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UNIQUENESS THEOREM AND SINGULAR SPECTRUM IN THE FRIEDRICHS MODEL NEAR A SINGULAR POINT

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Abstract. A uniqueness theorem is proved for a class of analytic functions with positive imaginary part that admit representation in a special form. This theorem imposes some restrictions on the character of decay of these functions in the vicinity of their zeros. As an application, the density of the point spectrum and the singular continuous spectrum are described for selfadjoint operators in the Friedrichs model near a singular point.

§1. Statement of the problem

On the domain formed by the functions $u(t) \in L_2(\mathbb{R})$ such that $t^2u(t) \in L_2(\mathbb{R})$, we consider a family of selfadjoint operators $A_m, m > 0$, given by the formula

$$A_m = |t|^m \cdot + (\cdot, \varphi)\varphi. \tag{1.1}$$

Here $\varphi \in L_2(\mathbb{R})$ and t is an independent variable. The action of these operators can be written as follows:

$$\big(A_m u\big)(t) = |t|^m \cdot u(t) + \varphi(t) \int_{\mathbb{R}} u(x)\overline{\varphi(x)}\, dx. \tag{1.2}$$

The function φ is assumed to satisfy the smoothness condition

$$|\varphi(t+h) - \varphi(t)| \le \omega\big(|h|\big), \quad |h| \le 1, \tag{1.3}$$

where the function $\omega(t)$ (the modulus of continuity of φ) is monotone and satisfies the Dini conditions

$$\omega(t) \downarrow 0 \text{ as } t \downarrow 0 \quad \text{and} \quad \int_0^1 \frac{\omega(t)}{t}\, dt < \infty. \tag{1.4}$$

The absolutely continuous spectrum of the operators A_m coincides with the interval $[0; +\infty)$. We are interested in the behavior of the singular spectrum of A_m. Note that by the singular spectrum we mean the union of the point spectrum and the singular continuous spectrum. The structure of $\sigma_{\rm sing}(S_1)$ (the singular spectrum of the operator $S_1 = t \cdot + (\cdot, \varphi)\varphi$) has been studied in detail in [1]–[9]. In particular, in the papers [7], [9] it was shown that for this operator there is an exact condition of finiteness of the singular spectrum. Namely, if $\omega(t) = O(\sqrt{t})$ as $t \to 0$, then $\sigma_{\rm sing}(S_1)$ is trivial, that is, it consists of at most finitely many eigenvalues of finite multiplicity (the singular continuous spectrum is absent). On the other hand, if $\limsup \omega(t)/\sqrt{t} = +\infty$ as $t \to 0^+$, then examples can be constructed to show that a nontrivial singular spectrum arises; in particular, the eigenvalues of S_1 have an accumulation point. Note that, for the first time, the true appearance of a nontrivial singular spectrum in the Friedrichs model for

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the operator S_1 was shown by Pavlov and Petras [2] in 1970. In the same paper, the Lebesgue measure of a δ-neighborhood of the singular spectrum was estimated in the case where $\omega(t) = t^\alpha$, $\alpha \in (0;1)$. The simple change of variables $|t|^m = x$ allows us to show that outside any neighborhood of the point $t = 0$ the structure of the spectrum $\sigma_{\mathrm{sing}}(A_m)$ is identical with that for the operator S_1. This result is explained by the smoothness of the above change of variables outside any neighborhood of the origin, and also by the local character of the main results of [1]–[9] relating to the structure of $\sigma_{\mathrm{sing}}(S_1)$. Here by locality we mean the following. Suppose that conditions (1.3) and (1.4) and also some additional conditions on the function φ are fulfilled only in some interval $(c;d) \subset \mathbb{R}$; then the main results of [1]–[9] about the structure of $\sigma_{\mathrm{sing}}(S_1)$ remain true in any closed subinterval $\Delta \subset (c;d)$.

However, it turns out that in a neighborhood of the origin the behavior of $\sigma_{\mathrm{sing}}(A_m)$ is quite different. For instance, it is easy to show that for $m \le 1$ the spectrum in question is always absolutely continuous in a neighborhood of zero on the interval $[0;+\infty)$. At the same time, for $m > 3$ examples show that a nontrivial singular spectrum of the operator A_m may arise in a neighborhood of the origin for any function $\omega(t)$ that satisfies (1.4) and the natural additional constraint of semiadditivity: $\omega(t_1+t_2) \le \omega(t_1) + \omega(t_2)$. The requirement $\omega(t) = O(\sqrt{t})$ as $t \to 0$ for $m = 2$ (see [10]), as well as the requirement $\omega(t) = O(t)$ as $t \to 0$ for $m = 3$ (see [11]), are sharp conditions guaranteeing the absolute continuity of the spectrum near the point zero. In a neighborhood of the origin we can still use the change of variables $|t|^m = x$ mentioned above, but, since, e.g., $\big(|t|^m\big)'|_0 = 0$ for $m > 1$, this change is not smooth (i.e., it is not a diffeomorphism) near zero. In this sense, the zero point is a singular point of the operators $A_m, m > 1$, so that it needs special inspection. Observe that the origin is also a boundary point of the continuous spectrum of A_m, which coincides with the interval $[0;+\infty)$. Thus, for $m > 1$ the operator A_m can have a nontrivial singular spectrum in a neighborhood of zero. This set has Lebesgue measure zero, and the problem of describing its structure is meaningful. According to Beurling [12], the structure of a set $E \subseteq \mathbb{R}$ of Lebesgue measure zero can be judged from the asymptotic behavior as $\delta \to 0^+$ of the Lebesgue measure of the δ-neighborhood $E^\delta := \big\{x \in \mathbb{R} : \operatorname{dist}(x,E) < \delta\big\}$. Thus, instead of E we investigate a family of sets $E^\delta, \delta > 0$, that have positive measure and satisfy the conditions

$$E^{\delta_1} \supseteq E^{\delta_2} \text{ if } \delta_1 \ge \delta_2, \quad \text{and} \quad \bigcap_{\delta>0} E^\delta = E. \tag{1.5}$$

Under a more general approach, the family of sets E^δ satisfying (1.5) can be constructed in various ways. For example, suppose that some nonnegative function $\varepsilon_x(\delta)$ is given for all sufficiently small values of $\delta > 0$ at every point $x \in E$. Consider the set

$$E_\varepsilon^\delta := \bigcup_{x\in E} \big(x - \varepsilon_x(\delta), x + \varepsilon_x(\delta)\big). \tag{1.6}$$

If $\varepsilon_x(\delta)$ tends to zero monotonically as $\delta \to 0$, then, obviously, conditions (1.5) are fulfilled. We can study the asymptotic behavior as $\delta \to 0^+$ of the function

$$\psi_\varepsilon(\delta) := \sigma\text{-meas} E_\varepsilon^\delta \equiv \int_{E_\varepsilon^\delta} d\sigma(t), \tag{1.7}$$

where $d\sigma$ is some nonnegative measure. It turns out that, in many cases, further information about the structure of the set E can be obtained by choosing the function $\varepsilon_x(\delta)$ in a specific way and by varying the measure $d\sigma$. In the case of the Friedrichs model the singular spectrum of A_m embeds in the set of the real zeros of some analytic function M with positive imaginary part. Accordingly, the choice of the function $\varepsilon_x(\delta)$ is determined by the character of decay of the function M in a neighborhood of its zeros. The theorems that establish a relationship between the character of decay of analytic functions near their zeros and the "thickness" of the set of zeros are usually called uniqueness theorems. In §2 we show how the description of the singular spectrum is reduced to the investigation of the set of real zeros of an analytic function. In §3 we prove some uniqueness theorems for analytic functions of that class. In §4 we restrict our attention to the operator A_2 and apply the uniqueness theorems obtained to describe the behavior of the singular spectrum near the origin depending on the smoothness of the function φ. Extension of these results to the case of an arbitrary $m > 1$ does not seem to be complicated.

