EXPONENTIAL GROWTH OF SPACES
WITHOUT CONJUGATE POINTS

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§1. INTRODUCTION

An \(n\)-dimensional polyhedral space is a length space \(M\) (with intrinsic metric) triangulated into \(n\)-simplexes with smooth Riemannian metrics. In the definitions below, we assume that the triangulation is fixed.

The boundary of \(M\) is the union of the \((n-1)\)-simplexes of the triangulation that are adjacent to only one \((n-1)\)-simplex.

As usual, a geodesic in \(M\) is a naturally parametrized locally shortest curve defined on an interval. We say that \(M\) has no conjugate points if any two points in the universal covering space \(\tilde{M}\) of \(M\) are joined by a unique geodesic.

We say that the volume entropy of \(\tilde{M}\) is positive if the volume of metric balls in \(\tilde{M}\) has at least exponential growth.

Now, we state the main result of this paper.

**Theorem 1.** Let \(M\) be an \(n\)-dimensional compact polyhedral space without boundary and without conjugate points. If the triangulation of \(M\) contains three \(n\)-simplexes with a common \((n-1)\)-face, then the volume entropy of \(\tilde{M}\) is positive.

**Corollary 1.** Under the assumptions of Theorem 1, the fundamental group \(\pi_1(M)\) of \(M\) has at least exponential growth.

§2. GEODESICS IN \(M\): \(G\), \(SM\), ETC.

A geodesic in \(M\) is complete if it is defined on the entire real line \(\mathbb{R}\). A geodesic is generic if it intersects no \((n-2)\)-simplexes and intersects \((n-1)\)-simplexes transversally.

We denote by \(G\) the set of complete generic geodesics in \(M\) and consider the geodesic flow transformation (GFT)

\[ \varphi_t : G \to G, \quad \varphi_t \gamma(s) = \gamma(t+s). \]

We observe that a generic geodesic \(\gamma : [a,b] \to M\) with \(\gamma(b) \notin M^{n-1}\) is uniquely continued beyond \(b\). If \(\gamma(b)\) belongs to a common \((n-1)\)-face \(F\) of \(n\)-simplexes \(\Delta_1, \ldots, \Delta_l\), then \(\gamma\) is continued beyond \(b\) in \(l-1\) different ways (uniquely into each of the remaining \(l-1\) simplexes by the rule “the angle of incidence is equal to the angle of reflection”).

The tangent space \(T_xM\) of \(M\) at a point \(x \in M\) is the tangent cone of \(M\) at \(x\). If \(x \in M \setminus M^{n-1}\), then \(T_xM\) is isometric to \(\mathbb{R}^n\).
If $x$ belongs to an $(n-1)$-simplex $F$ that is a common $(n-1)$-face of $n$-simplexes $\Delta_1, \ldots, \Delta_i$, then $T_xM$ is the union of the half-spaces $T_x\Delta_i$ with common boundary hyperplane $T_x F$. We say that the vectors in $T_x\Delta_i \setminus T_x F$ go in the direction of $\Delta_i$.

For each unit vector $e \in T_xM \setminus T_x F$, there exists a geodesic $\gamma$ with $\gamma'(0) = e$. Observe that if we “glue together” two geodesic segments with initial velocity vectors making an angle of $\pi$, then we obtain a geodesic segment.

In what follows, we consider only tangent vectors at points in $M \setminus M^{n-2}$. For $x \in M$, we let $S_x \subset T_xM$ be the set of unit tangent vectors in $T_xM$. For any set $K \subset M$, we define $SK := \bigcup_{x \in K} S_x$. Thus, $SM$ is the space of all unit tangent vectors of $M$.

§3. THE LIOUVILLE MEASURE $\mu_L$

Let $M$ be a polyhedral space. A canonical measure $\mu_L$ on the space $SM$ is defined in a standard way as the product of two measures: the Riemannian volume on $M$ and Lebesgue measure $\lambda_x$ on the unit $(n-1)$-sphere $S_x$, $x \in M$. This measure is called Liouville measure.

Let $A = \{ \gamma : [a, b] \to M \}$ be a set of generic geodesics. We define

$$A'(t) := \{ \gamma'(t) \mid \gamma \in A \} \subset SM, \quad t \in [a, b].$$

The multiset $A'(t)$ is the pair $(A'(t), 1_{A'(t)})$, where

$$1_{A'(t)} : SM \to \{0\} \cup \mathbb{N}$$

is the “indicator function” acting by the rule

$$e \mapsto \# \{ \gamma \in A \mid \gamma'(t) = e \}.$$

The measure $\mu_L(A'(t))$ of $A'(t)$ is the integral of $1_{A'(t)}$:

$$\mu_L(A'(t)) := \int_{SM} 1_{A'(t)} d\mu_L.$$

If for any two geodesics $\gamma_1, \gamma_2 \in A$ we have $\gamma_1'(t) \neq \gamma_2'(t)$, then $A'(t)$ may be regarded as the usual set $A'(t)$, and $1_{A'(t)}$ is the usual indicator function of $A'(t)$ (equal to 1 on $A'(t)$ and vanishing outside $A'(t)$): $1_{A'(t)} = 1_{A'(t)}$. Obviously, in this case, we have

$$\mu_L(A'(t)) = \mu_L(A'(t)).$$

We say that two generic geodesics defined on the segment $[a, b]$ have one combinatorial type if they traverse the simplexes in the same succession. (In particular, they pass the branchings in the same way.)

