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SPECTRAL THEORY OF OPERATOR MEASURES IN HILBERT SPACE

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ABSTRACT. In §2 the spaces $L^2(\Sigma, H)$ are described; this is a solution of a problem posed by M. G. Kreĭn.

In §3 unitary dilations are used to illustrate the techniques of operator measures. In particular, a simple proof of the Naĭmark dilation theorem is presented, together with an explicit construction of a resolution of the identity. In §4, the multiplicity function N_Σ is introduced for an arbitrary (nonorthogonal) operator measure in H. The description of $L^2(\Sigma, H)$ is employed to show that this notion is well defined. As a supplement to the Naĭmark dilation theorem, a criterion is found for an orthogonal measure E to be unitarily equivalent to the minimal (orthogonal) dilation of the measure Σ.

In §5 it is proved that the set Ω_Σ of all principal vectors of an arbitrary operator measure Σ in H is massive, i.e., it is a dense G_δ-set in H. In particular, it is shown that the set of principal vectors of a selfadjoint operator is massive in any cyclic subspace.

In §6, the Hellinger types are introduced for an arbitrary operator measure; it is proved that subspaces realizing these types exist and form a massive set.

In §7, a model of a symmetric operator in the space $L^2(\Sigma, H)$ is studied.

CONTENTS

§1. INTRODUCTION

The theory of orthogonal operator measures (spectral measures) is exhaustively complete; see [1], [6], [17], [27]. The principal objects studied in this paper are operator measures (not necessarily orthogonal) in a separable Hilbert space H.

Operator measures arise naturally in various questions of the spectral theory of selfadjoint operators (with spectrum of finite or infinite multiplicity), in integral representations of operator-valued functions of Herglotz and Nevanlinna classes, in the theory of models for symmetric operators, etc. In [10], operator measures were employed for estimating univalent functions.

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Such measures were also used in an essential way in the quantum measurement theory and quantum statistics; see [32]. For example, from quantum theory it is known [32] that every $\mathbb{R}^n$-measurement (an affine mapping $S \mapsto \mu_S(du)$ of the set of states to the set of probability measures on $\mathbb{R}^n$) is uniquely determined by a generalized (not necessarily orthogonal) resolution Σ of the identity in H by the formula

$$\mu_S(\delta) = \operatorname{Tr}(S \cdot \Sigma(\delta)), \quad \delta \in \mathcal{B}(\mathbb{R}^n),$$

in which S is a state ($S \in \mathfrak{S}_1(H)$, $S \geq 0$, $\operatorname{Tr} S = 1$).

We briefly describe the content of the paper.

Every operator measure Σ in H gives rise to a semi-Hilbert space $\widetilde{L}^2(\Sigma, H)$ (endowed with a seminorm instead of a norm) and to a Hilbert space $L^2(\Sigma, H)$. We remind the reader that $\widetilde{L}^2(\Sigma, H)$ is defined as follows. Consider the space $C_{00}(H)$ of H-valued strongly continuous functions with compact support and taking values in finite-dimensional subspaces of H. Then $\tilde{L}^2(\Sigma, H)$ is the completion of $C_{00}(H)$ with respect to the seminorm

$$\|f\|^2_{L^2(\Sigma,H)} := (f, f)_{L^2(\Sigma,H)} = \int_{\mathbb{R}} (d\Sigma(t) f(t), f(t))_H. \tag{1.1}$$

The integral in (1.1) is understood as the limit of the Riemann–Stieltjes sums. Furthermore, the Hilbert space $L^2(\Sigma, H)$ is the quotient of $\widetilde{L}^2(\Sigma, H)$ over the kernel of the seminorm (1.1):

$$L^2(\Sigma, H) = \widetilde{L}^2(\Sigma, H)/\{f \in \widetilde{L}^2(\Sigma, H) : \|f\|_{L^2(\Sigma,H)} = 0\}.$$

The problem of an intrinsic description (in function theory terms) of the spaces $\widetilde{L}^2(\Sigma, H)$ and $L^2(\Sigma, H)$ was posed by M. G. Kreĭn [22].

In the case where $\dim H < \infty$, this problem was solved by I. S. Kats [20] (see also [1], [17]). In §2 (see Theorem 2.14) we present a solution of this problem in the case of infinite dimension.

Both in the proof of Theorem 2.14 and in subsequent constructions, a crucial role is played by the Berezanskiĭ–Gel′fand–Kostyuchenko theorem (BGK-theorem) about the differentiability of an operator measure; see [2], [3], [13], [12].

We note also that the descriptions of the space $L^2(\Sigma, H)$ for $\dim H < \infty$ and for $\dim H = \infty$ differ in principle. If $\dim H < \infty$, the space $\widetilde{L}^2(\Sigma, H)$ identifies with a certain space of ρ-measurable H-valued functions, but if $\dim H = \infty$, this is not so even in simplest cases. For instance, let $0 < K \in \mathfrak{S}_\infty$, and let $\Sigma(t) = 0$ for $t \leq t_0$, $\Sigma(t) = K$ for $t > t_0$. Then $L^2(\Sigma, H) = H_-$, where H_- is the completion of H with respect to the negative norm $\|f\|_- = \|\Sigma_0^{1/2} f\|$.

Moreover (see Proposition 2.2), not all strongly continuous vector-valued functions with compact support belong to $L^2(\Sigma, H)$ if the variation of Σ is not bounded.

Also in §2, we present a new proof of the Kats theorem; this proof is much simpler than other known proofs. Next, Theorem 2.14 is applied to the study of the existence of the Riemann–Stieltjes integral in (1.1).

In §3, from Theorem 2.14 we deduce the well-known Naĭmark dilation theorem in an easy way. Specifically, it turns out that the resolution $E(t)$ of the identity for the operator Q of multiplication by t in $L^2(\Sigma, H)$ is a minimal (orthogonal) dilation of a generalized resolution $\Sigma(t)$ of the identity, i.e., $\Sigma(t) = P_H E(t) \lceil H$. However (see Theorem 3.3), we give quite an elementary proof of this fact; the argument only involves the definition of $L^2(\Sigma, H)$ and does not refer to the description of this space in Theorem 2.14.

Next, in §3 we use the Naĭmark theorem to deduce the Sz.-Nagy–Foiaş theorem [29] about the existence of a ρ-unitary dilation for an arbitrary operator T belonging to the class $\mathcal{C}_\rho(H)$. In the same section, with the help of a result due to T. Kato about U-smooth operators (U is unitary), we prove the known result about the absolute continuity of the spectrum of the unitary dilation U for simple operators $T \in \mathcal{C}_\rho(H)$.

In §4, starting with an arbitrary orthonormal basis $\{e_i\}_1^\infty$ in H and an arbitrary scalar measure ρ equivalent to Σ ($\rho \sim \Sigma$), we introduce the multiplicity function $N_\Sigma(t)$ of the operator measure Σ. Specifically, putting $\sigma_{ij}(t) := \big(\Sigma(t)e_i, e_j\big)$, we set

$$N_\Sigma(t) = \operatorname{rank}\big(d\sigma_{ij}(t)/d\rho(t)\big)_{i,j=1}^\infty \qquad (\rho\text{-a.e.}).$$

The definition itself of the multiplicity function N_Σ is quite elementary (and does not refer to Theorem 2.14); however, the proofs of the fact that N_Σ does not depend on the basis $\{e_i\}_1^\infty$ and of the fact that the new definition coincides with the classical one in the case of a spectral measure $\Sigma = E$ involve Theorem 2.4 and Theorem 3.3 (about the existence of a unitary dilation) in an essential way.

The multiplicity function plays a crucial role in the subsequent discussion. With the help of this function, in §4 we introduce the notion of spectral subordination: a measure Σ_1 is said to be *subordinate* to a measure Σ_2 ($\Sigma_1 \prec\prec \Sigma_2$) if Σ_1 is absolutely continuous with respect to Σ_2 and $N_{\Sigma_1}(t) \le N_{\Sigma_2}(t)$. Two measures Σ_1 and Σ_2 are said to be *spectrally equivalent* if $\Sigma_1 \prec\prec \Sigma_2$ and $\Sigma_2 \prec\prec \Sigma_1$.

This notion allows us to supplement Naĭmark's well-known theorem in the following way: a spectral measure E is isomorphic (unitarily equivalent) to the minimal orthogonal dilation of a generalized resolution Σ of the identity if and only if E and Σ are spectrally equivalent. Theorem 4.15 describes a criterion for Σ and E to be spectrally equivalent, which is expressed in terms of cyclic subspaces of E.

As an application of the function N_Σ, in Proposition 4.19 we calculate the multiplicity function for the ρ-unitary dilation of an operator $T \in \mathcal{C}_\rho(H)$ with $r(T) < 1$.

Next, in §5 we study the set

$$\Omega_\Sigma := \{f \in H : \Sigma \sim \mu_f,\ \mu_f : \delta \to \big(\Sigma(\delta)f, f\big),\ \delta \in \mathcal{B}_b(\mathbb{R})\}$$

of principal vectors for a nonorthogonal measure Σ. Two approaches to this problem will be proposed. The first approach is new even for the orthogonal measures and allows us to prove that Ω_Σ is dense in H and also to explicitly indicate a part of the set Ω_Σ.

The second approach is a development of a well-known method (see [1], [6]) that yields the nonemptiness of Ω_E in the case where the measure E is orthogonal. With the help of it, in Theorem 5.10 we show that Ω_Σ is a dense G_δ-set of second category in H. By the Naĭmark theorem, this result implies (among other things) that for every cyclic subspace L ($\subset H$) of an orthogonal measure E in H ($L \in \operatorname{Cyc} E$) the set $\Omega_E \cap L$ is dense in L and of type G_δ and, consequently, of second category. Apparently, this is new even for $L = H$.

In the same §5 we characterize the subspaces $L \subset H$ for which $\Omega_\Sigma \cap L \ne \varnothing$. It turns out that either $\Omega_\Sigma \cap L = \varnothing$, or "almost all" vectors in L (in the measure-theoretic and in the Baire category sense) are principal.

In §6 we develop the theory of Hellinger types for a nonorthogonal measure Σ. It is the multiplicity function N_Σ that will allow us to introduce the Hellinger types $\Gamma_i(\Sigma)$ precisely as in the case of an orthogonal measure (cf. [6]):

$$\Gamma_i(\Sigma) = \{t \in \mathbb{R} : N_\Sigma(t) \ge i\}, \qquad i \in \{1, \dots, m(\Sigma)\}. \tag{1.2}$$

Next, if ρ is a scalar measure equivalent to Σ, $\rho \sim \Sigma$, then the ith Hellinger type is defined to be the type $[\chi_i d\rho]$ of the measure $\chi_i d\rho$, where χ_i is the indicator of the set $\Gamma_i(\Sigma)$.

In view of this definition, the nonemptiness of the set Ω_Σ means that the 1st Hellinger type is realized by principal vectors in H. At the same time, simple examples show that, generally speaking, "junior" Hellinger types of a nonorthogonal measure Σ are not realized by vectors in H.

However, it turns out that the kth type can be realized by k-vectors, i.e., by elements of the kth exterior power $\wedge^k(H)$ of H, or by k-dimensional subspaces of H. Motivated by the desire to realize the "junior" Hellinger types, we introduce the notion of Hellinger subspaces.

Specifically, a subspace $L = L_k$ with $\dim L_k = k$ is called the kth Hellinger subspace of an operator measure Σ in H (in symbols: $L_k \in \mathrm{Hel}_k(\Sigma)$) if $\Gamma_i(P_L\Sigma\lceil L) = \Gamma_i(\Sigma)$ for all $i \le k$ (P_L is the orthogonal projection onto L). In Theorems 6.7 and 6.12 we show that for each vector $h \in \Omega_\Sigma$ there exists a Hellinger chain

$$\{\lambda h\} =: H_1 \subset H_2 \subset \cdots \subset H_k \subset \cdots \subset H_m \subset H, \quad H_k \in \mathrm{Hel}_k(\Sigma), \tag{1.3}$$

if m is finite, and an infinite chain of such subspaces if $m = \infty$.

Let $H_1 \subset \cdots \subset H_m$ be a chain of subspaces of the form (1.3), and let $\{e_i\}_1^m$ be an orthonormal basis in H_m such that $H_k = \mathrm{span}\{e_i\}_1^k$, $k \in \{1,\dots,m\}$. We put $\sigma_{ij}(t) := (\Sigma(t)e_i, e_j)$, $\psi_{ij}(t) := d\sigma_{ij}(t)/d\rho$, and $\Psi_k(t) := (\psi_{ij}(t))_{i,j=1}^k$. Then the chain $H_1 \subset \cdots \subset H_m$ is Hellinger if and only if

$$\Gamma_k(\Sigma) = \{t \in \mathbb{R} : \det \Psi_k(t) \ne 0\} \pmod{\Sigma}, \quad k \in \{1,\dots,m\}. \tag{1.4}$$

Thus, Theorems 6.7 and 6.12 mean that there exists an orthonormal system $\{e_i\}_1^m$ in H such that the kth Hellinger type is realized by the measure $(\wedge^k\Psi(t)\varphi_k, \varphi_k)d\mu$, where $\wedge^k\Psi(t)$ is the kth exterior power of the operator $\Psi(t) := d\Sigma_T/d\mu$, and $\varphi_k := e_1 \wedge \cdots \wedge e_k$ is a k-vector, $\varphi_k \in \wedge^k(H)$. Theorem 6.7 implies the following statement (compare with Theorem 6.10).

Corollary 1.1. *Suppose that A is a selfadjoint operator in H, $E(t)$ is the resolution of the identity corresponding to A, $L \in \mathrm{Cyc}(A)$, and $h \in \Omega_E \cap L$. Then L contains a Hellinger chain of subspaces of the form* (1.3) *if m is finite, and an infinite Hellinger chain if $m = \infty$.*

If the multiplicity $m = m(E)$ of the measure E is finite, then $H_m \in \mathrm{Cyc}\, A$.

If vectors $\{e_i\}_{i=1}^m$ are pairwise spectrally orthogonal relative to the measure E and $H_k := \mathrm{span}\{e_i\}_1^k$, then the chain $H_1 \subset \cdots \subset H_m$ of subspaces is Hellinger if and only if e_i realizes the ith Hellinger type: $\Gamma(e_i) = \Gamma_i(E)$, $i \le m$. Corollary 1.1 shows that the Hellinger types of the operator A (and consequently, a complete sequence of its unitary invariants) can be recovered by an arbitrary cyclic subspace $L \in \mathrm{Cyc}(A)$. It should be noted that, though L may fail to possess a spectrally orthogonal sequence of vectors that realize the Hellinger types, these types are realized by some subspaces $H_k \subset L$ or by k-vectors $\varphi_k = e_1 \wedge \cdots \wedge e_k \in \wedge^k(L)$ (see Corollary 1.1).

All results of §§5–6 are new also in the case where $\dim H < \infty$.

Finally, in §7 we construct a model of a symmetric operator in the space $L^2(\Sigma, H)$. The description of $L^2(\Sigma, H)$ in Theorem 2.14 allows us to give a consistent proof of Proposition 5.2 in [18]. In this connection, we mention the recent paper [15], in which a different model of a symmetric operator was studied with the aid of Herglotz functions.

The main results of the paper were announced in [23].

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Notation. • H is a separable Hilbert space;

• $B(H)$ is the algebra of bounded linear operators in H;

• $\mathfrak{S}_p$ are the Schatten–von Neumann ideals in H (in particular, $\mathfrak{S}_2$ is the ideal of Hilbert–Schmidt operators);
• $\operatorname{Lat} T$ is the lattice of invariant subspaces of an operator $T \in B(H)$;
• $\mathcal{D}(T)$ denotes the domain of T;
• $\mathcal{R}(T)$ is the range of a closed operator T;
• $\sigma(T)$ and $\rho(T)$ stand for the spectrum and the resolvent set of T;
• $r(T)$ is the spectral radius of T;
• $H_{\mathrm{ac}}(A)$ and $H_s(A)$ are the absolutely continuous and the singular subspace of a selfadjoint (unitary) operator A in H;
• $\mathcal{B}(X)$ is the Borel σ-algebra generated by the open subsets of a topological space X;
• $\mathbb{C}$ is the field of complex numbers;
• $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$; $\mathbb{T} = \{z : |z| = 1\}$.

§2. Functional description of the space $L^2(\Sigma, H)$

2.1. Operator measures. Let H be a separable Hilbert space, and let $\{e_k\}$ be an orthonormal basis in H. We denote by $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra of the line $\mathbb{R}$ and by $\mathcal{B}_b(\mathbb{R})$ the algebra of bounded sets in $\mathcal{B}(\mathbb{R})$.

Definition 2.1. a) A mapping $\Sigma : \mathcal{B}_b(\mathbb{R}) \to B(H)$ is called an *operator measure* in H if it satisfies the following conditions:

i) $\Sigma(\varnothing) = 0$; ii) $\Sigma(\delta) \geq 0$ for $\delta \in \mathcal{B}_b(\mathbb{R})$; iii) the function Σ is strongly countably additive, i.e., if $\delta = \bigcup_{j=1}^{\infty} \delta_j$ is a disjoint decomposition of a set $\delta \in \mathcal{B}_b(\mathbb{R})$ into the union of sets δ_j, then $\Sigma(\delta) = \text{s-}\sum_{j=1}^{\infty} \Sigma(\delta_j)$;

b) an operator measure Σ is said to be *bounded* if it is defined on the σ-algebra $\mathcal{B}(\mathbb{R})$ or, which is the same, $\Sigma(\mathbb{R}) \in B(H)$;

c) an operator measure $\Sigma =: E : \mathcal{B}(\mathbb{R}) \to B(H)$ is said to be *spectral* or *orthogonal* if it is projection-valued ($E(\delta) = E(\delta)^2$ for $\delta \in \mathcal{B}(\mathbb{R})$) and $E(\mathbb{R}) = I$.

We observe that the strong convergence of the series $\Sigma(\delta) = \text{s-}\sum_{j=1}^{\infty} \Sigma(\delta_j)$ in Definition 2.1 is equivalent to weak convergence. Indeed, since the operators $T_n := \sum_{j=1}^{n} \Sigma(\delta_j) = \Sigma(\bigcup_1^n \delta_j)$ form a monotone increasing upper bounded sequence (we have $T_n \leq \Sigma(\delta)$), the weak convergence of it implies strong convergence (see [1], [6]).

Every operator measure Σ in H generates a monotone nondecreasing and strongly left continuous operator-valued function, called the *distribution function*:

$$\Sigma(t) := \begin{cases} \Sigma([0,t)), & t > 0, \\ 0, & t = 0, \\ -\Sigma([t,0)), & t < 0. \end{cases} \tag{2.1}$$

If Σ is bounded, then it is often more convenient to define $\Sigma(t)$ in a different way, namely, $\Sigma(t) := \Sigma((-\infty, t))$. We show that, conversely, an arbitrary monotone nondecreasing strongly left continuous operator-valued function $\Sigma(t)$ generates a measure Σ. Indeed, fixing $k \in \mathbb{N}$, we introduce an operator measure Σ on the semiring of intervals $\{[a, b) \subset [-n, n]\}$ by putting $\Sigma([a, b)) = \Sigma(b) - \Sigma(a)$, and extend it to the algebra $\mathfrak{R}$ generated by this semiring. On $\mathfrak{R}$, we consider the finite measures $\mu_f := \mu_{f,f}(\delta) := (\Sigma(\delta)f, f) \geq 0$, $f \in H$, and $\rho : \mathfrak{R} \ni \delta \mapsto \sum_{k=1}^{\infty} 2^{-k} \big(\Sigma(\delta)e_k, e_k\big)$ and take their Lebesgue extensions to the entire Borel σ-algebra $\mathcal{B}([-n, n])$ generated by $\mathfrak{R}$. Clearly, these extensions are compatible for different n, so that, in fact, we obtain σ-finite measures on $\mathcal{B}(\mathbb{R})$.

By the polarization formula, on the interval $[-k, k]$ the complex measure $\mu_{f,g}(\delta) := (\Sigma(\delta)f, g)$ is expressed as a linear combination of four measures of the form $\mu_{f,f}$; therefore, it extends to a complex measure on the algebra $\mathcal{B}_b([-n,n])$. We show that, for fixed $\delta \in \mathcal{B}_b(\mathbb{R})$, the mapping $\mu(\delta) : H \oplus H \ni \{f,g\} \mapsto \mu_{f,g}(\delta) \in \mathbb{C}$ is bilinear.

For every $\delta \in \mathcal{B}_b(\mathbb{R})$, there exists a sequence $\{\delta_n\}$, $\delta_n \in \mathfrak{R}$, such that $\rho(\delta_n \Delta \delta) \to 0$ as $n \to \infty$. Then $\mu_{\lambda_1 f_1 + \lambda_2 f_2, g}(\delta_n) \to \mu_{\lambda_1 f_1 + \lambda_2 f_2, g}(\delta)$ as $n \to \infty$ for all $f_1, f_2, g \in H$.

On the other hand,

$$\mu_{\lambda_1 f_1 + \lambda_2 f_2, g}(\delta_n) = \lambda_1 \big(\Sigma(\delta_n) f_1, g\big) + \lambda_2 \big(\Sigma(\delta_n) f_2, g\big) \to \lambda_1 \mu_{f_1,g}(\delta) + \lambda_2 \mu_{f_2,g}(\delta),$$

and we have proved that the mapping $\mu(\delta)$ is bilinear.

Next, if $\delta \in \mathcal{B}_b(\mathbb{R})$, then $\delta \subset [-n,n)$ for some n, and by the Cauchy inequality we have $|\mu_{f,g}(\delta)|^2 \le \mu_{f,f}(\delta)\mu_{g,g}(\delta) \le (\Sigma_n f, f)(\Sigma_n g, g) \le c(n)\|f\|^2\|g\|^2$, where $\Sigma_n := \Sigma([-n,n))$. Therefore, the form $\mu_{f,g}(\delta)$ admits a representation $\mu_{f,g}(\delta) = \big(\Sigma(\delta)f, g\big)$ with some $\Sigma(\delta) \in B(H)$. Since $\mu_{f,g}(\delta) = \overline{\mu_{g,f}(\delta)}$, we have $\Sigma(\delta) = \Sigma(\delta)^* \ge 0$.

This is the required extension.

Definition 2.2. a) An operator measure Σ_1 is said to be *absolutely continuous* with respect to an operator measure Σ_2 (in symbols: $\Sigma_1 \prec \Sigma_2$) if $\Sigma_1(\delta) = 0$ whenever $\Sigma_2(\delta) = 0$ for $\delta \in \mathcal{B}(\mathbb{R})$.

b) Two operator measures Σ_i in H $(i = 1,2)$ are said to be *equivalent* ($\Sigma_1 \sim \Sigma_2$) if $\Sigma_1 \prec \Sigma_2$ and $\Sigma_2 \prec \Sigma_1$.

2.2. The space $L^2(\Sigma, H)$. We recall the definition of the space $L^2(\Sigma, H)$ as it was given in the book [3]. Let $C_{00}(H)$ be the set of strongly continuous functions f with values in a finite-dimensional subspace of H depending on f. On the set $C_{00}(H)$, consider the semiscalar product

$$(f,g)_{L^2(\Sigma,H)} = \int_{\mathbb{R}} (d\Sigma(t)f(t), g(t))_H = \lim_{d(\pi_n)\to 0} \sum_{k=1}^{n} (\Sigma(\Delta_k) f(\xi_k), g(\xi_k)), \tag{2.2}$$

where, as usual, $\pi_n = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of a segment $[a,b]$; this segment is chosen so as to include the supports of f and g. Next, $\Sigma(\Delta_k) := \Sigma(t_k) - \Sigma(t_{k-1})$, $\xi_k \in [t_{k-1}, t_k]$, and $d(\pi_n)$ is the diameter of the partition π_n. The limit in (2.2) is understood in the same sense as in the definition of the Riemann–Stieltjes integral, i.e., the particular choice of π_n with a given diameter and of $\xi_k \in [t_{k-1}, t_k]$ is irrelevant.

The completion of $C_{00}(H)$ with respect to the seminorm $p(f) := (f,f)^{1/2}_{L^2(\Sigma,H)}$ is a semi-Hilbert space $\widetilde{L}^2(\Sigma, H)$ (i.e., a complete space with a seminorm in place of a norm). The quotient of $\widetilde{L}^2(\Sigma, H)$ over the kernel $\ker p := \{f \in \widetilde{L}^2(\Sigma, H) : p(f) = 0\}$ of the seminorm is the Hilbert space $L^2(\Sigma, H)$.

Definition 2.3. a) A vector-valued function $f : \mathbb{R} \to H$ is said to have a 1*st order discontinuity* at a point t_0 if the strong limits

$$f(t_0 \pm 0) := \operatorname*{s-lim}_{\varepsilon \downarrow 0} f(t_0 \pm \varepsilon)$$

exist and $f(t_0 + 0) \ne f(t_0 - 0)$.

b) A vector-valued function f is said to be *piecewise continuous* if it has at most finitely many 1st order discontinuities and is strongly continuous at the other points.

Definition 2.4. An operator measure Σ in H is said to be *of locally bounded variation* in the norm $\|\cdot\|$ ($\Sigma \in BV_{\mathrm{loc}}(\mathbb{R})$) if for every $\delta \in \mathcal{B}_b(\mathbb{R})$ we have

$$\operatorname{Var}_\delta(\Sigma) := \sup\left\{ \sum_i \|\Sigma(\delta_i)\| : \delta = \bigcup_1^n \delta_i,\ \delta_i \cap \delta_j = \varnothing,\ \delta_i \in \mathcal{B}_b(\mathbb{R}) \right\} < \infty, \tag{2.3}$$

where the supremum is taken over all disjoint Borel partitions of δ.

Clearly, $\Sigma \in BV_{\mathrm{loc}}(\mathbb{R})$ if Σ has locally finite trace: $\mathrm{Var}_\delta(\Sigma) \le \mathrm{Tr}\,\Sigma(\delta)$.

The following lemma is quite elementary and its proof is omitted.

Lemma 2.5. *Let f and g be piecewise continuous H-valued functions with compact support, and let $\{t'_k\}_1^{n_1}$ and $\{t''_k\}_1^{n_2}$ be their respective points of discontinuity. If the distribution function Σ is continuous at the points $\{t''_k\}_1^{n_1}$ and $\{t''_k\}_1^{n_2}$ and is of norm-bounded variation, then*
i) *the Riemann–Stieltjes integral (2.2) exists*;
ii) $f, g \in L^2(\Sigma, H)$;
iii) *we have*

$$(f, g)_{L^2(\Sigma,H)} = \int_{\mathbb{R}} (d\Sigma(t)f(t), g(t)). \tag{2.4}$$

Corollary 2.6. *Suppose that all assumptions of Lemma* 2.5 *but the last are fulfilled, i.e., the measure Σ is not necessarily of bounded variation. If f and g take their values in a finite-dimensional subspace L ($\subset H$), then the conclusions of Lemma* 2.5 *are true.*

The next statement shows that the boundedness of the variation of Σ is essential in Lemma 2.5.

Proposition 2.7. *Let $t_k \in (a, b)$ for $k \in \mathbb{N}$, and let $\lim_{k\to\infty} t_k =: t_0$. Suppose that Σ is the discrete measure of bounded variation with jumps $A_k > 0$ at the points t_k, i.e.,*

$$\Sigma(t) = \text{s-}\sum_{t_k<t} A_k, \qquad \sum_k \|A_k\| = \infty. \tag{2.5}$$

Then there exist strongly continuous vector-valued functions $f : [a, b] \to H$ such that the Riemann–Stieltjes integral (2.2)) *diverges for $g = f$.*

Proof. There is no loss of generality in assuming that $t_0 = a$ and $t_k \downarrow a$. Indeed, the case of $t_0 = b$ is treated similarly, and the case of $t_0 \in (a, b)$ reduces to one of the preceding cases because the integral (2.2) is finitely additive if it exists on $[a, b]$.

We fix an arbitrary summable numerical sequence a_k, $0 \le a_k < \|A_k\|$, and choose $g_k \in H$ in such a way that $\|g_k\| = 1$ and $(A_k g_k, g_k) \ge \|A_k\| - a_k$. Next, we put

$$f_k := s_k^{-1/2} g_k \quad \text{with } s_k := \sum_{j=1}^{k} (\|A_j\| - a_j),\ k \in \mathbb{N}. \tag{2.6}$$

Now, we define a piecewise-linear function $f(t)$ by setting $f(a) = f(b) = 0$, $f(t_k) = f_k$ for $k \in \mathbb{N}$, and by making it linear on the segments $[t_{k+1}, t_k]$, $k \in \mathbb{N}$, and on $[t_1, b]$. Since $\lim_{k\to\infty} f(t_k) = \lim_{k\to\infty} f_k = 0 = f(a)$, the function f is continuous on $[a, b]$.

Consider a partition $\pi_n = \{a < t_n < t_{n-1} < \cdots < t_1 < b\}$. Taking $\xi_k = t_k$, we form the integral sum corresponding to π_n:

$$\sum_{k=1}^{n} \big(\Delta\Sigma_k f(\xi_k), f(\xi_k)\big) = \sum_{k=1}^{n} (A_k f_k, f_k) = \sum_{k=1}^{n} s_k^{-1}(A_k g_k, g_k) \ge \sum_{k=1}^{n} s_k^{-1}(\|A_k\| - a_k). \tag{2.7}$$

From the second identity in (2.5) and the definition (2.6) of the sequence s_k, we deduce that these integral sums are unbounded as $n \to \infty$.