§2. Analytic function $M(z)$ and the singular spectrum

One of the possible approaches to the investigation of the point spectrum and the singular continuous spectrum in the Friedrichs model is based on the study of some properties of analytic functions with positive imaginary part. The point is that a certain analytic function $M(z)$ can be defined in such a way that the singular spectrum of the perturbed operator embeds in the set of real roots of $M(z)$.

For $z \in \mathbb{C} \setminus [0; +\infty)$, we define the analytic function $M(z)$ as follows:

$$M(z) = 1 + \int_{-\infty}^{+\infty} \frac{|\varphi^2(t)|}{|t|^m - z}\, dt. \tag{2.1}$$

The proof of the following propositions for $m = 2$ can be found in [10].

Proposition 2.1. *If conditions* (1.3) *and* (1.4) *are fulfilled, then the analytic function* $M(z)$ *is defined on the complex plane with the slit* $(0; +\infty)$ *and has continuous boundary values on the edges of the slit.*

Let $M(\lambda) := M(\lambda + i0)$ for $\lambda > 0$, and let $N := \{\lambda > 0 : M(\lambda) = 0\}$ be the set of roots of $M(z)$. The set N is bounded (see [10]).

Proposition 2.2. *If* φ *satisfies* (1.3) *and* (1.4), *then the singular spectrum of the operator* A_m *defined by* (1.1) *embeds in the set* N *plus the origin, i.e.,* $\sigma_{\mathrm{sing}}(A_m) \subseteq N \cup \{0\}$.

Thus, the investigation of $\sigma_{\mathrm{sing}}(A_m)$ reduces to the description of the set of roots N. (It is not difficult to show that zero is not an eigenvalue of the operator $A_m = |t|^m \cdot +(\cdot, \varphi)\varphi$; see [10].) Thus, we need to study the behavior of the function $M(z)$ near its real roots. For this, we prove some uniqueness theorem for this function, which imposes some restrictions on the admissible structure of the roots of $M(z)$. In fact, this uniqueness theorem applies to a certain class of analytic functions. The functions of that class can be represented in a specific form. We start §3 with the description of this class of functions. Note that the behavior of the boundary functions and, in particular, their uniqueness sets were studied by many authors (see, e.g., [13]).

§3. A uniqueness theorem

Obviously, after the change of variables $|t|^m = \tau$ the function $M(z)$ takes the form

$$M(z) = 1 + \int_0^{+\infty} \frac{\psi(\tau)}{\tau - z}\, d\tau, \quad z \notin [0; +\infty), \tag{3.1}$$

where

$$\psi(\tau) = \frac{|\varphi^2(\tau^{1/m})| + |\varphi^2(-\tau^{1/m})|}{m\tau^{(m-1)/m}}. \tag{3.2}$$

In the following lemma we describe a class of analytic functions. The uniqueness theorem will be formulated for this class.

Lemma 3.1. *Suppose that*

$$f(z) = 1 + \int_0^{+\infty} \frac{d\nu(t)}{t-z}, \quad z \in \mathbb{C} \setminus [0; +\infty), \tag{3.3}$$

with a positive finite measure $d\nu(t)$:

$$d\nu(t) \geq 0, \quad \int_0^{+\infty} d\nu(t) < \infty. \tag{3.4}$$

Then the function $\big(f(z)\big)^{-1}$ *admits the representation*

$$\big(f(z)\big)^{-1} = 1 - \int_0^{+\infty} \frac{d\mu(t)}{t-z}, \tag{3.5}$$

where $d\mu(t)$ *is a positive finite measure with the following properties*:

$$\int_0^1 \frac{d\mu(t)}{t} \leq 1, \tag{3.6}$$

$$\int_0^{+\infty} \frac{y\, d\mu(t)}{t^2 + y^2} \leq 1, \quad y > 0. \tag{3.7}$$

Proof. For the function $\varphi(z) := f(z) - 1$ we can write

$$\varphi(z) = \int_{-\infty}^{+\infty} \frac{d\nu(t)}{t-z} \tag{3.8}$$

with a positive finite measure $d\nu(t)$ (in our case $d\nu(t) = 0$ for $t < 0$); in accordance with [3], [4], this means that $\varphi(z)$ is an analytic R_0-function. We recall that a function φ belongs to the class R_0 if and only if

$$\operatorname{Im} \varphi(z) \geq 0 \text{ for } \operatorname{Im} z > 0, \quad \varphi(iy) \to 0 \text{ as } y \to +\infty, \tag{3.9}$$

$$\lim_{y \to +\infty} y \operatorname{Im} \varphi(iy) < \infty. \tag{3.10}$$

If this is the case, then the following relation is easily verified:

$$\lim_{y \to +\infty} y \operatorname{Im} \varphi(iy) = \int_{-\infty}^{+\infty} d\nu(t). \tag{3.11}$$

Observe that $f(z)$ has no zeros in $\mathbb{C} \setminus [0; +\infty)$. Indeed, if $\operatorname{Im} z_0 > 0$ and $f(z_0) = 0$, then the maximum principle for harmonic functions shows that the function $\operatorname{Im} f(z) = \int_0^{+\infty} \frac{y}{(t-x)^2+y^2}\, d\nu(t) \geq 0$ is identically zero in $\mathbb{C}_+$. Then the spectral function $\nu(t)$ is constant, and from the integral representation we see that $f(z) = 1$ for all $z \in \mathbb{C}_+$. The case of $\mathbb{C}_-$ is treated similarly. Now, if $z = x_0 < 0$, then $f(x_0) = 1 + \int_0^{+\infty} \frac{d\nu(t)}{t-x_0} \geq 1$. At the same time, under certain smoothness conditions on $\nu(t)$ the function $f(z)$ can be continuously extended to the positive real semiaxis $(0, +\infty)$ where it can already have zeros. The study of the density of this zero set, which is a closed set of Lebesgue measure zero, is our main goal in this paper.

We check that the function

$$g(z) := 1 - 1/f(z) = \varphi(z)/\left(1 + \varphi(z)\right), \quad z \in \mathbb{C} \setminus [0; +\infty), \tag{3.12}$$

is also an analytic R_0-function. Obviously, $g(z)$ satisfies (3.9). As to condition (3.10), observe that $\operatorname{Im} g(z) = \operatorname{Im} \varphi(z)/|1 + \varphi(z)|^2$. Then, clearly,

$$\lim_{y \to +\infty} y \operatorname{Im} g(iy) = \lim_{y \to +\infty} y \frac{\operatorname{Im} \varphi(iy)}{|1 + \varphi(iy)|^2} = \lim_{y \to +\infty} y \operatorname{Im} \varphi(iy) < \infty. \tag{3.13}$$

Consequently,

$$g(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{t-z} \tag{3.14}$$

with a finite positive measure $d\mu(t)$. For $x > 0$ the function $g(-x) = 1 - 1/f(-x)$ takes real values; therefore, by the Stieltjes inversion formula, the spectral function $\mu(t)$ has no points of growth in $(-\infty; 0)$. Assuming that $\mu(t)$ is left continuous at zero, we can write

$$g(z) = \int_0^{+\infty} \frac{d\mu(t)}{t-z}. \tag{3.15}$$

Then

$$\frac{1}{f(z)} = 1 - \int_0^{+\infty} \frac{d\mu(t)}{t-z}. \tag{3.16}$$

Besides, relations (3.11) and (3.13) imply that

$$\int_0^{+\infty} d\mu(t) = \lim_{y\to+\infty} y \operatorname{Im} g(iy) = \lim_{y\to+\infty} y \operatorname{Im} \varphi(iy) = \int_0^{+\infty} d\nu(t). \tag{3.17}$$