**Claim.** Let $A = \{ \gamma : [a, b] \to M \}$ be a set of generic geodesics of one combinatorial type. Then

$$\mu_L(A'(a)) = \mu_L(A'(b)),$$

i.e., the “transformation of the geodesic flow along $A$” preserves Liouville measure.

To see this, it suffices to prove that Liouville measure is preserved in a neighborhood of a point of any $(n-1)$-simplex adjacent exactly to two $n$-simplexes. We give a precise statement.

**Lemma 1.** Let $F$ be a common $(n-1)$-face of two $n$-simplexes $\Delta_1$ and $\Delta_2$. Let $U \subset \Delta_1 \cup \Delta_2$ be an open ball with center in $F$, and let $B = \{ \gamma : [0, c] \to U \}$ be a set of generic geodesics. Then

$$\mu_L(B'(0)) = \mu_L(B'(c)).$$
Proof. The set $B$ is a countable union of sets $B_k$ such that each geodesic in $B_k$ intersects $F$ at an angle greater than $1/k$. It suffices to prove that $\mu_L(B_k(0)) = \mu_L(B_k(\epsilon))$ for each $k$. Therefore, we may assume that each geodesic in $B$ intersects $F$ once.

Let $dx$ be the volume element on $F$. We consider the measure $\mu_F$ on $SF$ with the density

$$d\mu_F(v) = |\cos \alpha(v)| d\lambda_\epsilon(v) dx,$$

where $\alpha(v)$ is the angle between $v$ and the normal to $F$ (see [4] and [4, Chapter 6]). For $C \subset SF$, $\mu_F(C)$ is equal to the flux across $C$ of the vector field generating the geodesic flow on $U$.

Let $t_\gamma$ be the value of the parameter for which $\gamma \in B$ intersects $F$: $\{t_\gamma\} = \gamma^{-1}(F)$. The mapping

$$B'(0) \to SF \times [0, c], \quad \gamma'(0) \mapsto (\gamma'(t_\gamma), t_\gamma),$$

determines coordinates $(v, t)$ on $B'(0)$.

Since the Liouville measure $\mu_L$ is preserved within one simplex, the density of $\mu_L$ in the coordinates $(v, t)$ has the form $d\mu_L(v, t) = d\mu_F(v) dt$ (see [4, Chapter 6]). Under the passage from $B'(0)$ to $B'(c)$, the vector $v \in SF$ changes to the opposite one, and the parameter along the flow changes by a constant. Hence, the Liouville measure $\mu_L$ is preserved.

\begin{proposition}
Let $A = \{\gamma : [a, b] \to M\}$ be a set of generic geodesics. Then

\begin{equation}
\mu_L(A^1(a)) = \mu_L(A^1(b)). \tag{3.2}
\end{equation}

\end{proposition}

Proof. 1) First, suppose that the geodesics in $A$ have one combinatorial type. Then the velocity vector at $t$ uniquely determines a geodesic in $A$, so that $1_{A^1(t)} = 1_{A^1(t)}$ for each $t \in [a, b]$, whence

$$\mu_L(A^1(a)) = \mu_L(A^1(a)) \equiv \mu_L(A^1(b)) = \mu_L(A^1(b)).$$

2) In the general case, $A$ splits into countably many subsets $A_k$ in each of which the geodesics have one combinatorial type. For each $k$, we have

$$\mu_L(A^1_k(a)) = \mu_L(A^1_k(b)).$$

Summing these relations, we obtain (3.2). \hfill \Box

\section{Proof of Theorem 1}

Let $X$ be an $(n - 1)$-simplex that is a common hyperface of $n$-simplexes $\Upsilon_1, \ldots, \Upsilon_d$, where $d \geq 3$.

\textbf{Notation.} Let $\gamma : (a, b) \to M$ be a generic geodesic. Suppose that $\gamma$ passes from an $n$-simplex $\Delta_1$ to an $n$-simplex $\Delta_2$ and intersects their common $(n - 1)$-face at a point $\gamma(c)$, $c \in (a, b)$. We define

$$\gamma_+(c) := \gamma'(c) \in T_{\gamma(c)} \Delta_2 \subset T_{\gamma(c)} M \quad \text{and} \quad \gamma_-(c) := \gamma'(-c) \in T_{\gamma(c)} \Delta_1 \subset T_{\gamma(c)} M,$$

where $\gamma(t) = \gamma(-t)$.

A tangent vector $v$ at a point of an $(n - 1)$-simplex $F$ is said to be almost orthogonal to $F$ if $v$ makes an angle less than $\pi/10$ with one of the normals of $F$. We denote by $(\tilde{M}, \tilde{\rho})$ a universal cover of $(M, \rho)$, where $\tilde{\rho}$ is the lifting of the metric $\rho$.

Let $\Omega \subset X$ be a (sufficiently small) region. We denote by $\tilde{\Omega}$ the preimage of $\Omega$ in $\tilde{M}$.

We recall that $\tilde{M}$ is isometric to the quotient space $\hat{M}/\Gamma$, where $\Gamma$ is a subgroup of the group of isometries of $\hat{M}$ isomorphic to $\pi_1(M)$. We denote by $\tilde{M}_0$ a fundamental domain in $\tilde{M}$.
4.1. A special set. For convenience, we introduce a certain subset \( G_{\text{reg}} \subset G \) within which all geodesics extend uniquely in both directions, except the branching on \( \Omega \).