Thus, the Riemann–Stieltjes integral (2.2) with $f = g$ does not exist and $\|f\|^2_{L^2(\Sigma,H)} = \sum_{k\in\mathbb{N}} (A_k f(t_k), f(t_k)) = \sum_{k\in\mathbb{N}} s_k^{-1}(A_k g_k, g_k) = \infty$. $\square$

In conclusion, we give an explicit example of a measure Σ and a continuous function f not belonging to $L^2(\Sigma, H)$.

Example 2.8. Let $\{e_k\}_1^\infty$ be an orthonormal basis in H, let $P_k = (\cdot, e_k)e_k$ be the orthogonal projection to e_k, and let $\Sigma(t) := \sum_{1/k<t} P_k$ be the orthogonal resolution of the identity with jumps P_k at the points $1/k$, $k \in \mathbb{N}$. It is easily seen that $\mathrm{Var}_{[0,1]}(\Sigma) = \sum_{k\in\mathbb{N}} \|P_k\| = \infty$. We define a vector-valued function $f : [0,1] \to H$ by putting

$$f(t) = \left(\frac{e_k}{\sqrt{k}}\left(t - \frac{1}{k+1}\right) + \frac{e_{k+1}}{\sqrt{k+1}}\left(\frac{1}{k} - t\right)\right)k(k+1), \qquad t \in [1/(k+1), 1/k].$$

Then $f(1/k) = \frac{e_k}{\sqrt{k}}$, and f is strongly continuous on $[0,1]$.

Consider the partition $\pi_n = \{0 < n^{-1} < (n-1)^{-1} < \cdots < 1\}$ of $[0,1]$. Taking $\xi_k = 1/k$, we calculate the corresponding integral sum related to π_n:

$$\sum_{k=1}^{n}\big(\Delta\Sigma_k f(\xi_k), f(\xi_k)\big) = \sum_{k=1}^{n}\big(P_k f(\xi_k), f(\xi_k)\big) = \sum_{k=1}^{n}\frac{1}{k}(P_k e_k, e_k) = \sum_{k=1}^{n}\frac{1}{k} \sim \ln n.$$

Since the integral sums are unbounded as $n \to \infty$, the Riemann–Stieltjes integral (2.2) does not exist, and $\|f\|^2_{L^2(\Sigma,H)} = \sum_{k\in\mathbb{N}}(P_k f(1/k), f(1/k)) = \sum_{k\in\mathbb{N}} 1/k = \infty$.

At the 11th Summer Meeting in Mathematical Analysis in St. Petersburg (August, 2002) V. I. Vasyunin and N. K. Nikol′skiĭ informed us that they also knew examples of continuous functions $f : [a,b] \to \mathbb{R}$ not belonging to $L^2(\Sigma, H)$; see [10].

2.3. Direct integral of Hilbert spaces. Following the pattern of [6], we recall some notions related to the definition of the direct integral and required in the sequel. Concerning the properties of the direct integral and its applications, we refer the reader to [6], [12], [28], [27].

Definition 2.9 (see [6])**.** a) Let $\{Y, \mathfrak{A}, \mu\}$ be a separable space with σ-finite measure. Suppose that for μ-a.e. $y \in Y$ some Hilbert space $G(y)$ is given in such a way that the dimension function $N(y) = \dim G(y)$ is μ-measurable. An at most countable set Ω_0 of vector-valued functions g with $g(y) \in G(y)$ is called a *measurable base* if it satisfies the following conditions:

$$\begin{aligned} &\mathrm{span}\{g(y) : g \in \Omega_0\} = G(y) \quad \text{for } \mu\text{-a.e. } y \in Y; \\ &(g_1(y), g_2(y))_{G(y)} \quad \text{is a } \mu\text{-measurable function for every } g_1, g_2 \in \Omega_0. \end{aligned} \tag{2.8}$$

b) If a measurable base Ω_0 is fixed, then a vector-valued function h with $h(y) \in G(y)$ is said to be *measurable* (with respect to Ω_0) if for every $g \in \Omega_0$ the scalar function $y \mapsto (h(y), g(y))_{G(y)}$ is μ-measurable. The set of all such vector-valued functions $h(y)$ is denoted by $\widehat{\Omega}_0$. Clearly, $\Omega = \widehat{\Omega}_0$ is a linear space, and $\Omega \supset \Omega_0$. The family $G(y)$ of Hilbert spaces together with the measurable structure Ω is called a *measurable Hilbert family* and is denoted by $(G(\cdot), \Omega)$.

c) A measurable base $\Omega_1 = \{e_i(y)\}_1^\infty$ is said to be *orthogonal* if the vectors $\{e_i(y)\}_1^{N(y)}$ form an orthonormal basis in $G(y)$ for μ-a.e. $y \in Y$ and $e_j(y) = 0$ for $j \geq N(y)$.

It is known (see [6]) that for every measurable base Ω_0 there exists an orthogonal measurable base Ω_1 such that $\Omega_1 \subset \Omega$ and $\widehat{\Omega}_1 = \Omega$, and that the function $(f(y), g(y))_{G(y)}$ is μ-measurable for $f, g \in \Omega$.

Definition 2.10. Let $(G(y), \Omega)$ be a measurable Hilbert family on a separable space $(Y, \mathfrak{A}, \mu)$ with σ-finite measure. On the set $\widetilde{\mathfrak{H}}$ of vector-valued functions $h(y) \in \Omega$ for which $\|h\|^2 := \int_Y \|h(y)\|^2_{G(y)}\, d\mu(y) < \infty$, we introduce the semiscalar product

$$(h_1(y), h_2(y)) = \int_Y (h_1(y), h_2(y))_{G(y)}\, d\mu(y). \tag{2.9}$$

The space obtained from $\widetilde{\mathfrak{H}}$ after factorization by the kernel of the seminorm is called the *direct integral* of Hilbert spaces and is denoted by $\mathfrak{H} = \int_Y \oplus_\Omega G(y)\, d\mu(y)$.

The space $\mathfrak{H}$ is complete (see [6]), and so it is a Hilbert space. It is customary not to specify the measurable structure explicitly and not to reflect the dependence on Ω in the notation. Therefore, we shall write $\mathfrak{H} = \int_Y \oplus G(y)\, d\mu(y)$.

2.4. The Kats theorem. The problem of an intrinsic functional description of $L^2(\Sigma, H)$ was posed by M. G. Kreĭn [22] and, in the case where $\dim H = n < \infty$, was solved completely by Kats [20] (see also [1] and [17]). Here we present a simple proof of that result.

Let $\Sigma = (\sigma_{ij})_{i,j=1}^n$ be a matrix measure with values in $\mathbb{C}^{n\times n}$, and let $\sigma = \sum \sigma_{ii}$. Since the measure Σ is absolutely continuous with respect to σ, by the Radon–Nikodym theorem there exists a σ-measurable matrix density $\Psi = (\psi_{ij})_{i,j=1}^n$ satisfying

$$\Sigma(\Delta) = \int_\Delta \Psi(t)\, d\sigma(t), \qquad \Psi(t) := (\psi_{ij})_{i,j=1}^n = (d\sigma_{ij}/d\sigma)_{i,j=1}^n. \tag{2.10}$$

Let $\widetilde{L}_0^2(\Sigma, \mathbb{C}^n)$ be the set of σ-measurable vector-valued functions $f : \mathbb{R} \to \mathbb{C}^n$ such that

$$\|f\|^2_{\widetilde{L}_0^2(\Sigma,\mathbb{C}^n)} := \int_{\mathbb{R}} (\Psi(t)f(t), f(t))\, d\sigma(t) < \infty. \tag{2.11}$$

Theorem 2.11 (see [20])**.** *The spaces $\widetilde{L}^2(\Sigma, \mathbb{C}^n)$ and $L^2(\Sigma, \mathbb{C}^n)$ are identified isometrically with the spaces $\widetilde{L}_0^2(\Sigma, \mathbb{C}^n)$ and $L_0^2(\Sigma, \mathbb{C}^n) := \widetilde{L}_0^2(\Sigma, \mathbb{C}^n)/N_0$, respectively, where $N_0 = \{f \in \widetilde{L}_0^2(\Sigma, \mathbb{C}^n) : \|f\|_{\widetilde{L}_0^2(\Sigma,\mathbb{C}^n)} = 0\}$ is the kernel of the seminorm. Therefore, a vector-valued function f belongs to $\widetilde{L}^2(\Sigma, \mathbb{C}^n)$ if and only if f is σ-measurable and the norm* (2.11) *is finite.*

Proof. Let $\widetilde{G}(t)$ be the n-dimensional Euclidean space with the semiscalar product $\langle f, g\rangle := (\Psi(t)f, g)$. Putting $N_0(t) := \{f \in \widetilde{G}(t) : (\Psi(t)f, f) = 0\}$ and $G(t) := \widetilde{G}(t)/N_0(t)$, we obtain a k-dimensional Euclidean space, where $k = \operatorname{rank} \Psi(t) \le n$. Comparing (2.11) and (2.9), we arrive at the relations

$$\widetilde{L}_0^2(\Sigma, \mathbb{C}^n) = \int_{\mathbb{R}} \oplus \widetilde{G}(t)\, d\sigma(t) =: \widetilde{\mathfrak{H}}, \qquad L_0^2(\Sigma, \mathbb{C}^n) = \int_{\mathbb{R}} \oplus G(t)\, d\sigma(t) =: \mathfrak{H}. \tag{2.12}$$

By (2.12), the completeness of $\widetilde{L}_0^2(\Sigma, \mathbb{C}^n)$ and $L_0^2(\Sigma, \mathbb{C}^n)$ follows from that of $\widetilde{\mathfrak{H}}$ and $\mathfrak{H}$.

Let $f = \Sigma \chi_{\Delta_k} h_k$ be a vector-valued step function, where $\Delta_k = [a_k, a_{k+1})$, $h_k \in \mathbb{C}^n$, and Σ is continuous at the points $\{a_k\}$. Then by (2.10) and Lemma 2.5 the seminorms (2.2) and (2.11) coincide for such functions f and, consequently, for $f \in C_{00}(\mathbb{C}^n)$ (by approximation). Since $C_{00}(C^n)$ is dense in $\widetilde{L}^2(\Sigma, \mathbb{C}^n)$, the spaces $\widetilde{L}^2(\Sigma, \mathbb{C}^n)$ and $\widetilde{L}_0^2(\Sigma, \mathbb{C}^n)$ coincide isometrically. □

Remark 2.1. The relationship between direct integrals and the spaces $\widetilde{L}^2(\Sigma, \mathbb{C}^n)$ and $L^2(\Sigma, \mathbb{C}^n)$ (i.e., formulas (2.12)) was not mentioned in [20].

2.5. Theorem on the differentiability of an operator measure. In the next subsection we shall give a similar description of the spaces $\widetilde{L}^2(\Sigma, H)$ and $L^2(\Sigma, H)$ in the case of $\dim H = \infty$. For this, we need the following theorem due to Berezanskiĭ [2] and Gel′fand and A. G. Kostyuchenko [13]; see also [3] and [12] (the BGK-theorem).

Theorem 2.12. *Let Σ be an operator measure in H, let ρ be a scalar measure, and let $\Sigma \prec \rho$. Suppose that $K \in \mathfrak{S}_2(H)$, $\ker K = \ker K^* = \{0\}$, and let $T := K^{-1}$. Then:*

i) *there exists a weakly ρ-measurable nonnegative operator-valued function* (*an operator-valued density*) $\Psi(t) := \Psi_K(t) \geq 0$ *with values in* $\mathfrak{S}_1(H)$ *and such that*

$$(\Sigma(\delta)f, g) = \int_\delta (\Psi(t)Tf, Tg)\, d\rho(t), \quad f, g \in \mathcal{D}(T), \ \ \delta \in \mathcal{B}_b(\mathbb{R}); \tag{2.13}$$

ii) *the density* $\Psi(t)$ *is the weak ρ-a.e. limit in* H *of the sequence* $K^*\Sigma(\Delta_n)K/\rho(\Delta_n)$ *of operators, where* Δ_n *is a sequence of intervals shrinking to* t, *i.e., in the weak sense, we have*

$$\operatorname*{w-lim}_{n\to\infty} \frac{K^*\Sigma(\Delta_n)K}{\rho(\Delta_n)} =: \frac{dK^*\Sigma K}{d\rho}(t) =: \Psi(t) \quad \rho\text{-}a.e. \tag{2.14}$$

Remark 2.2. Since for $\delta \in \mathcal{B}_b(\mathbb{R})$ the operator $K^*\Sigma(\delta)K$ is of trace class, we can introduce a scalar σ-finite measure $\rho \sim \Sigma$ by setting

$$\rho(\delta) = \mathrm{Tr}(K^*\Sigma(\delta)K), \qquad \delta \in \mathcal{B}_b(\mathbb{R}). \tag{2.15}$$

In the sequel we assume that $\Psi(t)$ exists everywhere, putting (if necessary) $\Psi(t) = 0$ at the points t where the derivative does not exist.

Birman and Èntina [7] supplemented Theorem 2.12 for the case of a spectral measure $\Sigma = E$ by proving that the derivative in (2.14) exists in $\mathfrak{S}_1$. Their proof extends easily to the case of nonorthogonal measures Σ. Specifically, we have the following statement.

Proposition 2.13 (see [7]). *Under the assumptions of Theorem* 2.12, *the derivative* $\Psi(t)$ *exists in* $\mathfrak{S}_1$.

Proof. Clearly, K admits a factorization $K = K_1K_2$ with $K_1 \in \mathfrak{S}_2$ and $K_2 \in \mathfrak{S}_\infty$. Then by Theorem 2.12 the derivative of the operator-valued function $K_1^*\Sigma K_1$ exists in the weak sense. Since $K_2 \in \mathfrak{S}_\infty$, by [7, Lemma 2.1] the derivative also exists in the norm of $\mathfrak{S}_1$. □

2.6. Description of the space $L^2(\Sigma, H)$. Passing to the description of the space $L^2(\Sigma, H)$ in the case where $\dim H = \infty$, we mention a fundamental difference from the case of $\dim H < \infty$. Namely, if H is infinite-dimensional, even in the simplest situations the space $L^2(\Sigma, H)$ contains functions some values of which do not belong to H.

Indeed, let $\{t_i\}_{i=1}^n$ be a collection of points in $\mathbb{R}$, let Σ_i be nonnegative compact operators in H, and let $\Sigma(t) := \sum_{t_i<t} \Sigma_i$ be a jump operator-valued function. Then

$$\|f\|^2_{L^2(\Sigma,H)} = \sum_{i=1}^n (\Sigma_i f(t_i), f(t_i)) = \sum_{i=1}^n \|\Sigma_i^{1/2} f(t_i)\|^2.$$

Therefore, $L^2(\Sigma, H) = \bigoplus_1^n H_-^i$, where H_-^i is the completion of H in the "negative" norm $\|f\|_{i,-} = \|\Sigma_i^{1/2} f\|$ (about spaces with negative norm see, e.g., [3]).

Denote by $\widetilde{H}_t$ the semi-Hilbert space obtained by completing the space $\mathcal{D}(T) = \mathcal{R}(K)$ with respect to the semiscalar product defined by the formula

$$(f, g)_{\widetilde{H}_t} := (\Psi_K(t)Tf, Tg)_H = (\Psi_K(t)^{1/2}Tf, \Psi_K(t)^{1/2}Tg)_H, \qquad f, g \in \mathcal{D}(T). \tag{2.16}$$

Factorization of $\tilde{H}_t$ by the kernel of the seminorm yields a Hilbert space $\widetilde{H}_t$. The next theorem is the main result of this section.

Theorem 2.14. *Let* Σ *be an operator measure in* H, *let* ρ *be a scalar measure, and let* $\Sigma \prec \rho$. *Then for every* $K \in \mathfrak{S}_2(H)$ *with* $\ker K = \ker K^* = \{0\}$ *the spaces* $\widetilde{L}^2(\Sigma, H)$ *and* $L^2(\Sigma, H)$ *coincide, respectively, with the direct integrals of* $\widetilde{H}_t$ *and* H_t *against the measure* ρ:

$$\widetilde{L}^2(\Sigma, H) = \int_{\mathbb{R}} \oplus \widetilde{H}_t\, d\rho(t) =: \widetilde{\mathfrak{H}}, \quad L^2(\Sigma, H) = \int_{\mathbb{R}} \oplus H_t\, d\rho(t) =: \mathfrak{H}. \tag{2.17}$$

Moreover, for f and g with values in $\widetilde{H}_+ := \mathcal{R}(K)$ (such functions form a dense subset of $\widetilde{L}_2(\Sigma, H)$) we have

$$(f, g)_{L_2(\Sigma,H)} = \int_{\mathbb{R}} (\Psi_K(t)^{1/2} K^{-1} f(t), \Psi_K(t)^{1/2} K^{-1} g(t))_H \, d\rho(t). \tag{2.18}$$

Furthermore, the expansions (2.17) do not depend on the choice of a measure ρ ($\sim \Sigma$) and an operator $K \in \mathfrak{S}_2(H)$. In particular, $\widetilde{H}_t \supset H$ for ρ-a.e. $t \in \mathbb{R}$.

Proof. It suffices to verify the first identity in (2.17).

i) Let $\{e_i\}_1^\infty$ be an orthonormal basis in H, and let $g_i := Ke_i$. We show that the system $\{g_i\}_1^\infty$ is complete in $\widetilde{H}_t$. Let $T := K^{-1}$, and let $f \in \mathcal{D}(T) = \mathcal{R}(K)$. Then for every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and $\alpha_i \in \mathbb{C}$, $1 \le i \le n$, such that

$$\|Tf - (\alpha_1 e_1 + \dots + \alpha_n e_n)\| \le \varepsilon.$$

Consequently,

$$\begin{aligned} \Big\| \Big(f - \sum_i \alpha_i g_i\Big) \Big\|^2_{\widetilde{H}_t} &= \Big(\Psi(t) T\Big(f - \sum_i \alpha_i g_i\Big), T\Big(f - \sum_i \alpha_i g_i\Big)\Big) \\ &\le \|\Psi(t)\| \cdot \Big\| T\Big(f - \sum_i \alpha_i g_i\Big)\Big\|^2 = \|\Psi(t)\| \cdot \Big\| Tf - \sum_i \alpha_i e_i \Big\|^2 \\ &\le \|\Psi(t)\| \varepsilon^2. \end{aligned} \tag{2.19}$$

Therefore, the system $\{g_i\}_1^\infty$ of constant vector-valued functions constitutes a measurable base for the space $\widetilde{\mathfrak{H}} = \int_{\mathbb{R}} \oplus \widetilde{H}_t \, d\rho(t)$. We put

$$\mathrm{Int_c} := \{\Delta = [\alpha, \beta) : \alpha, \beta \in \mathbb{R} \text{ and the function } \Sigma(t) \text{ is continuous at } \alpha \text{ and } \beta\}. \tag{2.20}$$

Since the set of discontinuity points for $\Sigma(t)$ is at most countable, the minimal σ-algebra generated by $\mathrm{Int_c}$ coincides with $\mathcal{B}(\mathbb{R})$. Using this, we prove in an elementary way (compare with the proof of Lemma 7.1.5 in [6]) that the set

$$S'_\Sigma := \{h_{j,\Delta} := \chi_\Delta(t) g_j : \Delta \in \mathrm{Int_c}\,,\ j \in \mathbb{N}\} \tag{2.21}$$

of vector-valued functions is dense in $\mathfrak{H}$.

ii) We show that the vector-valued functions $\chi_\delta(t) g$ belong to the space $\widetilde{L}^2(\Sigma, H)$ for every $\delta \in \mathcal{B}_b(\mathbb{R})$ and every $g \in H$.

Indeed, let $\widetilde{C}_0(\mathbb{R})$ be the set of continuous scalar functions on $\mathbb{R}$ with compact support, let $\varphi \in \widetilde{C}_0(\mathbb{R})$, and let $g \in H$. By the definition of $\widetilde{L}^2(\Sigma, H)$, the vector-valued function $\varphi(t) g$ belongs to $\widetilde{L}^2(\Sigma, H)$ and

$$\|\varphi(t) g\|^2_{L^2(\Sigma,H)} = \int_{\mathbb{R}} |\varphi(t)|^2 \, d(\Sigma(t) g, g). \tag{2.22}$$

We put $\Sigma_g(t) := (\Sigma(t) g, g)$. By (2.22), the operator

$$\widetilde{j}_g : \widetilde{C}_0(\mathbb{R}) \to \widetilde{L}^2(\Sigma, H), \qquad \widetilde{j}_g \psi(t) = \psi(t) g \tag{2.23}$$

(defined initially on $\mathcal{D}(\widetilde{j}_g) := \widetilde{C}_0(\mathbb{R})$) is an isometry. Since the set $\widetilde{C}_0(\mathbb{R})$ is dense in $\widetilde{L}^2(\Sigma_g) := \widetilde{L}^2(\Sigma_g, \mathbb{R})$, the mapping $\widetilde{j}_g$ extends by continuity to an isometric embedding $j_g : \widetilde{L}^2(\Sigma_g) \to \widetilde{L}^2(\Sigma, H)$ (this embedding acts in accordance with the same formula (2.23)). Therefore, $\chi_\delta(t) g \in \widetilde{L}^2(\Sigma, H)$ for all $g \in H$ and all $\delta \in \mathcal{B}_b(\mathbb{R})$.

iii) At this stage we prove that the set of step functions

$$S_\Sigma := \Big\{ f = \sum_{j=1}^n \chi_{\Delta_j} f_j : f_j \in H, \Delta_j \in \mathrm{Int_c}, \Delta_i \cap \Delta_j = \varnothing \text{ for } i \ne j, n \in \mathbb{Z}_+ \Big\} \tag{2.24}$$

is dense in $\widetilde{L}^2(\Sigma, H)$. It suffices to show that S_Σ is dense in $C_{00}(H)$.

Suppose $f \in C_{00}(H)$ and $\operatorname{supp} f \subset [a,b]$. Since f is strongly continuous, for every $\varepsilon > 0$ there exists $\varepsilon_1 > 0$ such that $\|f(t) - f(s)\| < \varepsilon$ as soon as $|t-s| < \varepsilon_1$. Let $\pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a,b]$ such that $|t_j - t_{j-1}| < \varepsilon_1$ and $\Delta_j = [t_j, t_{j-1}) \in \mathrm{Int}_{\mathrm{c}}$. By Corollary 2.6, the vector-valued function $f_\varepsilon(t) := \sum_{j=1}^n \chi_{\Delta_j}(t) f(t_j)$ belongs to $\widetilde{L}^2(\Sigma, H)$, and we have

$$\begin{aligned}
\|f(t) - f_\varepsilon(t)\|^2_{L^2(\Sigma,H)} &= \int_{\mathbb{R}} (d\Sigma(t)(f(t) - f_\varepsilon(t)), f(t) - f_\varepsilon(t)) \\
&= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (d\Sigma(t)(f(t) - f(t_j)), f(t) - f(t_j)).
\end{aligned} \tag{2.25}$$

We estimate the norm of each summand in (2.25). Since f belongs to $C_{00}(H)$, this function takes values in some finite-dimensional subspace $H_f \subset H$.

Let $P := P_{H_f}$ be the orthogonal projection of H onto H_f, and let $\Sigma_f := P\Sigma\lceil H_f$. Next, let $\pi_{jm} = \{t_j = t_{j0} < t_{j1} < \cdots < t_{jm} = t_{j+1}\}$ be a partition of $[t_j, t_{j+1}]$, and let $\Delta_{jk} := [t_{j(k-1)}, t_{jk}) \in \mathrm{Int}_{\mathrm{c}}$. Then

$$\begin{aligned}
&\int_{\Delta_j} (d\Sigma(t)(f - f(t_j)), f - f(t_j)) \\
&\qquad = \int_{\Delta_j} (d\Sigma_f(t)(f - f(t_j)), f - f(t_j)) \\
&\qquad = \lim_{d(\pi_{mj})\to 0} \sum_{k=1}^m (\Sigma_f(\Delta_{jk})(f - f(t_j)), f - f(t_j)) \\
&\qquad \le \lim_{d(\pi_{mj})\to 0} \sum_{k=1}^m \|\Sigma_f(\Delta_{jk})\| \cdot \|f - f(t_j)\|^2_H \le \lim_{d(\pi_{mj})\to 0} \sum_{k=1}^m \operatorname{Tr} \Sigma_f(\Delta_{jk}) \cdot \varepsilon^2 \\
&\qquad \le \varepsilon^2 \cdot \operatorname{Tr} \Sigma_f(\Delta_j).
\end{aligned} \tag{2.26}$$

Combining this estimate with (2.25), we arrive at the inequality

$$\|f(t) - f_\varepsilon(t)\|^2_{L^2(\Sigma,H)} \le \sum_{j=1}^n \operatorname{Tr} \Sigma_f(\Delta_j)\varepsilon^2 = \operatorname{Tr} \Sigma_f([a,b])\varepsilon^2,$$

which proves that the family S_Σ defined in (2.24) is dense in $\widetilde{L}^2(\Sigma, H)$.

iv) Now, we show that the spaces $\widetilde{L}^2(\Sigma, H)$ and $\widetilde{\mathfrak{H}}$ coincide isometrically. Let $\Delta_i = [\alpha_i, \beta_i) \in \mathrm{Int}_{\mathrm{c}}$ $(i = 1, 2)$, let $a = \min \alpha_i$, $b = \max \beta_i$, and let $h_{k,\Delta_i} := \chi_{\Delta_i} g_k$, where $g_k = Ke_k$ $(i = 1, 2)$.

If $\pi_n = \{a = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = b\}$ is a sequence of partitions of $[a,b]$ and $\Delta_{kn} := [t_{k-1}^{(n)}, t_k^{(n)})$, by Lemma 5 and formulas (2.4) and (2.13) we obtain

$$\begin{aligned}
&(h_{i,\Delta_1}, h_{j,\Delta_2})_{\widetilde{L}^2(\Sigma,H)} \\
&\qquad = \lim_{d(\pi_n)\to 0} \sum_{k=1}^n (\Sigma(\Delta_{kn})\chi_{\Delta_1}(t_k) g_i, \chi_{\Delta_2}(t_k) g_j) = (\Sigma(\Delta_1 \cap \Delta_2) g_i, g_j) \\
&\qquad = \int_{\Delta_1\cap\Delta_2} (\Psi(t) T g_i, T g_j)_H \, d\rho(t) = \int_{\mathbb{R}} (\chi_{\Delta_1}(t)\Psi(t) T g_i, \chi_{\Delta_2}(t) T g_j)_H \, d\rho(t) \\
&\qquad = \int_{\mathbb{R}} (\chi_{\Delta_1}(t) g_i, \chi_{\Delta_2}(t) g_j)_{H_t} \, d\rho(t) = (h_{i,\Delta_1}, h_{j,\Delta_2})_{\widetilde{\mathfrak{H}}}.
\end{aligned} \tag{2.27}$$

By linearity, (2.27) extends to the vector-valued functions f $(\in S_\Sigma)$ of the form (2.24) with $f_j = \sum_{i=1}^{n_j} \alpha_{ij} g_i$, $\alpha_{ij} \in \mathbb{C}$. Next, since $\ker K^* = \{0\}$, the system $\{g_i\}_1^\infty$ is complete in H. Thus, each vector $f \in H$ can be approximated by finite linear combinations of the vectors $\{g_i\}_1^\infty$ and, consequently, the system S'_Σ of the form (2.21) is dense in $\widetilde{L}^2(\Sigma, H)$. Now, by continuity, (2.27) implies that the semi-Hilbert spaces $\widetilde{\mathfrak{H}}$ and $\widetilde{L}^2(\Sigma, H)$ coincide isometrically.

v) We prove the uniqueness of the expansion (2.17). Let $K_1 \in \mathfrak{S}_2$ with $\ker K_1 = \ker K_1^* = \{0\}$, let ρ_1 $(\sim \Sigma)$ be a scalar measure, and let $\Psi_1 := \Psi_{K_1}$ be the density of the form (2.14) constructed by K_1 and ρ_1. Starting with Ψ_1, we define the subspaces $\widetilde{H}_t^1$ and H_t^1 of the form (2.16). By the statements already proved, along with (2.17) we have the expansion $L^2(\Sigma, H) = \int_{\mathbb{R}} \oplus H_t^1 \, d\rho_1(t) =: \mathfrak{H}_1$. Since $\mathfrak{H} = \mathfrak{H}_1$ $\big(= L^2(\Sigma, H)\big)$ and $\rho \sim \rho_1$, by Theorem 7.2.2 in [6] the identity mapping $I_{\mathfrak{H}}$ of the space $\mathfrak{H}$ expands in a "direct integral" of the mappings $p(t)v(t) : H_t \to H_t^1$, i.e., $(I_{\mathfrak{H}} h)(t) = p(t)v(t)h(t)$, where $p = \sqrt{d\rho/d\rho_1}$, and $v(t)$ is a measurable operator-valued function that maps H_t onto H_t^1 unitarily. It follows that $v(t) = p(t)^{-1} I_{H_t}$ and $H_t^1 = H_t$ for ρ-a.e. $t \in \mathbb{R}$.

Now it is clear that $\widetilde{H}_t \supset H$ because for every $h \in H$ there exists an operator $K = K^* \in \mathfrak{S}_2$ $(\ker K = \{0\})$ such that $h \in \mathcal{D}(K^{-1})$. $\square$

Remark 2.3. If $\Sigma = E$ is a resolution of the identity in H and $f = E(\delta)h$, where $h \in H_+ := \mathcal{D}(T)$ and $\delta \in \mathcal{B}_b(\mathbb{R})$, then (2.18) takes the form

$$(E(\delta)h, h) \big(= \|f(t)\|^2_{L_2(E,H)} \big) = \int_\delta \|\Psi_K(t)^{1/2} T h\|_H^2 \, d\rho(t).$$

This is equivalent to the BGK-theorem for E (in the form of a direct integral; see [4]).