Since $g(-x) = 1 - 1/f(-x)$ for $x > 0$, we have

$$\int_0^{+\infty} \frac{d\mu(t)}{t+x} = g(-x) = 1 - \frac{1}{1 + \int_0^{+\infty} \frac{d\nu(t)}{t+x}}. \tag{3.18}$$

Letting $x \to 0^+$, we obtain

$$\int_0^{+\infty} \frac{d\mu(t)}{t} = 1 - \frac{1}{1 + \int_0^{+\infty} \frac{d\nu(t)}{t}} \le 1. \tag{3.19}$$

Since $\operatorname{Re} \varphi(iy) = \int_0^{+\infty} \frac{t}{t^2+y^2}\, d\nu(t) \ge 0$ for $y > 0$, we obviously have

$$\operatorname{Im} g(iy) = \frac{\operatorname{Im} \varphi(iy)}{|1+\varphi(iy)|^2} \le 1. \tag{3.20}$$

It follows that

$$\int_0^{+\infty} \frac{y}{t^2+y^2}\, d\mu(t) = \operatorname{Im} g(iy) \le 1, \tag{3.21}$$

which completes the proof. □

The proof of our uniqueness theorem is based Lemma 3.1 and the following remark. As was shown in [14], if a positive locally integrable (with respect to Lebesgue measure) function $\sigma(t)$ is defined on the real axis and satisfies the condition

$$\sup_{I\subset\mathbb{R}} \left\{ \left(\frac{1}{|I|} \int_I \sigma(x)\, dx \right) \cdot \operatorname*{ess\,sup}_{x\in I} \frac{1}{\sigma(x)} \right\} < \infty, \tag{3.22}$$

where I is an arbitrary finite interval, then for the Hilbert transform $\widehat{H}$ of any function $g \in L_{1,\sigma}(\mathbb{R})$ we have the following weighted norm inequality:

$$\int_{\{|\widehat{H}g|>a\}} \sigma(t)\, dt \le \frac{C}{a} \cdot \int_{-\infty}^{+\infty} |g(t)|\sigma(t)\, dt, \quad a > 0, \tag{3.23}$$

where C is a constant independent of g and a. (From now on, C will denote various universal constants.)

In what follows we use the notation $\sigma\text{-meas} I := \int_I \sigma(x)\, dx$ for $I \subset \mathbb{R}$.

Theorem 3.1 (uniqueness theorem)**.** *Let $\sigma(t)\,dt$ be a measure on the real axis. Suppose that the weight function $\sigma(t)$ is positive, even, and monotone decreasing on the positive real semiaxis*:

$$\sigma(t)=\sigma(-t);\quad \sigma(t)\downarrow \ \text{ as } t\in(0;+\infty). \tag{3.24}$$

Also, assume that $\sigma(t)$ satisfies (3.22)*. Let $f(z)$ be an analytic function written as in* (3.3) *and* (3.4)*, i.e.,*

$$f(z)=1+\int_0^{+\infty}\frac{d\nu(t)}{t-z},\quad z\in\mathbb{C}\setminus[0;+\infty),$$

where

$$d\nu(t)\ge 0,\quad \int_0^{+\infty}d\nu(t)<\infty.$$

Then

$$\sigma\text{-meas}\,\{x>0:|f(x+iy)|<d\}\le Cd \tag{3.25}$$

for all sufficiently small $d>0$, where C is a constant independent of $y>0$.

Proof. For $a=1/d$ we have

$$\sigma\text{-meas}\,\{x>0:|f(x+iy)|<d\}=\sigma\text{-meas}\left\{x>0:|f^{-1}(x+iy)|>a\right\}. \tag{3.26}$$

By Lemma 3.1,

$$\begin{aligned} f^{-1}(x+iy)&=1-\int_0^{+\infty}\frac{(t-x)\,d\mu(t)}{(t-x)^2+y^2}-i\int_0^{+\infty}\frac{y\,d\mu(t)}{(t-x)^2+y^2}\\ &=:1-u(x+iy)-iv(x+iy). \end{aligned}\tag{3.27}$$

Clearly,

$$\int_{\mathbb{R}}v(x+iy)\frac{dx}{1+|x|}\le\int_0^{+\infty}d\mu(t)\int_{\mathbb{R}}\frac{y}{(t-x)^2+y^2}\,dx=\pi\int_0^{+\infty}d\mu(t)<+\infty. \tag{3.28}$$

By the properties of the Poisson kernel (see, e.g., [15, Chapter 6]), for $\tau\in\mathbb{R}$ and $\delta>0$ we have

$$\begin{aligned} &\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\tau-x}{(\tau-x)^2+\delta^2}v(x+iy)\,dx\\ &\qquad=\int_0^{+\infty}d\mu(t)\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\tau-x}{(\tau-x)^2+\delta^2}\cdot\frac{y}{(t-x)^2+y^2}\,dx\\ &\qquad=\int_0^{+\infty}\frac{\tau-t}{(\tau-t)^2+(y+\delta)^2}\,d\mu(t)\\ &\qquad=u\big(\tau+i(y+\delta)\big). \end{aligned}\tag{3.29}$$

Consequently,

$$u(\tau+iy)=\lim_{\delta\to0+}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\tau-x}{(\tau-x)^2+\delta^2}v(x+iy)\,dx=\widehat{H}_{x\to\tau}v(x+iy). \tag{3.30}$$

Thus, for any fixed $y>0$ we obtain the relation

$$(\widehat{H}_xv)(x+iy)=u(x+iy). \tag{3.31}$$

Now, by (3.23),

$$\sigma\text{-meas}\,\{x:|u(x+iy)|>a\}\le\frac{C}{a}\int_{\mathbb{R}}v(x+iy)\sigma(x)\,dx \tag{3.32}$$

for every $y > 0$. We shall estimate the integral

$$\int_{\mathbb{R}} v(x+iy)\sigma(x)\,dx = \int_0^{+\infty} d\mu(t) \int_{\mathbb{R}} \frac{y\sigma(x)}{(t-x)^2+y^2}\,dx. \tag{3.33}$$

For this, we split the domain of the inner integration into three parts as follows:

$$\int_{\mathbb{R}} \frac{y\sigma(x)}{(t-x)^2+y^2}\,dx = \left(\int_{-\infty}^{-1} + \int_{-1}^{t/2} + \int_{t/2}^{+\infty}\right) \frac{y\sigma(x)}{(t-x)^2+y^2}\,dx. \tag{3.34}$$