For this, we introduce the following structure. For each \((n-1)\)-simplex \( F \), we cyclically order the \( n \)-simplexes adjacent to \( F \). We say that a generic geodesic \( \gamma \) is regular if for every \( x \in \gamma^{-1}(\mathcal{M}^{n-1}) \) one of the following two conditions is fulfilled:

1) \( \gamma(x) \in \Omega \), and the vector \( \gamma_-'(x) \) is almost orthogonal to \( X \);
2) the point \( \gamma(x) \) belongs to a common \((n-1)\)-face \( F \) of \( n \)-simplexes \( \Delta_1, \ldots, \Delta_l \), and

\[
\gamma_-'(x) \in T_{\gamma(x)} \Delta_i, \quad \gamma_+(x) \in T_{\gamma(x)} \Delta_{i+1}.
\]

(As usual, we set \( \Delta_{l+1} := \Delta_1 \).

We denote by \( G_{\text{reg}} \) the set of complete regular geodesics.

**Remark 1.**
1) For “almost every” unit tangent vector \( v \in \mathcal{M} \), each regular geodesic with initial velocity vector \( v \) can be continued to a generic regular complete geodesic. (Cf. the Appendix.)
2) The set \( G_{\text{reg}} \) is GFT-invariant.

For \( V \subset \mathcal{M} \), we define

\[
G(V) := \{ \gamma \in G : \gamma'(0) \in V \}, \quad G_{\text{reg}}(V) := \{ \gamma \in G_{\text{reg}} : \gamma'(0) \in V \}.
\]

4.2. A special measure on \( G_{\text{reg}} \). In the Appendix it is proved that there is a measure \( m \) on \( G \) such that \( G_{\text{reg}} \) has full measure in \( G \) and the following properties 1)–3) are fulfilled (an invariant measure on \( G \) satisfying property 2) is described in detail in \([\Pi]\)):

1) the measure \( m \) is GFT-invariant;
2) if \( V \subset \mathcal{M} \) is a measurable subset, then \( m(G_{\text{reg}}(V)) = \mu_L(V) \);
3) let \( j \in \{1, \ldots, d\} \), and let \( \Psi = \{ \gamma : (-\infty, x_\gamma) \to \mathcal{M} \} \) be a set of one-sided regular geodesics such that for each \( \gamma \in \Psi \) we have \( \gamma(x_\gamma) \in \Omega \), the vector \( \gamma_-'(x_\gamma) \) is almost orthogonal to \( X \), and \( \gamma_+(x_\gamma) \in T_{\gamma(x_\gamma)} Y_j \). Furthermore, we assume that none of the geodesics in \( \Psi \) is a continuation of another geodesic. For \( i \in \{1, \ldots, d\} \setminus \{j\} \), let \( \Psi_i \) denote the set of continuations of all geodesics in \( \Psi \) to the simplex \( Y_i \), and further, in all possible ways, up to complete regular geodesics in \( G_{\text{reg}} \), i.e.,

\[
\Psi_i := \{ \gamma \in G_{\text{reg}} : \gamma|_{(-\infty, x_\gamma)} \in \Psi \& \gamma_+(x_\gamma) \in T_{\gamma(x_\gamma)} Y_i \}.
\]

Thus, \( \Psi := \bigcup_i \Psi_i \) is the set of all possible continuations of geodesics in \( \Psi \) to complete geodesics in \( G_{\text{reg}} \). Then

\[
m(\Psi_i) = \frac{m(\Psi)}{d - 1}, \quad i \in \{1, \ldots, d\} \setminus \{j\}.
\]

4.3. Estimating the measure of a set of geodesics. We denote by \( \tilde{G} \) the space of complete generic geodesics \( \gamma : \mathbb{R} \to \mathcal{M} \), we denote by \( \tilde{G}_{\text{reg}} \) the preimage of the set \( G_{\text{reg}} \) in \( \tilde{G} \), and we let \( \pi : \tilde{G}_{\text{reg}} \to G_{\text{reg}} \) be the projection map, which is a covering.

We say that a geodesic \( \gamma \) in \( \mathcal{M} \) is regular if \( \gamma \) is a lifting of a regular geodesic.

Let \( \tilde{m} \) denote the lifting of the measure \( m \) to \( \tilde{G}_{\text{reg}} \) under the covering \( \pi \).

For a set \( A \) of regular generic geodesics in \( \mathcal{M} \) that are defined on segments, we denote by \( \tilde{m}(A) \) the \( \tilde{m} \)-measure of the set of all possible continuations of the geodesics in \( A \) to complete geodesics in \( \tilde{G}_{\text{reg}} \). If for \( \gamma \in A \) we have \( \gamma(a) \in \mathcal{M}_0 \), then the projection of \( A \) to the space of geodesics in \( \mathcal{M} \) is injective, and the measure \( \tilde{\mu}_L(A^l(a)) \) of the multiset \( A^l(a) \) is well defined as the measure of the projection. The following lemma relates the measure \( \tilde{m}(A) \) of some set \( A \) of geodesics defined on a segment to \( \tilde{\mu}_L(A^l) \).
Lemma 2. Let \( l \in \mathbb{N} \), and let \( \mathcal{A} = \{ \gamma : [a, b] \to \tilde{\mathfrak{M}} \} \) be a set of regular generic geodesics with initial points in a fundamental domain \( \mathfrak{M}_0 \). Suppose that for every \( \gamma \in \mathcal{A} \) there exist parameters \( t_1(\gamma) < \cdots < t_i(\gamma) \in (a, b) \) such that \( \gamma(t_i(\gamma)) \in \Omega \) for each \( i \), and \( \gamma'_i(t_i(\gamma)) \) is almost orthogonal to \( \mathfrak{X} \) and goes in the direction of \( \tilde{\mathfrak{X}}_1 \). Then

\[
(\ast) \quad \bar{m}(\mathcal{A}) \leq \frac{\tilde{\mu}_L(A^i(a))}{(d-1)^l} = \frac{\bar{\mu}_L(A^i(b))}{(d-1)^l}.
\]

