i}\right)-\left(\frac{1}{T_j}-\frac{1}{\bar{T}_j}\right)\right](T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}\\
&+ \frac{p\kappa_2}{N}\sum_{i,j\notin S_2}\zeta(\|\bar{x}_i-\bar{x}_j\|_p)\left[\left(\frac{1}{T_i}-\frac{1}{\bar{T}_i}\right)-\left(\frac{1}{T_j}-\frac{1}{\bar{T}_j}\right)\right](T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}\\
&=: \mathcal{I}_{421}+\mathcal{I}_{422}.
\end{aligned} \tag{B.22}$$

◇ Case D.2.1 (Estimate of $\mathcal{I}_{421}$): By the same argument as $\mathcal{I}_{41}$, the estimates of $\mathcal{I}_{421}$ are as follows:

$$\begin{aligned}
\mathcal{I}_{421} &= \frac{p\kappa_2}{N}\sum_{i,j\in S_2}\zeta(\|\bar{x}_i-\bar{x}_j\|_p)\left[\left(\frac{1}{T_i}-\frac{1}{\bar{T}_i}\right)-\left(\frac{1}{T_j}-\frac{1}{\bar{T}_j}\right)\right](T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}\\
&\le \frac{p\kappa_2}{2N}\sum_{i,j\in S_2}\zeta(\|\bar{x}_i-\bar{x}_j\|_p)\left[\left(\frac{1}{T_i}-\frac{1}{\bar{T}_i}\right)-\left(\frac{1}{T_j}-\frac{1}{\bar{T}_j}\right)\right]\\
&\times\left((T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}-(T_j-\bar{T}_j)|T_j-\bar{T}_j|^{p-2}\right)\\
&= -\frac{p\kappa_2}{2N}\sum_{i,j\in S_2}\zeta(\|\bar{x}_i-\bar{x}_j\|_p)\left(\frac{T_i-\bar{T}_i}{T_i\bar{T}_i}-\frac{T_j-\bar{T}_j}{T_j\bar{T}_j}\right)\\
&\times\left((T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}-(T_j-\bar{T}_j)|T_j-\bar{T}_j|^{p-2}\right)\\
&\le -\frac{p\kappa_2\zeta(\mathcal{D}_x)}{2NT_M^2}\sum_{i,j\in S}\left((T_i-\bar{T}_i)-(T_j-\bar{T}_j)\right)\\
&\times\left((T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}-(T_j-\bar{T}_j)|T_j-\bar{T}_j|^{p-2}\right).
\end{aligned} \tag{B.23}$$

◇ Case D.2.2 (Estimate of $\mathcal{I}_{422}$): By direct calculation, we have

$$\begin{aligned}
\mathcal{I}_{422} &= \frac{p\kappa_2}{N}\sum_{i,j\notin S_2}\zeta(\|\bar{x}_i-\bar{x}_j\|_p)\left[\left(\frac{1}{T_i}-\frac{1}{\bar{T}_i}\right)-\left(\frac{1}{T_j}-\frac{1}{\bar{T}_j}\right)\right](T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}\\
&= -\frac{p\kappa_2}{N}\sum_{i,j\notin S_2}\zeta(\|\bar{x}_i-\bar{x}_j\|_p)\left(\frac{T_i-\bar{T}_i}{T_i\bar{T}_i}-\frac{T_j-\bar{T}_j}{T_j\bar{T}_j}\right)(T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}\\
&= -\frac{p\kappa_2}{N}\sum_{i,j\notin S_2}\zeta(\|\bar{x}_i-\bar{x}_j\|_p)\left(\frac{T_i-\bar{T}_i}{T_i\bar{T}_i}-\frac{T_i-\bar{T}_i}{T_j\bar{T}_j}\right)(T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}\\
&- \frac{p\kappa_2}{N}\sum_{i,j\notin S_2}\zeta(\|\bar{x}_i-\bar{x}_j\|_p)\left(\frac{T_i-\bar{T}_i}{T_j\bar{T}_j}-\frac{T_j-\bar{T}_j}{T_j\bar{T}_j}\right)(T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}\\
&=: \mathcal{I}_{4221}+\mathcal{I}_{4222}.
\end{aligned} \tag{B.24}$$

The estimate for $\mathcal{I}_{4221}$ can be done as follows:

$$\begin{aligned}\mathcal{I}_{4221} &\le \frac{p\kappa_2\zeta(0)}{N}\sum_{i,j\notin S_2}\left|\frac{1}{T_i\bar{T}_i}-\frac{1}{T_j\bar{T}_j}\right||T_i-\bar{T}_i|^p\\ &\le \frac{p\kappa_2\zeta(0)}{N}\sum_{i,j\notin S_2}\left(\frac{T_j\bar{T}_j-T_i\bar{T}_i}{T_i\bar{T}_iT_j\bar{T}_j}\right)|T_i-\bar{T}_i|^p\\ &\le \frac{p\kappa_2\zeta(0)}{N}\sum_{i,j\notin S_2}\left(\frac{\mathcal{D}_T\mathcal{D}_{\bar{T}}+T_M\mathcal{D}_T+T_M\mathcal{D}_{\bar{T}}}{T_m^4}\right)|T_i-\bar{T}_i|^p\\ &\le pN\kappa_2\zeta(0)\left(\frac{\mathcal{D}_T\mathcal{D}_{\bar{T}}+T_M\mathcal{D}_T+T_M\mathcal{D}_{\bar{T}}}{T_m^4}\right)\|T-\bar{T}\|_p^p,\end{aligned}\tag{B.25}$$