2.7. An application of Theorem 2.14. The following analog of the BGK-theorem is true; see [3].

Proposition 2.15. *Let Σ be an operator measure in H with norm locally bounded variation ($\Sigma \in BV_{\mathrm{loc}}(\mathbb{R})$), and let ρ be a scalar measure, $\rho \sim \Sigma$. Then*:

i) *the measure Σ is differentiable in the weak sense with respect to ρ and its derivative $\Psi(t) = d\Sigma(t)/d\rho$ takes values in $B(H)$ for ρ-a.e. t*;

ii) *we have*

$$\Sigma(\delta) = \int_\delta \Psi(t) \, d\rho(t) \quad \textit{and} \quad \mathrm{Var}_\delta(\Sigma) = \int_\delta \|\Psi(t)\| \, d\rho(t); \tag{2.28}$$

moreover, the first integral in (2.28) *is a Bochner integral convergent in the norm of $B(H)$.*

Proposition 2.15 allows us to refine Theorem 2.14 for the measures $\Sigma \in BV_{\mathrm{loc}}(\mathbb{R})$.

Theorem 2.16. *Let Σ, ρ, and Ψ be the same as in Proposition* 2.15. *Then for the spaces $\widetilde{L}^2(\Sigma, H)$ and $L^2(\Sigma, H)$ the expansions* (2.17) *are valid in which $\widetilde{H}_t$ is the semi-Hilbert space obtained by completion of H with respect to the seminorm p ($p(f) := \|f\|_{\widetilde{H}_t} := \|\Psi(t)^{1/2} f\|$), and $H_t := \widetilde{H}_t / \ker p$.*

Proposition 2.17. *Let Σ be an operator measure in H with $\Sigma \in BV_{\mathrm{loc}}(\mathbb{R})$, and let $\Psi = d\Sigma/d\rho$ be its density. Next, let f be a bounded H-valued function with compact support that is continuous a.e. with respect to Σ. Then the Riemann–Stieltjes integral* (2.2) *exists, $f \in \widetilde{L}^2(\Sigma, H)$, and*

$$\int_{\mathbb{R}} (d\Sigma(t) f(t), f(t)) = \|f\|^2_{L^2(\Sigma,H)}. \tag{2.29}$$

Proof. Suppose $\operatorname{supp} f \subset [a,b]$. Let $\pi_n = \{a = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = b\}$ be a sequence of partitions of $[a,b]$ such that $\pi_n \subset \pi_{n+1}$ and $d(\pi_n) \to 0$ as $n \to \infty$. It is easily seen that f is continuous at a point x_0 if and only if the sequence of the step functions

$$S_n(t) = \sum_{j=1}^{n} \chi_{\Delta_{jn}} f(\xi_j) \quad \text{with } \Delta_{jn} = [t_{j-1}^{(n)}, t_j^{(n)}),\ \xi_j \in \Delta_j, \tag{2.30}$$

converges strongly to f at x_0 as $d(\pi_n) \to 0$: $\|S_n(x_0) - f(x_0)\| \to 0$ as $n \to \infty$.

It can easily be checked that for S_n of the form (2.30) we have

$$\|S_n\|^2_{L^2(\Sigma,H)} = \sum_{j=1}^{n} \big(\Sigma(\Delta_{jn}) f(\xi_j), f(\xi_j)\big), \qquad n \in \mathbb{Z}_+. \tag{2.31}$$

Indeed, if $\Delta_j \in \mathrm{Int_c}$ (see (2.20)) for $j \in \{1, \dots, n\}$, then this identity follows from Lemma 2.5. The general case is obtained by passage to the limit.

On the other hand, by Theorem 2.16 we have

$$\|S_n - S_m\|^2_{L^2(\Sigma,H)} = \int_a^b \|\Psi^{1/2}(t)\big(S_n(t) - S_m(t)\big)\|^2_H \, d\rho(t), \tag{2.32}$$

where $\Psi = \frac{d\Sigma}{d\rho}$. Since f is continuous ρ-a.e., we see that $\lim_{n\to\infty} \|f(t) - S_n(t)\|_H = 0$ for ρ-a.e. t. Therefore,

$$\lim_{m,n\to\infty} \|\Psi^{1/2}(t)(S_n(t) - S_m(t))\|^2 = 0 \quad \text{for } \rho\text{-a.e. } t \in [a,b]. \tag{2.33}$$

Moreover, the relations $\|S_n(t) - S_m(t)\|_H \le 2\|f(t)\| \le 2c$ and $\|\Psi(t)^{1/2}\| \in L^2[a,b]$ show that the integrable function $c^2\|\Psi^{1/2}(t)\|^2$ majorizes the integrand in (2.32). By the Lebesgue dominated convergence theorem, (2.32) and (2.33) imply that

$$\lim_{m,n\to\infty} (\|S_n\|_{L^2(\Sigma,H)} - \|S_m\|_{L^2(\Sigma,H)})^2 \le \lim_{m,n\to\infty} \|S_n - S_m\|^2_{L^2(\Sigma,H)} = 0;$$

consequently, the limit $\lim_{n\to\infty} \|S_n\|_{L^2(\Sigma,H)}$ exists.

To complete the proof, it remains to observe that, by (2.31), the norm $\|S_n\|^2_{L^2(\Sigma,H)}$ coincides with the Riemann–Stieltjes sum constructed by means of the function f and partition π_n. By (2.32), the limit is independent of the choice of a sequence S_n. □

§3. The Naĭmark theorem and ρ-unitary dilations

3.1. Orthogonal dilations. In this subsection we use the definition of the space $L^2(\Sigma, H)$ to give a short proof of the Naĭmark dilation theorem and also to supplement that statement. We recall the following definition.

Definition 3.1 (see [1]). A monotone nondecreasing operator function $\Sigma = \Sigma^* : \mathbb{R} \to B(H)$ is called a *generalized resolution of the identity* if it is left strongly continuous and

$$\Sigma(-\infty) := \operatorname*{s-lim}_{t\to-\infty} \Sigma(t) = 0, \qquad \Sigma(+\infty) := \operatorname*{s-lim}_{t\to+\infty} \Sigma(t) = I. \tag{3.1}$$

The well-known Naĭmark theorem [25, 29] makes the following definition natural.

Definition 3.2. a) Let $\Sigma(t)$ be a generalized resolution of the identity in H. A left continuous resolution $E(t)$ of the identity in a Hilbert space $\widetilde{H} \supset H$ is called an *orthogonal dilation* of $\Sigma(t)$ if

$$\Sigma(t) = P_H E(t)\lceil H, \qquad t \in \mathbb{R}, \tag{3.2}$$

where P_H is the orthogonal projection of $\widetilde{H}$ onto H;

b) the above $E(t)$ is called a *minimal orthogonal dilation* for $\Sigma(t)$ if

$$\widetilde{H} = \operatorname{span}\{E(\delta)H : \delta \in \mathcal{B}(\mathbb{R})\} = \operatorname{span}\{E(\Delta)H : \Delta \in \mathrm{Int_c}\}. \tag{3.3}$$

We recall that $C_{00}(H)$ is the set of strongly continuous functions f with compact support and with values in finite-dimensional subspaces of H depending on f, and that $C_{00}(H)$ is dense in $\mathfrak{H} := L^2(\Sigma, H)$.

Theorem 3.3. *Let Σ be a generalized resolution of the identity in H, and let $E_0(x)$ be the family of operators in $\mathfrak{H} = L^2(\Sigma, H)$ defined by*

$$E_0(x) : C_{00}(H) \to \mathfrak{H}, \quad E_0(x) : h(t) \mapsto \chi_\Delta(t)h(t), \ \ \Delta = (-\infty, x). \tag{3.4}$$

Then:

i) *the family $E_0(x)$ is well defined and extends up to a continuous resolution $E(x)$ of the identity in $\mathfrak{H}$*;

ii) *$E(x)$ is a minimal* (*orthogonal*) *dilation for $\Sigma(x)$*;

iii) *the operator*

$$Q : f(t) \mapsto tf(t), \quad \mathcal{D}(Q) = \{f(t) \in L^2(\Sigma, H) : tf(t) \in L^2(\Sigma, H)\},$$

is selfadjoint in $\mathfrak{H}$, and $E(x)$ is its resolution of the identity; *furthermore*, *the corresponding* (*orthogonal*) *spectral measure E has the form*

$$\delta \mapsto E(\delta), \ \ E(\delta) : h(t) \mapsto \chi_\delta(t)h(t), \quad \delta \in \mathcal{B}(\mathbb{R}). \tag{3.5}$$

Proof. i) First, let x be a continuity point for Σ. By Corollary 2.6, the operator $E_0(x)$ is well defined on the vector-valued functions $h \in C_{00}(H)$, i.e., $\chi_\Delta h \in L^2(\Sigma, H)$ for $\Delta = (-\infty, x)$ and, moreover, $\chi_\Delta h$ is Riemann–Stieltjes integrable. Inspection of the integral sums yields

$$\|\chi_\Delta h\|^2_{L^2(\Sigma)} = \int_{\mathbb{R}} (d\Sigma(t)\chi_\Delta(t)h(t), \chi_\Delta(t)h(t)) \le \int_{\mathbb{R}} (d\Sigma(t)h(t), h(t)) = \|h\|^2_{L^2(\Sigma)}. \tag{3.6}$$

This means that $E_0(x)$ is a contraction. Since $E_0(x)$ is a symmetric operator on $C_{00}(H)$ (which also follows from the inspection of the integral sums), we see that $E_0(x)$ extends by continuity to the entire space $\mathfrak{H}$, becoming a selfadjoint contraction $E(x)$. Since $\chi^2_\Delta(t) = \chi_\Delta(t)$, $E(x)$ is an orthogonal projection, $E(x) = E(x)^* = E(x)^2$. Moreover, from the definition of $L^2(\Sigma, H)$ it easily follows that $E(x)$ extends from $C_{00}(H)$ to $\mathfrak{H}$ in accordance with the same formula (3.4).

Now, if $x_1 < x_2$ are points of continuity for Σ and $\Delta_i := (-\infty, x_i)$, then, by analogy with (3.6), we obtain the inequality

$$\|E(x_1)h\|^2 = \|\chi_{\Delta_1} h\|^2_{L^2(\Sigma,H)} \le \|\chi_{\Delta_2} h\|^2_{L^2(\Sigma,H)} = \|E(x_2)h\|^2, \quad h \in C_{00}(H).$$

Consequently, $E(x_1) \le E(x_2)$. This inequality permits us to extend the projection-valued function $E(x)$ to all discontinuity points $\{t_j\}_1^N$ ($N \le \infty$) of Σ. Namely, we put $E(t_j) = \sup\{E(x) : x < t_j, \Sigma(x) = \Sigma(x+0)\}$. It is easily seen that for every $j \le N$ the operators $E(t_j)$ also act on $C_{00}(H)$ in accordance with (3.4).

Now it is clear that the projection-valued function $E(x)$ is monotone nondecreasing on $\mathbb{R}$.

Next, the definition (3.4) implies the relations

$$E(x-0)h = E(x)h, \ \ \lim_{x\uparrow\infty} E(x)h = h, \ \ \lim_{x\downarrow-\infty} E(x)h = 0, \quad h \in C_{00}(H).$$

Since $C_{00}(H)$ is a dense subset of $\mathfrak{H}$, we see that the function E is left continuous and that the completeness relations s-$\lim_{x\uparrow\infty} E(x) = I$, s-$\lim_{x\downarrow-\infty} E(x) = 0$ are fulfilled. Thus, E is a resolution of the identity.

ii) Consider the trivial embedding $j : H \hookrightarrow \mathfrak{H}$ defined by the formula

$$jh = h =: h(t)^q, \quad t \in \mathbb{R}, \tag{3.7}$$

i.e., h is identified with a constant vector-valued function. Clearly,

$$(jh, jh)_{\mathfrak{H}} = \int_{\mathbb{R}} (d\Sigma h, h) = (\Sigma(+\infty)h, h) - (\Sigma(-\infty)h, h) = (h, h)_H; \tag{3.8}$$

therefore, j is an isometry. This enables us to view H as a subspace of $\mathfrak{H}$. Let $P := P_H$ be the orthogonal projection of $\mathfrak{H}$ onto H. We show that $\Sigma(x) = PE(x)\lceil H$ for every $x \in \mathbb{R}$. Indeed, since $\Sigma(-\infty) = 0$ (see (3.1)), it follows that

$$((PE(x)\lceil H)h, h) = (E(x)h, h)_{L^2(\Sigma,H)} = \int_{\mathbb{R}} (d\Sigma(t)\chi_\Delta(t)h, h) = (\Sigma(x)h, h) \tag{3.9}$$

for every $h \in H$, where $\Delta = (-\infty, x)$. So, E is a dilation of Σ.

E is a minimal dilation because the set S_Σ of step functions defined in (2.24) is dense in $L^2(\Sigma, H)$; see item iii) of the proof of Theorem 2.14.

iii) By Theorem 2.14, the space $L^2(\Sigma, H)$ is a direct integral of Hilbert spaces. Therefore, by Theorem 6.2.1 in [6], the operator Q is selfadjoint and E is its spectral measure. □

In particular, statements i) and ii) of Theorem 3.3 imply the Naĭmark theorem [25].

Theorem 3.4 (Naĭmark dilation theorem)**.** *Let $\Sigma(t)$ be a generalized resolution of the identity in H. Then*:

i) *$\Sigma(t)$ has an orthogonal dilation $E(t)$ in a Hilbert space $\widetilde{H} \supset H$*;

ii) *the dilation E can be taken minimal*;

iii) *every two minimal dilations for Σ are unitarily equivalent*;

iv) *if E is a minimal dilation for Σ, then $\Sigma \sim E$.*

Proof. Statements i) and ii) follow from Theorem 3.3.

iii) For $k = 1, 2$, let E_k be a minimal dilation of Σ in the space $\widetilde{H}_k \supset H$ $(k = 1, 2)$. Then for every $\delta_i \in \mathcal{B}(\mathbb{R})$ and every $h_i \in H$ $(1 \le i \le n,\ n \in \mathbb{N})$ we have

$$\Big\| \sum_i E_1(\delta_i)h_i \Big\|^2 = \sum_{i,j} (E_1(\delta_i)h_i, E_1(\delta_j)h_j) = \sum_{i,j} (\Sigma(\delta_i \cap \delta_j)h_i, h_j) = \Big\| \sum_i E_2(\delta_i)h_i \Big\|^2.$$

Therefore, the operator U defined on the linear space $\{\sum_i E_1(\delta_i)h_i\}$ by the formula $U(\sum_i E_1(\delta_i)h_i) = \sum_i E_2(\delta_i)h_i$ is an isometry. Since the dilations E_1 and E_2 are minimal, from (3.3) it follows that the operator U extends by continuity to a unitary operator from $\widetilde{H}_1$ onto $\widetilde{H}_2$, $U : \widetilde{H}_1 \to \widetilde{H}_2$. Taking $n = 1$ and $\delta_1 = \mathbb{R}$ in the definition of U, we see that U is the identity on H $(Uh = h,\ h \in H)$. Now, the relation $UE_1 = E_2U$ is obvious.

iv) By (3.2), $\Sigma(\delta) = P_H E(\delta)\lceil H$, $\delta \in \mathcal{B}(\mathbb{R})$. Consequently, $\Sigma(\delta) = 0$ if $E(\delta) = 0$.

Conversely, if $\Sigma(\delta) = 0$, then by (3.2) we obtain $\|E(\delta)h\| = (E(\delta)h, h)^{1/2} = 0$ for $h \in H$, i.e., $E(\delta)H = 0$. Therefore, $E(\delta)E(\Delta)H = 0$ for all . But then $E(\delta)\widetilde{H} = 0$ by (3.3), i.e., $E(\delta) = 0$. □

Naĭmark′s Theorem 3.4 extends easily to the case of an arbitrary bounded operator measure; see [8]. We present an analog of Theorem 3.3 for that case.

Theorem 3.5. *Let Σ be a bounded monotone nondecreasing operator-valued function in H satisfying the conditions $\Sigma(-\infty) = 0$ and $\Sigma(+\infty) = K \ge 0$, let $\mathfrak{H} := L^2(\Sigma, H)$, and let $E(t)$ be defined by* (3.4) *and* (3.5)*. Then*:

i) *$E(t)$ is a resolution of the identity in $\mathfrak{H}$ that satisfies the minimality condition* (3.3) *with $\widetilde{H} = \mathfrak{H}$*;

ii) *there exists a bounded operator $T : H \to L^2(\Sigma, H)$ such that*

$$\Sigma(t) = T^*E(t) \qquad T^*T = K; \tag{3.10}$$

iii) *the spectral measure E is equivalent to Σ, $E \sim \Sigma$.*

Proof. Statements i) and iii) were proved in Theorem 3.3, because the arguments given there are applicable to arbitrary (even not necessarily bounded) measures.

ii) In the present setting, the embedding (3.7) is no longer an isometry because now we have $(jh, jh)_{\mathfrak{H}} = (Kh, h)_H$ in place of (3.8), i.e., $j^*j = K$. Putting $T = j$ and taking relation (3.9) with $T^* = j^*$ into account, we get the required result. □

Remark 3.1. a) Statement i) of Theorem 3.3 is covered by statement iii). However, we have presented an elementary proof not involving Theorem 2.14 because, in passing, we have obtained also an elementary proof of the Naĭmark theorem.

b) Statement iv) of Theorem 3.4 is known to experts, though in the literature only the weaker equivalence $\Sigma(t_1) = \Sigma(t_2) \iff E(t_1) = E(t_2)$ can be found; see [8].

3.2. Unitary ρ-dilations.

Definition 3.6 (see [29]). a) An operator $T \in B(H)$ is attributed to the class $\mathfrak{C}_\rho(H)$ if there exists a unitary operator U in a Hilbert space $\widetilde{H} \supset H$ such that

$$\rho P_H U^n \lceil H = T^n, \qquad n \in \mathbb{N}, \ \rho > 0. \tag{3.11}$$

The operator U is called a unitary ρ-dilation of T. Clearly, $\|T\| \le \rho$ for $T \in \mathfrak{C}_\rho(H)$. If $\rho = 1$, then T is a contraction, and its unitary 1-dilation is called simply a unitary dilation.

b) A dilation U is said to be minimal if $\operatorname{span}_{n=-\infty}^{+\infty}\{U^n H\} = \widetilde{H}$.

From Theorem 3.4 it is easy to deduce the well-known result due to B. Szőkefalvi-Hagy and C. Foiaş [29].

Theorem 3.7 (see [29]). *Let $T \in B(H)$. Then $T \in \mathfrak{C}_\rho$ if and only if for all $h \in H$ we have*

$$(\rho - 2)\|(1 - zT)h\|^2 + 2\operatorname{Re}((I - zT)h, h) \ge 0, \quad z \in \mathbb{D}. \tag{3.12}$$

In particular, every contraction $T \in B(H)$ has a unitary dilation.

Proof. From (3.12) it easily follows that $\sigma(T) \subset \mathbb{D}$ (see [29]).

Put $\Phi(z) := I - 2\rho^{-1} + 2\rho^{-1}(I - zT)^{-1}$. Then by (3.12) we have

$$2^{-1}\rho[\Phi(z) + \Phi^*(z)] = (\rho - 2)I + 2\operatorname{Re}(I - zT)^{-1} \ge 0, \quad z \in \mathbb{D}. \tag{3.13}$$

By the Herglotz theorem (see [11]; its extension to operator-valued functions can be found, e.g., in [8]), $\Phi(z)$ has an integral representation of the form

$$\Phi(z) = iK + \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, d\Sigma(t), \qquad |z| < 1, \tag{3.14}$$

in which $K = K^*$ and $\Sigma(t)$ is a monotone nondecreasing $B(H)$-valued function such that $\Sigma(0) = 0$. Since $\Phi(0) = I$, we have $K = 0$ and $\Sigma(2\pi) = \operatorname{Re}\Phi(0) = I$. By (3.14) and the definition of Φ, we obtain

$$\frac{2}{\rho}(I - zT)^{-1} = \frac{2}{\rho} - 2 + \int_0^{2\pi} \frac{2e^{it}\, d\Sigma(t)}{e^{it} - z} = \frac{2}{\rho} - 2 + 2\int_0^{2\pi} \frac{d\Sigma(t)}{1 - ze^{-it}}, \quad z \in \mathbb{D}. \tag{3.15}$$

By the Naĭmark theorem, there exists a Hilbert space $\widetilde{H} \supset H$ and a resolution $E(t)$ of the identity in $\widetilde{H}$ such that $\Sigma(t) = P_H E(t)\lceil H$. Putting $U = \int_0^{2\pi} e^{-it}\, dE(t)$, we rewrite (3.15) in the form $(I - zT)^{-1} = (1 - \rho)I + \rho P_H (I - zU)^{-1}\lceil H$, $z \in \mathbb{D}$, which means that U is a unitary ρ-dilation for T. □

Remark 3.2. For $\rho = 1$ condition (3.12) means that T is a contraction, and Theorem 3.7 coincides with the well-known Szőkefalvi-Nagy theorem. For $\rho = 2$ condition (3.12) means that the numerical radius $w(T) := \sup\{|(Th, h)| : h \in H, \|h\| \le 1\}$ of T does not exceed 1. In this case, Theorem 3.7 was proved by Berger [5] and Halmos (see [30]). The general case of $\rho > 0$ was studied by Szőkefalvi-Nagy and Foiaş [30].

Corollary 3.8. *Suppose $T \in \mathfrak{C}_\rho$ and $\rho(T) < 1$. Then*

$$T^n = \rho \int_0^{2\pi} e^{int}\, d\Sigma(t), \qquad n \in \mathbb{N}, \tag{3.16}$$

where Σ is the spectral measure of T defined by

$$\Sigma(t) = \frac{1}{2\pi\rho} \int_0^t [(I - e^{is}T)^{-1} + (I - e^{-is}T^*)^{-1} + \rho - 2]\, ds. \tag{3.17}$$

The proof follows from (3.15) and the Stieltjes inversion formula. □

For the study of the spectral function of the unitary ρ-dilation, we recall the following definition due to T. Kato [19].

Definition 3.9 (see [19])**.** Let U be a unitary operator in a Hilbert space H. An operator $A \in B(H)$ is said be U-*smooth* if

$$\int_0^{2\pi} \|A(I - re^{it}U)^{-1} f\|^2\, dt \le c\|f\|^2 \quad \text{for } r \ne 1 \tag{3.18}$$

with a constant c independent of f and r.

Clearly, the U-smoothness of an operator A is equivalent to its U^{-1}-smoothness.

We begin with the following statement.

Proposition 3.10. *Let T be a contraction in H, let U be the unitary dilation of T in $\widetilde{H} \supset H$, and let $D_T = (I - T^*T)^{1/2}$ is the defect operator of T. Then the operators $D_T P_H$ and $D_{T^*} P_H$ are U-smooth, where P_H is the orthogonal projection of $\widetilde{H}$ onto H.*

Proof. Let $0 \le r < 1$. By the Parseval identity, we have

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \|D_T P_H (I - re^{it}U)^{-1} f\|^2\, dt &= \frac{1}{2\pi} \int_0^{2\pi} \|D_T (I - re^{it}T)^{-1} f\|^2\, dt \\
&= \sum_{n=0}^{\infty} r^{2n} \|D_T T^n f\|^2 = \sum_{n=0}^{\infty} r^{2n} ((I - T^*T)T^n f, T^n f) \\
&= \sum_{n=0}^{\infty} r^{2n} \|T^n f\|^2 - \sum_{n=0}^{\infty} r^{2n} \|T^{n+1} f\|^2 = \|f\|^2 - (1 - r^2) \sum_{n=0}^{\infty} r^{2n} \|T^{n+1} f\|^2 \\
&\le \|f\|^2.
\end{aligned} \tag{3.19}$$

Next, $(I - r^{-1}e^{it}U)^{-1} = re^{-it}U^*(re^{-it}U^* - I)^{-1}$. Using (3.19) with T^* in place of T and the simple identity $D_T T^* = T^* D_{T^*}$ (see [29]), we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \|D_T P_H (I - r^{-1}e^{it}U)^{-1} f\|^2\,dt \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi} \|D_T P_H re^{-it}U^*(re^{-it}U^* - I)^{-1} f\|^2\,dt \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi} \|re^{-it}D_T T^*(re^{-it}T^* - I)^{-1} f\|^2\,dt \\
&\quad = r^2\frac{1}{2\pi}\int_0^{2\pi} \|T^* D_{T^*}(re^{-it}T^* - I)^{-1} f\|^2\,dt \\
&\quad \le \|T^*\|^2\cdot\frac{1}{2\pi}\int_0^{2\pi} \|D_{T^*}(re^{-it}T^* - I)^{-1} f\|^2\,dt \le \|T^*\|^2\|f\|^2.
\end{aligned} \tag{3.20}$$

Combining (3.19) and (3.20), we conclude that $D_{T^*}P_H$ is U-smooth.

In a similar way it is proved that $D_T P_H$ is U-smooth. □

Definition 3.11. A contraction T in a Hilbert space H is said to be *simple* (*completely nonunitary*) if there is no invariant (equivalently, reducing) subspace $L \in \operatorname{Lat} T$ such that $T\lceil L$ is a unitary operator.

We need the following known fact.

Lemma 3.12. *Let T be a completely nonunitary contraction in H, and U its minimal unitary dilation in $\widetilde{H}$. Then*

$$\operatorname{span}\{U^n(I - TT^*)H, U^{*n}(I - T^*T)H\}_{n=-\infty}^{+\infty} = \widetilde{H}. \tag{3.21}$$

We pass to the proof of the well-known theorem due to B. Szőkefalvi-Nagy and C. Foiaş.

Theorem 3.13 (see [29]). *Let T be a simple contraction in H. Then the spectral measure E_U of its minimal unitary dilation is equivalent to the Lebesgue measure m, $E_U \sim m$. Moreover, the scalar measure $\mu_f := (E_U f, f)$ is equivalent to m for every $f \in H$.*

Proof. i) By Proposition 3.10, the operators $D_T P_H$ and $D_{T^*}P_H$ are U-smooth. Therefore, by the Kato theorem [19], we have $\mathcal{R}((D_T P_H)^*)$, $\mathcal{R}((D_{T^*}P_H)^*) \subset \widetilde{H}_{\mathrm{ac}}(U) = \widetilde{H}_{\mathrm{ac}}(E_U)$. But $(D_T P_H)^* = D_T P_H$, and consequently, $\mathcal{R}(I - T^*T), \mathcal{R}(I - TT^*) \subset \widetilde{H}_{\mathrm{ac}}(U)$. By Lemma 3.12, we have $\widetilde{H} = \widetilde{H}_{\mathrm{ac}}(U)$, whence $E_U \prec m$.

ii) We show that $E_U \sim m$. If E_U is not equivalent to m ($E_U \nsim m$), then $\delta := \mathbb{T}\setminus\sigma(U) \ne \varnothing$, whence it follows that $m(\delta) > 0$ but $E(\delta) = 0$. But the operator U^{-1} can be approximated uniformly by polynomials in U, $\|U^{-1} - p_n(U)\| \to 0$ as $n \to \infty$; therefore, $\operatorname{Lat} U \subset \operatorname{Lat} U^*$.

However, for every $f \in H$ the subspace $L_f := \operatorname{span}\{U^n f : n \ge 0\}$ is an element of $\operatorname{Lat} U$, but $L_f \notin \operatorname{Lat} U^*$. Indeed, if L_f is a reducing subspace for U, then $T^n f = P_H U^n f = U^n f$. The latter means that $L_f \in \operatorname{Lat} T$, so that $T\lceil L_f = U\lceil L_f$, which is impossible because T is simple. □

§4. The multiplicity function of a measure Σ

4.1. Cyclic subspaces and the multiplicity function of an operator measure. We recall a series of well-known definitions and facts.

Definition 4.1. Let A be a selfadjoint operator in H, let E be its resolution of the identity, and let L be a linear subset of H (closed or not). We denote by H_L the minimal A-invariant subspace in H containing L, i.e.,

$$H_L := H_L^A = \operatorname{span}\{E(\delta)L : \delta \in \mathcal{B}(\mathbb{R})\} = \operatorname{span}\{(A-\lambda)^{-1}L : \lambda \in \mathbb{C} \setminus \mathbb{R}\}. \tag{4.1}$$

If A is bounded, then $H_L = \operatorname{span}\{A^n L : n \geq 0\}$. If $L = \operatorname{span}\{g_1, \dots, g_n\}$, we put $H_{\{g_1,\dots,g_n\}} := H_L$. In particular, $H_g = H_L$, where $L = \{\lambda g : \lambda \in \mathbb{C}\}$.

Definition 4.2. i) A linear set $L \subset H$ is said to be *cyclic* for A (for E) if $H_L = H$. The set of all cyclic subspaces of an operator A (a measure E) is denoted by $\operatorname{Cyc}(A)$ ($\operatorname{Cyc}(E)$). The number

$$n(A) := \min\{\dim L : L \in \operatorname{Cyc}(A)\} =: n(E) \tag{4.2}$$

is called the *total multiplicity* of the spectrum of A (of E).

ii) The multiplicity of the spectrum of A (of E) at a point $t \in \mathbb{R}$ is defined as follows:

$$n_A(t) := \inf_{t\in\Delta}(n(A\lceil E(\Delta)H)) := n_E(t). \tag{4.3}$$

In the sequel, the spectral theorem will be used repeatedly in the following form (see [6], [12], [28]).