(The identity \( \bar{\mu}_L(A^i(a)) = \tilde{\mu}_L(A^i(b)) \) was proved earlier.)

Proof. At a point \( t_0(\gamma) \), the geodesic \( \gamma \in \mathcal{A} \) passes to one of the simplices \( \tilde{\mathfrak{X}}_2, \ldots, \tilde{\mathfrak{X}}_d \).
With each geodesic \( \gamma \in \mathcal{A} \) we associate a sequence \( \{i_1, \ldots, i_l\} \) so that \( \gamma \) passes to \( \tilde{\mathfrak{X}}_{i_k} \) at the point \( t_k(\gamma) \). Since the number of such sequences is finite, \( \mathcal{A} \) splits into finitely many subsets \( \mathcal{A}_i \) such that for each \( i \) the geodesics in \( \mathcal{A}_i \) determine one and the same sequence. \( \Box \)

Claim. The subsets \( \mathcal{A}_i \) satisfy the required inequality, namely,

\[
(\ast_i) \quad \bar{m}(\mathcal{A}_i) \leq \frac{\tilde{\mu}_L(A^i(a))}{(d-1)^l}
\]

for each \( i \).

We denote by \( \mathcal{A}_{i,s} \) the set consisting of all complete geodesics in \( \tilde{\mathfrak{G}}_{\text{reg}} \) that are continuations of the restrictions of the geodesics \( \gamma \in \mathcal{A}_i \) to the interval \( (a, t_s(\gamma)) \) (this interval is specific for each geodesic \( \gamma \)):

\[
\mathcal{A}_{i,s} := \{ \gamma|(a, t_s(\gamma)) : \gamma \in \tilde{\mathfrak{G}}_{\text{reg}} \}.
\]

Since the initial points of the geodesics in \( \mathcal{A} \) lie in \( \mathfrak{M}_0 \), it follows that the projection \( \pi|_\mathcal{A} : \mathcal{A} \to \mathfrak{G}_{\text{reg}} \) is an injective mapping, whence \( \bar{m}(\pi(\mathcal{A})) = \bar{m}(\mathcal{A}) \). Therefore, by property 3) of the measure \( \bar{m} \), where we put \( \Psi_i := \mathcal{A}_{i,s+1} \) and \( \Psi := \mathcal{A}_{i,s} \), we have

\[
\bar{m}(\mathcal{A}_{i,s+1}) \leq \frac{\bar{m}(\mathcal{A}_{i,s})}{d-1}, \quad s = 1, \ldots, l.
\]

Property 2) of the measure \( \bar{m} \) implies that

\[
\bar{m}(\mathcal{A}_{i,1}) \leq \tilde{\mu}_L(A^i(a)).
\]

Combining the above \( l+1 \) inequalities, we obtain

\[
\bar{m}(\mathcal{A}_{i,l}) \leq \tilde{\mu}_L(A^i(a)).
\]

Since \( \bar{m}(\mathcal{A}_i) \leq \bar{m}(\mathcal{A}_{i,l}) \), we arrive at \((\ast_i)\).

Summing the inequalities \((\ast_i)\), \( i \in \mathbb{N} \), we obtain

\[
\bar{m}(\mathcal{A}) \leq \frac{1}{(d-1)^l} \sum_i \tilde{\mu}_L(A^i(a)).
\]

Since \( \mathcal{A} = \bigcup_i \mathcal{A}_i \), we have

\[
\sum_i 1_{A^i(a)}(e) \leq 1_{A^1(a)}(e), \quad e \in \mathfrak{M}.
\]

Integration over \( \mathfrak{M} \) yields

\[
\sum_i \tilde{\mu}_L(A^i(a)) \leq \tilde{\mu}_L(A^1(a)),
\]

which proves \((\ast)\).
4.4. The set $A_\varepsilon$ of often-branching geodesics. The next step of the proof is construction of the set of sufficiently-often-branching geodesics, to which we apply Lemma 2.

We describe an auxiliary subset of $G_{\text{reg}}$. Let $\Theta \subset \Omega$ be the set of all unit tangent vectors at points in $\Omega$ that go in the direction of $T_1$ and are almost orthogonal to $X$. We assume that the closure of $\Theta$ lies strictly inside $X$.