First, we observe that, by (3.24), $\sigma(|x|) \le \sigma(1)$ for $|x| \ge 1$. Next, plugging $I = (0;1)$ in (3.22), we see that

$$C \ge \int_0^1 \sigma(t)\,dt \cdot \frac{1}{\sigma(1)} \ge \int_0^x \sigma(t)\,dt \cdot \frac{1}{\sigma(1)} \ge \frac{\sigma(x)\cdot x}{\sigma(1)} \tag{3.35}$$

for $x \in (0;1)$, i.e.,

$$\sigma(x) \le C\sigma(1)\cdot\frac{1}{x}, \quad x \in (0;1). \tag{3.36}$$

Using the first inequality, we obtain

$$\begin{aligned}
&\int_{-\infty}^{-1} \frac{y\sigma(x)}{(t-x)^2+y^2}\,dx \le \sigma(1)\int_{-\infty}^{+\infty} \frac{y}{(t-x)^2+y^2}\,dx = \pi\sigma(1),\\
&\int_{-1}^{t/2} \frac{y\sigma(x)}{(t-x)^2+y^2}\,dx \le \frac{y}{(t/2)^2+y^2}\int_{-1}^{t/2}\sigma(x)\,dx\\
&\qquad\qquad \le \frac{y}{(t/2)^2+y^2}\Big(2\int_0^1 \sigma(x)\,dx + \sigma(1)\cdot\frac{t}{2}\Big),\\
&\int_{t/2}^{+\infty} \frac{y\sigma(x)}{(t-x)^2+y^2}\,dx \le \sigma(t/2)\int_{-\infty}^{+\infty} \frac{y}{(t-x)^2+y^2}\,dx = \pi\sigma(t/2).
\end{aligned} \tag{3.37}$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}} v(x+iy)\sigma(x)\,dx\\
&\quad \le \int_0^{+\infty} d\mu(t)\Big[\pi\sigma(1) + \pi\sigma(t/2)\\
&\qquad\qquad + \frac{y}{(t/2)^2+y^2}\Big(2\int_0^1 \sigma(x)\,dx + \sigma(1)\frac{t}{2}\Big)\Big].
\end{aligned} \tag{3.38}$$

We estimate each summand separately, invoking the properties of $d\mu(t)$ proved in Lemma 3.1. Combining (3.4) for $d\mu(t)$ and (3.7), we get

$$\pi\sigma(1)\int_0^{+\infty} d\mu(t) + 2\int_0^1 \sigma(x)\,dx \cdot 2\int_0^{+\infty} \frac{2y}{t^2+(2y)^2}\,d\mu(t) < \infty. \tag{3.39}$$

Since $\sigma(t)$ is monotone for $t > 0$, from (3.36) it follows that

$$\int_0^{+\infty} d\mu(t)\sigma(t/2) \le 2C\sigma(1)\int_0^1 \frac{d\mu(t)}{t} + \sigma(1/2)\int_0^{+\infty} d\mu(t) < \infty \tag{3.40}$$

(the last inequality is a consequence of (3.6) and (3.4)). Since $(t/2)y \le \big((t/2)^2+y^2\big)/2$, we have

$$\sigma(1)\int_0^{+\infty} \frac{y(t/2)}{(t/2)^2+y^2}\,d\mu(t) \le \frac{\sigma(1)}{2}\int_0^{+\infty} d\mu(t) < \infty. \tag{3.41}$$

Finally, we see that

$$\int_{\mathbb{R}} v(x+iy)\sigma(x)\,dx \le C \tag{3.42}$$

uniformly for $y > 0$. Therefore, by Chebyshev's inequality,

$$\sigma\text{-meas}\,\{x : v(x+iy) > a\} \le \frac{1}{a}\int_{\mathbb{R}} v(x+iy)\sigma(x)\,dx \le \frac{C}{a}. \tag{3.43}$$

Obviously, for $a > 4$ we have

$$\left\{x > 0 : |1 + u(x+iy)| > \frac{a}{2}\right\} \subseteq \left\{x > 0 : |u(x+iy)| > \frac{a}{4}\right\}. \tag{3.44}$$

On the other hand,

$$\begin{aligned} &\sigma\text{-meas}\left\{x > 0 : |f^{-1}(x+iy)| > a\right\} \\ &\quad \le \sigma\text{-meas}\left\{x > 0 : |\operatorname{Re} f^{-1}(x+iy)| > \frac{a}{2}\right\} \\ &\quad\quad + \sigma\text{-meas}\left\{x > 0 : |\operatorname{Im} f^{-1}(x+iy)| > \frac{a}{2}\right\}, \end{aligned} \tag{3.45}$$

that is,

$$\begin{aligned} &\sigma\text{-meas}\left\{x > 0 : |f^{-1}(x+iy)| > a\right\} \\ &\quad \le \sigma\text{-meas}\left\{x > 0 : |u(x+iy)| > \frac{a}{4}\right\} \\ &\quad\quad + \sigma\text{-meas}\left\{x > 0 : |v(x+iy)| > \frac{a}{2}\right\}. \end{aligned} \tag{3.46}$$

However, (3.32) and (3.42) imply that

$$\sigma\text{-meas}\,\{x > 0 : |u(x+iy)| > a\} \le \frac{C}{a}\int_{\mathbb{R}} v(x+iy)\sigma(x)\,dx \le \frac{C}{a}. \tag{3.47}$$

As a result, we obtain

$$\sigma\text{-meas}\left\{x > 0 : |f^{-1}(x+iy)| > a\right\} \le \frac{C}{a}, \tag{3.48}$$

and it remains to recall (3.26). □

We also need a uniqueness theorem in the limit form. Since $f(x+iy)$ has positive imaginary part in the upper half-plane, this function admits nontangential limits almost everywhere in $(0; +\infty)$. Let $f(x) := \lim_{y\downarrow 0} f(x+iy)$. The following theorem shows that estimate (3.25) remains valid for the limit function $f(x)$. Namely,

$$\sigma\text{-meas}\,\{x > 0 : |f(x)| < d\} \le Cd. \tag{3.49}$$

Theorem 3.2. *Let $(\mathcal{U}, \Sigma, \rho)$ be a measure space (with positive measure), and let $\{\varphi_n\}$ be a sequence of measurable functions defined on a set $\mathcal{E} \in \Sigma$. Suppose that*

$$\rho\,\{x \in \mathcal{E} : \varphi_n(x) < d\} \le Cd \tag{3.50}$$

for all sufficiently small $d > 0$, where $C > 0$ is a constant independent of n.

If the limit $\lim_{n\to+\infty} \varphi_n(x) =: \varphi(x)$ exists for a.e. $x \in \mathcal{E}$ with respect to ρ, then a similar inequality is true for the limit function $\varphi(x)$. Namely,

$$\rho\,\{x \in \mathcal{E} : \varphi(x) < d\} \le Cd \tag{3.51}$$

with the same constant $C > 0$.

Proof. If $\chi_{\{\varphi<d\}}(t)$ is the indicator function of the set $\{\varphi < d\} \equiv \{x \in \mathcal{E} : \varphi(x) < d\}$, then

$$\rho\{\varphi < d\} = \int_{\mathcal{E}} \chi_{\{\varphi<d\}}(t)\, d\rho(t). \tag{3.52}$$

Suppose that for some $t_0 \in \mathcal{E}$ the number $\chi_{\{\varphi<d\}}(t_0)$ is equal to 1, i.e., $\varphi(t_0) < d$. If $\varphi(t_0) = \lim_{n\to+\infty} \varphi_n(t_0)$, then $\varphi_n(t_0) < d$ for all sufficiently large n. Thus, $\chi_{\{\varphi_n<d\}}(t_0) = 1$ for such values of n. Therefore,

$$\chi_{\{\varphi_n<d\}}(t_0) \to \chi_{\{\varphi<d\}}(t_0) \quad \text{as } n \to +\infty. \tag{3.53}$$

Hence, on $\mathcal{E}$ we have

$$\chi_{\{\varphi<d\}}(t) \le \liminf_{n\to\infty} \chi_{\{\varphi_n<d\}}(t) \tag{3.54}$$

almost everywhere with respect to ρ. It follows that

$$\rho\{\varphi < d\} = \int_{\mathcal{E}} \chi_{\{\varphi<d\}}(t)\, d\rho(t) \le \int_{\mathcal{E}} \liminf_{n\to\infty} \chi_{\{\varphi_n<d\}}(t)\, d\rho(t). \tag{3.55}$$

By Fatou's lemma,

$$\begin{aligned} &\int_{\mathcal{E}} \liminf_{n\to\infty} \chi_{\{\varphi_n<d\}}(t)\, d\rho(t) \\ &\qquad \le \liminf_{n\to\infty} \int_{\mathcal{E}} \chi_{\{\varphi_n<d\}}(t)\, d\rho(t) \equiv \liminf_{n\to\infty} \rho\{\varphi_n < d\} \\ &\qquad \le Cd, \end{aligned} \tag{3.56}$$

and the proof is complete. □

Corollary 3.1. *Estimate* (3.49) *is valid for the boundary values $f(x)$ of any function $f(z)$ that belongs to the class described by conditions* (3.3) *and* (3.4).