Theorem 4.3. *Let A be a selfadjoint operator in H, let $E := E_A$ be its spectral measure, and let ρ be a scalar Borel measure on $\mathbb{R}$ equivalent to E. Then the following statements are true.*

i) *A is unitarily equivalent to the operator Q of multiplication by t in some direct integral $\mathfrak{H} = \int_{\mathbb{R}} \oplus G(t)\, d\rho(t)$ of Hilbert spaces*:

$$Q : f(t) \mapsto tf(t), \qquad \mathcal{D}(Q) = \{f(t) \in \mathfrak{H} : tf(t) \in \mathfrak{H}\}. \tag{4.4}$$

ii) *Two operators Q_i of multiplication by t in two distinct direct integrals $\int \oplus G_i(t)\, d\rho_i(t)$ $(i = 1, 2)$ are unitarily equivalent if and only if*

$$\rho_1 \sim \rho_2 \quad \textit{and} \quad \dim G_1(t) = \dim G_2(t) \pmod{\rho_1}.$$

Corollary 4.4. *The equivalence class E of the measure $E = E_A$ and the multiplicity function*

$$\widetilde{N}_A(t) := \dim G(t) \,(\mathrm{mod}\, \rho) =: \widetilde{N}_E(t) \tag{4.5}$$

form a complete system of unitary invariants of the operator A (the measure E).

In other words, two selfadjoint operators A_i in H_i $(i = 1, 2)$ are unitarily equivalent if and only if $[E_{A_1}] \sim [E_{A_2}]$ and $\widetilde{N}_{A_1}(t) = \widetilde{N}_{A_2}(t) \,(\mathrm{mod}\, E_{A_1})$.

It can easily be shown that

$$n_A(t) = \widetilde{N}_A(t) \ (E\text{-a.e.}) \quad \text{and} \quad n(A) = \operatorname*{ess\,sup}_{t\in\mathbb{R}} \widetilde{N}_A(t) \,(\mathrm{mod}\, \rho), \tag{4.6}$$

i.e., $\widetilde{N}_A(t)$ coincides E-a.e. with the multiplicity $n_A(t)$ of the spectrum of A at the point t.

4.2. The multiplicity function of a nonorthogonal measure. The following definition plays a crucial role in the sequel.

Definition 4.5. Suppose Σ is an operator measure in H, ρ is a scalar measure with $\Sigma \prec \rho$, and $e := \{e_j\}_{j=1}^\infty$ is an orthonormal basis in H. Let

$$\begin{aligned} &\sigma_{ij}(t) := (\Sigma(t)e_i, e_j), \qquad \psi_{ij}(t) := d\sigma_{ij}(t)/d\rho, \\ &\Psi_n^e(t) := (\psi_{ij}(t))_{i,j=1}^n, \quad \Psi^e(t) := (\psi_{ij}(t))_{i,j=1}^\infty. \end{aligned} \tag{4.7}$$

The multiplicity function $N_\Sigma := N^e_\Sigma$ and the total multiplicity $m := m(\Sigma) := m^e(\Sigma)$ of Σ are defined by the formulas

$$N^e_\Sigma(t) := \operatorname{rank} \Psi^e(t) := \sup_{n\geq 1} \operatorname{rank} \Psi_n(t), \quad m^e := \operatorname*{ess\,sup}_{t\in\mathbb{R}} N_\Sigma(t) \pmod{\rho}. \tag{4.8}$$

Remark 4.1. a) Denote by $M_n(\Psi^e)$ the collection of all principal (not necessarily corner) submatrices of Ψ^e of size $n \times n$. It is easily seen that

$$\operatorname{rank} \Psi^e(t) := \sup\{\operatorname{rank} \Psi^e_n(t) : \Psi^e_n \in M_n(\Psi^e),\ n \geq 1\}.$$

b) It should be noted that the functions ψ_{ij} and the density Ψ^e are defined only ρ-a.e. At each point where Ψ^e exists, we have

$$\operatorname{rank} \Psi^e(t) \leq \inf_n \operatorname{rank} \Sigma(\Delta_n), \qquad \Delta_n = (t - 1/n, t + 1/n). \tag{4.9}$$

Furthermore, inequality (4.9) may be strict even if $\dim H < \infty$. For instance, let $\{f_j\}_1^n$ be a system of continuous linearly independent functions on $[0, 1]$ such that $\Pi_1^n f_j(t) \neq 0$ for $t \in [0, 1]$. Putting

$$\Psi(t) = \Psi_n(t) := \big(f_i(t)f_j(t)\big)_{i,j=1}^n \quad \text{and} \quad \Sigma(t) = \int_0^t \Psi(s)\, ds,$$

we see that $\operatorname{rank} \Psi(t) = 1$ for $t \in [0, 1]$, though $\operatorname{rank} \Sigma(\Delta) = n$ for every interval $\Delta = (a, b) \subset [0, 1]$.

Among other things, the following statement shows that Definition 4.5 does not depend of the choice of the basis e.

Proposition 4.6. *Suppose Σ is an operator measure in H, ρ is a scalar measure with $\Sigma \prec \rho$, and $L^2(\Sigma, H) = \int_{\mathbb{R}} \oplus H_t\, d\rho(t)$ is the expansion* (2.17) *of the space $L^2(\Sigma, H)$ in a direct integral. Then for every orthonormal basis $e := \{e_j\}_1^\infty$ in H we have*

$$N^e_\Sigma(t) = \dim H_t \pmod{\rho}. \tag{4.10}$$

In particular, the definition (4.8) *does not depend on the choice of an orthonormal basis.*

Proof. We introduce a diagonal operator K by putting $Ke_j = e_j/j$. Clearly, $K = K^* \in \mathfrak{S}_2$. Next, let $\Psi_K(t) = dK^*\Sigma(t)K/d\rho$ be the density of the form (2.14), and let the H_t be the Hilbert spaces constructed in Theorem 2.14 starting with Ψ_K (see (2.16)). It is easily seen that Ψ_K is related to the density Ψ^e of the form (4.7) as follows:

$$\big(\Psi_K(t)e_i, e_j\big)_{i,j=1}^\infty = K\Psi^e(t)K = \big(i^{-1}j^{-1}\psi_{ij}(t)\big)_{i,j=1}^\infty. \tag{4.11}$$

This yields

$$N^e_\Sigma(t) := \operatorname{rank} \Psi^e(t) = \operatorname{rank} \Psi_K(t) = \dim H_t =: N^K_\Sigma \pmod{\rho}. \tag{4.12}$$

But, by Theorem 2.14, the expansion (2.17) does not depend on the choice of K. Therefore, by (4.12), N^e_Σ does not depend on the choice of the basis e. □

Proposition 4.6 allows us to omit the index e in the definition (4.8). So, in what follows we write $N_\Sigma(t) := N^e_\Sigma(t)$ and $m := m^e$.

Corollary 4.7. *Let Σ and ρ be the same as in Proposition* 4.6. *Suppose $K \in \mathfrak{S}_2$ and $\ker K = \ker K^* = \{0\}$. Denote by $\Psi_K(t) := dK^*\Sigma(t)K/d\rho$ the density of the form* (2.14). *Then*

$$N_\Sigma(t) = \operatorname{rank} \Psi_K(t) =: N^K_\Sigma(t) \pmod{\rho}.$$

Remark 4.2. It can be shown that the total multiplicity $m(\Sigma)$ of the spectral measure Σ of the contraction $T = \int_0^{2\pi} e^{it}\, d\Sigma(t)$ coincides with the multiplicity $n(T)$ of T, $n(T) := \min\{\dim L : L \in \mathrm{Cyc}(T)\}$.

We note also that in [33], for an arbitrary linear operator T (in particular, for an arbitrary contraction), the so-called local spectral multiplicity function $m_{T,\mu}(z)$ with respect to a measure μ was introduced. That function vanishes off the spectrum $\sigma(T)$. If $\sigma(T)$ is a subset of the unit circle, the relationship between $N_\Sigma(t)$ and $m_{T,\mu}(z)$ is unclear.

However, if T is a strict contraction, these functions are in no way related to each other: $m_{T,\mu}(e^{it}) = 0$ for $t \in [0, 2\pi]$, but the function N_Σ is defined only on the unit circle.

The multiplicity function allows us to introduce the following definition.

Definition 4.8. i) An operator measure Σ_1 is said to be *spectrally subordinate* to an operator measure Σ_2 (in symbols: $\Sigma_1 \prec\!\prec \Sigma_2$) if $\Sigma_1 \prec \Sigma_2$ and $N_{\Sigma_1}(t) \le N_{\Sigma_2}(t)$ (Σ_2-a.e.).

ii) Two measures Σ_1 and Σ_2 are said to be *spectrally equivalent* if $\Sigma_1 \sim \Sigma_2$ and $N_{\Sigma_1} = N_{\Sigma_2}$ (Σ_1-a.e.).

We show that for an orthogonal measure E the new definition of the multiplicity function coincides with the classical definition (4.5).

Proposition 4.9. i) *Let E be an orthogonal measure in H, and let N_E be its multiplicity function in the sense of Definition* 4.5. *Then*

$$\widetilde{N}_E(t) = N_E(t) \ \ (E\text{-}a.e.) \quad \text{and} \quad n(E) = m(E), \tag{4.13}$$

where $\widetilde{N}_E(t)$, $n(E)$, and $m(E)$ are defined by (4.5), (4.2), *and* (4.8).

ii) *Let Σ be a generalized resolution of the identity, and let E be its minimal orthogonal dilation. Then Σ and E are spectrally equivalent, in particular,*

$$N_\Sigma(t) = N_E(t) = \widetilde{N}_E(t) \ \ (E\text{-}a.e.) \quad \text{and} \quad m(\Sigma) = m(E). \tag{4.14}$$

Proof. Since E coincides with its minimal orthogonal dilation, it suffices to prove ii).

But by Theorem 3.3 the dilation E is unitarily equivalent to the resolution E_Q of the identity for the operator Q of multiplication by t in $L^2(\Sigma, H)$. Consequently, the corresponding multiplicity functions coincide: $\widetilde{N}_E = \widetilde{N}_{E_Q}$. On the other hand, by Theorem 2.14 we have $\widetilde{N}_Q(t) = \dim H_t$. Comparing this with (4.5), we obtain

$$\widetilde{N}_Q(t) = \dim H_t = \operatorname{rank} \Psi(t) = N_\Sigma(t) \qquad (\Sigma\text{-a.e.}).$$

It remains to observe that $\widetilde{N}_Q(t) = \widetilde{N}_{E_Q}(t)$ by $\square$

Corollary 4.10. *Two multiplication operators $Q_i : f \mapsto tf$ in the spaces $L^2(\Sigma_i, H_i)$ $(i = 1, 2)$ are unitarily equivalent if and only if Σ_1 and Σ_2 are spectrally equivalent.*

To state another corollary, we recall the following definitions.

Let ρ be a scalar measure, $\rho \sim \Sigma$, and let $\rho = \rho_{\mathrm{ac}} + \rho_{\mathrm{sc}} + \rho_d$ be the Lebesgue decomposition of ρ into absolutely continuous, singular continuous and discrete parts. Let $\Delta_{\mathrm{ac}}, \Delta_{\mathrm{sc}}$, and Δ_d be mutually disjoint (not necessarily topological) supports of $\rho_{\mathrm{ac}}, \rho_{\mathrm{sc}}$, and ρ_d, respectively. Then, putting

$$\Sigma_{\mathrm{ac}}(\delta) := \Sigma(\delta \cap \Delta_{\mathrm{ac}}), \quad \Sigma_{\mathrm{sc}}(\delta) := \Sigma(\delta \cap \Delta_{\mathrm{sc}}), \quad \Sigma_d(\delta) := \Sigma(\delta \cap \Delta_d),$$

we obtain the Lebesgue decomposition of Σ.

Corollary 4.11. *Let $\Sigma = \Sigma_{\mathrm{ac}} + \Sigma_s + \Sigma_d$ be the Lebesgue decomposition of the measure Σ, and let $\widetilde{\Sigma} := \Sigma_{\mathrm{ac}} \oplus \Sigma_{\mathrm{sc}} \oplus \Sigma_d$. Then the operator measures Σ and $\widetilde{\Sigma}$ are spectrally equivalent.*

Proof. Consider the operator

$$V : L^2(\Sigma, H) \to L^2(\widetilde{\Sigma}, H), \qquad V : f \to \{f\chi_{\Delta_{\rm ac}}, f\chi_{\Delta_{\rm sc}}, f\chi_{\Delta_d}\}.$$

Since the supports of the measures $\Delta_{\rm ac}$, $\Delta_{\rm sc}$, and Δ_d are pairwise disjoint, we have

$$\|f\|^2_{L^2(\Sigma,H)} = \|f\|^2_{L^2(\Sigma_{\rm ac},H)} + \|f\|^2_{L^2(\Sigma_{\rm sc},H)} + \|f\|^2_{L^2(\Sigma_d,H)} = \|f\|^2_{L^2(\widetilde{\Sigma},H)},$$

and V maps $L^2(\Sigma, H)$ onto $L^2(\widetilde{\Sigma}, H)$ isometrically. Clearly, V intertwines the operators Q_1 and Q_2 of multiplication by t in these spaces: $VQ_1 = Q_2V$. By Corollary 4.10, the measures Σ and $\widetilde{\Sigma}$ are spectrally equivalent. □

In conclusion of this subsection, we present yet another criterion of spectral equivalence for two operator measures.

Proposition 4.12. *Suppose Σ_1 and Σ_2 are two operator measures in H, ρ is a scalar measure, $\Sigma_1 + \Sigma_2 \prec \rho$, and $\Psi_j = dK_j^*\Sigma_jK_j/d\rho$ is the operator density of Σ_j calculated in accordance with relation* (2.14), *in which K is replaced by K_j. Then*:

i) *if there exists a ρ-measurable operator-valued function $C(\cdot) : \mathbb{R} \to B(H)$ such that the densities Ψ_1 and Ψ_2 obey the relations*

$$\Psi_2(t) = C^*(t)\Psi_1(t)C(t) \ (\mathrm{mod}\,\rho), \quad \ker C(t) = \{0\} \ (\mathrm{mod}\,\rho), \tag{4.15}$$

then Σ_1 and Σ_2 are spectrally equivalent;

ii) *conversely, if Σ_1 and Σ_2 are spectrally equivalent and $N(t) := N_{\Sigma_1}(t) = N_{\Sigma_2}(t) < \infty$ $(\mathrm{mod}\,\rho)$, then there exists a ρ-measurable operator-valued function $C(\cdot) : \mathbb{R} \to B(H)$ such that Ψ_1 and Ψ_2 obey* (4.15).

Proof. i) Suppose (4.15) is fulfilled. By Proposition 4.6 and the definition (4.8), we have $N_{\Sigma_1}(t) = N_{\Sigma_2}(t)$ $(\mathrm{mod}\,\rho)$. Next, by (2.13), for every $j = 1, 2$ we have the following equivalence:

$$\Sigma_j(\delta) = 0,\ \delta \in \mathcal{B}(\mathbb{R}) \iff \Psi_j(t) = 0 \text{ for } \rho\text{-a.e. } t \in \delta. \tag{4.16}$$

On the other hand, (4.15) implies that $\Psi_1(t) = 0$ for ρ-a.e. $t \in \delta$ if and only if $\Psi_2(t) = 0$ for ρ-a.e. $t \in \delta$. By (4.16), this means that the measures Σ_1 and Σ_2 are equivalent.

ii) Let $E_j(\cdot, t)$ be the resolution of the identity for the operator $\Psi_j(t)$ $(j = 1, 2)$, and let

$$I - P_j(t) := \ker \Psi_j(t) = E_j(+0, t) - E_j(-0, t) = E_j\big((-\infty, 0], t\big), \quad j = 1, 2. \tag{4.17}$$

From (4.17) it is clear that $P_j(\cdot)$ is a ρ-measurable projection-valued function. Since $N(t) < \infty$ (mod ρ), we have $H_j(t) := P_j(t)H = \mathcal{R}(\Psi_j(t))$.

We choose an orthonormal basis $\{e_k\}_1^\infty$ in H and put $e_{kj}(t) := P_j(t)e_k$ and $\varphi_{kj}(t) := e_k - e_{kj}(t)$ $(j = 1, 2)$. It is easily seen that $\Omega_j := \{e_{kj}\}_{k=1}^\infty$ and $\Omega'_j := \{\varphi_{kj}\}_{k=1}^\infty$ form measurable bases in the families $H_j(t)$ and $H_j^\perp(t)$, respectively. This turns $H_j(t)$ and $H_j^\perp(t)$ into measurable Hilbert families on the measure space $\{\mathbb{R}, \mathcal{B}(\mathbb{R}), \rho\}$. By [6, Lemma 7.1.3], there are two operator-valued functions $U(t)$ and $V(t)$ that are defined ρ-a.e. on $\mathbb{R}$ and take $H_2(t)$ and $H_1(t)$ onto $H_2^\perp(t)$ and $H_1^\perp(t)$, respectively. Therefore, the operator-valued function $\widetilde{\Psi}_1(t) := \big(U^*(t) \oplus V^*(t)\big)\Psi_1(t)\big(U(t) \oplus V(t)\big)$ is also ρ-measurable and satisfies the relations

$$\ker \widetilde{\Psi}_1(t) = H_2^\perp(t), \qquad \mathcal{R}\big(\widetilde{\Psi}_1(t)\big) = H_2(t) \quad (\mathrm{mod}\ \rho). \tag{4.18}$$

By (4.18), the operator-valued function $C_1(t) := \widetilde{\Psi}_1(t)^{-1/2}\Psi_2(t)^{1/2}\lceil H_2(t)$ is well defined and takes values in $B\big(H_2(t)\big)$. Putting $\widetilde{C}(t) := C_1(t) \oplus I_t$, where I_t is the identity mapping of $H_2^\perp(t)$, we arrive at the identity $\widetilde{C}^*(t)\widetilde{\Psi}_1(t)\widetilde{C}(t) = \Psi_2(t)$ (mod ρ). Taking the definition of $\widetilde{\Psi}_1$ into account, we conclude that the operator-valued function $C(t) := U(t)\widetilde{C}(t)$ satisfies (4.15). □

The following example shows that the finite multiplicity condition for $N(t)$ is essential for the validity of statement ii) in Proposition 4.12.

Example 4.13. Suppose $T_j = T_j^* \in \mathfrak{S}_\infty(H)$ and $\ker T_j = \{0\}$ $(j = 1,2)$. Consider two discrete operator measures Σ_1 and Σ_2 in H with jumps at some points t_1 and t_2 that satisfy

$$\Sigma_j(t_1+0) - \Sigma_j(t_1-0) = T_1^j, \ \ \Sigma_j(t_2+0) - \Sigma_j(t_2-0) = T_2^{3-j} \quad (j = 1,2). \tag{4.19}$$

It is easily seen that (4.15) with a bounded operator-valued function $C(t)$ is impossible in this case.

4.3. A criterion for spectral equivalence, and cyclic subspaces. We start with the following result.

Proposition 4.14. *Let A_i be selfadjoint operators in H_i, and let $E_i := E_{A_i}$ be their resolutions of the identity. Then A_2 is unitarily equivalent to the restriction of A_1 to some subspace $L \in \operatorname{Lat} A_1$ if and only if $E_2 \prec\prec E_1$.*

Proof. i) Suppose A_2 is unitarily equivalent to $A_1 \lceil L$, i.e., there is an isometry U of L onto H_2 such that $U^{-1}A_2U = A\lceil L$. Since L reduces A_1, we have $E_2(t) = U\big(E_1(t)\lceil L\big)U^*$. Consequently, $E_2 \prec E_1$ and $n_{A_2}(t) \le n_{A_1}(t)$ for all $t \in \mathbb{R}$. But then $N_{E_2}(t) = n_{A_2}(t) \le n_{A_1}(t) = N_{E_1}(t)$; consequently, $E_2 \prec\prec E_1$.

ii) Suppose $E_2 \prec\prec E_1$. Let ρ_i be scalar Borel measures on $\mathbb{R}$ with $\rho_i \sim E_i$ $(i = 1,2)$. By Theorem 4.3, the operators A_1 and A_2 can be identified with the operators of multiplication by t of the form (4.4) in direct integrals

$$\mathfrak{H}_1 = \int_{\mathbb{R}} \oplus G_1(t)\, d\rho_1(t) \quad \text{and} \quad \mathfrak{H}_2 = \int_{\mathbb{R}} \oplus G_2(t)\, d\rho_2(t). \tag{4.20}$$

Since $\rho_2 \prec \rho_1$, we have $d\rho_2 = \varphi d\rho_1$ with $\varphi \ge 0$. Therefore, the second direct integral in (4.20) can be written in the form $\mathfrak{H}_2 = \int_{\mathbb{R}} \oplus G_2'(t)\, d\rho_1(t)$, where

$$G_2'(t) := \begin{cases} G_2(t) & \text{if } \varphi(t) > 0, \\ 0 & \text{if } \varphi(t) = 0. \end{cases}$$

Next, we choose an arbitrary ρ_1-measurable Hilbert family of subspaces $L(t)\big(\subset G_1(t)\big)$ such that $\dim L(t) = \dim G_2'(t)$, $t \in \mathbb{R}$, $t \in \mathbb{R}$. Then the subspace $L := \int_{\mathbb{R}} \oplus L(t)\, d\rho_1(t)$ is invariant under A_1 $(L \in \operatorname{Lat} A_1)$. By Theorem 7.2.2 in [6], there exists a measurable operator-valued function $U(t) : G_2'(t) \to L(t)$ that is unitary-valued for ρ_1-a.e. $t \in \mathbb{R}$. Then the operator $U := \int_{\mathbb{R}} \oplus U(t)\, d\rho_1(t) : (Uh)(t) = U(t)h(t)$ maps $\mathfrak{H}_2$ onto L unitarily, and we have $UA_2U^* = A_1\lceil L\cdot$. □

Theorem 4.15. *Let Σ be a generalized resolution of the identity in a Hilbert space H_1, let A be a selfadjoint operator in another Hilbert space H, and let $E := E_A$ be its spectral measure. Then the following statements are true.*

i) *The subordination $\Sigma \prec\prec E$ occurs if and only if there exists a Hilbert space $\widetilde{H} \supset H_1$ and a unitary operator $U : H \to \widetilde{H}$ such that*

$$\Sigma(t) := P_{H_1} U(E(t)\lceil U^* H_1), \qquad t \in \mathbb{R}, \tag{4.21}$$

where P_{H_1} is the orthogonal projection of $\widetilde{H}$ onto H_1.

ii) *If U^*H_1 is a cyclic subspace for A, then the measures Σ and E are spectrally equivalent. In particular* (*for $H = \widetilde{H}$ and $U = I$*), *the measure Σ and its minimal* (*orthogonal*) *dilation E are spectrally equivalent.*

iii) *If* Σ *and* E *are spectrally equivalent and* $N_E(t) < \infty$ E*-a.e.* (*for example, if* $n(A) < \infty$), *then the subspace* U^*H_1 *is cyclic for* E.

Proof. i) Clearly, (4.21) implies $\Sigma \prec E$.

Next, choosing an orthonormal basis in H_1 and extending it to a basis in $\widetilde{H}$, and then applying Definition 4.5 and Proposition 4.9 i), we deduce that $N_\Sigma(t) \le N_E(t)$ E-a.e.

Conversely, suppose $\Sigma \prec\prec E$. Let E_Σ be a minimal dilation for Σ in a space $\widetilde{H} \supset H_1$, and let A_Σ be the selfadjoint operator the spectral measure of which is E_Σ. By Proposition 4.9 ii), the measures E_Σ and Σ are spectrally equivalent, whence $E_\Sigma \prec\prec E$. Therefore, by Proposition 4.14, the operator A_Σ is unitarily equivalent to the restriction of A to some subspace $L \in \operatorname{Lat} A$, $L \subset H$, i.e., $U^*A_\Sigma U = A\lceil L$, where U maps L onto $\widetilde{H}$ unitarily. Consequently, $E_\Sigma(t) = U(E(t)\lceil L)U^*$. Together with the obvious inclusion $U^*H_1 \subset L$, this implies the relations

$$\Sigma(t) = P_{H_1}E_\Sigma(t)\lceil H_1 = P_{H_1}(U(E(t)\lceil L)U^*)\lceil H_1 = P_{H_1}U(E(t)\lceil U^*H_1),$$

as required.

ii) Suppose U^*H_1 is cyclic for A ($U^*H_1 \in \operatorname{Cyc} A$) and (4.21) is fulfilled. Then UEU^* is a minimal orthogonal dilation for Σ. By Proposition 4.9 ii), the measures E and Σ are spectrally equivalent.

iii) Conversely, suppose that (4.21) is fulfilled, the measures Σ and E are spectrally equivalent, and $N_A(t) < \infty$ E-a.e. We show that $U^*H_1 \in \operatorname{Cyc} A$. For this, we denote by L the minimal A-invariant subspace containing U^*H, and put $A_1 := A\lceil L$.

By Theorem 4.3, A is unitarily equivalent to the operator Q of multiplication by t of the form (4.4) in some direct integral $\mathfrak{H} = \int_{\mathbb{R}} \oplus G(t)\, d\rho(t)$ of Hilbert spaces, i.e., $U_1AU_1^* = Q$, where U_1 is a unitary mapping of H onto $\mathfrak{H}$. Then $L_1 := U_1L \in \operatorname{Lat} Q$ and, as is well known (see [6]), we have

$$L_1 = \int_{\mathbb{R}} \oplus G_1(t)\, d\rho(t), \qquad G_1(t) := \pi(t)G(t),$$

where $\pi(t)$ is a measurable projection-valued function.

By assumption, the measures E and Σ are spectrally equivalent. On the other hand, (4.21) and the condition $U^*H \in \operatorname{Cyc} A$ together mean that the spectral measure E_{A_1} of A_1 is an orthogonal dilation of Σ. By Proposition 4.9, the measures E_{A_1} and Σ are also spectrally equivalent.

Consequently, E and E_{A_1} are spectrally equivalent and, in particular, $\widetilde{N}_A(t) = \widetilde{N}_{A_1}(t)$. But then

$$\dim G(t) = \widetilde{N}_A(t) = \widetilde{N}_{A_1}(t) = \dim\bigl(\pi(t)G(t)\bigr) \quad \rho\text{-a.e.}$$

Therefore, $G(t) = G_1(t) = \pi(t)G(t)$ ρ-a.e., i.e., $\pi(t) = I_{G(t)}$ ρ-a.e. and $\mathfrak{H} = L_1$, yielding $L = H$ and $U^*H_1 \in \operatorname{Cyc} A$. □

Corollary 4.16. *Under the assumptions of Theorem* 4.15, *suppose that* $\widetilde{N}_A(t) < \infty$ E*-a.e. and that* (4.21) *is fulfilled. Then the operator measures* Σ *and* E *are spectrally equivalent if and only if* $U^*H_1 \in \operatorname{Cyc} A$.

Remark 4.3. In Theorem 4.15 and in Corollary 4.16, the condition $N_E(t) < \infty$ E-a.e. is essential for the relation $U^*H_1 \in \operatorname{Cyc} A$.

For example, if $A = I$ is the identity operator in H, then for every infinite-dimensional subspace L ($\subset H$) the spectral measures $E_A\lceil L$ and E_A are unitarily equivalent (therefore, spectrally equivalent).

In fact, it can be proved that the condition $N_E(t) < \infty$ E-a.e. is necessary and sufficient for the validity of condition iii) in Theorem 4.15.

Corollary 4.17. *Consider a selfadjoint operator A in H with spectral measure E_A. Let $L \in \mathrm{Cyc}(A)$, let $P := P_L$ be the orthogonal projection of H onto L, and let $\Sigma := PE\lceil L$. Then A is unitarily equivalent to the operator Q of multiplication by t of the form (4.4) in the space $L^2(\Sigma, H)$.*

Proof. The condition $L \in \mathrm{Cyc}\, A$ means that E_A is a minimal orthogonal dilation for Σ. By Theorem 3.3, the spectral measures E_A and E_Q of the operators A and Q are unitarily equivalent. Consequently, A and Q are unitarily equivalent. □

In conclusion of this subsection, we present some conditions that guarantee the spectral subordination of measures.

Proposition 4.18. *Let Σ_i be an operator measure in H_i $(i = 1, 2)$. Then*:

i) *if*

$$\Sigma_1(\delta) \le T^*\Sigma_2(\delta)T, \qquad \delta \in \mathcal{B}_b(\mathbb{R}), \tag{4.22}$$

for some operator $T \in B(H_1, H_2)$, then $\Sigma_1 \prec\prec \Sigma_2$;

ii) *if $\Sigma_2 =: E$ is an orthogonal resolution of the identity*, *then* (4.22) *implies the relation $\Sigma_1 = K^*\Sigma_2 K$ with some $K \in B(H_1, H_2)$;*

iii) *if there exists a nonnegative function $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}, d\Sigma_2)$ with*

$$\Sigma_1(\delta) = \int_\delta f(t)\, d\Sigma_2(t), \qquad \delta \in \mathcal{B}_b(\mathbb{R}) \tag{4.23}$$

(*the integral is understood in the weak sense*), *then $\Sigma_1 \prec\prec \Sigma_2$.*

Proof. i) Clearly, (4.22) implies that $\Sigma_1 \prec \Sigma_2$. Let ρ be a scalar measure equivalent to Σ_2, and let $\Psi_{1n}(t)$ and $\Psi_{2n}(t)$ be principal $(n \times n)$-submatrices of the density matrices $\Psi_1(t)$ and $\Psi_2(t)$ (the latter matrices are constructed starting with ρ and, respectively, Σ_1 and Σ_2 in accordance with Definition 4.5). Then

$$\Psi_{1n}(t) \le \Psi_{2n}(t) \,(\mathrm{mod}\, \rho) \quad \text{for all } n \ge 1.$$

Therefore, $N_{\Sigma_1}(t) \le N_{\Sigma_2}(t)$ (mod ρ) and, consequently, $\Sigma_1 \prec\prec \Sigma_2$.

ii) The required statement follows from i) and Theorem 4.15 i).

iii) For every n, formula (4.23) implies the estimate $\Sigma_1(\delta) \le C_n\Sigma_2(\delta)$, $\delta \in \mathcal{B}([-n, n])$, with a constant $C_n > 0$ independent of δ. It remains to apply i). □

4.4. Application to operators of class $\mathfrak{C}_\rho(H)$. In this subsection we supplement Theorem 3.7 in the case where $r(T) < 1$ by finding the multiplicity function of the minimal unitary ρ-dilation U for an operator $T \in \mathfrak{C}_\rho(H)$.