There exists $\delta_0 > 0$ such that for every $e \in \Theta$ there exists a unique geodesic $\gamma_e : [0, \delta_0] \to \mathfrak{M}$ with $\gamma(0) = e$, i.e., $\gamma_e$ does not intersect the $(n - 1)$-simplexes on which branching is possible. We define

$$g_0 := \{ -\gamma'_e(t) \mid e \in \Theta, 0 < t < \delta_0 \} \subset \mathfrak{M}$$

and set

$$G_0 := G_{\text{reg}}(g_0).$$

We have $\mu_L(g_0) \neq 0$. Then property 2) of the measure $m$ and Remark 2 imply that $m(G_0) \neq 0$. For $\gamma \in G_{\text{reg}}$ and $k > 0$, we let $N_\gamma(k)$ be the number of connected components of the set $[0, k] \cap (\gamma)^{-1}(g_0)$, i.e., $N_\gamma(k)$ is the number of crossings of $\gamma$ into $G_0$ under the action of the geodesic flow transformation within the time $k$.

The set $g_0$ is chosen so that the duration of the stay of a geodesic in the set $G_0$ under the action of the geodesic flow be at least $\delta_0$. Then, by the definition of $N_\gamma(k)$, we have

$$(4.1) \quad N_\gamma(k)\delta_0 \geq \int_0^k 1_{g_0}(\varphi_s \gamma)ds.$$

**Lemma 3.** For any $\varepsilon > 0$, there is a set $A_\varepsilon \subset G_{\text{reg}}$ and positive numbers $N$ and $\delta$ with the following properties:

1) $m(A_\varepsilon) \neq 0$;
2) $\text{diam}(A_\varepsilon(0)) < \varepsilon$. (Here and below, we use the natural notation $A_\varepsilon(0) := \{ \gamma(0) \mid \gamma \in A_\varepsilon \} \subset \mathfrak{M}$, etc.);
3) $N_\gamma(k) > \delta k$ for all $k > N$ and all $\gamma \in A_\varepsilon$.

**Proof.** We use a general result for measure spaces, the proof of which involves the ergodic theorem. \hfill $\Box$

**Proposition 2.** Suppose $D$ is a space with a measure $m$ and $\{T_t\}$ is a one-parametric semigroup of measure-preserving transformations of $D$, where $t$ takes nonnegative real values and $T_{s+t} = T_s \cdot T_t$. Furthermore, suppose that

$$D \times \mathbb{R}_{\geq 0} \to D, \quad (x, t) \mapsto T_t(x)$$

is a measurable mapping.

Then for every set $\Delta \subset D$ of nonzero finite measure there is a set $D_0 \subset D$ of nonzero measure such that for the points in $D_0$ the average duration of the stay in $\Delta$ under the action of the transformation $T_t$ during the time $t$ is uniformly bounded away from zero as $t \to \infty$. This means that there exist $s_0 > 0$ and $\varepsilon_0 > 0$ such that for any $x \in D_0$ and any $s > s_0$ we have

$$\frac{1}{s} \int_0^s 1_{\Delta}(T_t(x))dt > \varepsilon_0.$$

**Proof.** Let $\text{Ave}_\Delta(x)$ denote the average value of $1_{\Delta}$ on the trajectory of the geodesic flow with initial value $x$:

$$\text{Ave}_\Delta(x) := \lim_{t \to \infty} \frac{1}{t} \int_0^t 1_{\Delta}(T_s(x))ds.$$

Applying the ergodic theorem, we obtain

$$\int_D \text{Ave}_\Delta(x)dm(x) = \int_D 1_{\Delta}(x)dm(x) = m(\Delta) > 0.$$
Consequently, there is $\varepsilon_2 > 0$ and a set $D_1 \subset D$ of nonzero measure such that $\text{Ave}_\Delta(x) > \varepsilon_2$ for each $x \in D_1$. Then there exist numbers $s_0 > 0$ and $\varepsilon_0 > 0$ and a set $D_0 \subset D_1$ such that for all $x \in D_0$ and $s > s_0$ we have
\[
\frac{1}{s} \int_0^s 1_{\Delta}(T_ix)dt > \varepsilon_0.
\]

Since the measure $m$ is invariant under the geodesic flow on $G_{\text{reg}}$, and $m(G_0) \neq 0$, we can apply Proposition 2 to the case where

\[(D, m, \Delta, T_t) := (G_{\text{reg}}, m, G_0, \varphi_t).
\]

Thus, there is a set $A_0 \subset G_{\text{reg}}$ of nonzero measure and positive numbers $s_0$ and $\varepsilon_0$ such that
\[
\frac{1}{T} \int_0^T 1_{\varphi_s\gamma}(\varphi_s\gamma)ds > \varepsilon_0
\]
for any $\gamma \in A_0$ and any $T > s_0$. Applying inequality (4.1) and letting $\delta = \varepsilon_0/\delta_0$ and $N = s_0$, we see that $N_\gamma(k) > \delta k$ for each geodesic $\gamma \in A_0$ and each $k > N$. Moreover, passing if necessary to a subset of $A_0$ of nonzero measure, we may assume that $\text{diam}(A_0(0)) < \varepsilon$. Lemma 3 is proved.

**Proposition 3.** If the volume entropy of $\overline{M}$ is zero, then for every $\varepsilon > 0$ there are two complete generic geodesics $\gamma_1$ and $\gamma_2$ in $\overline{M}$ and a number $t_0 > 1$ with the following properties:

1) $\overline{\rho}(\gamma_1(0), \gamma_2(0)) < \varepsilon$;

2) $\gamma_1(t_0) = \gamma_2(t_0) \in \Omega$;

3) $\gamma_{1+}(t_0) = \gamma_{2+}(t_0)$, and the vector $\gamma_{1+}(t_0)$ is almost orthogonal to $\overline{X}$ and goes in the direction of $\overline{T}_j$ for some $j \in \{1, \ldots, d\}$;

4) at the point $t_0$, the geodesics $\gamma_1$ and $\gamma_2$ pass to $\overline{T}_j$ from distinct $n$-simplexes adjacent to $\overline{X}$, i.e., $\gamma_{1+}(t_0) \neq \gamma_{2+}(t_0)$.