Proof. We introduce the sequence $\varphi_n(x) := |f(x + iy_n)|$, where $y_n \downarrow 0$. By the absolute continuity of the measure $d\rho(t) := \sigma(t)\, dt$, the limit $\lim_{n\to\infty} \varphi_n(x) = |f(x)|$ also exists ρ-a.e. in $(0; +\infty)$. □

Relations (3.1) and (3.2) show that the function $M(\lambda)$ belongs to the class of analytic functions described in Lemma 3.1. Therefore, Theorem 3.1 applies to $M(\lambda)$, which implies that estimate (3.49) is valid, i.e.,

$$\sigma\text{-meas}\, \{\lambda > 0 : |M(\lambda)| < d\} \le Cd. \tag{3.57}$$

Clearly, estimate (3.57) imposes some restrictions on the character of decay of $M(\lambda)$ near the real roots of $M(\lambda)$ and, thereby, on the structure of this set of zeros.

For the first time, a uniqueness theorem of this type for functions with positive imaginary part was obtained by Pavlov in [16]. Then Naboko established a series of similar theorems for operator-valued functions (see [3], [4]). These theorems can be applied in the case we consider, but this does not lead to a sharp description of the structure of the zero set in the vicinity of the singular point $t = 0$. This effect is due to some special restrictions on the weight function $\sigma(t)$: the uniqueness theorems proved earlier allowed one to use only Lebesgue measure, that is, to consider only the weight function $\sigma(t) = 1$. Our theorem provides an opportunity to consider different measures: in this paper we use the function $\sigma(t) = 1/|t|^q$, where $q \in [0; 1)$. This permits us to obtain sharp results concerning the structure of the set of roots N.

§4. Description of the singular spectrum of the operator $A_2 = t^2 \cdot + (\cdot, \varphi)\varphi$ near the singular point $t = 0$

In order to apply our uniqueness theorem (estimate (3.57)) to the description of the structure of the set N corresponding to the operator A_2 near the singular point zero, we need to know the behavior of the function $M(\lambda)$ near its roots. In what follows we restrict our consideration to the case where the function φ belongs to the class $\operatorname{Lip}\alpha$, $\alpha \in (0; 1/2)$; in other words, we assume that

$$|\varphi(x+h) - \varphi(x)| \le C|h|^\alpha, \quad |h| < 1, \tag{4.1}$$

with $\alpha \in (0; 1/2)$. If $\alpha \ge 1/2$, then the set N is empty near zero, and $\sigma_{\text{sing}}(A_2)$ consists of at most finitely many eigenvalues of finite multiplicity (see [10]).

We need the following refinement of the Pavlov and Petras lemma (see [2] and also [4] or [6], [8]).

Lemma 4.1 (on the smoothness of $M(\lambda)$)**.** *Suppose that $\varphi \in L_2(\mathbb{R}) \cap \operatorname{Lip}\alpha$, $\alpha \in (0; 1/2)$. If $\lambda_0 \in N$, then in the ε-neighborhood of λ_0 with $0 \le \varepsilon \le \lambda_0/4$ we have*

$$|M(\lambda)| = |M(\lambda) - M(\lambda_0)| \le C \frac{|\lambda - \lambda_0|^{2\alpha}}{\lambda_0^{1/2+\alpha}}. \tag{4.2}$$

Proof. Using the Sokhotskiĭ formulas, from the representation (3.1) for $m = 2$ we deduce that, for $\lambda > 0$,

$$M(\lambda) = 1 + \text{P.V.} \int_0^{+\infty} \frac{|\varphi^2(\sqrt{\tau})| + |\varphi^2(-\sqrt{\tau})|}{\tau - \lambda} \frac{d\tau}{2\sqrt{\tau}} + i\pi \frac{|\varphi^2(\sqrt{\lambda})| + |\varphi^2(-\sqrt{\lambda})|}{2\sqrt{\lambda}}. \tag{4.3}$$

Hence, if $M(\lambda_0) = 0$, then $\varphi(\sqrt{\lambda_0}) = \varphi(-\sqrt{\lambda_0}) = 0$. Obviously, it suffices to check (4.2) for the function

$$f(\lambda) := \text{P.V.} \int_{-\infty}^{+\infty} \frac{\eta(t)}{t - \lambda}\, dt + i\eta(\lambda), \tag{4.4}$$

where

$$\eta(t) := \begin{cases} 0 & \text{if } t \le 0, \\ \frac{|\varphi^2(\sqrt{t})|}{\sqrt{t}} & \text{if } t > 0. \end{cases} \tag{4.5}$$

It is easily seen that $\eta(t)$ satisfies the local Lipschitz condition with exponent α in the interval $(0; +\infty)$; therefore, understanding $\int_{-\infty}^{+\infty} (\dots)\, dt$ as $\lim_{N\to+\infty} \int_{-N}^{N} (\dots)\, dt$, we can write

$$f(\lambda) - f(\lambda_0) = \int_{-\infty}^{+\infty} \frac{\eta(t) - \eta(\lambda)}{t - \lambda}\, dt - \int_{-\infty}^{+\infty} \frac{\eta(t) - \eta(\lambda_0)}{t - \lambda_0}\, dt + i\eta(\lambda). \tag{4.6}$$

Putting $\delta := |\lambda - \lambda_0|$, we introduce the interval $S := (\lambda_0 - 2\delta, \lambda_0 + 2\delta)$. Then, since $\eta(\lambda_0) = 0$, the difference $f(\lambda) - f(\lambda_0)$ can be rewritten as

$$\begin{aligned} f(\lambda) &- f(\lambda_0) \\ &= \int_S \frac{\eta(t) - \eta(\lambda)}{t - \lambda}\, dt + \int_{\mathbb{R}\setminus S} \frac{\eta(t) - \eta(\lambda_0)}{t - \lambda}\, dt - \int_{\mathbb{R}\setminus S} \frac{\eta(\lambda)}{t - \lambda}\, dt \\ &\quad - \int_S \frac{\eta(t) - \eta(\lambda_0)}{t - \lambda_0}\, dt - \int_{\mathbb{R}\setminus S} \frac{\eta(t) - \eta(\lambda_0)}{t - \lambda_0}\, dt + i\eta(\lambda) \end{aligned} \tag{4.7}$$

(see [2]). Combining the second integral with the last one and calculating the third integral, we see that

$$
\begin{aligned}
&f(\lambda) - f(\lambda_0) \\
&\quad = \int_S \frac{\eta(t) - \eta(\lambda)}{t - \lambda}\, dt - \int_S \frac{\eta(t) - \eta(\lambda_0)}{t - \lambda_0}\, dt \\
&\quad + \int_{\mathbb{R}\setminus S} (\lambda - \lambda_0) \frac{\eta(t) - \eta(\lambda_0)}{(t - \lambda)(t - \lambda_0)}\, dt + (i + \operatorname{sgn}(\lambda - \lambda_0) \ln 3)\, \eta(\lambda) \\
&\quad \equiv I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4.8}
$$