Proposition 4.19. *If $T \in \mathfrak{C}_\rho(H)$ and $r(T) < 1$, then the minimal unitary ρ-dilation U of T is absolutely continuous, and its multiplicity function N_U is given by the formula*

$$N_U(t) = \mathrm{rank}[2 \cdot I + (\rho - 2)(I + T^*T) + (1 - \rho)e^{it}T + (1 - \rho)e^{-it}T^*]. \tag{4.24}$$

Proof. Since $r(T) < 1$, relation (3.17) is fulfilled, in which Σ is a generalized resolution of the identity. On the other hand, from the proof of Theorem 3.7 it follows that the resolution $E(t)$ of identity for the minimal ρ-dilation U is a minimal orthogonal dilation for $\Sigma(t)$. By Proposition 4.9, $N_E = N_\Sigma$ E-a.e., that is, $N_U = N_E$ coincides E-a.e. with the rank of the density $d\Sigma(t)/dt$, i.e.,

$$N_U(t) = \mathrm{rank}[(I - e^{it}T)^{-1} + (I - e^{-it}T^*)^{-1} + (\rho - 2)]. \tag{4.25}$$

Combining (4.25) with the obvious identity

$$\begin{aligned} &(I - zT)^{-1} + (I - \bar{z}T^*)^{-1} + (\rho - 2) \cdot I \\ &\qquad = (I - \bar{z}T^*)^{-1}[2 \cdot I + (\rho - 2)(I + |z|^2T^*T) + (1 - \rho)zT + (1 - \rho)\bar{z}T^*](I - zT)^{-1}, \end{aligned} \tag{4.26}$$

we arrive at the required formula (4.24). □

Corollary 4.20. *Suppose $T \in B(H)$ and $r(T) < 1$. Then*

i) *if $T \in \mathfrak{C}_2(H)$, i.e., the numerical radius $w(T)$ of T does not exceed 1, then the multiplicity function $N_U(t)$ of the 2-unitary dilation U for T is given by the formula*

$$N_U(t) = \operatorname{rank}(2 \cdot I - e^{it}T + e^{-it}T^*); \tag{4.27}$$

ii) *if T is a contraction, then the multiplicity function $N_U(t)$ of its unitary dilation U is constant and is given by*

$$N_U(t) = \operatorname{rank}(I - T^*T) = \operatorname{rank}(I - TT^*). \tag{4.28}$$

Remark 4.4. Formula (4.24) shows that, in general, for $\rho \neq 1$ the multiplicity function $N_U(t)$ of the ρ-unitary dilation U depends on t.

On the contrary, (4.28) shows that for $\rho = 1$ this function does not depend on t.

As in [20], we attribute a contraction T in H to the class $C_{0\cdot}$ ($C_{\cdot 0}$) if $\text{s-lim}_{n\to\infty} T^n = 0$ ($\text{s-lim}_{n\to\infty} T^{*n} = 0$). Clearly, $T \in C_{0\cdot} \cap C_{\cdot 0}$ if $r(T) < 1$.

In the following proposition, we supplement Corollary 3.8 and Proposition 4.19 for operators $T \in C_{0\cdot}$ by dropping the condition $r(T) < 1$.

Proposition 4.21. *Let $T \in C_{0\cdot}$. Then*:

i) *the spectral measure Σ of T (see the representation (3.15)) has the form*

$$\big(\Sigma(\delta)f, f\big) = \int_\delta \|\Psi(e^{it})f\|^2\,dt, \qquad \delta \in \mathcal{B}([0, 2\pi]),\ \ f \in H, \tag{4.29}$$

where $\Psi(e^{it})f := \lim_{r\uparrow 1} \Psi(re^{it})f$ and $\Psi(re^{it}) := D_T(I - re^{it}T)^{-1}$;

ii) *$N_U(t) = N_\Sigma(t) = \operatorname{rank}(I - T^*T) =: \delta_T$ for Σ-a.e. $t \in [0, 2\pi]$.*

Proof. i) By Proposition 3.10 (see relation (3.19)), the vector-valued function $\Psi(z)f := D_T(I - zT)^{-1}f$ belongs to the vector Hardy class $H^2(\mathbb{D}, H_T)$ for every $f \in H$ (here $H_T := \overline{D_T H}$)). Therefore, the boundary values $\Psi(e^{it})f$ exist a.e. as well as in $L^2((0, 2\pi), H_T)$: $\|\Psi(re^{it})f - \Psi(e^{it})f\|_{L^2} \to 0$ as $r \to 1$.

Applying the Stieltjes inversion formula to (3.14) and taking (4.26) with $\rho = 1$ into account, we obtain

$$\begin{aligned}\big((\Sigma(b) - \Sigma(a))f, f\big) &= \lim_{r\uparrow 1} \int_a^b \|(1 - r^2T^*T)^{1/2}(I - re^{it}T)^{-1}f\|^2\,dt \\ &= \lim_{r\uparrow 1}(1 - r^2)\int_a^b \|(1 - re^{it}T)^{-1}f\|^2\,dt + \lim_{r\uparrow 1} r^2 \int_a^b \|D_T(I - re^{it}T)^{-1}f\|^2\,dt.\end{aligned} \tag{4.30}$$

Since $T \in C_{0\cdot}$, by the Parseval identity and the Abel theorem we obtain

$$\begin{aligned}\lim_{r\uparrow 1}(1 - r^2)\int_0^{2\pi} \|(1 - re^{it}T)^{-1}f\|^2\,dt &= \lim_{r\uparrow 1}(1 - r^2)\sum_{k=1}^\infty r^{2k}\|T^kf\|^2 \\ &= \|f\|^2 + \lim_{r\uparrow 1}\sum_{k=1}^\infty r^{2k}(\|T^kf\|^2 - \|T^{k-1}f\|^2) \\ &= \|f\|^2 + \sum_{k=1}^\infty(\|T^kf\|^2 - \|T^{k-1}f\|^2) = \lim_{k\to\infty}\|T^kf\|^2 \\ &= 0.\end{aligned} \tag{4.31}$$

Combining (4.30) and (4.31), we arrive at (4.29) with $\Delta = (a, b) \subset [0, 2\pi]$. Now the measure Σ extends to the entire σ-algebra $\mathcal{B}([0, 2\pi])$ by formula (4.29).

ii) Since the operator T is completely nonunitary, we have $\ker(I - e^{it}T) = \{0\}$, $t \in [0, 2\pi]$. Therefore, by (4.29), the density Ψ of Σ (in the sense of Definition 4.5) has the form $\Psi^{1/2}(t) = V(t)D_T(I - e^{it}T)^{-1}$, where $V(t)$ is a partial isometry, $t \in [0, 2\pi]$. Now it is clear that

$$\begin{aligned}\operatorname{rank}\Psi^{1/2}(t) &= \operatorname{rank} D_T(I - e^{it}T)^{-1}\\ &= \operatorname{rank}(I - e^{-it}T^*)^{-1}D_T = \delta_T \quad \text{for a.e. } t \in [0, 2\pi].\end{aligned}$$

To clarify, we mention that if $\delta_T = \infty$, then the basis $\{e_i\}_1^\infty$ in Definition 4.5 must be taken in the linear space $\mathcal{D}\big((I - e^{-it}T^*)^{-1}D_T\big)$. □

Remark 4.5. a) Proposition 4.21 gives yet another (and even still more elementary) proof of Theorem 3.13, but only for the operators $T \in C_{0\cdot}$ (resp. $C_{\cdot 0}$). Moreover, comparison of Proposition 4.21 and Corollary 4.4 gives an elementary proof of the following well-known statement (see [26], [29]): the minimal unitary dilation U of an operator $T \in C_{0\cdot}$ (resp. $C_{\cdot 0}$) is the bilateral shift of multiplicity $\operatorname{rank} D_T$ (resp. $\operatorname{rank} D_{T^*}$).

b) Since T is a contraction, the limit $\lim_{k\to\infty}\|T^k f\|$ exists for every $f \in H$. Therefore, by (4.31) and the Abel theorem, we obtain

$$T \in C_{0\cdot} \iff \lim_{r\uparrow 1}(1 - r^2)\int_0^{2\pi}\|(I - re^{it}T)^{-1}f\|^2\,dt = 0, \quad f \in H. \tag{4.31a}$$

Comparison with the identity

$$\|f\|^2 - \|\theta_{T^*}(re^{it})f\|^2 = (1 - r^2)\|(I - re^{it}T)^{-1}D_{T^*}f\|^2 =: \Phi_{*f}(re^{it}) \tag{4.31b}$$

for the characteristic function θ_{T^*} of T^* proves the "only if" part of the following well-known (see [26], [29]) equivalence:

$$T \in C_{0\cdot} \text{ (resp. } C_{\cdot 0}) \quad \text{if and only if} \quad \theta_{T^*} \text{ (resp. } \theta_T) \text{ is inner.}$$

Indeed, since θ_{T^*} has strong radial limits a.e., by (4.31b) the limit $\lim_{r\uparrow 1}\Phi_{*f}(re^{it}) := \Phi_{*f}(e^{it})$ exists. By (4.31a), $\Phi_{*f}(e^{it}) = 0 \pmod{m}$, i.e., θ_{T^*} is inner.

The proof of the "if" part of the above equivalence (for a simple contraction T) is only slightly more difficult.

We note also that the canonical triangulation of T (see [20]) implies that the integral condition (4.31a) characterizes the singular subspace H_1 of T, i.e., the maximal subspace $H_1 \in \operatorname{Lat} T$ for which $T\lceil H_1 \in C_{0\cdot}$.

Theorem 4.22 (see [29])**.** *Suppose T is a completely nonunitary contraction with finite defect numbers $\delta_T = n$ and $\delta_{T^*} = m$. Then the multiplicity function $N_U(t)$ of the minimal unitary dilation U of T is computed by the formula*

$$N_U(t) = m + \operatorname{rank}\Delta_T(t), \tag{4.32}$$

where $\Delta_T(t) := \big(I - \theta_T^*(e^{it})\theta_T(e^{it})\big)^{1/2}$ and $\theta_T(\lambda)$ is the characteristic function of T.

Proof. By the model theorem [29, Theorem 6.3.1], U is unitarily equivalent to the operator of multiplication by e^{it} in the space

$$\mathfrak{H} = L_m^2(0, 2\pi) \oplus \overline{\Delta L_n^2(0, 2\pi)} =: \mathfrak{H}_1 \oplus \mathfrak{H}_2, \tag{4.33}$$

where $L_n^2(0, 2\pi) = L^2(0, 2\pi) \otimes \mathbb{C}^n$.

Putting $\Sigma(t) := \int_0^t \Delta^2(s)\,ds$, we consider the spaces $\widetilde{L}^2(\Sigma, \mathbb{C}^n)$ and $L^2(\Sigma, \mathbb{C}^n)$. Since $0 \le \Delta(t) \le I_n$, we have $L_n^2(0, 2\pi) \subset \widetilde{L}^2(\Sigma, \mathbb{C}^n)$. Moreover, since the seminorms (2.2) and (2.11) coincide (see the proof of Theorem 2.11), we arrive at the identity

$$\|\Delta f\|^2_{L_n^2(0,2\pi)} = \int_0^{2\pi}\|\Delta(t)f(t)\|^2_{\mathbb{C}^n}\,dt = \|f\|^2_{L^2(\Sigma,\mathbb{C}^n)}, \qquad f \in L_n^2(0, 2\pi). \tag{4.34}$$

Therefore, the mapping

$$\widetilde{V} : \mathfrak{H}_2' := \Delta L_n^2(0, 2\pi) \to L^2(\Sigma, \mathbb{C}^n), \qquad \widetilde{V} : \Delta f \mapsto f,$$

is well defined: if $\Delta f = 0$, then the function f is equivalent to 0 in $\widetilde{L}^2(\Sigma, H)$. Moreover, (4.34) means that the mapping $V := \pi \cdot \widetilde{V}$, where π is the quotient map of $\widetilde{L}^2(\Sigma, H)$ into $L^2(\Sigma, H)$, is an isometry of $\mathfrak{H}_2'$ into $L^2(\Sigma, \mathbb{C}^n)$. Since the space $\widetilde{V}(\mathfrak{H}_2') = L_n^2(0, 2\pi)$ is dense in $L^2(\Sigma, \mathbb{C}^n)$, the mapping V extends by continuity to a unitary operator V from $\mathfrak{H}_2$ onto $L^2(\Sigma, \mathbb{C}^n)$.

Next, clearly, U is unitarily equivalent to the operator Q of multiplication by e^{it} in $L^2(\Sigma_1, \mathbb{C}^{n+m})$, where $\Sigma_1(t) = I_m t \oplus \Sigma(t)$, $t \in [0, 2\pi]$. Consequently, $N_U(t) = N_Q(t)$. But the second identity in (2.12) yields

$$N_Q(t) = N_{\Sigma_1}(t) = \operatorname{rank}\big(d\Sigma_1(t)/dt\big) = m + \operatorname{rank} \Delta^2(t) = m + \operatorname{rank} \Delta(t) \text{ a.e.}$$

□

Remark 4.6. Comparison of Theorem 4.22 with Theorem 4.3 shows that U is unitarily equivalent to the operator of multiplication by e^{it} on the space $L_m^2(0, 2\pi) \bigoplus_{k=1}^n L^2(Y_k)$, where $Y_k := \{t \in (0, 2\pi) : \operatorname{rank} \Delta(t) \geq k\}$. This result coincides with [29, Theorem 6], where a different proof was given, though the functional model was also involved.

§5. Vectors of maximal type

An arbitrary operator measure Σ in H generates the family $\mu_f : \delta \mapsto (\Sigma(\delta) f, f)$ of σ-finite scalar measures on the σ-algebra $\mathcal{B}(\mathbb{R})$. Clearly, $\mu_f \prec \Sigma$ for all $f \in H$. It is well known [1], [6] that any orthogonal measure E in H possesses elements f of maximal type (principal vectors), i.e., elements f with the property $\mu_f \sim E$.

In this section we study the structure of the set

$$\Omega_\Sigma := \{f \in H : \Sigma \sim \mu_f,\ \mu_f(\delta) := \big(\Sigma(\delta) f, f\big),\ \delta \in \mathcal{B}_b(\mathbb{R})\} \tag{5.1}$$

of vectors of maximal type (principal vectors) for a not necessarily orthogonal operator measure Σ. In particular, we shall prove that this set is nonempty.

Some of the results are new even for the orthogonal measures.

We present two approaches to the problem, which lead to different results. Each of the methods has its own merits.

5.1. The set Ω_Σ. The first approach. This method is elementary and it is new even for an orthogonal measure. It gives some principal vectors in an "explicit" form.

We need a series of auxiliary lemmas.

Lemma 5.1. *Let H be a separable Hilbert space. There exists a system $\{h_\alpha\}_{\alpha \in I}$, $I = [0, 1]$, of vectors in H that has the power of the continuum and possesses the property that each countable subsystem of it is complete in H.*

Proof. Let $\{e_j\}_{j=0}^\infty$ be an orthonormal basis in H, and let $\varepsilon > 0$. We claim that the formula

$$f_\alpha := \sum_{j=0}^\infty \alpha^j e_j, \qquad \alpha \in [0, 1 - \varepsilon], \tag{5.2}$$

defines a required system, i.e., the system $\{f_{\alpha_i}\}_{i=0}^\infty$ is complete in H for every collection of pairwise different $\alpha_i \in [0, 1 - \varepsilon]$. Suppose a vector $x = \sum_{j=0}^\infty x_j e_j$ is orthogonal to the system $\{f_{\alpha_i}\}_{i=0}^\infty$, i.e.,

$$\sum_{j=0}^\infty x_j \alpha_i^j = 0, \qquad i \in \mathbb{N}. \tag{5.3}$$

Since $\sum_{j=1}^{\infty}|x_j|^2<\infty$, the function $g(z)=\sum_{j=0}^{\infty}x_jz^j$ is analytic in $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ (and even $g\in H^2(\mathbb{D})$). Condition (5.3) means that $g(\alpha_i)=0$, $i\in\mathbb{N}$. Since the sequence $\{\alpha_i\}$ has a limit point $\alpha^0\in[0,1-\varepsilon]$, we have $g\equiv 0$ by the uniqueness theorem. So, $x=0$, as required. □

Originally, the following lemma was proposed by the authors as a conjecture to L. L. Oridoroga, to whom we are indebted for an initial proof of it; see [24].

Lemma 5.2. *Let μ be a scalar measure on $\mathbb{R}$, and let $\delta_\alpha\in\mathcal{B}(\mathbb{R})$ for $\alpha\in I$, where I is an uncountable set. If $0<\mu(\delta_\alpha)<\infty$ for all $\alpha\in I$, then among these sets there is a countable subsystem $\{\delta_{\alpha_j}\}_{j=1}^{\infty}$, $\alpha_j\in I$, such that $\mu\big(\bigcap_{j=1}^{\infty}\delta_{\alpha_j}\big)>0$.*

Proof. Clearly, $\|\chi_{\delta_1}-\chi_{\delta_2}\|_{L_1(\mu,\mathbb{R})}=\mu(\delta_1\triangle\delta_2)$, where χ_δ is the indicator of the set δ and $A\triangle B$ is the symmetric difference of the sets A and B.

Since the space $L_1(\mu,\mathbb{R})$ is separable, every uncountable subset of it contains a non-isolated point. In particular, the set $\{\chi_\alpha:=\chi_{\delta_\alpha}\}_{\alpha\in I}$ contains a condensation point, say, $\chi_\beta:=\chi_{\delta_\beta}$. Therefore, since $L_1(\mu,\mathbb{R})$ is a metric space, there is a sequence $\alpha_j\in I$ such that

$$\|\chi_j-\chi_\beta\|_{L_1(\mu,\mathbb{R})}=\mu(\delta_\beta\triangle\delta_{\alpha_j})\to 0\quad \text{as } j\to\infty\ (\chi_j:=\chi_{\alpha_j}).$$

Passing, if necessary, to a subsequence, we can assume that $\mu(\delta_\beta\triangle\delta_{\alpha_j})\le 2^{-j-1}\mu(\delta_\beta)$. Therefore,

$$\begin{aligned}\mu\Big(\bigcap_{j=1}^{\infty}\delta_{\alpha_j}\Big)&\ge\mu\Big(\delta_\beta\bigcap_{j=1}^{\infty}\delta_{\alpha_j}\Big)\\&\ge\mu(\delta_\beta)-\sum_{j=1}^{\infty}\mu(\delta_\beta\triangle\delta_{\alpha_j})\\&\ge\mu(\delta_\beta)\Big[1-\sum_{j=1}^{\infty}2^{-j-1}\Big]=\frac{1}{2}\mu(\delta_\beta).\end{aligned}$$

So, $\{\delta_{\alpha_j}\}_{j=1}^{\infty}$ is the required sequence. □

Proposition 5.3. *Suppose Σ is an operator measure in H and $\{e_j\}_0^{\infty}$ is an orthonormal basis in H. Then the set Ω_Σ defined by* (5.1) *contains all vectors of the form* (5.2) *except for an at most countable subset.*

Proof. Let ρ be a scalar measure equivalent to Σ (see (2.15)).

Denoting by M the set of vectors of the form (5.2), we show that the set $M\setminus\Omega_\Sigma=:\{h_\alpha\}_{\alpha\in I}$ is at most countable. Assume the contrary. For every $h_\alpha\in M\setminus\Omega_\Sigma$ ($\alpha\in I$), we find a set $\delta_\alpha\in\mathcal{B}_b$ of positive Σ-measure ($\Sigma(\delta_\alpha)\ne 0$) such that $\mu_h(\delta_\alpha):=(\Sigma(\delta_\alpha)h_\alpha,h_\alpha)=0$, i.e., $\Sigma(\delta_\alpha)h_\alpha=0$.

Since, by assumption, $M\setminus\Omega_\Sigma$ is uncountable, by Lemma 5.2 there exists a sequence $\alpha_j\in I$ such that

$$\delta_0:=\bigcap_1^{\infty}\delta_{\alpha_j}\qquad \textit{and}\qquad \rho(\delta_0)>0.$$

Since $\rho\sim\Sigma$, we have $\Sigma(\delta_0)\ne 0$. Next, $\Sigma(\delta_{\alpha_j})\ge\Sigma(\delta_0)$ and $\Sigma(\delta_{\alpha_j})h_{\alpha_j}=0$. Therefore,

$$0\le(\Sigma(\delta_0)h_{\alpha_j},h_{\alpha_j})\le(\Sigma(\delta_{\alpha_j})h_{\alpha_j},h_{\alpha_j})=0,\qquad j\in\mathbb{N},$$

whence it follows that $\Sigma(\delta_0)h_{\alpha_j}=0$ for all $j\in\mathbb{N}$. But by Lemma 5.1 the system $\{h_{\alpha_j}\}_{j=1}^{\infty}$ is complete in H; consequently, $\Sigma(\delta_0)=0$. This contradicts the fact that $\Sigma(\delta_0)\ne 0$. □

5.2. The set Ω_E. The case of an orthogonal measure $\Sigma = E$. Our second approach is less elementary, but yields more complete information about the structure of Ω_Σ.

We need several easy lemmas.

Lemma 5.4. *Suppose μ is a finite Borel measure on $\mathbb{R}$, $X = L^p(\mu, \mathbb{R})$ ($p \in [1, \infty]$), and L is a subspace of X. Then for every $a > 0$ the subset*

$$S_a(L) := \{f \in L : \mu\{x \in \mathbb{R} : f(x) = 0\} \geq a\} \tag{5.4}$$

is closed in X.

Proof. Assume the contrary. Then there exists a sequence $f_n \in S_a(L)$ with $f_n \to \varphi \in L^p(\mu, \mathbb{R})$ as $n \to \infty$ but $\varphi \notin S_a(L)$. This means that

$$\mu(K) = \mu(\mathbb{R}) - a + \varepsilon, \qquad \text{where} \;\; K := \{x \in \mathbb{R} : \varphi(x) \neq 0\}.$$

Since $\mu(\mathbb{R}) < \infty$, from the Lusin and Egorov theorems we deduce the existence of a closed set $K_1 \subset K$ such that $\mu(K_1) = \mu(\mathbb{R}) - a + \varepsilon/2$, the functions $f_n \lceil K_1$ and $\varphi \lceil K_1$ are continuous, and the sequence f_n converges uniformly on K_1. Consequently, there exists a number N such that $f_n(x) \neq 0$ for all $n \geq N$ and all $x \in K_1$. But this contradicts the assumption $f_n \in S_a(L)$ because $\mu(K_1) = \mu(\mathbb{R}) - a + \varepsilon/2$. □

For the spaces $L^p(\mu, \mathbb{R})$ with an infinite measure, Lemma 4.5 is not true in general. The next counterexample was indicated to us by V. I. Vasyunin.

Example 5.5. Consider the subspace $L = \operatorname{span}\{\chi_k := \chi_{[k,k+1]} : k \in \mathbb{Z}\}$ of $L^2(m, \mathbb{R})$, where m is the Lebesgue measure and χ_k is the characteristic function of the segment $[k, k+1]$. Clearly, $L = \{\sum_{k\in\mathbb{Z}} c_k\chi_k : \sum_{k\in\mathbb{Z}} |c_k|^2 < \infty\}$, and the linear set $L_0 = \{\sum_m^n c_k\chi_k : m, n \in \mathbb{Z}\}$ is dense in L. At the same time, $S_a(L) \supset L_0$ for every $a > 0$ and, therefore, $S_a(L)$ is not closed.

Lemma 5.6. *Suppose μ is a finite Borel measure on $\mathbb{R}$. Let $X = L^p(\mu, \mathbb{R})$, and let L be a subspace of X containing a function $g \neq 0 \pmod{\mu}$. Then the set*

$$S(L) := \{f \in L : \mu\{x \in \mathbb{R} : f(x) = 0\} > 0\} \tag{5.5}$$

is of the first category in L and is an F_σ-set.

Proof. Clearly,

$$S(L) = \bigcup_{a\in\mathbb{R}_+} S_a(L) = \bigcup_{n=1}^{\infty} S_{1/n}(L), \tag{5.6}$$

where the sets $S_a(L)$ were defined in (5.4). Consequently, $S(L)$ is an F_σ-set.

Next, let $f \in S_a(L)$. Since the measure μ is finite, the function $f + \varepsilon g = (\varepsilon + f/g)g$ may vanish on a set of positive measure only for at most countably many values of ε. So, $f + \varepsilon g \in S_a(X)$ for at most countably many values of ε. Therefore, $S_a(L)$ is contained in the closure of $L \setminus S_a(L)$. Since the set $S_a(L)$ is closed in L, it is nowhere dense in L. By (5.6), $S(L)$ is of the first category in L. □

Now, we are able to prove the first main result of the present section.

Theorem 5.7. *For a selfadjoint operator A in H, let E be its spectral measure, and let $\Omega_A := \Omega_E$ be the set of principal vectors for E. If $L \in \operatorname{Cyc}(A)$, then:*

i) *$\Omega_A \cap L (= \Omega_E \cap L)$ is a G_δ-subset of the second category and is dense in L;*

ii) *$\omega(\Omega_A \cap L) = 1$ for every Gaussian measure ω in L.*

Proof. We split the proof in several steps, restricting ourselves to the case where $\dim L = \infty$. The case where $\dim L < \infty$ is much simpler.

1) At the first step we restate the condition $f \in \Omega_E \cap L$ in the language of function theory. Let $\{f_j\}_{j=1}^{\infty}$ be an orthonormal basis in L.

Putting $g_1 := f_1$, we use induction to define a system $\{g_j\}_1^N$ of pairwise spectrally orthogonal vectors with $n(A) \le N \le \infty$. Specifically, we put $g_k := I - P_{k-1} f_{n_k}$, where $n_1 = 1$ and, for $k \ge 2$, n_k is the minimal natural number satisfying

$$f_{n_k} \notin \mathfrak{H}_{k-1} := \operatorname{span}\{H_{f_j} : j \in \{n_1, \dots, n_{k-1}\}\} = \operatorname{span}\{H_{f_j} : 1 \le j \le n_{k-1}\},$$

and P_{k-1} is the orthogonal projection of H onto $\mathfrak{H}_{k-1}$. We recall that

$$H_f := \operatorname{span}\{(A - \lambda)^{-1} f : \lambda \in \mathbb{C} \setminus \mathbb{R}\}.$$

It is easily seen that g_i and g_j are spectrally orthogonal if $i \ne j$, and that the system $\{g_j\}_1^N$ is equivalent to $\{f_j\}_1^{\infty}$, i.e.,

$$\mathfrak{H}_k = \operatorname{span}\{H_{f_j} : 1 \le j \le n_k\} = \bigoplus_{j=1}^{k} H_{g_j}, \quad k \le N.$$

Since $L \in \operatorname{Cyc} A$, we have $H = \bigoplus_{j=1}^N H_{g_j}$.

We denote by g_{kj} the orthogonal projection of f_k to H_{g_j}, $1 \le j \le k$ (by definition, $g_{kj} = 0$ for $j > \min\{n_k, N\}$) and observe that $g_{n_k k} = g_k$.