**Proof.** Applying Lemma 3, we obtain a set $A_\varepsilon \subset G_{\text{reg}}$ and numbers $N$ and $\delta$.

We fix a fundamental domain $M_0 \subset \overline{M}$ and a point $x_0 \in M_0$. Let $SB_r(x_0)$ denote the set of unit tangent vectors at the points of the ball $B_r(x_0)$, and let $A$ be the set of the geodesics in $G_{\text{reg}}$ that are the liftings of the geodesics in $A_\varepsilon$ with initial points in $M_0$.

For $k > N$, we define
\[
A_k := \{\gamma|_{[0, k]} : \gamma \in A\}.
\]
Assertion 3) of Lemma 3 implies that $N_\gamma(k) \geq \delta k$ for $\gamma \in A$. We apply Lemma 2 to the set $A_k$, letting $l := \lceil \delta k \rceil + 1$. Lemma 3 and the inequality $\overline{m}(A) \leq \overline{m}(A_k)$ show that
\[
(4.2)
\]
We assume that $\text{diam} M_0 < 1$. Then the function $1_{A_k^+(v)}$ vanishes outside $SB_{k+1}(x_0)$, and (4.2) takes the form
\[
\frac{1}{(d-1)^{\delta k}} \int_{SB_{k+1}(x_0)} 1_{A_k^+(v)}d\overline{m} \geq \overline{m}(A).
\]
We define
\[
f(k) := \max\{1_{A_k^+(v)} : v \in SB_{k+1}(x_0)\}.
\]
Since the volume entropy of \( \tilde{\mathfrak{M}} \) is zero, we have \( \tilde{\mu}_L(\mathbb{S}B_{k+1}(x_0)) = o((d - 1)^{\delta k}) \). Then
\[
\frac{o((d - 1)^{\delta k})f(k)}{(d - 1)^{\delta k}} \geq \tilde{\mathfrak{m}}(A) =: c > 0,
\]
whence
\[
f(k) \geq c \frac{(d - 1)^{\delta k}}{o((d - 1)^{\delta k})}.
\]

Estimate (4.3) implies the existence of \( k_1 > N \) such that \( f(k_1) \geq f(N) + 1 \). This means that \( A \) contains \( f(k_1) \) geodesics that are distinct on the interval \((0, k_1)\) and have equal velocity vectors at \( k_1 \). At least two of them are distinct on the interval \((N, k_1)\). (Indeed, otherwise there were \( f(k_1) \) geodesics that are distinct on the interval \((0, N)\) and have equal velocity vectors at \( N \). This would mean that \( f(N) \geq f(k_1) \), which contradicts our choice of \( k_1 \).) Consequently, these geodesics meet at a point \( t_0 \in (N, k_1) \). These two geodesics and the parameter \( t_0 \) satisfy all the requirements of the proposition. \( \square \)

4.5. End of the proof of Theorem 1. Suppose that the volume entropy of the universal cover \( \tilde{\mathfrak{M}} \) is zero. The remaining part of the proof proceeds in \( \tilde{\mathfrak{M}} \). For short, the distance function in \( \tilde{\mathfrak{M}} \) is denoted by \( | \cdot | \). For \( a, b \in \tilde{\mathfrak{M}} \), we denote by \([a, b]\) a unique geodesic segment joining \( a \) and \( b \). Since \( \tilde{\mathfrak{M}} \) contains no conjugate points, the initial velocity vector \( e_{ab} \) of \([a, b]\) depends continuously on \( a \) and \( b \). Furthermore, since \([a, b]\) is a shortest curve, the length of \([a, b]\) is equal to \(|ab|\).

Applying Proposition 3 to a sufficiently small \( \varepsilon \), we obtain geodesics \( \gamma_1 \) and \( \gamma_2 \) and a number \( t_0 \). We define
\[
\begin{align*}
c &:= \gamma_1(t_0) = \gamma_2(t_0), \\
a &:= \gamma_1(0), \quad b := \gamma_2(0), \\
d &:= \gamma_1(t_0 + 1/2) = \gamma_2(t_0 + 1/2).
\end{align*}
\]
In this notation, the geodesic segments \([a, d]\) and \([b, d]\) are the restrictions of the geodesics \( \gamma_1 \) and \( \gamma_2 \) to the interval \([0, t_0 + 1/2]\). We have \(|ab| < \varepsilon\).

Our purpose is to join \( a \) and \( d \) by a polygonal line of length less than \(|ad|\), which is a contradiction. More precisely, we find a point \( f^* \in [a, b] \) such that \(|df^*|\) is less than \(|da|\) by a constant depending only on the geometry of \( \tilde{\mathfrak{M}} \). Now, we can make \(|af^*|\) arbitrarily less than this constant by taking \( \varepsilon \) sufficiently small.