We estimate each summand separately. For all $t > 0$ we have

$$
|\eta(t)| \le C \frac{|t - \lambda_0|^{2\alpha}}{\sqrt{t} \cdot \lambda_0^{\alpha}}, \tag{4.9}
$$

where the constant C is independent of $\lambda_0 \in N$. Indeed, by (4.1), if $|\sqrt{t} - \sqrt{\lambda_0}| < 1$, then

$$
|\varphi(\sqrt{t})| = |\varphi(\sqrt{t}) - \varphi(\sqrt{\lambda_0})| \le C|\sqrt{t} - \sqrt{\lambda_0}|^{\alpha}. \tag{4.10}
$$

Since $\varphi \in L_2(\mathbb{R})$, from (4.1) it follows that $\varphi(t) \to 0$ as $t \to +\infty$. Therefore, replacing the constant C in (4.10) with $(\max_{\mathbb{R}} |\varphi(\tau)| + C)$, we see that (4.10) remains true for $|\sqrt{t} - \sqrt{\lambda_0}| \ge 1$. Consequently,

$$
\begin{aligned}
|\eta(t)| &= \frac{|\varphi(\sqrt{t}) - \varphi(\sqrt{\lambda_0})|^2}{\sqrt{t}} \le C \frac{|\sqrt{t} - \sqrt{\lambda_0}|^{2\alpha}}{\sqrt{t}} \\
&= C \frac{|t - \lambda_0|^{2\alpha}}{\sqrt{t}(\sqrt{t} + \sqrt{\lambda_0})^{2\alpha}} \le C \frac{|t - \lambda_0|^{2\alpha}}{\sqrt{t} \cdot \lambda_0^{\alpha}}.
\end{aligned} \tag{4.11}
$$

Since $|\lambda - \lambda_0| \le \lambda_0/4$, for $t \in S$ we have $1/t \le 2/\lambda_0$. Therefore,

$$
|\eta(t) - \eta(\lambda_0)| = |\eta(t)| \le C \frac{|t - \lambda_0|^{2\alpha}}{\lambda_0^{1/2+\alpha}}, \quad t \in S. \tag{4.12}
$$

Now, we immediately deduce that

$$
\begin{aligned}
|I_2| &\le \frac{C}{\lambda_0^{1/2+\alpha}} \int_0^{2\delta} \frac{t^{2\alpha}}{t}\, dt \le C \frac{\delta^{2\alpha}}{\lambda_0^{1/2+\alpha}}, \\
|I_4| &\le C|\eta(\lambda)| = C|\eta(t)|\,\big|_{t=\lambda} \le C \frac{\delta^{2\alpha}}{\lambda_0^{1/2+\alpha}}.
\end{aligned} \tag{4.13}
$$

Since $1/|t - \lambda| \le 2/|t - \lambda_0|$ for $t \notin S$, inequality (4.9) allows us to write

$$
\begin{aligned}
|I_3| &\le C\left(\int_0^{\lambda_0/2} + \int_{\lambda_0/2}^{\lambda_0 - 2\delta} + \int_{\lambda_0 + 2\delta}^{+\infty} \right) \delta \frac{|t - \lambda_0|^{2\alpha - 2}}{\sqrt{t} \cdot \lambda_0^{\alpha}}\, dt \\
&\le C\delta \left(\lambda_0^{\alpha - 2} \int_0^{\lambda_0/2} \frac{dt}{\sqrt{t}} + \int_{\lambda_0/2}^{\lambda_0 - 2\delta} \frac{|t - \lambda_0|^{2\alpha - 2}}{\lambda_0^{1/2+\alpha}}\, dt + \int_{2\delta}^{+\infty} \frac{t^{2\alpha - 2}}{\lambda_0^{1/2+\alpha}}\, dt \right).
\end{aligned} \tag{4.14}
$$

Hence,

$$
|I_3| \le C \left(\frac{\delta}{\lambda_0^{3/2-\alpha}} + \frac{\delta^{2\alpha}}{\lambda_0^{1/2+\alpha}} \right). \tag{4.15}
$$

To estimate I_1, we need to consider the difference $\eta(t) - \eta(\lambda)$ for $t \in S$:

$$
|\eta(t) - \eta(\lambda)| \le \frac{\big| |\varphi^2(\sqrt{t})| - |\varphi^2(\sqrt{\lambda})| \big|}{\sqrt{t}} + \frac{|\varphi(\sqrt{\lambda}) - \varphi(\sqrt{\lambda_0})|^2}{\sqrt{t}\sqrt{\lambda}} \cdot |\sqrt{t} - \sqrt{\lambda}|. \tag{4.16}
$$

Using (4.1) and (4.9), we obtain the inequalities

$$\begin{aligned}
&\frac{\big||\varphi^2(\sqrt{t})|-|\varphi^2(\sqrt{\lambda})|\big|}{\sqrt{t}}\\
&\qquad\le C\frac{|\varphi(\sqrt{t})-\varphi(\sqrt{\lambda})|}{\sqrt{\lambda_0}}\big(|\varphi(\sqrt{t})-\varphi(\sqrt{\lambda_0})|+|\varphi(\sqrt{\lambda})-\varphi(\sqrt{\lambda_0})|\big)\\
&\qquad\le C\frac{|t-\lambda|^\alpha}{\lambda_0^{1/2+\alpha/2}}\cdot\frac{|\lambda-\lambda_0|^\alpha}{\lambda_0^{\alpha/2}},
\end{aligned}\tag{4.17}$$

$$\begin{aligned}
\frac{|\varphi(\sqrt{\lambda})-\varphi(\sqrt{\lambda_0})|^2}{\sqrt{t}\sqrt{\lambda}}\cdot|\sqrt{t}-\sqrt{\lambda}|&\le C\frac{|(\sqrt{\lambda})-(\sqrt{\lambda_0})|^{2\alpha}}{\lambda_0}\cdot\sqrt{|t-\lambda|}\\
&\le C\frac{|\lambda-\lambda_0|^\alpha}{\lambda_0}\cdot|t-\lambda|^{1/2}.
\end{aligned}\tag{4.18}$$

Thus,

$$\begin{aligned}
|I_1|&\le C\int_0^{2\delta}\frac{dt}{t}\left(t^\alpha\cdot\frac{|\lambda-\lambda_0|^\alpha}{\lambda_0^{1/2+\alpha}}+t^{1/2}\cdot\frac{|\lambda-\lambda_0|^\alpha}{\lambda_0}\right)\\
&\le C(\delta^{2\alpha}/\lambda_0^{1/2+\alpha}+\delta^{1/2+\alpha}/\lambda_0).
\end{aligned}\tag{4.19}$$

Finally, for $\lambda_0\in N$, $\delta\le\lambda_0/4$, and $\alpha\in(0;1/2)$ we obtain

$$|f(\lambda)-f(\lambda_0)|\le C\left(\frac{\delta^{2\alpha}}{\lambda_0^{1/2+\alpha}}+\frac{\delta}{\lambda_0^{3/2-\alpha}}+\frac{\delta^{1/2+\alpha}}{\lambda_0}\right)\le 3C\frac{\delta^{2\alpha}}{\lambda_0^{1/2+\alpha}}.\tag{4.20}$$

This completes the proof of Lemma 4.1. □

For positive δ and x, we introduce the function $\varepsilon_x(\delta):=2\delta x^\gamma$ with an arbitrarily fixed $\gamma\ge(1/2+\alpha)/(2\alpha)$ and put

$$N^\delta:=\bigcup_{x\in N}\big(x-\varepsilon_x(\delta),x+\varepsilon_x(\delta)\big).\tag{4.21}$$

The following theorem is proved by combining our uniqueness theorem with Lemma 4.1.