Since $g_{kj} \in H_{g_j}$, there exists a function $\varphi_{kj} \in L^2(\mu_j, \mathbb{R})$ (here $\mu_j := \mu_{g_j} := (E_t g_j, g_j)$) with $g_{kj} = \varphi_{kj}(A) g_j$ and $\varphi_{n_k k} = 1$. But the system $\{f_k\}_1^{\infty}$ is orthonormal, whence we see that the series $\sum_k \lambda_k f_k$ converges in H if $\overrightarrow{\lambda} = \{\lambda_k\}_1^{\infty} \in l^2$. Since $f_k = \sum_{j=1}^k g_{kj}$, we have

(5.7)
$$f(\overrightarrow{\lambda}) := \sum_{k=1}^{\infty} \lambda_k f_k = \sum_{k=1}^{\infty} \lambda_k \left(\sum_{j=1}^{k} g_{kj} \right) = \sum_{j=1}^{\infty} \left(\sum_{k=j}^{\infty} \lambda_k g_{kj} \right) = \sum_{j=1}^{\infty} h_j(\overrightarrow{\lambda}), \quad \overrightarrow{\lambda} \in l^2,$$

where

$$h_j(\overrightarrow{\lambda}) := h_j(A, \overrightarrow{\lambda}) g_j \in H_{g_j} \text{ and } h_j(t, \overrightarrow{\lambda}) := \sum_{k=j}^{\infty} \lambda_k \varphi_{kj}(t), \quad j \in \mathbb{N}, \tag{5.8}$$

and $\varphi_{n_j j} = 1$. Next, let μ be a finite Borel measure equivalent to E ($\mu \sim E$ and $\mu(\mathbb{R}) < \infty$). By the Radon–Nikodym theorem, every measure μ_j is of the form $\mu_j(\delta) = \int_\delta p_j \, d\mu$, where $p_j \ge 0$ and $p_j \in L^1(\mu, \mathbb{R})$. Since the elements h_j are pairwise spectrally orthogonal for every $\overrightarrow{\lambda} \in l^2$, from (5.7) it follows that for every $\delta \in \mathcal{B}(\mathbb{R})$ and $h_j(t) := h_j(t, \overrightarrow{\lambda})$,

$$\begin{aligned} \mu_f(\delta) &= \sum_{j=1}^{\infty} \mu_{h_j}(\delta) = \sum_{j=1}^{\infty} \big(E(\delta) h_j, h_j\big) \\ &= \sum_{j=1}^{\infty} \int_\delta |h_j(t)|^2 \, d\big(E(t) g_j, g_j\big) = \sum_{j=1}^{\infty} \int_\delta |h_j(t)|^2 \, d\mu_j(t) \\ &= \int_\delta \sum_{j=1}^{\infty} |h_j(t)|^2 p_j(t) \, d\mu(t). \end{aligned} \tag{5.9}$$

Since $\mu \sim E$, formula (5.9) implies the required description:

$$f = \sum_{1}^{\infty} \lambda_k f_k \in \Omega_E \cap L \iff \sum_{j=1}^{\infty} |h_j(t, \overrightarrow{\lambda})|^2 p_j(t) \ne 0 \pmod{\mu}. \tag{5.10}$$

2) Since the mapping $\overrightarrow{\lambda} \mapsto \sum \lambda_k f_k$ is an isometry of l^2 onto L, (5.10) implies that the set $L \setminus \Omega_E$ is isometrically isomorphic to $\Lambda := \bigcup_{a\in\mathbb{R}_+} \Lambda_a$, where

$$\Lambda_a := \Big\{\overrightarrow{\lambda} \in l^2 : \mu\Big\{t \in \mathbb{R} : \sum_{j=1}^{\infty} |h_j(t, \overrightarrow{\lambda})|^2 p_j(t) = 0\Big\} \geq a\Big\}, \quad a \in \mathbb{R}_+. \tag{5.11}$$

We show that the sets Λ_a are closed. We consider the nonlinear mapping

$$\Phi : l^2 \to L_1(\mu, \mathbb{R}), \quad \Phi : \overrightarrow{\lambda} \mapsto \sum_{j=1}^{\infty} |h_j(t, \overrightarrow{\lambda})|^2 p_j(t) \tag{5.12}$$

and show that it is locally Lipschitz. By (5.7)–(5.9),

$$\|\overrightarrow{\lambda} - \overrightarrow{\lambda}^0\|_{l_2}^2 = \int_{\mathbb{R}} \sum_{j=1}^{\infty} |h_j(t, \overrightarrow{\lambda}) - h_j(t, \overrightarrow{\lambda}^0)|^2 p_j(t)\, d\mu(t), \qquad \overrightarrow{\lambda}, \overrightarrow{\lambda}^0 \in l^2. \tag{5.13}$$

From (5.12), (5.13), and the Cauchy inequality we deduce that

$$\begin{aligned}
\|\Phi(\overrightarrow{\lambda}) - \Phi(\overrightarrow{\lambda}^0)\|_{L_1(\mu_1,\mathbb{R})} &\leq \sum_{j=1}^{\infty} \int_{\mathbb{R}} \big||h_j(t, \overrightarrow{\lambda})|^2 - |h_j(t, \overrightarrow{\lambda}^0)|^2\big| p_j(t)\, d\mu \\
&\leq \bigg(\sum_{j=1}^{\infty} \int_{\mathbb{R}} \big||h_j(t, \overrightarrow{\lambda})| - |h_j(t, \overrightarrow{\lambda}^0)|\big|^2 p_j(t)\, d\mu\bigg)^{1/2} \\
&\quad \cdot \bigg(\sum_{j=1}^{\infty} \int_{\mathbb{R}} (|h_j(t, \overrightarrow{\lambda})| + |h_j(t, \overrightarrow{\lambda}^0)|)^2 p_j(t)\, d\mu\bigg)^{1/2} \\
&\leq 2\|\overrightarrow{\lambda} - \overrightarrow{\lambda}^0\|_{l^2} (\|\overrightarrow{\lambda}\|_{l_2}^2 + \|\overrightarrow{\lambda}^0\|_{l_2}^2)^{1/2}.
\end{aligned}$$

So, the mapping Φ is continuous. By Lemma 5.4, the set $\Lambda_a = \Phi^{-1}\big(S_a(L_1(\mu, \mathbb{R}))\big)$ is closed.

Now, the identity

$$\Lambda := \bigcup_{a\in\mathbb{R}_+} \Lambda_a = \bigcup_{n=1}^{\infty} \Lambda_{1/n} \tag{5.14}$$

implies that Λ is an F_σ-set.

3) At this step we show that Λ is of the first category in l^2. Put

$$\Lambda_a^j := \Big\{\overrightarrow{\lambda} \in l^2 : \mu\Big\{t \in \mathbb{R} : \sum_{k=j}^{\infty} \lambda_k \varphi_{kj}(t) = 0\Big\} \geq a\Big\}, \quad a \in \mathbb{R}_+, \;\; j \in N. \tag{5.15}$$

Since the mappings $\Phi_j : \overrightarrow{\lambda} \to h_j(t, \overrightarrow{\lambda})$ from l^2 to $L^2(\mu, \mathbb{R})$ are continuous, the sets Λ_a^j are closed by Lemma 5.4. Next, we introduce the vector $\lambda^j := \overrightarrow{\lambda} - \lambda_j e_j \in l_j^2$, where $l_j^2 = l^2 \ominus \{e_j\} = \operatorname{span}\{e_k : k \neq j\}$. Then the proof of Lemma 5.6 shows that for fixed λ^j the one-dimensional section

$$Q_a^j(\lambda^j) := \{\gamma \in \mathbb{C} : \{\lambda_k + \gamma\delta_{jk}\}_{k=1}^{\infty} \in \Lambda_a^j\} \tag{5.16}$$

(δ_{jk} is Kronecker's delta) is at most countable, because $\varphi_{n_j j} = 1 \neq 0 \pmod{\mu}$. Therefore, this (closed) set is nowhere dense in l^2. It follows that the sets

$$\Lambda'(j) := \bigcup_{a>0} \Lambda_a^j = \bigcup_{n=1}^{\infty} \Lambda_{1/n}^j \quad \text{and} \quad \Lambda' := \bigcup_{j=1}^{\infty} \Lambda'(j) \tag{5.17}$$

are of the first category in l^2.

Next, it is easily seen that $\Lambda \subset \Lambda'$. Indeed, let $\overrightarrow{\lambda} \in \Lambda$. Then $\overrightarrow{\lambda} \in \Lambda_a$ for some $a > 0$ and, by (5.11), for some μ-measurable set K we have

$$|h_j(t, \overrightarrow{\lambda})|^2 p_j(t) = 0 \quad \text{for } t \in K,\ j \in N, \text{ and } \mu(K) \geq a. \tag{5.18}$$

Since $H = \bigoplus_{j=1}^N H_{g_j}$ (the system $\{g_j\}_1^N$ is cyclic!), we have $\mu\bigl(\bigcap_{j=1}^\infty \{t : p_j(t) = 0\}\bigr) = 0$. Therefore, some of the sets $K_j := K \cap \{t \in \mathbb{R} : p_j(t) \neq 0\}$ (say, K_i) is of positive μ-measure, and the ith identity in (5.18) takes the form

$$h_i(t, \overrightarrow{\lambda}) = 0 \quad \text{for } t \in K_i \text{ and } \mu(K_i) = b > 0. \tag{5.19}$$

By (5.15) and (5.17), identity (5.19) means that $\overrightarrow{\lambda} \in \Lambda_b^i \subset \Lambda'$, and the inclusion $\Lambda \subset \Lambda'$ is proved. Therefore, Λ is of the first category in l^2.

By (5.14), the set $\Omega_E \cap L$ is isomorphic to $l^2 \setminus \Lambda = \bigcap_n (l^2 \setminus \Lambda_{1/n})$. Thus, statement i) follows from the Baire category theorem.

4) We prove statement ii). As an orthonormal basis $\{f_j\}_1^\infty$ in L, we take the basis of eigenvectors of the correlation operator B of the Gaussian measure w in $L : Bf_j = b_j f_j$, $b_j \geq 0$, $j \in \mathbb{N}$.

Along with w, consider the Gaussian measure wg in l^2 with the correlation operator Bg, $Bge_k = b_k e_k$. The isomorphism $\overrightarrow{\lambda} = \{\lambda_k\}_1^\infty \leftrightarrow \sum \lambda_k f_k$ between l^2 and L shows that $w(L \setminus \Omega_E) = wg(\Lambda)$ (the sets $L \setminus \Omega_E$ and Λ are of type F_σ and, thus, Borel-measurable). In the basis $\{e_k\}_1^\infty$, the measure wg is a product measure (see the definitions in [31]); therefore, for every $j \in N$ it is representable in the form $w' = w_j \otimes w^j$, where w_j and w^j are the projections of w' onto $l^2 \ominus l_j^2$ and l_j^2, respectively, where $l_j^2 = \operatorname{span}\{e_k : k \neq j\}$.

Since for every $\lambda^j := \overrightarrow{\lambda} - \lambda_j e_j \in l_j^2$ the one-dimensional section $Q_a^j(\lambda^j)$ of the form (5.16) is at most countable, from (5.17) and the Fubini theorem we obtain

$$w'\bigl(\Lambda'(j)\bigr) = \lim_{n\to\infty} w'(\Lambda_{1/n}^j) \leq \sum_{n=1}^\infty \int_{l_j^2} w_j\bigl(Q_{1/n}^j(\lambda^j)\bigr)\, dw^j = 0.$$

Therefore, $w(L \setminus \Omega_E) = w'(\Lambda) \leq w'(\Lambda') \leq \sum_{j=1}^\infty w'\bigl(\Lambda'(j)\bigr) = 0$, as required. □

Example 5.8. Let T be a contraction in H with simple spectrum, and let U be the minimal unitary dilation of T in $\widetilde{H} \supset H$. By Proposition 3.6, $H \in \operatorname{Cyc} E_U$. In this case Theorem 3.13 says more than Theorem 5.7: $H \subset \Omega_E$.

Remark 5.1. a) If $n(A) = 1$, i.e., the operator A is cyclic, then $\Omega_A = \{h : \{\lambda h\} \in \operatorname{Cyc} A\}$, and by Theorem 5.7, in each subspace $L \in \operatorname{Cyc} A$, the set of all cyclic vectors is dense, of type G_δ, etc.

b) The set $\Lambda \setminus \Omega_E$ may happen to be everywhere dense in L, though it is of the first category in L. For example, $H \setminus \Omega_E$ is dense in $H := L^2(\mu, \mathbb{R})$, because every $f \in H$ can be approximated by functions of the form $f\chi_{\delta_n}$.

c) It can be shown that the sets Λ_a of the form (5.11) are nowhere dense in l^2. Therefore, (5.14) yields a representation of Λ as a countable union of closed nowhere dense subsets.

d) Statement ii) in Theorem 5.7 is true for any continuous product measure in L. This is clear from the proof. It should also be noted that the main difficulty in the proof of statement ii) is to show that Ω_E (or, which is the same, $\Lambda = L \setminus \Omega_E$)) is Borel measurable.

Moreover, if we already know that Ω_E is a Borel set, then the fact that $L \setminus \Omega_E$ is of the first category in L (i.e., a part of statement (i) in Theorem 5.7) is an easy consequence of the following theorem due to Kuratowski and Ulam:

A set M in the Cartesian product $X \times Y$ of two metric spaces X and Y is of the first category if M has the Baire property and all sections M_x are of the first category in Y.

Indeed, since the sections $Q_a^j(\lambda^j)$ are at most countable, from (5.17) it easily follows that the sections of $L \setminus \Omega_E$ are of the first category in L.

Corollary 5.9. *Let A be a selfadjoint operator in H, and let H_+ be a (nonclosed) linear set in H that is a Hilbert space with respect to a norm $\|\cdot\|_+$, and the embedding $H_+ \to H$ is continuous: $\|\cdot\|_+ \geq c\|\cdot\|_H$. If the space H_+ is cyclic for E, then:*
i) $\Omega_A \cap H_+$ *is a dense G_δ-subset of the second category in H_+;*
ii) $\omega(\Omega_A \cap H_+) = 1$ *for every Gaussian measure ω in H_+.*

Proof. The arguments differ from the proof of Theorem 5.7 only in certain details. We give the necessary comments only.

As the system $\{f_j\}_1^\infty$ we take an orthonormal basis in H_+ (relative to the scalar product $(\cdot,\cdot)_+$). Since the identity embedding $J : H_+ \to H$ is continuous, the series $\sum \lambda_k f_k$ converges in H for every $\overrightarrow{\lambda} \in l^2$ because it converges in H_+. For the same reason, the mapping Φ of the form (5.12) remains continuous. Beyond these differences, the proof of Theorem 5.7 applies literally. □

Remark 5.2. It is well known that every linear space H_+ continuously embedded in H is of the form $H_+ = \mathcal{D}(S)$, where $S = S^* \geq \varepsilon I$ is a selfadjoint operator in $Hg := \overline{H}_+$; in this situation we have $\|f\|_+ = \|Sf\|$.

5.3. The set Ω_Σ. The second approach. From Theorem 5.7 and the Naĭmark theorem we deduce the following refinement of Proposition 5.3.

Theorem 5.10. *Let Σ be an operator measure in H. Then:*
i) Ω_Σ *is a dense G_δ-subset of the second category in H;*
ii) $\omega(\Omega_\Sigma) = 1$ *for every Gaussian measure ω in H.*

Proof. We split the arguments into several steps.

1) First, let $\Sigma(t)$ be a generalized resolution of the identity in H (see (3.1)), i.e., $\Sigma(-\infty) = 0$, $\Sigma(+\infty) = I$. By the Naĭmark theorem, $\Sigma(t) = P_H E(t)\lceil H$, where E is a minimal orthogonal dilation for Σ in a space $\widetilde{H} \supset H$.

Since E is minimal, we have $\Sigma \sim E$ and $H \in \mathrm{Cyc}(E)$ by (3.3) and (4.1). It remains to apply Theorem 5.7 to the measure E in $\widetilde{H}$ and to the subspace $L = H$.

2) Let $\Sigma(t)$ be a finite operator measure, i.e., $\Sigma(\mathbb{R}) \in B(H)$. In this case the strong limits $\Sigma(\pm\infty) := \text{s-lim}_{t\to\pm\infty} \Sigma(t)$ exist. We normalize Σ by the condition $\Sigma(-\infty) = 0$. Then $\Sigma(+\infty) \geq 0$.

By Theorem 3.5, $\Sigma(t) := T^*E(t)T$ (see (3.10)), where $E(t)$ is a resolution of the identity in $\widetilde{H}$ ($\supset H$) that satisfies the minimality condition (3.3), and T is a bounded operator from H to $\widetilde{H}$, $T^*T = \Sigma(+\infty)$. For simplicity, we restrict ourselves to the case where $\ker T = \{0\}$.

Since T^{-1} is a closed operator, the linear set $H_+ := TH = \mathcal{D}(T^{-1})$ ($\subset \widetilde{H}$) is a Hilbert space with the scalar product $(f,g)_+ := (T^{-1}f, T^{-1}g)$. Since T is bounded, we have $\|f\|_+ \geq c\|f\|_{\widetilde{H}}$. Therefore, $H_+ \in \mathrm{Cyc}\, E$ and the embedding $J : H_+ \to \widetilde{H}$ is continuous. By Corollary 5.9, the set $H_+ \setminus \Omega_E$ is of the first category and of type F_σ in the Hilbert space H_+, and $\omega(H_+ \setminus \Omega_E) = 0$ for every Gaussian measure ω in H_+.

Since T maps H onto H_+ isometrically ($\|Th\|_+ = \|h\|, h \in H$), we see that $H \setminus \Omega_\Sigma = T^{-1}(H_+ \setminus \Omega_E)$ is also an F_σ-set of the first category in H, and $\omega(H \setminus \Omega_\Sigma) = 0$ for every Gaussian measure in H.

3) Let Σ be an arbitrary unbounded measure in H. Since Σ is defined on $\mathcal{B}_b(\mathbb{R})$, the measures

$$\Sigma_n : \mathcal{B}(\mathbb{R}) \to B(H),\ \ \Sigma_n(\delta) := \Sigma(\delta \cap [-n,n]),\quad \delta \in \mathcal{B}(\mathbb{R}),\quad n \in \mathbb{N}, \tag{5.20}$$

are bounded: $\Sigma_n(\mathbb{R}) \in B(H)$, $n \in \mathbb{N}$.

Next, clearly, we have $\Omega_\Sigma = \bigcap_{n=1}^{\infty} \Omega_{\Sigma_n}$. Indeed, if $\Sigma(\delta) = 0$, then $\Sigma_n(\delta) = 0$ for all $n \in \mathbb{N}$ by (5.20). Conversely, if $\Sigma_n(\delta) = 0$ for all $n \in \mathbb{N}$, then $\Sigma(\delta) = \text{s-lim}_{n\to\infty} \Sigma_n(\delta) = 0$. Now statements i)–ii) follow from the identity $H \setminus \Omega_\Sigma = \bigcup_{n=1}^{\infty}(H \setminus \Omega_{\Sigma_n})$ and the corresponding statements for the measures Σ_n already verified at step 2). □

In conclusion, we give a criterion for a subspace $L \subset H$ to contain a principal vector of an operator measure Σ, i.e., for $\Omega_\Sigma \cap L \neq \varnothing$.

Definition 5.11. A subspace $L \subset H$ is said to be *principal* for a measure Σ if

$$L \not\subset \ker \Sigma(\delta) \quad \text{for every } \delta \in \mathcal{B}(\mathbb{R}) \text{ with } \Sigma(\delta) \neq \{0\}. \tag{5.21}$$

Proposition 5.12. *Let Σ be an operator measure in H, and let L be a subspace of H. The following statements are equivalent*:

i) *L is a principal subspace for Σ*;
ii) $\Sigma_L := P_L \Sigma\lceil L \sim \Sigma$;
iii) $\Omega_\Sigma \cap L \neq \varnothing$;
iv) *$\Omega_\Sigma \cap L$ is a dense G_δ-subset in L of the second category*;
v) *$\omega(\Omega_\Sigma \cap L) = 1$ for every Gaussian measure ω in L.*

Proof. i) $\iff$ ii). Clearly, $\Sigma_L \prec \Sigma$. If the measures Σ_L and Σ are not equivalent, then there exists a set $\delta \in \mathcal{B}(\mathbb{R})$ such that $\Sigma_L(\delta) = 0$ and $\Sigma(\delta) \neq 0$. But the first relation shows that $L \in \ker \Sigma(\delta)$. The reverse implication is proved similarly.

ii) $\iff$ iii). Let $\Sigma_L \sim \Sigma$. By Proposition 5.3, $\Omega_{\Sigma_L} \neq \varnothing$, i.e., there exists $f \in L$ such that $\mu_f \sim \Sigma_L$. But then $\mu_f \sim \Sigma$ and $f \in \Omega_\Sigma \cap L$.

Conversely, if $f \in \Omega_\Sigma \cap L$ ($\neq \varnothing$), then $\mu_f \sim \Sigma$. Since $f \in L$, we have $\mu_f \prec \Sigma_L$, whence $\Sigma \prec \Sigma_L$.

iii) $\iff$ iv) and iii) $\iff$ v). Since $\Omega_\Sigma \cap L \neq \varnothing$, the equivalence i) $\iff$ ii) (already proved) yields $\Sigma_L \sim \Sigma$. Therefore, $\Omega_\Sigma \cap L = \Omega_{\Sigma_L}$. But Theorem 5.10 applied to the measure Σ_L in L shows that $\Omega_\Sigma \cap L$ is a G_δ-subset of the second category in L, and $\omega(\Omega_\Sigma \cap L) = 1$.

The equivalences iv) $\iff$ i) and v) $\iff$ i) follow from i) $\iff$ iii) and Theorem 5.10. □

Corollary 5.13. *Suppose A is a selfadjoint operator in H and L is a subspace of H. Let $E(t)$ be the resolution of the identity for A. Then*:

1) *either $L \cap \Omega_E = \varnothing$, or L is a principal subspace and $\Omega_\Sigma \cap L$ is a dense G_δ-subset of L of the second category*;

2) *if $L \in \operatorname{Cyc} A$, then L is a principal subspace in H.*

§6. Hellinger types and Hellinger subspaces

6.1. Hellinger subspaces. The first approach. Let μ be a scalar measure. The class of all measures equivalent to μ is called the *type* of μ and is denoted by $[\mu]$.

Definition 6.1 (see [6])**.** a) Let $A = A^*$ be a selfadjoint operator in H, and let $E := E_A$ be its spectral measure. For every $g \in H$, the type of the measure μ_g is called the *type of the vector* g and is denoted by $[g] := [\mu_g]$. We recall that $\mu_g(\delta) := (E(\delta)g, g)$.

b) Let vectors $\{g_i\}_{i=1}^m$ ($m \leq \infty$) be spectrally orthogonal ($H_{g_i} \perp H_{g_j}$), and let

$$H = \bigoplus_{i=1}^{m} H_{g_i}, \quad [g_{i+1}] \prec [g_i], \; i < m, \tag{6.1}$$

i.e., the types $[g_i]$ decrease monotonically. Then the sequence $\{g_i\}_1^m$ is called a *Hellinger sequence*, and the types $[g_i]$ are called the *Hellinger types* of the measure E.

In this case, the number m ($m \le \infty$) and the types $[g_i]$ in the decomposition (6.1) are uniquely determined (see [6], [27]) and $m = n(A) = n(E)$, where $n(A)$ is defined by (4.2).

The collection of Hellinger types forms a complete system of unitary invariants of a selfadjoint operator A (equivalently, of the measure E); see [6], [27].

Let $\Omega_A := \Omega_E := \{g \in H : \mu_g \sim E\}$ be the set of vectors of maximal type, and let $g_1 \in \Omega_E$. Then $\mu_g \prec \mu_{g_1} := \mu$ for all $g \in H$ and, consequently, the type $[g]$ is uniquely determined by the set

$$\Gamma(g) := \{t \in \mathbb{R} : d\mu_g/d\mu > 0\}. \tag{6.2}$$

Since $\mu_g \prec \mu$, we see that $\Gamma(g)$ is a (nontopological) support of μ.

It follows that the Hellinger types are uniquely determined by the senior type $[\mu_{g_1}] = [g_1]$ and by the supports $\Gamma_i(E) := \Gamma(g_i)$, $i \le m$.

The sets $\Gamma_i(E)$ are determined by the multiplicity function $N_E = N_A$:

$$\Gamma_i(E) := \{t \in \mathbb{R} : N_E(t) \ge i\}. \tag{6.3}$$

Remark 6.1. The Hellinger types can be defined in a different way. For example, let $\{g_i\}_1^k$ be a system of spectrally orthogonal vectors ($H_{g_i} \perp H_{g_j}, 1 \le i \ne j \le k$) that satisfies the condition $\Gamma(g_i) = \Gamma_i(E)$, $i \le k$. Next, putting $Hg := H \ominus (\bigoplus_{i=1}^k H_{g_i})$, we take $g_{k+1} \in Hg$ such that $[g] \prec [g_{k+1}]$ for all $g \in Hg$ (i.e., g_{k+1} is a principal vector for $A\lceil Hg$). Then $[g_{k+1}]$ is the $(k+1)$st Hellinger type for E.

The Hellinger types admit the following invariant description (see [21]):

$$[g_{k+1}] = \min_{g_1,\dots,g_k} \max_{g \in H \ominus H_{g_1,\dots,g_k}} [g], \tag{6.4}$$

where the maximum and minimum are understood in the sense of the partial order $\prec$. Relation (6.4) can rewritten in terms of the supports $\Gamma(g)$:

$$\Gamma_{k+1}(E) = \min_{g_1,\dots,g_k} \max_{g \in H \ominus H_{g_1,\dots,g_k}} \Gamma(g), \tag{6.5}$$

where the maximum and minimum are taken in the sense of the partial order $\subset$. Invoking the multiplicity function N_Σ described in Definition 4.5 (see formula (4.8)), we see that, for an arbitrary (not necessarily orthogonal) operator measure Σ, it is natural to define the supports $\Gamma_i(\Sigma)$ of the Hellinger types by analogy with (6.3):

$$\Gamma_i(\Sigma) = \{t \in \mathbb{R} : N_\Sigma(t) \ge i\}, \qquad i \in \{1, \dots, m\}. \tag{6.6}$$

Next, the ith Hellinger type of Σ is defined to be the type $[\mu_i]$ of the scalar measure $d\mu_i := \chi_i d\rho$, where $\rho \sim \Sigma$ and χ_i is the indicator of $\Gamma_i(\Sigma)$, $1 \le i \le m(\Sigma)$.

By (4.13), the definitions (6.3) and (6.6) coincide for an orthogonal measure $\Sigma = E$. On the other hand, by Theorem 5.10, $\Omega_\Sigma \ne \varnothing$ and for every $g_1 \in \Omega_\Sigma$ we have $\mu_g \prec \mu_{g_1} =: \mu \sim \Sigma$, where $\mu_g(\delta) := (\Sigma(\delta)g, g)$ and $g \in H$. Therefore, the set $\Gamma(g)$ defined by (6.2) is a (nontopological) support of μ_g.

It turns out that, although $\Omega_\Sigma \ne \varnothing$, elements of "junior" types (i.e., vectors $g \in H$ with $\Gamma(g) = \Gamma_i(\Sigma)$ for some $i \ge 2$) may be absent in general.

Example 6.2. Let $\Sigma(t)$ be a (2×2)-matrix-valued discrete measure with jumps at 1, 2, and 3, i.e., $\Sigma(t) = \sum_{i<t} P_i$, where

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly,

$$N_\Sigma(t) = \begin{cases} 1 & \text{if } t = 1, 2, \\ 2 & \text{if } t = 3, \end{cases}$$

and $\Gamma_1(\Sigma) = \{1, 2, 3\}$, $\Gamma_2(\Sigma) = \{3\}$.

Let $h = \{h_1, h_2\}$. If $h_1 h_2 \neq 0$, then $\Gamma(h) := \operatorname{supp}(\Sigma h, h) = \{1,2,3\} = \Gamma_1(\Sigma)$. But if $h_1 = 0$, we have $\Gamma(h) := \operatorname{supp}(\Sigma h, h) = \{2,3\}$, and if $h_2 = 0$, then $\Gamma(h) := \operatorname{supp}(\Sigma h, h) = \{1,3\}$. Therefore, the relation $\Gamma(h) := \operatorname{supp}(\Sigma h, h) = \{3\} = \Gamma_2(\Sigma)$ is impossible.

The following simple statement will be used below.

Lemma 6.3. *If* $\delta \subset \Gamma_n(\Sigma), \delta \in \mathcal{B}(\mathbb{R})$, *and* $\Sigma(\delta) \neq 0$, *then* $\operatorname{rank} \Sigma(\delta) \geq n$.

Proof. Assume the contrary. Let $\mathcal{H} := \mathcal{R}\big(\Sigma(\delta)\big)$ with $\dim \mathcal{H} = \operatorname{rank}\Sigma(\delta) = n_1 < n$. We choose an orthonormal basis $\{e_j\}_1^\infty$ in H in such a way that $\{e_j\}_1^{n_1}$ is a basis in $\mathcal{H}$, and $\{e_j\}_{j>n_1}$ is a basis in $\mathcal{H}^\perp$. Next, we introduce a diagonal operator $K = K^* \in \mathfrak{S}_2$ by putting $Ke_j = e_j/j$, $j \in \mathbb{N}$. If $\rho \sim \Sigma$ and $\Psi_K := dK^*\Sigma K/d\rho$, then

$$K^*\Sigma(\delta)K = \int_\delta \Psi_K(t)\, d\rho(t) \tag{6.7}$$

by Theorem 2.12. Since $K^*\Sigma(\delta)Ke_j = 0$ for $j > n_1$, by (6.7) there exists a set $\delta_1 \subset \delta$, $\delta_1 \in \mathcal{B}(\mathbb{R})$, such that

$$\Psi_K(t)e_j = 0, \quad t \in \delta_1,\ \ j > n_1,\ \ \rho(\delta \setminus \delta_1) = 0. \tag{6.8}$$

Consequently, $\mathcal{R}\big(\Psi_K(t)\big) \subset \mathcal{H}$ for $t \in \delta_1$, and $N_\Sigma(t) = \operatorname{rank}\Psi_K(t) \leq n_1$ for $t \in \delta_1$ by Corollary 4.7, which contradicts the definition of $\Gamma_n(\Sigma)$. $\square$

We denote by $G_k(H)$ the collection of all k-dimensional subspaces of H (a "Grassmann manifold").

An operator measure Σ in H generates the family of $(k \times k)$-matrix-valued measures of the form $P_L\Sigma\lceil L$, where $L \in G_k(H)$. Clearly, we have $P_L\Sigma\lceil L \prec\prec \Sigma$; therefore, $N_{P_L\Sigma\lceil L}(t) \leq N_\Sigma(t)$ for a.e. $t \in \mathbb{R}$.