where we used the following relation:

$$\begin{aligned}|T_j\bar{T}_j-T_i\bar{T}_i| &= |(T_j-T_i)(\bar{T}_j-\bar{T}_i)+T_i\bar{T}_j+T_j\bar{T}_i-2T_i\bar{T}_i|\\ &\le |(T_j-T_i)(\bar{T}_j-\bar{T}_i)|+|T_i(\bar{T}_j-\bar{T}_i)|+|(T_j-T_i)\bar{T}_i|\\ &\le \mathcal{D}_T\mathcal{D}_{\bar{T}}+T_M\mathcal{D}_{\bar{T}}+T_M\mathcal{D}_T.\end{aligned}$$

The estimate for $\mathcal{I}_{4222}$ can be done as before:

$$\begin{aligned}\mathcal{I}_{4222} \le -\frac{p\kappa_2\zeta(\mathcal{D}_{\bar{X}})}{2NT_M^2}&\sum_{i,j\notin S_2}\big((T_i-\bar{T}_i)-(T_j-\bar{T}_j)\big)\\ &\times\big((T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}-(T_j-\bar{T}_j)|T_j-\bar{T}_j|^{p-2}\big).\end{aligned}\tag{B.26}$$

Finally, we combine all estimatees (B.22), (B.23), (B.24), (B.25), and (B.26) to obtain

$$\begin{aligned}\mathcal{I}_{42} &\le -\frac{p\kappa_2\zeta(\mathcal{D}_{\bar{X}})}{2NT_M^2}\sum_{i,j\in S_2}\big((T_i-\bar{T}_i)-(T_j-\bar{T}_j)\big)\\ &\quad\times\big((T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}-(T_j-\bar{T}_j)|T_j-\bar{T}_j|^{p-2}\big)\\ &\quad-\frac{p\kappa_2\zeta(\mathcal{D}_{\bar{X}})}{2NT_M^2}\sum_{i,j\notin S_2}\big((T_i-\bar{T}_i)-(T_j-\bar{T}_j)\big)\\ &\quad\times\big((T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}-(T_j-\bar{T}_j)|T_j-\bar{T}_j|^{p-2}\big)\\ &\quad+pN\kappa_2\zeta(0)\left(\frac{\mathcal{D}_T\mathcal{D}_{\bar{T}}+T_M\mathcal{D}_T+T_M\mathcal{D}_{\bar{T}}}{T_m^4}\right)\|T-\bar{T}\|_p^p\\ &\le -\frac{p\kappa_2\zeta(\mathcal{D}_{\bar{X}})}{2NT_M^2}\sum_{i,j}\big((T_i-\bar{T}_i)-(T_j-\bar{T}_j)\big)\\ &\quad\times\big((T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}-(T_j-\bar{T}_j)|T_j-\bar{T}_j|^{p-2}\big)\\ &\quad+pN\kappa_2\zeta(0)\left(\frac{\mathcal{D}_T\mathcal{D}_{\bar{T}}+T_M\mathcal{D}_T+T_M\mathcal{D}_{\bar{T}}}{T_m^4}\right)\|T-\bar{T}\|_p^p\\ &\le +pN\kappa_2\zeta(0)\left(\frac{\mathcal{D}_T\mathcal{D}_{\bar{T}}+T_M\mathcal{D}_T+T_M\mathcal{D}_{\bar{T}}}{T_m^4}\right)\|T-\bar{T}\|_p^p.\end{aligned}\tag{B.27}$$

Here we use the fact that

$$\big((T_i-\bar{T}_i)-(T_j-\bar{T}_j)\big)\times\big((T_i-\bar{T}_i)|T_i-\bar{T}_i|^{p-2}-(T_j-\bar{T}_j)|T_j-\bar{T}_j|^{p-2}\big)\ge 0.$$

Now, we collect the estimates (B.20) and (B.27) to obtain

$$\begin{aligned} N\frac{d}{dt}\|T-\bar{T}\|_p^p &= \mathcal{I}_{41}+\mathcal{I}_{42} \\ &\le 2pN\kappa_2\|\zeta\|_{Lip}\frac{\mathcal{D}_T}{T_m^2}\|X-\bar{X}\|_p\|T-\bar{T}\|_p^{p-1} \\ &\quad + pN\kappa_2\zeta(0)\left(\frac{\mathcal{D}_T\mathcal{D}_{\bar{T}}+T_M\mathcal{D}_T+T_M\mathcal{D}_{\bar{T}}}{T_m^4}\right)\|T-\bar{T}\|_p^p. \end{aligned}$$

This yields

$$\begin{aligned} \frac{d}{dt}\|T-\bar{T}\|_p &\le 2\kappa_2\|\zeta\|_{Lip}\frac{\mathcal{D}_T}{T_m^2}\|X-\bar{X}\|_p \\ &\quad + \kappa_2\zeta(0)\left(\frac{\mathcal{D}_T\mathcal{D}_{\bar{T}}+T_M\mathcal{D}_T+T_M\mathcal{D}_{\bar{T}}}{T_m^4}\right)\|T-\bar{T}\|_p. \end{aligned} \tag{B.28}$$

Finally, we use (B.28), $T_m \ge T_m^0$, and $T_M \le T_M^0$ to obtain the desired estimate.

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