Theorem 4.1. *Let $\phi\in L_2(\mathbb{R})\cap\operatorname{Lip}\alpha$, $\alpha\in(0;1/2)$. Let $\sigma(x)dx$ be a measure with a positive weight function $\sigma(t)$ satisfying conditions* (3.22) *and* (3.24). *Then for all sufficiently small $\delta>0$ we have*

$$\sigma\text{-meas}N^\delta\le C\delta^{2\alpha},\tag{4.22}$$

where C is a constant depending only on the weight function $\sigma(x)$.

Proof. Since $\gamma>1$ and the set N is bounded, for all sufficiently small δ we have $\varepsilon_x(\delta)<x/4$. Thus, $N^\delta\subseteq(0;+\infty)$, and inequality (4.2) is fulfilled in the $\varepsilon_x(\delta)$-neighborhood of $x\in N$. If $\lambda\in N^\delta$, then there exists $x\in N$ such that $\lambda\in\big(x-\varepsilon_x(\delta),x+\varepsilon_x(\delta)\big)$. By (4.2),

$$|M(\lambda)|\le C\frac{|\lambda-x|^{2\alpha}}{x^{1/2+\alpha}}\le C\frac{\varepsilon_x^{2\alpha}}{x^{1/2+\alpha}}\le C\delta^{2\alpha}x^{2\alpha\gamma-(1/2+\alpha)}.\tag{4.23}$$

Since the exponent $2\alpha\gamma-(1/2+\alpha)$ is nonnegative and the set N is bounded, we see that $|M(\lambda)|\le C\delta^{2\alpha}$. Consequently,

$$N^\delta\subset\left\{\lambda>0:|M(\lambda)|<C\delta^{2\alpha}\right\}.\tag{4.24}$$

Putting $d=C\delta^{2\alpha}$ in (3.57), we get

$$\sigma\text{-meas}N^\delta\le\sigma\text{-meas}\left\{\lambda>0:|M(\lambda)|<C\delta^{2\alpha}\right\}\le C\delta^{2\alpha}.\tag{4.25}$$

The theorem is proved. □

Observe that estimate (2) is best possible for $\gamma = (1/2+\alpha)/(2\alpha)$ in the vicinity of the origin because the set N^δ becomes wider in this case.

The set N^δ (see (1)) turns out to differ little from the δ-neighborhood of N with respect to some special metric. For $\gamma > 1$, we define the metric ρ_γ on the positive real semiaxis as follows:

$$\rho_\gamma(x,y) := \left| \int_x^y \frac{du}{u^\gamma} \right|, \quad x,y \in (0;+\infty). \tag{4.26}$$

Let $B_\delta(x) := \{y > 0 : \rho_\gamma(y,x) < \delta\}$ be the ball of radius δ with center at x. The following lemma provides information on the structure of the set $B_\delta(x)$ from the point of view of the Euclidean metric.

Lemma 4.2. *If $\varepsilon_x = 2\delta x^\gamma, x > 0$, then for any x in any finite interval $(0;a)$, $a > 0$, we have*

$$(x - \varepsilon_x/3, x + \varepsilon_x/3) \subseteq B_\delta(x) \subseteq (x - \varepsilon_x, x + \varepsilon_x) \tag{4.27}$$

for all sufficiently small δ (the required smallness of δ depends on γ and a, but is independent of x).

Proof. Obviously, to check the inclusion $B_\delta(x) \subseteq (x - \varepsilon_x, x + \varepsilon_x)$ it suffices to show that, uniformly in $x \in (0;a)$, for all sufficiently small δ the inequality

$$\delta > \int_x^{x+\varepsilon} \frac{du}{u^\gamma} \tag{4.28}$$

implies that $\varepsilon < \varepsilon_x$. From (4.22) we see that $\varepsilon \to 0$ as $\delta \to 0$. Putting $u = xt$, we obtain

$$\int_x^{x+\varepsilon} \frac{du}{u^\gamma} = \frac{1}{x^{\gamma-1}} \int_1^{1+\varepsilon/x} \frac{dt}{t^\gamma}. \tag{4.29}$$

Therefore,

$$\delta > \frac{1}{a^{\gamma-1}} \int_1^{1+\varepsilon/x} \frac{dt}{t^\gamma}. \tag{4.30}$$

Hence, uniformly in $x \in (0;a)$, $\varepsilon/x \to 0$ as $\delta \to 0$. Now,

$$\begin{aligned} \delta > \int_x^{x+\varepsilon} \frac{du}{u^\gamma} &= \frac{1}{(\gamma-1)x^{\gamma-1}} \left[1 - \left(1 + \frac{\varepsilon}{x}\right)^{1-\gamma} \right] \\ &= \frac{1}{(\gamma-1)x^{\gamma-1}} \left[(\gamma-1)\frac{\varepsilon}{x} + o\left(\frac{\varepsilon}{x}\right) \right]. \end{aligned} \tag{4.31}$$

Uniformly in $x \in (0;a)$, for sufficiently small δ we have

$$\left| o\left(\frac{\varepsilon}{x}\right) \right| < \frac{1}{2}(\gamma-1)\frac{\varepsilon}{x}. \tag{4.32}$$

Consequently,

$$\delta > \frac{\varepsilon}{2x^\gamma}, \quad \text{i.e.,} \quad \varepsilon < 2\delta x^\gamma \equiv \varepsilon_x. \tag{4.33}$$

Next,

$$\int_{x-\varepsilon_x/3}^{x} \frac{du}{u^\gamma} = \frac{1}{(\gamma-1)x^{\gamma-1}} \left[(\gamma-1)\frac{\varepsilon_x}{3x} + o\left(\frac{\varepsilon_x}{x}\right) \right], \tag{4.34}$$

where $\varepsilon_x/x = 2\delta x^{\gamma-1} \le 2\delta a^{\gamma-1}$ tends to 0 as $\delta \to 0$, uniformly for $x \in (0;a)$. Therefore, again uniformly in $x \in (0;a)$, for all sufficiently small δ we have

$$\left| o\left(\frac{\varepsilon_x}{x}\right) \right| < \frac{1}{2}(\gamma-1)\frac{\varepsilon_x}{3x}. \tag{4.35}$$

Thus,

$$\int_{x-\varepsilon_x/3}^{x} \frac{du}{u^\gamma} < \frac{\varepsilon_x}{2x^\gamma} = \delta, \tag{4.36}$$

and the proof is complete. □

Now Theorem 4.1 can be written in the following equivalent form.

Theorem 4.2. *Let $\phi \in L_2(\mathbb{R}) \cap \operatorname{Lip}\alpha$, $\alpha \in (0;1/2)$, and let $\sigma(x)dx$ be a measure with a positive weight function $\sigma(t)$ satisfying* (3.22) *and* (3.24)*. Then for all sufficiently small $\delta > 0$ we have*

$$\sigma\text{-meas}\left\{\lambda > 0 : \rho_{(1/2+\alpha)/2\alpha}(\lambda, N) < \delta\right\} \le C\delta^{2\alpha}, \tag{4.37}$$

where C is a constant depending only on the weight function $\sigma(x)$.