Consider geodesic segments \([c, f]\), where \( f \in [a, b]\). The space \( \mathbb{S}_c \) is the union of \((n - 1)\)-hemispheres glued together along their common \((n - 2)\)-sphere. By assertion 4) of Proposition 3, the vectors \( e_{ca} \) and \( e_{ab} \) lie in distinct \((n - 1)\)-hemispheres. By continuity, there is \( f^* \in [a, b] \) such that \( e_{cf^*} \in T_f\tilde{\Omega}, \) i.e., \( e_{cf^*} \) is tangent to an \((n - 1)\)-simplex. There is \( e \in [c, f^*] \) such that \(|ce| = 1/2\). By compactness, there is a number \( q > 0 \) depending only on \( \tilde{\mathfrak{M}} \) and \( \tilde{\Omega} \) and such that
\[
|dc| + |ce| - |de| > q.
\]
In more detail, we consider the set \( \mathfrak{A} \) of triplets of points \((x, y, z)\) belonging to \( \mathfrak{M}_0 \) and such that the following conditions 1)–3) are fulfilled:
1) \( x \in \tilde{\Omega} \) and \(|xy| = |xz| = 1/2|; 
2) \( e_{xy} \) is almost orthogonal to \( \tilde{X} \);
3) \( e_{xz} \) is tangent to \( \tilde{X} \).

The set \( \mathfrak{A} \) is compact. Therefore, the function \(|xy| + |xz| - |yz|\) attains a minimum, which is positive because of the absence of conjugate points (since \( x \notin [yz]\)).
So, in Proposition 3 we put $\varepsilon := q/4$. The inequality $|dc| + |ce| - |de| > q$ and the triangle inequality for $\triangle de f^*$ imply that

$$|df^*| < |dc| + |cf^*| - q.$$

By our choice of $\varepsilon$ in Proposition 3, we have $|ab| < q/4$, and $|af^*| < q/4$ because $f^* \in ab$. By the triangle inequality,

$$|cf^*| < |ac| + \frac{q}{4}.$$

Adding the last two inequalities, we obtain

$$|ac| + |dc| > |df^*| + \frac{3}{4}q.$$

Again by the triangle inequality,

$$|df^*| > |ad| - \frac{q}{4}.$$

Adding the two inequalities obtained, we get

$$|ac| + |dc| > |ad| + \frac{q}{2},$$

i.e., $a$, $c$, and $d$ do not lie on one shortest curve. Therefore, $\gamma_1$ is not a geodesic, a contradiction. Theorem 1 is proved.

§5. Appendix

5.1. Notation. Two tangent vectors at one point are opposite if they make an angle of $\pi$.

We set $\mathfrak{M}^{n-1} := \mathfrak{M}^{n-2} \setminus \mathfrak{M}^{n-2}$.

Let $\Lambda(\mathfrak{M}) \subset \mathfrak{M}$ denote the set of all unit vectors tangent to $\mathfrak{M}$ strictly inside the $(n-1)$-faces and transversal to these faces, and let $D \subset \Lambda(\mathfrak{M}) \times \Lambda(\mathfrak{M})$ be the set of all pairs of opposite vectors.

5.2. Definitions. 1) A measurable function

$$p : D \rightarrow [0, 1]$$

is called a transition probability function if it possesses the following property: if $v_1 \in \Lambda$ is a tangent vector at a point of an $(n-1)$-face to which $m$ faces of dimension $n$ are adjacent, and $v_2, \ldots, v_m$ are all the vectors opposite to the vector $v_1$, then

$$p(v_1, v_2) + \cdots + p(v_1, v_m) = 1 = p(v_2, v_1) + \cdots + p(v_m, v_1).$$

2) We say that $p$ is single-valued on a subset $D_0 \subset D \subset \Lambda \times \Lambda$ if $p(D_0) \subset \{0, 1\}$.

3) We say that a geodesic $\gamma$ obeys $p$ if for each point $\gamma(c)$ of transversal intersection with an $(n-1)$-face we have $p(\gamma^{-}_c(c), \gamma^{+}_c(c)) = 0$.

**Proposition 4.** Let $D_0 \subset D$ be the subset consisting of all pairs of vectors such that the angle that they make with the corresponding $(n-1)$-faces is less than some positive constant $\theta$, and of all pairs of tangent vectors at the points lying in the $\tau$-neighborhood of the $(n-2)$-skeleton for some $\tau > 0$.

Suppose that $p : D \rightarrow [0, 1]$ is a transition function single-valued on $D_0$.

Then the set of unit vectors $v$ for which there exists a nongeneric geodesic with the initial velocity vector $v$ and obeying the transition function $p$ has zero Liouville measure in $\mathfrak{M}$.
This is similar to the corresponding fact of the theory of billiard dynamical systems, and we do not present a complete proof here. To prove this fact it suffices to estimate the measure of the \( \varepsilon \)-neighborhood of the tangent vectors of \((n - 1)\)-faces starting at the points of \((n - 2)\)-faces (this measure is \( O(\varepsilon^2) \)) and the duration of nongeneric geodesics in this neighborhood (it is at least \( \text{const} \cdot \varepsilon \)).