Below we shall show that the kth Hellinger type of Σ can be realized by a subspace $L \in G_k(H)$. This makes the following definition natural.

Definition 6.4. Let Σ be an operator measure in H, and let m be its total multiplicity.

a) A subspace $L = L_k$ $(\in G_k(H))$ is called a kth *Hellinger subspace* (in symbols: $L \in \operatorname{Hel}_k(\Sigma)$) if

$$\Gamma_i(L) := \Gamma_i(P_L\Sigma\lceil L) = \Gamma_i(\Sigma), \qquad i \in \{1,\dots,k\}. \tag{6.9}$$

b) A *Hellinger chain* is any finite chain of the form

$$\{\lambda h\} =: L_1 \subset L_2 \subset \cdots \subset L_k \subset \cdots \subset L_m \subset H, \quad L_k \in \operatorname{Hel}_k(\Sigma),\ \ k \in \{1,\dots,m\}, \tag{6.10}$$

if the total multiplicity $m := m(\Sigma)$ is finite, and any infinite chain of the form

$$\{\lambda h\} =: L_1 \subset L_2 \subset \cdots \subset L_k \subset \cdots \subset H, \qquad L_k \in \operatorname{Hel}_k(\Sigma),\ \ k \in \mathbb{N}, \tag{6.11}$$

if $m = m(\Sigma) = \infty$.

c) A basis $\{e_i\}_1^k$ in L_k $\big(\in \operatorname{Hel}_k(\Sigma)\big)$ is called a *Hellinger basis* if the spaces $L_j := \operatorname{span}\{e_i : 1 \leq i \leq j\}$ constitute a Hellinger chain of the form (6.10) with $h = e_1$.

Remark 6.2. a) In particular, $H_1 = \{\lambda h\} \in \operatorname{Hel}_1(\Sigma)$ if and only if $h \in \Omega_\Sigma$.

b) Let $L_1 \subset \cdots \subset L_k$ be a chain of subspaces, and let $\{e_i\}_1^k$ be an orthonormal basis in L_k such that $L_j = \operatorname{span}\{e_i : 1 \leq i \leq j\}$, $j \leq k$. We put $\sigma_{ij}(t) := (\Sigma(t)e_i, e_j)$, $\psi_{ij}(t) := d\sigma_{ij}(t)/d\rho$, and $\Psi_p(t) := (\psi_{ij}(t))_{i,j=1}^p$. Then $L_1 \subset \cdots \subset L_k$ is a Hellinger chain if and only if

$$\Gamma_p(\Sigma) = \{t \in \mathbb{R} : \det \Psi_p(t) \neq 0\} \pmod{\Sigma}, \quad p \in \{1,\dots,k\}. \tag{6.12}$$

By Definition 4.5, for ρ-almost every $t \in \Gamma_p(\Sigma)$ there exists a nonzero minor of order p for $\Psi_k(t)$. However, this minor may depend on t. The meaning of the notion of a Hellinger chain is that nonzero minors can be chosen independent of t (see (6.12)).

c) If the density matrix $\Psi_k(t)$ is diagonal, $\Psi_k(t) = \operatorname{diag}\bigl(\psi_1(t), \dots, \psi_k(t)\bigr)$, and the supports $\operatorname{supp}\psi_j$ decrease ($\operatorname{supp}\psi_j \supseteq \operatorname{supp}\psi_{j+1}$), then $\Gamma_n(\Sigma) = \operatorname{supp}\psi_n$, $n \in \{1, \dots, k\}$.

In particular, this situation occurs if the measure $\Sigma = E$ is orthogonal and the basis $\{e_i\}_1^k$ consists of pairwise spectrally orthogonal vectors (relative to E) and, moreover, e_i realizes the ith Hellinger type: $\Gamma(e_i) = \Gamma_i(E)$, $i \in \{1, \dots, k\}$.

This observation shows that a Hellinger chain of a measure Σ is a natural analog of (and a substitute for) a Hellinger sequence in the case of an orthogonal measure E.

Example 6.5. Let $H = \mathbb{C}^3$. Suppose that a matrix $\Psi(t)$ is diagonal in the basis $\{e_j\}_1^3$, i.e., $\Psi(t) = \Psi_3(t) = \operatorname{diag}(\psi_1, \psi_2, \psi_3)$. Moreover, suppose $\psi_j \in C[0,2]$ $(1 \le j \le 3)$ and

$$\psi_1(x) > 0,\ x \in [0, 3/2), \quad \psi_2(x) > 0,\ x \in (1/2, 2], \quad \psi_3(x) > 0,\ x \in (1/4, 3/2).$$

If the function ψ_j $(1 \le j \le 3)$ vanishes off the segment indicated, then the measure Σ ($\Sigma(\delta) = \int_\delta \Psi(t)\,dt$) satisfies $\Gamma_1(\Sigma) = [0,2]$, $\Gamma_2(\Sigma) = [1/4, 3/2]$, and $\Gamma_3(\Sigma) = [1/2, 3/2]$.

a) The space $L^2 = \operatorname{span}\{e_1, e_3\}$ is not Hellinger because

$$\Gamma_1(L^2) = [0, 3/2] \ne \Gamma_1(\Sigma), \quad \text{though } \Gamma_2(L^2) = [1/4, 3/2] = \Gamma_2(\Sigma). \tag{6.13}$$

Relations (6.13) show that the jth identity in (6.9) does not imply the other relations with $i < j$.

b) The subspace $L^3 = \operatorname{span}\{e_1, e_2\}$ is not Hellinger, because

$$\Gamma_1(L^3) = [0,2] = \Gamma_1(\Sigma) \quad \text{but} \quad \Gamma_2(L^3) = [1/2, 3/2] \ne \Gamma_2(\Sigma).$$

This example is interesting because L^3 contains a continuum of principal vectors ($\lambda_1 e_1 + \lambda_2 e_2 \in \Omega_\Sigma$ with $\lambda_1\lambda_2 \ne 0$), but $L^3 \notin \operatorname{Hel}_2(\Sigma)$.

c) Since $L^3 \notin \operatorname{Hel}_2(\Sigma)$, neither the chain $\{\lambda e_1\} \subset L^3 \subset H$, nor the basis $\{e_j\}_1^3$ is Hellinger. At the same time, the basis $\{g_1 := (e_1 + e_2)/\sqrt{2},\ g_2 := e_3,\ g_3 := (e_1 - e_2)/\sqrt{2}\}$ and the chain $\{\lambda g_1\} \subset \operatorname{span}\{g_1, g_2\} \subset H$ are Hellinger.

Proposition 6.6. *Suppose Σ is an operator measure in H, $L := L_k \in G_k(H)$, $T \in B(H)$, and $\ker T = \ker T^* = \{0\}$. Then*:

i) $\Gamma_i(T^*\Sigma T) = \Gamma_i(\Sigma)$ *for all* $i \in \mathbb{N}$;

ii) $L \in \operatorname{Hel}_k(T^*\Sigma T)$ *if and only if* $TL \in \operatorname{Hel}_k(\Sigma)$;

iii) *if $\Sigma(t)$ is of norm locally bounded variation and $\Psi(t) = d\Sigma(t)/d\rho$ is the density of Σ, i.e., $\Sigma(\delta) = \int_\delta \Psi(t)\,d\rho(t)$ (see Proposition* 2.15), *then $L := L_k \in \operatorname{Hel}_k(\Sigma)$ if and only if*

$$\dim(L/(L \cap \ker\Psi(t))) \ge i \quad \textit{for } \rho\textit{-a.e. } t \in \Gamma_i(\Sigma),\ i \in \{1, 2, \dots, k\}. \tag{6.14}$$

Moreover, the first (*last*) *condition in* (6.14) *is equivalent to the condition $L \not\subset \ker\Psi(t)$ for $t \in \Gamma_1(\Sigma)$ ($L \cap \ker\Psi(t) = \{0\}$ for $t \in \Gamma_k(\Sigma)$).*

Proof. Statements i) and ii) are obvious, and iii) follows from the identity $\dim(L/(L \cap \ker\Psi(t))) = \operatorname{rank}\bigl(P_L\Psi(t)\lceil L\bigr)$. $\square$

We prove the existence of Hellinger chains (see Proposition 6.4).

Theorem 6.7. *Let Σ be an operator measure in H, and let $m = m(\Sigma)$ be its total multiplicity. Suppose $T \in \mathfrak{S}_2(H)$, $\ker T = \ker T^* = \{0\}$, and $TH_k \in \operatorname{Hel}_k(\Sigma)$. If $M := \{f_\alpha\}_{\alpha \in I}$ is a family of vectors of the form* (5.2) *with $I := [0,1]$, then*:

i) *the subspace* $\operatorname{span}\{TH_k, Tf_\alpha\}$ *belongs to* $\operatorname{Hel}_{k+1}(\Sigma)$ *for all but at most countably many* $\alpha \in [0,1)$;

ii) *for every $h \in \Omega_\Sigma$ there exists a Hellinger chain of the form* (6.10) *or* (6.11);

iii) *every subspace $L \in \operatorname{Hel}_k(\Sigma)$ can be included in a Hellinger chain of the form* (6.10) *or* (6.11).

Proof. i) Consider the operator measure $\Sigma_T(t) := T^*\Sigma(t)T$. Define a scalar measure ρ by $\rho(\delta) = \operatorname{Tr}\Sigma_T(\delta)$ and put $\Psi_T := d\Sigma_T/d\rho$. Then

$$\Gamma_i(\Sigma) = \Gamma_i(\Sigma_T) := \{t \in \mathbb{R} : \operatorname{rank}\Psi_T(t) \geq i\}, \qquad i \in \mathbb{N}. \tag{6.15}$$

For $k = 1$, statement i) is the content of Proposition 5.3. Assuming that i) is proved for $k = n$, we verify it for $k = n + 1$.

By assumption, there exists an n-dimensional Hellinger subspace $H_n^T \in \operatorname{Hel}_n(\Sigma_T)$. Consider the following family of $(n + 1)$-dimensional subspaces:

$$H_\alpha^T := \operatorname{span}\{H_n^T, f_\alpha\} \supset H_n^T, \quad \alpha \in I, \tag{6.16}$$

where $\{f_\alpha\}$ is the set of vectors of the form (5.2).

By Proposition 6.6, $L_1 \subset L_2 \subset \cdots \subset L_k$ is a Hellinger chain for Σ_T if and only if

$$L_i \cap \ker \Psi_T(t) = \{0\}, \qquad t \in \Gamma_i(\Sigma_T) = \Gamma_i(\Sigma),\ \ i \in \{1, \ldots, k\}. \tag{6.17}$$

We denote by $\Omega_\Sigma^k(H_n^T)$ the set of all vectors $f_\alpha \in M$ for which the subspace H_α^T defined in (6.16) is a Hellinger subspace for Σ_T.

We show that the set $M \setminus \Omega_\Sigma^k(H_n^T)$ is at most countable. Supposing the contrary, in accordance with (6.17), for every $f_\alpha \in M \setminus \Omega_\Sigma^k(H_n^T)$ we find a measurable subset $\delta_\alpha \subset \Gamma_{n+1}$ such that

$$H_\alpha^T \cap \ker \Psi_T(t) \neq \{0\}, \qquad t \in \delta_\alpha,\ \ \rho(\delta_\alpha) > 0. \tag{6.18}$$

By Lemma 5.2, there exists a sequence $\{\alpha_i\}_1^\infty$ with $\rho(\bigcap \delta_{\alpha_i}) > 0$. Therefore,

$$H_{\alpha_i}^T \cap \ker \Psi_T(t) \neq \{0\}, \qquad i \in \mathbb{N}, \qquad t \in \delta := \bigcap_i \delta_{\alpha_i}\ (\subset \Gamma_{n+1}). \tag{6.19}$$

On the other hand, since $H_n^T \in \operatorname{Hel}_n(\Sigma_T)$, Proposition 6.6 yields

$$H_n^T \cap \ker \Psi_T(t) = \{0\}, \qquad t \in \Gamma_n \supset \Gamma_{n+1}. \tag{6.20}$$

By (6.19) and (6.20), we have

$$f_{\alpha_i} \in \operatorname{span}\{H_n^T, \ker \Psi_T(t)\}, \qquad t \in \delta\ (\subset \Gamma_{n+1}),\ \ i \in \mathbb{N}. \tag{6.21}$$

But Lemma 5.1 shows that the sequence $\{f_{\alpha_i}\}_{i=1}^\infty$ is complete in H and, consequently, $\operatorname{span}\{H_n^T, \ker \Psi_T(t)\} = H$ for all $t \in \delta$. This is impossible because $\operatorname{codim}\ker \Psi_T(t) \geq n + 1$ for ρ-a.e. $t \in \Gamma_{n+1} \supset \delta$.

Thus, for all $\alpha \in I$ save an at most countable set, we have $H_\alpha^T \in \operatorname{Hel}_{n+1}(\Sigma_T)$, and Proposition 6.6 ii) implies also that $\operatorname{span}\{TH_n, Tf_\alpha\} \in \operatorname{Hel}_{n+1}(\Sigma)$. □

Remark 6.3. By Remark 6.2, Theorem 6.7 means that there exists an orthonormal system $\{e_i\}_1^m$ in H such that conditions (6.12) are fulfilled for every $k \leq m$. In other words, the kth Hellinger type of Σ is realized by the measure $(\wedge^k \Psi(t)\varphi_k, \varphi_k)d\rho$, where $\wedge^k\Psi(t)$ denotes the kth exterior power of the operator $\Psi(t) := \Psi_K(t) = dK^*\Sigma K/d\rho$ ($K \in \mathfrak{S}_2(H)$), and $\varphi_k := e_1 \wedge \cdots \wedge e_k$ is a k-vector, $\varphi_k \in \wedge^k(H)$.

Example 6.8. Let $\Sigma(t) = \sum_{i<t} P_i$ be a matrix-valued discrete measure in $\mathbb{C}^3$, where

$$P_1 = \operatorname{diag}(1,0,0), \quad P_2 = \operatorname{diag}(0,1,0), \quad P_3 = \operatorname{diag}(0,0,1),$$
$$P_4 = \operatorname{diag}(1,1,0), \quad P_5 = \operatorname{diag}(1,1,1).$$

Then $\Gamma_1(\Sigma) = \{1,2,3,4,5\}$, $\Gamma_2(\Sigma) = \{4,5\}$, $\Gamma_3(\Sigma) = \{5\}$. The vector $h = (h_1, h_2, h_3)$ is principal for Σ $\big(h \in \Omega_\Sigma\big)$ if and only if $h_1h_2h_3 \neq 0$.

At the same time, the supports $\Gamma_2(\Sigma)$ and $\Gamma_3(\Sigma)$ are not realized by vectors $h \in \mathbb{C}^3$. By Proposition 6.6, the subspace L belongs to $\operatorname{Hel}_2(\Sigma)$ if

$$L \not\subset \ker P_i,\ i \in \{1,2,3\}, \quad \text{and} \quad L \cap \ker P_4 = L \cap \ker P_5 = \{0\}. \tag{6.22}$$

For example, $L := \operatorname{span}\{(0, 1/\sqrt{2}, 1/\sqrt{2}), (1, 0, 0)\} \in \mathrm{Hel}_2(\Sigma)$, because (6.22) is fulfilled and $\Sigma_L(t) := P_L\Sigma(t)\lceil L = \sum_{i<t} P_i^L$.

Writing $P_i^L = P_L P_i \lceil L$ in the orthonormal basis $(0, 1/\sqrt{2}, 1/\sqrt{2})$, $(1, 0, 0)$, we obtain

$$P_1^L = \mathrm{diag}(0, 1), \quad P_2^L = P_3^L = \mathrm{diag}(1/2, 0), \quad P_4^L = \mathrm{diag}(1/2, 1), \quad P_5^L = \mathrm{diag}(1, 1).$$

Clearly, this implies the relations $\Gamma_1(L) = \Gamma_1(\Sigma) = \{1, 2, 3\}$ and $\Gamma_2(L) = \Gamma_2(\Sigma) = \{4, 5\}$.

6.2. Hellinger subspaces. The case of an orthogonal measure.

Definition 6.9. Let A be a selfadjoint operator in H, $n(A)$ its total multiplicity, $E = E_A$ its spectral measure, and let L be a subspace in H, $\dim L \le \infty$.

a) A subspace L_k ($\in G_k(H)$) is called a kth Hellinger subspace for A (for E) (in symbols: $L_k \in \mathrm{Hel}_k(A)$ ($L_k \in \mathrm{Hel}_k(E)$)) if the first k Hellinger types of the operators A and $A_1 := A\lceil H_{L_k} = A_1^*$ coincide:

$$\Gamma_i(A_1) = \Gamma_i(A) \pmod{E}, \quad i \in \{1, \dots, k\}. \tag{6.23}$$

b) A *Hellinger chain* for A in L is any finite chain of subspaces in L of the form

$$\{\lambda h\} =: L_1 \subset L_2 \subset \cdots \subset L_{n(A)} \subseteq L, \quad L_j \in \mathrm{Hel}_j(A), \ j \in \{1, \dots, n(A)\}, \tag{6.24}$$

if $n(A) < \infty$, and any infinite chain of subspaces in L of the form (6.11) if $n(A) = \infty$.

For example, $L \in \mathrm{Hel}_k(A)$ if $L = \operatorname{span}\{g_i\}_{i=1}^k$, where the g_i are spectrally orthogonal and g_i realizes the ith Hellinger type for A.

By Proposition 4.9, we have $N_{A_1}(t) = N_{E_{A_1}}(t) = N_{P_L E\lceil L}(t)$ and $n(A) = n(E) = m(E)$; so, for an orthogonal measure E Definitions 6.4 and 6.9 coincide.

Theorem 6.10. *Suppose A is a selfadjoint operator in H, $L \in \mathrm{Cyc}(A)$, $L_k \subset L$, $L_k \in \mathrm{Hel}_k(A)$, and $k < n(A)$. Also, let*

$$\Omega_A^L(L_k) := \{f \in L : \operatorname{span}\{L_k, f\}\} \in \mathrm{Hel}_{k+1}(A). \tag{6.25}$$

Then:

i) *$\Omega_A^L(L_k)$ is a dense G_δ-subset in L of the second category*;

ii) *$\omega(\Omega_A^L(L_k)) = 1$ for every Gaussian measure ω in L*;

iii) *in L there exists a Hellinger chain of the form* (6.24) *or* (6.11), *$L_{n(A)} \in \mathrm{Cyc}\, A$ if $n(A) < \infty$; conversely, if $\dim L = n(A)$, then $L \in \mathrm{Hel}_{n(A)}(A)$*;

iv) *every Hellinger subspace $L_k \in \mathrm{Hel}_k(A)$ ($L_k \subset L$) can be incorporated in a Hellinger chain of the form* (6.24) *or* (6.11).

Proof. Since $k < n(A)$, we have $H \ominus H_k \ne \{0\}$ (where $H_k := H_{L_k}$) and $M_k := L \ominus (L \cap H_k) \ne \{0\}$. Consider the operator $A_{k+1} := A\lceil H \ominus H_k$. Next, let P_k be the orthogonal projection onto H_k, and let $Lg_{k+1} := (I - P_k)L$. Since $\ker(I - P_k) = H_k$, the projection $I - P_k$ maps M_k onto Lg_{k+1} injectively: $\ker\big((I - P_k)\lceil M_k\big) = \{0\}$. However, we observe that the linear sets Lg_{k+1} and $L + H_k = M_k + H_k$ may be closed only simultaneously. Nevertheless, the scalar product

$$(f', g')_+ := (f, g), \quad \text{where } fg = (I - P_k)f, \ gg = (I - P_k)g, \tag{6.26}$$

makes L'_{k+1} a Hilbert space. Moreover, $\|f'\| = \|(I - P_k)f\| \le \|f\| = \|f'\|_+$, so that the Hilbert space L'_{k+1} with the norm $\|\cdot\|_+$ is embedded in H continuously.

Finally, since $L \in \mathrm{Cyc}\, A$ and $H_k \in \mathrm{Lat}\, A$, we see that L'_{k+1} is a cyclic linear set for A_{k+1}. Thus, L'_{k+1} satisfies all assumptions of Corollary 5.9. Consequently, by Corollary 5.9 (or Theorem 5.7 if L'_{k+1} is closed), in L'_{k+1} there is a principal vector $g'_{k+1} \in L'_{k+1}$ for A_{k+1}. We show that g'_{k+1} realizes the $(k+1)$st Hellinger type for A. Indeed, by definition, the first k Hellinger types for A_k and A are the same. Therefore, $H_{L_k} = \bigoplus_{i=1}^k H_{g_i}$, where the vector g_i realizes the ith Hellinger type for A. Consequently, $g'_{k+1} \perp H_{L_k}$, and gg_{k+1} realizes the kth Hellinger type for A.

Now, let $g_{k+1} \in L$ be an arbitrary vector satisfying $(I - P_k)g_{k+1} = g'_{k+1}$. We show that $L_{k+1} := \operatorname{span}\{L_k, g_{k+1}\} \in \operatorname{Hel}_{k+1}(E)$, i.e., $g_{k+1} \in \Omega_A^L(L_k)$. Indeed,

$$H_{L_{k+1}} = \operatorname{span}(H_{L_k}, H_{g_{k+1}}) = H_{L_k} \oplus H_{g_{k+1}g} = \bigoplus_{i=1}^{k} H_{g_1} \oplus H_{g'_{k+1}}.$$

Putting $\widetilde{A} := A\lceil H_{L_{k+1}}$, we arrive at the identities

$$\Gamma_i(\widetilde{A}) = \Gamma_i(A), \qquad i \in \{1, \dots, k, k+1\},$$

which mean that $L_{k+1} \in \operatorname{Hel}_{k+1}(A)$.

Summarizing, we conclude that $f \in \Omega_A^L(L_k)$ if and only if $(I - P_k)f$ is a principal vector for A_{k+1}. Therefore, $\Omega_A^L(L_k)$ is the inverse image of $\Omega_{A_{k+1}} \cap L'_{k+1}$ under the mapping $I - P_k$.

Since Lg_{k+1} is cyclic for A_{k+1}, from Corollary 5.9 (or Theorem 5.7 if Lg_{k+1} is closed) we see that $\Omega_{A_{k+1}} \cap Lg_{k+1}$ is a dense G_δ-subset of L'_{k+1}.

But the definition of the norm in Lg_{k+1} (see (6.26)) readily shows that $I - P_k$ maps M_k onto L'_{k+1} (with the norm $\|\cdot\|_+$) isometrically. Therefore, $\Omega_A^{M_k}(L_k)$ is a dense G_δ-subset of M_k (here $\Omega_A^{M_k}(L_k)$ is defined by (6.25) in which L is replaced by M_k).

Now, the relation

$$\Omega_A^L(L_k) = \Omega_A^{M_k}(L_k) \oplus (L \cap H_k) = \Omega_A^{M_k}(L_k) \times (L \cap H_k) \tag{6.27}$$

implies that $\Omega_A^L(L_k)$ is a dense G_δ-subset in L; consequently, it is of the second category.

ii) Let w be a Gaussian measure in L with correlation operator B. The arguments are particularly simple if $M_k \in \operatorname{Lat} B$. Indeed, in this case $w = w_1 w_2$, where w_1 and w_2 are the projections of w to the subspaces M_k and $L \cap H_k$ in the decomposition $L = M_k \oplus (L \cap H_k)$. Since the measures w_i $(i = 1, 2)$ are Gaussian and the operators $I - P_k$ map M_k onto Lg_{k+1} isometrically, by Theorem 5.7 ii) we have $w_1\big(\Omega_A^{M_k}(L_k)\big) = 1$. Since, moreover, $w_2(L \cap H_k) = 1$, from (6.27) and the product nature of the measure w it follows that $w\big(\Omega_A^L(L_k)\big) = w_1\big(\Omega_A^{M_k}(L_k)\big) w_2(L \cap H_k) = 1$.

The case where $M_k \notin \operatorname{Lat} B$ is treated by a slight modification of the arguments employed at step 4) of the proof of Theorem 5.7.

iii) By Theorem 5.7, there exists a principal vector $h \in \Omega_\Sigma \cap L$. Assuming that a Hellinger chain $L_1 \subset \cdots \subset L_k$ has already been constructed, we put $L_{k+1} = \operatorname{span}\{L_k, f\}$, where $f \in \Omega_A^L(L_k)$. By step i), we have $L_{k+1} \in \operatorname{Hel}_{k+1}(A)$, which yields the required one-dimensional extension.

If $n(A) < \infty$ and $L_{n(A)} \notin \operatorname{Cyc} A$, then $H \ominus H_{L_{n(A)}} \neq \{0\}$. But this contradicts the condition $L_{n(A)} \in \operatorname{Hel}_{n(A)}(A)$ (the latter means that the first $n(A)$ Hellinger types for A and for $A_{n(A)} := A\lceil H_{L_{n(A)}}$ are the same; see (6.23)).

iv) Let $A_k := A\lceil H_{L_k}$. Then $n(A_k) = k$. Statement iii) applied to L_k $(\subset H_{L_k})$ and A_k yields a chain $L_1 \subset L_2 \subset \cdots \subset L_k$ in which $L_j \in \operatorname{Hel}_j(A_k) = \operatorname{Hel}_j(A)$, $j \in \{1, \dots, k\}$. The extendibility to the right to a chain of the form (6.24) or (6.11) follows from i). □

Remark 6.4. In particular, Theorem 6.10 shows that the Hellinger types of the operator A (consequently, a complete system of its unitary invariants) can be recovered from each of its cyclic subspaces $L \in \operatorname{Cyc}(A)$.

Namely, although in general a subspace $L \in \operatorname{Cyc}(A)$ may fail to possess a spectrally orthogonal system of vectors that realize the Hellinger types, Theorem 6.10 and Remark 6.3 show that these types are realized by subspaces $H_k \subset L$ or by k-vectors $\varphi_k = e_1 \wedge \cdots \wedge e_k \in \wedge^k(L)$: the kth Hellinger type is realized by the measure $(\wedge^k \Psi(t)\varphi_k, \varphi_k)d\rho$, where $\Psi(t) := \Psi_K(t) := dK^*\Sigma K/d\rho$, $K \in \mathfrak{S}_2(H)$, and $\Sigma = P_L E_A\lceil L$.

The next result is deduced from Theorem 6.10 in the same way as Corollary 5.9 is deduced from Theorem 5.7.

Corollary 6.11. *Let A be a selfadjoint operator in H, and let H_+ be a* (*nonclosed*) *linear set in H such that H_+ is a Hilbert space under a norm $\|\cdot\|_+$ and the embedding $H_+ \to H$ is continuous. If H_+ is cyclic for A, then all the statements of Theorem* 6.10 *are valid for H_+.*

Theorem 6.10, Corollary 6.11, and the Naĭmark theorem imply the following refinement of Theorem 6.7. The proof is entirely similar to that of Theorem 5.10.

Theorem 6.12. *Let Σ be an operator measure in H, let $H_k \in \mathrm{Hel}_k(\Sigma)$, and let*

$$\Omega_\Sigma(H_k) := \Omega_\Sigma^H(H_k) := \{f \in H : \mathrm{span}\{H_k, f\}\} \in \mathrm{Hel}_{k+1}(H). \tag{6.28}$$

Then the set $\Omega_\Sigma(H_k)$ satisfies the following conditions:

i) *$\Omega_\Sigma(H_k)$ is a dense G_δ-subset in H of the second category*;

ii) *$\omega(\Omega_\Sigma(H_k)) = 1$ for every Gaussian measure ω in H.*

In conclusion we give yet another result about the set $\mathrm{Hel}_k(\Sigma)$ (without proof).

Theorem 6.13. *Let Σ be an operator measure in H. Then*:

i) *$\mathrm{Hel}_k(\Sigma)$ is a dense subset in $G_k(H)$ of the second category*;

ii) *if $\dim H < \infty$, then $\tau_n\big(\mathrm{Hel}_k(\Sigma)\big) = 1$, where τ_n is the normalized measure on $G_k(H)$ invariant under the unitary operators on H.*

6.3. Realization of Hellinger types by vectors in H. If E is an orthogonal measure in H, then any finite scalar measure $\mu \prec E$ is realized by some vector $g \in H : \mu = \mu_g$, $\mu_g(\delta) = (\Sigma(\delta)g, g)$.

We do not know whether the same is true for a nonorthogonal measure Σ in H. However, below we describe the types $[\mu_g]$ of the scalar measures μ_g. As has already been noted, for this it suffices to describe their supports $\Gamma(g)$ of the form (6.2).

Proposition 6.14. *Let Σ be an operator measure in H, and let $X \in \mathcal{B}(\mathbb{R})$. Denote by Σ^X the restriction of Σ to X ($\Sigma^X : \ \delta \mapsto \Sigma(\delta \cap X)$). The following conditions are equivalent*:

i) *there exists a vector $h \in H$ with $\Gamma(h) = X$*;

ii) *$L_X := \ker \Sigma(\mathbb{R} \setminus X)$ is a principal subspace for the measure Σ^X in the sense of Definition* 5.11, *i.e., $L_X \not\subset \ker \Sigma(\delta)$ for every $\delta \in \mathcal{B}(\mathbb{R})$ such that $\Sigma(\delta) \neq \{0\}$*;

iii) *$\Sigma_{L_X} := P_{L_X}\Sigma\lceil L_X \sim \Sigma^X$*;

iv) *$\Omega^X \cap L_X \neq \varnothing$, where $\Omega^X := \{h \in H : \Gamma(h) = X\} = \Omega_{\Sigma^X}$*;

v) *$\Omega^X \cap L_X$ is a dense G_δ-subset in L_X of the second category*;

vi) *$\omega(\Omega^X(\bigcap L_X)) = 1$ for every Gaussian measure ω in L_X.*

Proof. This follows from Proposition 5.12. □

In particular, Proposition 6.14 implies a criterion for the existence of vectors that realize "junior" Hellinger types. We observe that if $g_1 \in \Omega_\Sigma$ and the set $M_2 = \{h \in H : \Gamma(h) = \Gamma_2(\Sigma)\}$ is not empty, it is not always possible to choose $g_2 \in M_2$ orthogonal to g_1. But for every $g_2 \in M_2$ it is always possible to choose a principal vector $g_1 \perp g_2$.