Proof. Consider the set $N_{\rho_\gamma}^\delta := \{\lambda > 0 : \rho_\gamma(\lambda, N) < \delta\}$, which is the δ-neighborhood of the set N in the metric ρ_γ. Obviously, $N_{\rho_\gamma}^\delta = \bigcup_{x\in N} B_\delta(x)$. Since the set N is bounded, Lemma 4.2 shows that the set $\bigcup_{x\in N} B_\delta(x)$ embeds in $\bigcup_{x\in N}(x-\varepsilon_x, x+\varepsilon_x)$ with $\varepsilon_x = 2\delta x^\gamma$, that is, $N_{\rho_\gamma}^\delta \subseteq N^\delta$. Consequently, by (4.22),

$$\sigma\text{-meas} N_{\rho_\gamma}^\delta \le \sigma\text{-meas} N^\delta \le C\delta^{2\alpha}. \tag{4.38}$$

The theorem is proved. □

Thus, for $\gamma = (1/2+\alpha)/2\alpha$, the σ-measure of the δ-neighborhood (in the metric ρ_γ) of the set N of roots is $O\left(\delta^{2\alpha}\right)$ as $\delta \to 0$. Clearly, estimate (4.37) imposes some restrictions on the possible structure of the set N, and thereby, on the structure of the set $\sigma_{\text{sing}}(A_2) \subseteq N \cup \{0\}$.

Corollary 4.1. *Let $\varphi \in L_2(\mathbb{R}) \cap \operatorname{Lip}\alpha$, $\alpha \in (0;1/2)$. Suppose that the sequence $\{\lambda_k\}_{k=1}^\infty$ of singular spectrum points of the operator $A_2 = t^2 \cdot + (\cdot, \varphi)\varphi$ decreases to zero in a power scale, that is, $\lambda_k = 1/k^\beta$; then from estimate* (4.37) *it follows that the exponent β satisfies the inequality*

$$\beta \ge 4\alpha/(1-2\alpha). \tag{4.39}$$

Proof. Suppose that

$$\beta < 4\alpha/(1-2\alpha). \tag{4.40}$$

Then the intervals $I_k := (\lambda_k - \varepsilon_{\lambda_k}/3, \lambda_k + \varepsilon_{\lambda_k}/3)$ will overlap for all sufficiently large k. Indeed, if $\lambda_k = 1/k^\beta$, then $\Delta\lambda_k := \lambda_k - \lambda_{k+1} \le C/k^{\beta+1}$. The intervals I_k and I_{k+1} will intersect provided $\Delta\lambda_k \le \varepsilon_{\lambda_k}/3$. Since $\varepsilon_{\lambda_k} = 2\delta\lambda_k^\gamma$ with $\gamma = (1/2+\alpha)/2\alpha$, the latter inequality is necessarily fulfilled if

$$\frac{C}{k^{\beta+1}} \le \frac{2\delta}{3k^{\beta\gamma}}. \tag{4.41}$$

Consequently, for $k \ge C\,(1/\delta)^{\frac{1}{1-\beta(\gamma-1)}} =: s$ the $\varepsilon_{\lambda_k}/3$-neighborhoods of the points λ_k overlap, and so if $[s]$ is the integral part of s, then $(0, \lambda_{[s]}) \subseteq \bigcup_{k=1}^{+\infty}(\lambda_k - \varepsilon_{\lambda_k}/3, \lambda_k + \varepsilon_{\lambda_k}/3)$. (Observe that $1 - \beta(\gamma-1) > 0$ if and only if $\beta < 4\alpha/(1-2\alpha)$.) By (4.27),

$$\bigcup_{x\in N}(x - \varepsilon_x/3, x + \varepsilon_x/3) \subseteq \bigcup_{x\in N} B_\delta(x) = N_{\rho_\gamma}^\delta. \tag{4.42}$$

Combined with (4.37), this yields

$$\begin{aligned} C\delta^{2\alpha} &\geq \sigma\text{-meas}\,\{\lambda > 0 : \rho_\gamma\,(\lambda, N) < \delta\} \\ &\geq \sigma\text{-meas}\left(\bigcup_{k=1}^{+\infty} \left(\lambda_k - \frac{\varepsilon_{\lambda_k}}{3}, \lambda_k + \frac{\varepsilon_{\lambda_k}}{3} \right) \right) \geq \int_0^{\lambda_{[s]}} \sigma(t)\,dt. \end{aligned} \tag{4.43}$$

Consequently, for $\sigma(t) = 1/|t|^q$, $q \in [0;1)$, we have

$$C\delta^{2\alpha} \geq \int_0^{\lambda_{[s]}} \frac{dt}{t^q}, \tag{4.44}$$

with a constant C independent of δ (possibly, C depends on q). Next,

$$\int_0^{\lambda_{[s]}} \frac{dt}{t^q} = \frac{1}{1-q} \cdot \frac{1}{[s]^{\beta(1-q)}} \geq C\delta^{\frac{\beta(1-q)}{1-\beta(\gamma-1)}}. \tag{4.45}$$

Thus, the following inequality must be fulfilled for all sufficiently small $\delta > 0$:

$$\delta^{\frac{\beta(1-q)}{1-\beta(\gamma-1)}} \leq C\delta^{2\alpha}. \tag{4.46}$$

It follows that $\frac{\beta(1-q)}{1-\beta(\gamma-1)} \geq 2\alpha$ for all $q \in [0;1)$. Since $\gamma = (1/2+\alpha)/(2\alpha)$, this implies that

$$\beta \geq \frac{4\alpha}{3-2q-2\alpha} \quad \text{for all} \quad q \in [0;1). \tag{4.47}$$

Letting $q \to 1^-$ yields $\beta \geq 4\alpha/(1-2\alpha)$, which contradicts (4.40). The proof is complete. □

The exponent β has the meaning of the rate of convergence of λ_k to zero. Estimate (4.39) implies that the points of N (and, in particular, the eigenvalues of the operator A_2) cannot tend to zero too slowly. Slower accumulation corresponds to a greater density of N and, hence, to a greater value of the measure of N. Since the function $4\alpha/(1-2\alpha)$ is monotone increasing for $\alpha \in (0;1/2)$, a better smoothness of the perturbation operator $V = (\cdot,\varphi)\varphi$ corresponds to a greater lower bound of the admissible values of β, that is, to a greater sparseness of N. Next, since $\beta \uparrow +\infty$ as $\alpha \uparrow 1/2$, the smoothness exponent $\alpha = 1/2$ is critical. This fact is consistent with the finiteness of the set N for $\alpha \geq 1/2$ (see [10]). Theorem 4.2 can also be used for the description of the structure of the set N off any neighborhood of zero, that is, the structure of the set $N_b := N \cap [b;+\infty)$ for any $b > 0$. In this case estimate (4.37) coincides with the result of [2] (in §1 it was already mentioned that the structure of the set of roots for the operator $S_1 = t\cdot + (\cdot,\varphi)\varphi$ is identical with that of the set N_b). Indeed, the set N is bounded, in every finite interval bounded away from zero we have $\varepsilon_x \geq C\delta$, and the measures dt/t^q are equivalent for different q. Putting $q = 0$, we obtain the following estimate of the Lebesgue measure of the δ-neighborhood of the set N_b:

$$\text{meas}\{\lambda > 0 : \text{dist}(\lambda, N_b) < \delta\} \leq C\delta^{2\alpha}. \tag{4.48}$$

For the eigenvalues (points of the singular spectrum, roots) $\lambda_k = \lambda_0 + 1/k^\beta$, $\lambda_0 > 0$, of the operator A_2, estimate (4.48) leads to the restriction $\beta \geq 2\alpha/(1-2\alpha)$. Therefore, (4.39) shows that we deal with duplication of the admissible rate of convergence of the eigenvalues to the limit point $\lambda_0 = 0$.

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