**Theorem 2.** Let \( p \) be as in Proposition 4. Then there exists a measure \( m_p \) on \( G \) with the following properties:

1. The measure \( m_p \) is GTF-invariant.
2. If \( V \subset \mathcal{M} \) is a measurable subset, then \( m_p(G(V)) = \mu_L(V) \).
3. Let \( \Psi = \{ \gamma : (-\infty, \infty) \to \mathcal{M} \} \) be a set of one-sided geodesics none of which is a continuation of another geodesic, where \( \gamma(\tau) \in \mathcal{M}^{n-1} \). Suppose that for each vector \( \gamma'_\tau(\tau) \) one opposite vector \( \gamma'_\tau(\tau) \) is chosen. Furthermore, suppose that for all \( \gamma \in \Psi \) we have

\[
p(\gamma'_\tau(\tau), \gamma'_\tau(\tau)) = \text{const} = p' \cdot m_p(\Psi) \cdot m_p(\Psi_+),
\]

where \( \Psi_+ \) is the set of all possible continuations of the geodesics \( \gamma \in \Psi \) to complete generic geodesics that have the chosen velocity vectors \( \gamma'_\tau(\tau) \) at the points \( \tau \), and \( \Psi \) is the set of all possible continuations of the geodesics in \( \Psi \) to complete generic geodesics.

**Proof.** Suppose that \( A \subset G \) is an open subset. We define

\[
m_p(A) = \int_{\mathcal{M}} h^A(v) d\mu_L(v),
\]

where \( h^A \) is the function with values in \([0, 1]\) and defined \( \mu_L \)-almost everywhere on \( \mathcal{M} \) as follows. Let \( v \in \mathcal{S}(\mathcal{M} \setminus \mathcal{M}^{n-1}) \) be a vector such that every complete geodesic \( \gamma \) in \( \mathcal{M} \) with \( \gamma'(0) = v \) and obeying the transition function \( p \) is generic. (See Proposition 4.) We fix some lifting of \( v \) to \( \widetilde{\mathcal{M}} \), which we also denote by \( v \), and consider the set \( G_v \subset G \) of all generic geodesics with initial velocity vector \( v \). The support (the union of the images of geodesics) \( T_v \) of \( G_v \) is a tree with an oriented distinguished edge \( e_0 \).

The function \( p \) (more precisely, a lifting of \( p \) to \( \widetilde{\mathcal{M}} \)) is defined at the points of intersection of the geodesics in \( G_v \) with \( \mathcal{M}^{n-1} \). Therefore, for all paths in \( T_v \) passing through \( e_0 \) in the given direction there are probabilities \( p(w, w') \) of the passage from an edge \( w \subset T_v \) to the subsequent edge \( w' \). For a path \( W = w_1 \cdots w_k \subset T_v \) containing \( e_0 \), let \( \mathcal{P}_W \subset G_v \) be the set of all paths containing \( W \). We define

\[
m_{p,v}(\mathcal{P}_W) := p(w_1, w_2) \cdots p(w_{k-1}, w_k).
\]

The properties of \( p \) imply that \( m_{p,v} \) is canonically extended to a probability measure on the space \( G_v \) of paths in \( T_v \); this extension is also denoted by \( m_{p,v} \).

Every geodesic \( \gamma \in A \) with \( \gamma'(0) = v \) is lifted to \( T_v \) canonically as one of the paths described above. Let \( A_v \subset G_v \) be the set of such liftings for all corresponding geodesics in \( A \). We set \( h^A(v) = m_{p,v}(A_v) \).

We check the properties of the measure \( m_p \).

1. Let \( t > 0 \), let \( \Delta_1 \) and \( \Delta_2 \) be two neighboring \( n \)-simplices, and let \( A \) be the set of the geodesics \( \gamma \) with \( \gamma(0) \in \Delta_1 \) and \( \gamma(t) \in \Delta_2 \). We assume that the geodesics in \( A \) intersect \( \mathcal{M}^{n-1} \) once. If \( \gamma \in A \) and \( \nu_t := \gamma'(t) \), then \( h^{A}(\nu_0) = h^{\nu_1}(\nu_t) \). Now, since the
Liouville measure is GFT-invariant, we obtain
\[ \int_{S\mathbb{R}} h^A(v)d\mu_L(v) = \int_{S\Delta_1} h^A(v)d\mu_L(v) \]
\[ = \int_{S\Delta_2} h^{\varphi_1 A}(v)d\mu_L(v) \]
\[ = \int_{S\mathbb{R}} h^{\varphi_1 A}(v)d\mu_L(v), \]
whence
\[ m_p(A) = m_p(\varphi_1 A). \]

In the general case, we prove the invariance of the measure \( m_p \) by splitting \( A \) and the parameter along the flow.

2. If \( V \subset S\mathbb{R} \), then \( h^{G(V)} \) is the indicator function of \( V \). This implies property 2) of the measure \( m_p \).

3. We may assume that \( x_\gamma > 0 \) for all \( \gamma \in \Psi \). Fixing a vector \( v \in S\mathbb{R} \), we put \( \Psi_v := \{ \Psi \cap G(\{v\}) \} \). The set \( \Psi_v \) splits into a countable number of subsets \( \Psi_v^i \) such that the point \( \gamma(x_\gamma) \) and the vector \( \gamma'_v(x_\gamma) \) are the same for the geodesics \( \gamma \in \Psi_v^i \). For each \( i \), by the definition of the measure \( m_{p,v} \), we have
\[ h^i(\Psi_v^i) = p'h^i(\Psi_v^i). \]

Since none of the geodesics in \( \Psi \) is a continuation of another geodesic, it follows that the sets \( (\Psi_v^i)' \) are disjoint, whence
\[ h^{\Psi^i} = p'h^{\Psi^i}. \]
This implies property 3) of \( m_p \).

In order to apply this theorem in our case (see Subsection 4.2), it suffices to let the function \( p \) be equal to 0, 1, and 1/(d − 1) at the corresponding points.

REFERENCES


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