Example 6.15. Denote by $\Sigma(t) = \sum_{i<t} P_i$ the discrete measure with the following jumps:

$$P_1 = \mathrm{diag}(1,0,0), \quad P_2 = \mathrm{diag}(0,1,0), \quad P_3 = \mathrm{diag}(0,1,1), \quad P_4 = \mathrm{diag}(1,1,1).$$

Then $\Gamma_1(\Sigma) = \mathrm{supp}\,\Sigma = \{1,2,3,4\}$, $\Gamma_2(\Sigma) = \{3,4\}$, and $\Gamma_3 = \{4\}$. If $h = (h_1,h_2,h_3) \in \mathbb{C}^3$, then $\Gamma(h) = \Gamma_2(\Sigma) = \{3,4\}$ precisely when $h_1 = h_2 = 0$. Next, if $g_1 = (x_1,x_2,x_3)$ is a principal vector and $g_1 \perp h$, then $x_3 = 0$. Therefore, if $x_3 \neq 0$, then there are no vectors h realizing $\Gamma_2(\Sigma)$ and orthogonal to g_1. At the same time, for the vector $h = (0, 0, h_3)$, which realizes $\Gamma_2(\Sigma)$, the vector $g_1 = (1, 1, 0)$ belongs to Ω_Σ and is orthogonal to h.

Proposition 6.16. *Let Σ be an operator measure in H, and let $g \in H$ satisfy $\Gamma(g) \subset \Gamma_2(\Sigma)$. Then the subspace $H \ominus g$ is principal, i.e., there exists a principal vector for Σ orthogonal to g.*

Proof. Assume the contrary. Then, by Definition 5.11, there exists $\delta \in \mathcal{B}(\mathbb{R})$ such that $\Sigma(\delta) \neq 0$ and $H \ominus g \subset \ker \Sigma(\delta)$. *A fortiori*, we have $H \ominus g \subset \ker \Sigma(\delta \setminus \Gamma_2(\Sigma))$.

First we suppose that $\Sigma(\delta \setminus \Gamma_2(\Sigma)) \neq 0$. By assumption, $\Gamma(g) \subset \Gamma_2(\Sigma)$ and $\Gamma(g)$ is a support of the measure $\mu_g := (\Sigma g, g)$; therefore, $g \in \ker \Sigma(\delta \setminus \Gamma_2(\Sigma))$. Consequently, $\Sigma = 0$, a contradiction.

But if $\Sigma(\delta \setminus \Gamma_2(\Sigma)) = 0$, then, replacing δ by $\delta \cap \Gamma_2(\Sigma)$ if necessary, we may assume that $\delta \subset \Gamma_2(\Sigma)$. By Lemma 6.3, $\operatorname{rank} \Sigma(\delta) \geq 2$. On the other hand, $H \ominus g \subset \ker \Sigma(\delta)$ by assumption; consequently, $\operatorname{rank} \Sigma(\delta) \leq 1$, a contradiction. □

It is natural to expect that if $g_1 \perp g_2$ and $\Gamma(g_i) = \Gamma_i(\Sigma)$ $(i = 1, 2)$, then $H_2 = \operatorname{span}\{g_1, g_2\} \in \operatorname{Hel}_2(\Sigma)$. The following example shows that this is not the case.

Example 6.17. Let $\Sigma(t) := \sum_{i<t} P_i$, where $i \in \{1, 2\}$ and

$$P_1 = \operatorname{diag}(0, 1, 0), \quad P_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\Gamma_1(\Sigma) = \{1, 2\}$ and $\Gamma_2(\Sigma) = \{2\}$. Clearly, $\Gamma(e_1) = \Gamma_2(\Sigma)$, where $e_1 = (1, 0, 0)$. Let $g_1 := e_2 = (0, 1, 0)$. Then g_1 is a principal vector for Σ and $g_1 \perp e_1$. Let $L = \operatorname{span}\{g_1, e_1\}$.

We put $h = e_1 - g_1 = (1, -1, 0) \in L$. Then $h \in \ker P_2$ and, consequently, $L \cap \ker P_2 \neq \{0\}$. By Proposition 6.6 iii), we have $L \notin \operatorname{Hel}_2(\Sigma)$.

6.4. Hellinger subspaces for a system of operators. It is easily seen that nearly all results of the paper remain true for the operator measures defined on the Borel subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$.

For example, we present analogs of Theorems 5.7 and 6.10 for a system $\mathbf{a} = \{A_j\}_1^m$ of pairwise commuting selfadjoint operators in H. Let $E = E_{\mathbf{a}}$ be the joint spectral measure of the system $\mathbf{a}$, defined on the σ-algebra $\mathcal{B}(\mathbb{R}^n)$ (see [6]).

For the system $\mathbf{a} = \{A_j\}_1^m$, the sets Ω_E of principal vectors, $\operatorname{Cyc}(\mathbf{a})$ of cyclic subspaces, and $\operatorname{Hel}_k(E) = \operatorname{Hel}_k(\mathbf{a})$ of Hellinger subspaces (see formula (5.1) and Definitions 4.2 and 6.9) are introduced as before, but relative to the measure $E_{\mathbf{a}}$.

Theorem 6.18. *Let $\mathbf{a} = \{A_j\}_1^m$ be a system of pairwise commuting selfadjoint operators in H, $E = E_{\mathbf{a}}$ their joint spectral measure, and $\Omega_{\mathbf{a}} := \Omega_E$ the set of principal vectors of $\mathbf{a}$. If $L \in \operatorname{Cyc}(\mathbf{a})$, then*:

i) $\Omega_{\mathbf{a}} \cap L (= \Omega_E \cap L)$ *is a dense G_δ-subset in L of the second category*;

ii) $\omega(\Omega_{\mathbf{a}} \cap L) = 1$ *for every Gaussian measure ω in L.*

The joint spectral multiplicity of the system $\mathbf{a} = \{A_j\}_1^m$ is defined to be the number

$$n(\mathbf{a}) := n(A_1, \ldots, A_m) := \min\{\dim L : L \in \operatorname{Cyc}(\mathbf{a})\} =: n(E_{\mathbf{a}}).$$

Corollary 6.19. *Let $\mathbf{a} = \{A_j\}_1^m$ be a system of pairwise commuting selfadjoint operators in H with simple joint spectrum, i.e., $n(\mathbf{a}) = n(E_{\mathbf{a}}) = 1$. Suppose $L \in \operatorname{Cyc}(\mathbf{a})$ and denote by $\Omega_{\mathbf{a}}$ the set of cyclic vectors of $\mathbf{a}$. Then*:

i) $\Omega_{\mathbf{a}}$ *is a dense G_δ-subset in L of the second category*;

ii) $\omega(\Omega_{\mathbf{a}} \cap L) = 1$ *for every Gaussian measure ω in L.*

Theorem 6.20. *Suppose* $\mathbf{a} = \{A_j\}_1^m$ *is a system of pairwise commuting selfadjoint operators in* H, $L \in \mathrm{Cyc}(\mathbf{a})$, $L_k \subset L$, $L_k \in \mathrm{Hel}_k(\mathbf{a})$, *and* $k < n(\mathbf{a})$. *If* $\Omega_{\mathbf{a}}^L(L_k) := \{f \in L : \mathrm{span}\{L_k, f\}\} \in \mathrm{Hel}_{k+1}(\mathbf{a})$, *then*:

i) $\Omega_{\mathbf{a}}^L(L_k)$ *is a dense* G_δ*-subset in* L *of the second category*;

ii) $\omega(\Omega_{\mathbf{a}}^L(L_k)) = 1$ *for every Gaussian measure* ω *in* L;

iii) *in* L *there exists a Hellinger chain of the form* (6.24) *or* (6.11); *moreover*, $L_{n(\mathbf{a})} \in \mathrm{Cyc}(\mathbf{a})$ *if* $n(\mathbf{a}) < \infty$, *and conversely, if* $\dim L = n(\mathbf{a})$, *then* $L \in \mathrm{Hel}_{n(\mathbf{a})}(\mathbf{a})$;

iv) *every Hellinger subspace* $L_k \in \mathrm{Hel}_k(\mathbf{a})$ *can be incorporated in a Hellinger chain of the form* (6.24) *or* (6.11).

Vasyunin and Nikol′skiĭ [9] introduced the following characteristic of a system $\mathbf{a} = \{A_j\}_1^m$:

$$\text{(6.29)} \qquad \mathrm{disc}(\mathbf{a}) := \mathrm{disc}(A_1, \dots, A_m) := \sup_{L \in \mathrm{Cyc}(\mathbf{a})} \min\{\dim Lg : L' \in \mathrm{Cyc}(\mathbf{a}),\ Lg \subset L\}.$$

It is easily seen that $\mathrm{disc}(\mathbf{a}) \geq n(\mathbf{a})$.

Corollary 6.21 ([9]). *Let* $\mathbf{a} = \{A_j\}_1^m$ *be a system of pairwise commuting operators in* H. *Then*

$$\text{(6.30)} \qquad \mathrm{disc}(\mathbf{a}) = \mathrm{disc}(A_1, \dots, A_m) = n(A_1, \dots, A_m) = n(\mathbf{a}).$$

Proof. The case where $n(\mathbf{a}) = \infty$ is trivial. Suppose $n(\mathbf{a}) < \infty$ and $L \in \mathrm{Cyc}(\mathbf{a})$. By Theorem 6.20 iii), in L there is a Hellinger subspace $L_{n(\mathbf{a})} \in \mathrm{Hel}_{n(\mathbf{a})}(\mathbf{a})$ $\big(\subset \mathrm{Cyc}(\mathbf{a})\big)$. Therefore, putting $Lg = L_{n(\mathbf{a})}$ in (6.29), we arrive at (6.30). □

Remark 6.5. Suppose an operator T $\big(\in B(H)\big)$ is normal ($T^*T = TT^*$) and reductive ($\mathrm{Lat}\, T = \mathrm{Lat}\, T^*$). Let $A_1 = (T + T^*)/2$, $A_2 = (T - T^*)/2i$. Then $\mathrm{Cyc}\, T = \mathrm{Cyc}(T, T^*) = \mathrm{Cyc}(A_1, A_2)$ and, consequently, $\mathrm{disc}\, T = \mathrm{disc}(A_1, A_2)$. In this case the relation $\mathrm{disc}\, T = n(T)$ $(= n(E_T))$, which coincides with (6.30) if $n = 2$, was proved in (9) by a different method.

Since $\mathrm{Cyc}\, T \subset \mathrm{Cyc}(E_T)$, every subspace $L \in \mathrm{Cyc}\, T$ contains a subspace $L_{n(T)} \in \mathrm{Hel}_{n(T)}(E_T)$ $(\subset \mathrm{Cyc}(E_T))$; however, this subspace may fail to be cyclic for T ($L_{n(T)} \notin \mathrm{Cyc}\, T$). Precisely this fact is an obstruction for the proof of the relation $\mathrm{disc}\, T = \mathrm{disc}(A_1, A_2)$ $\big(= n(T)\big)$ in the general case, and indeed, this identity fails for some operators. Namely, in [9] it was shown that this identity is true if and only if T is reductive.

It should be noted that if $n(\mathbf{a}) = \infty$, then the statement (6.30) is empty, and Theorem 6.20 guarantees the existence of Hellinger chains, vectors $f \in \Omega_E$, etc. in L $\big(\in \mathrm{Cyc}(\mathbf{a})\big)$.

§7. Functional model of a symmetric operator

7.1. Vector Riemann–Stieltjes and Lebesgue–Stieltjes integrals. Let $f : [a, b] \to H$ be a bounded function, and let Σ be an operator measure in H. As in (2.2), we define the vector Riemann–Stieltjes integral

$$\text{(7.1)} \qquad I_{RS}(\Sigma, f) := \int_a^b d\Sigma(t) f(t) := \lim_{d(\pi_n) \to 0} \sum_{k=1}^n \Sigma(\Delta_k) f(\xi_k).$$

We say that f is *Riemann-Stieltjes integrable* with respect to Σ if the limit in (7.1) exists (a particular choice of partitions with a given diameter and of points $\xi_k \in [t_{k-1}, t_k]$ is irrelevant; $\Sigma(\Delta_k) := \Sigma(t_k) - \Sigma(t_{k-1})$).

For a bounded function $f : \mathbb{R} \to H$ not having compact support, we put

$$\text{(7.2)} \qquad I_{RS}(\Sigma, f) := \int_{\mathbb{R}} d\Sigma(t) f(t) := \operatorname*{s\text{-}lim}_{n \to \infty} \int_{-n}^n d\Sigma(t) f(t)$$

if the integrals $\int_{-n}^n d\Sigma(t) f(t)$ and the limit in (7.2) exist.

Lemma 7.1. *Let f be a piecewise continuous H-valued function with compact support. If Σ is continuous at the discontinuities of f and is of locally bounded variation on $\mathbb{R}$, then the vector integral $I_{RS}(\Sigma, f)$ exists.*

Corollary 7.2. *Suppose all assumptions of Lemma* 7.1 *but the last are fulfilled, i.e., Σ may fail to be of locally bounded variation. Then*:

i) *if f takes its values in a finite-dimensional subspace $L \subset H$, then the integral $I_{RS}(\Sigma, f)$ of the form* (7.1) *exists*;

ii) *in particular, if $f = \sum_{k=1}^{n} \chi_{\Delta_k} f_k \in S_\Sigma$ (see the definition* (2.24)), *where $\Delta_k = [t_{k-1}, t_k)$ and $f_k \in H$, then $I_{RS}(\Sigma, f) = \sum_{k=1}^{n} \Sigma(\Delta_k) f_k$.*

The following statement is proved much as Proposition 2.7.

Proposition 7.3. *Let Σ be a discrete measure on $[a, b]$ of the form* (2.6), *i.e., $\Sigma \notin BV[a, b]$. Then there exist strongly continuous vector-valued functions f $(\in C([a, b]; H))$ for which the integral $I_{RS}(\Sigma, f)$ diverges.*

Example 7.4. Let f $\big(\in C([a, b]; H)\big)$ and let Σ be the same as in Example 2.8. The integral sum $S(\pi_n)$ corresponding to the partition $\pi_n = \{0 < n^{-1} < \cdots < 1/2 < 1\}$ and to the points $\xi_k = 1/k$ is of the form $S(\pi_n) = \sum_{k=1}^{n} P_k f(1/k) = \sum_{k=1}^{n} P_k k^{-1/2} e_k = \sum_{k=1}^{n} k^{-1/2} e_k$.

Since $\|S(\pi_n)\|^2 = \sum_{k=1}^{n} k^{-1} \to \infty$ as $n \to \infty$, the integral $I_{RS}(\Sigma, f)$ diverges.

Consider the operator

$$I'_\Sigma : S_\Sigma \to H, \quad I'_\Sigma : f = \sum \chi_{\Delta_k} f_k \mapsto \sum \Sigma(\Delta_k) f_k. \tag{7.3}$$

By Corollary 7.2, $I'_\Sigma f = I_{RS} f$ for $f \in S_\Sigma$. In the next statement, which plays a principal role in what follows, we indicate some conditions on the measure Σ ensuring the extendibility of I'_Σ by continuity to the domain $\mathcal{D}(Q_\Sigma)$ of the multiplication operator $Q_\Sigma : f \to tf$ in $\mathfrak{H} = L^2(\Sigma, H)$.

Proposition 7.5. *Let Σ be an operator measure in H such that*

$$K_\Sigma := \int_{\mathbb{R}} (1 + t^2)^{-1} \, d\Sigma(t) \in [H]. \tag{7.4}$$

Then the operator I'_Σ of the form (7.3) *extends by continuity up to an operator $I_\Sigma : \mathcal{D}(Q_\Sigma) \to H$, and we have*

$$\|I_\Sigma f\|_H \le \|K_\Sigma\|^{1/2} \cdot \|(1 + t^2)^{1/2} f\|_{L^2(\Sigma, H)}, \quad f \in \mathcal{D}(Q_\Sigma). \tag{7.5}$$

Proof. Along with Σ, we consider the operator measure $\Sigma_1 : \delta \mapsto \Sigma_1(\delta) = \int_\delta (1+t^2) \, d\Sigma(t)$. Then the corresponding densities Ψ and Ψ_1 (see (2.14)) obey the formula $\Psi_1(t) = (1 + t^2)\Psi(t)$. Therefore, Theorem 2.14 implies the identities

$$\|f\|^2_{Q_\Sigma} := \|(1 + t^2)^{1/2} f\|^2_{L^2(\Sigma, H)} = \|f\|^2_{L^2(\Sigma_1, H)}, \quad f \in \mathcal{D}(Q_\Sigma), \tag{7.6}$$

i.e., the space $\mathcal{D}(Q_\Sigma)$ with the graph norm coincides isometrically with $L^2(\Sigma_1, H)$. Let $f = \sum_{k=1}^{n} \chi_{\Delta_k} f_k \in S_\Sigma$, where $\Delta_k = [t_{k-1}, t_k)$ and $f_k \in H$. For every $h \in H$, the definition (7.3) implies that

$$
\begin{aligned}
|(I'_\Sigma f, h)| &= \left|\sum_{k=1}^{n}(\Sigma(\Delta_k)f_k, h)\right| \le \sum_{k=1}^{n}\left|\left(\int_\Delta (1+t^2)^{-1}\, d\Sigma_1(t) f_k, h\right)\right| \\
&\le \sum_{k=1}^{n}\left(\int_{\Delta_k}(d\Sigma_1(t)f_k, f_k)\right)^{1/2}\cdot\left(\int_{\Delta_k}(1+t^2)^{-2}\,(d\Sigma_1(t)h, h)\right)^{1/2} \\
&\le \left(\sum_{k=1}^{n}(\Sigma_1(\Delta_k)f_k, f_k)\right)^{1/2}\cdot\left(\sum_{k=1}^{n}\int_{\Delta_k}(1+t^2)^{-2}\, d(\Sigma_1(t)h, h)\right)^{1/2} \qquad (7.7)\\
&= \|f\|_{L^2(\Sigma_1,H)}\cdot\left(\int_{\mathbb{R}}(1+t^2)^{-1}\, d(\Sigma(t)h, h)\right)^{1/2} \\
&= \|f\|_{L^2(\Sigma_1,H)}\cdot (K_\Sigma h, h)^{1/2} \\
&\le \|f\|_{L^2(\Sigma_1,H)}\cdot \|K_\Sigma\|^{1/2}\cdot\|h\|.
\end{aligned}
$$

Since h is arbitrary, by (7.6) the weak estimate (7.7) coincides with (7.5) for $f \in S_\Sigma$. Since $S_{\Sigma_1} = S_\Sigma$ is dense in $L^2(\Sigma_1, H)$ (see step iii) of the proof of Theorem 2.14), we see that for every $g \in L^2(\Sigma_1, H)$ there exists an approximating sequence $g_j \in S_\Sigma$ with $\|g - g_j\|_{L^2(\Sigma_1,H)} \to 0$ as $j \to \infty$. Since S_Σ is a linear set, (7.7) yields

$$\|I'_\Sigma(g_i - g_j)\|_H \le \|K_\Sigma\|^{1/2}\|(1+t^2)^{1/2}(g_i - g_j)\|_{L^2(\Sigma,H)}, \qquad i, j \in N. \tag{7.8}$$

Therefore, the sequence $\{Ig_\Sigma g_j\}_1^\infty$ converges in H. Putting $I_\Sigma g := \lim_{j\to\infty} Ig_\Sigma g_j$, we obtain the required extension of Ig_Σ to $L^2(\Sigma_1, H)$.

By (7.8), the definition of $I_\Sigma g$ does not depend on the choice of an approximating sequence $g_j \in S_\Sigma$. □

Remark 7.1. A natural name for the vector $I_\Sigma f$ in (7.5) is the *Lebesgue–Stieltjes integral* of the function $f \in \mathcal{D}(Q_\Sigma)$ with respect to the operator measure $\Sigma : I_{LS}(\Sigma, f) := I_\Sigma f$. The relationship between $I_\Sigma f$ and the Bochner integral will be treated in another paper.

7.2. Functional model of a symmetric operator. Proposition 7.5 allows us to introduce a "model" symmetric operator A_Σ in $L^2(\Sigma, H)$:

$$A_\Sigma := Q_\Sigma\lceil\mathcal{D}(A_\Sigma), \quad \mathcal{D}(A) = \{f \in \mathcal{D}(Q_\Sigma) : I_\Sigma f = 0\}. \tag{7.9}$$

Now we can give a consistent proof of Proposition 5.2 in [18].

For this, we recall the following definitions.

Definition 7.6. An operator-valued function $F : \mathbb{C}_+ \cup \mathbb{C}_- \to [H]$ is called an R-function ($F \in R_H$) if: a) F is holomorphic in $\mathbb{C}_+ \cup \mathbb{C}_-$; b) $F(\lambda) = F(\overline{\lambda})^*$; and c) $\big(F(\lambda) - F^*(\lambda)\big)(\lambda - \overline{\lambda})^{-1} \ge 0$ for $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$.

By the Nevanlinna theorem, an R-function admits the following integral representation:

$$F(\lambda) = B\lambda + C + \int_{\mathbb{R}}\big((t-\lambda)^{-1} - t(1+t^2)^{-1}\big)\, d\Sigma(t), \tag{7.10}$$

where B, $C = C^* \in [H]$, $B \ge 0$, and the operator measure Σ satisfies (7.4).

Definition 7.7 ([16])**.** Let A be a symmetric operator in H with deficiency indices $n_+(A) = n_-(A)$. A triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is a separable Hilbert space and $\Gamma_i \in [\mathcal{D}(A^*), \mathcal{H}]$, is called a *boundary triple* for A^* if:

a)

$$(A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \mathcal{D}(A^*); \tag{7.11}$$

b) the mapping $\Gamma : f \mapsto \{\Gamma_0 f, \Gamma_1 f\}$ from $\mathcal{D}(A^*)$ to $\mathcal{H} \oplus \mathcal{H}$ is surjective.

In a natural way, every triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ generates two selfadjoint extensions $A_i = A_i^*$ for which $\mathcal{D}(A_i) = \ker \Gamma_i$ $(i = 1, 2)$.

Definition 7.8 ([18])**.** The *Weyl function* of the operator A corresponding to the triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is defined to be the operator-valued function $M(\lambda)$ that acts in accordance with the formula

$$M(\lambda)\Gamma_0 f_\lambda = \Gamma_1 f_\lambda, \quad f_\lambda \in \mathfrak{N}_\lambda := \ker(A^* - \lambda). \tag{7.12}$$

It is known (see [18]) that $M(\lambda)$ is an $R_{\mathcal{H}}$-function, and it determines the pair $\{A, A_0\}$ uniquely up to unitary equivalence.

Proposition 7.9 ([18])**.** *Let F be an operator-valued R_H-function of the form* (7.10) *that satisfies the conditions*

$$B = 0, \quad 0 \in \rho\bigl(\operatorname{Im} F(i)\bigr), \quad \text{and} \quad \lim_{y\uparrow\infty} y\bigl(F(iy)h, h\bigr) = \infty \text{ for } h \in H \setminus \{0\}. \tag{7.13}$$

Then:

i) *the operator A_Σ of the form* (7.9) *is a simple symmetric operator in $\mathfrak{H} = L^2(\Sigma, H)$, and its domain $\mathcal{D}(A_\Sigma)$ is dense in $\mathfrak{H}$;*

ii) *the deficiency indices $n_\pm(A_\Sigma)$ are equal to $\dim H$, and the defect subspace $\mathfrak{N}_\lambda(A_\Sigma)$ has the form*

$$\mathfrak{N}_\lambda(A_\Sigma) = \{(t-\lambda)^{-1}h : h \in H\}; \tag{7.14}$$

iii) *the adjoint operator A_Σ^* has the form*

$$\mathcal{D}(A_\Sigma^*) = \{f = f_Q + t(t^2+1)^{-1}h : f_Q \in \mathcal{D}(Q_\Sigma),\ h \in H\}, \quad A_\Sigma^* f = t f_Q - (t^2+1)^{-1}h; \tag{7.15}$$

iv) *the collection $\Pi = \{H, \Gamma_0, \Gamma_1\}$ in which Γ_0 and Γ_1 are defined by the formulas*

$$\Gamma_0 f = h, \quad \Gamma_1 f = Ch + I_\Sigma f_Q, \qquad f = f_Q + t(t^2+1)^{-1}h \in \mathcal{D}(A^*), \tag{7.16}$$

is a boundary triple for A_Σ^*;

v) *for the γ-field $\gamma(\lambda) := \bigl(\Gamma_0 \lceil \mathfrak{N}_\lambda(A_\Sigma)\bigr)^{-1} : H \to \mathfrak{N}_\lambda(A_\Sigma)$ we have the formulas*

$$\gamma(\lambda)h = (t-\lambda)^{-1}h, \quad \gamma^*(\lambda)f = I_\Sigma\bigl(f(t)(t-\lambda)^{-1}\bigr), \qquad f \in \mathfrak{H}; \tag{7.17}$$

vi) *the Weyl function $M(\lambda)$ corresponding to Π coincides with $F(\lambda)$.*

Proof. i) The condition $\overline{\mathcal{D}(A)} = \mathfrak{H}$ is a consequence of the last condition in (7.13).

Since $\mathfrak{N}'_\lambda(A) := \{(t-\lambda)^{-1}h : h \in H\} \subset \mathfrak{N}_\lambda(A)$, to prove that A is simple it suffices to show that $\mathfrak{H}' := \operatorname{span}\{\mathfrak{N}''_\lambda(A_\Sigma) : \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-\}$ coincides with $\mathfrak{H}$. Suppose a vector f $\bigl(\in L^2(\Sigma, H)\bigr)$ is orthogonal to $\mathfrak{H}'$. Since $\widetilde{H}_t \supset H$ for ρ-a.e. $t \in \mathbb{R}$, Theorem 2.14 shows that

$$0 = \bigl(f, (t-\overline{\lambda})^{-1}h\bigr)_{L^2(\Sigma,h)} = \int_{\mathbb{R}} (t-\lambda)^{-1}\bigl(f(t), h\bigr)_{\widetilde{H}_t}\, d\rho(t), \quad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-,\ \ h \in H.$$

Applying the uniqueness theorem for the Cauchy transforms of measures, we obtain $\bigl(f(t), h\bigr)_{\widetilde{H}_t} = 0 \pmod{\rho}$, $h \in H$. But H is dense in $\widetilde{H}_t$ for $t \in \mathbb{R}$ (see (2.16)), whence $f(t) = 0 \pmod{\rho}$.

ii) We show that $\mathfrak{N}'_\lambda(A_\Sigma)$ is dense in $\mathfrak{N}_\lambda(A_\Sigma)$. Indeed, suppose a vector $g \in \mathfrak{H}$ is orthogonal to $\mathfrak{N}'_\lambda(A_\Sigma)$. Then for every $h \in H$ we have

$$0 = \bigl(g, (t-\overline{\lambda})^{-1}h\bigr)_{L^2(\Sigma,H)} = \int_{\mathbb{R}} \bigl(d\Sigma(t)g(t)(t-\lambda)^{-1}, h\bigr) = \bigl(I_\Sigma(g(t)(t-\lambda)^{-1}), h\bigr)_H. \tag{7.18}$$

Therefore, $I_\Sigma f = 0$ for $f(t) := g(t)(t-\lambda)^{-1}$, and consequently, $\mathfrak{N}'_\lambda(A_\Sigma)$ is dense in $\mathfrak{N}_\lambda(A_\Sigma)$.

Next, we have $K_\Sigma = \operatorname{Im}\big(F(i)\big)$, and $0 \in \rho(K_\Sigma)$ by assumption. Thus, the following formula (which is a consequence of (2.22)) proves (7.14):

$$\|(t-i)^{-1}h\|^2_{L^2(\Sigma,H)} = \int_{\mathbb{R}} \big(d\Sigma(t)(t-i)^{-1}h, (t-i)^{-1}h\big) = \int_{\mathbb{R}} \frac{d\big(\Sigma(t)h,h\big)}{t^2+1} = \|K_\Sigma^{1/2}h\|^2_H.$$

iii) Identities (7.15) follow from (7.14) and the von Neumann formula.

iv) Suppose $f = f_Q + t(t^2+1)^{-1}h_1$, $g = g_Q + t(t^2+1)^{-1}h \in \mathcal{D}(A_\Sigma^*)$. By (7.15) and (7.16), the Green formula (7.11) to be verified reshapes as follows:

$$\begin{aligned} &\big(tf_Q, t(t^2+1)^{-1}h_2\big)_{L^2(\Sigma,H)} + \big(f_Q, (t^2+1)^{-1}h_2\big)_{L^2(\Sigma,H)} \\ &\qquad - \big((t^2+1)^{-1}h_1, g_Q\big)_{L^2(\Sigma,H)} - \big(t(t^2+1)^{-1}h_1, tg_Q\big)_{L^2(\Sigma,H)} \\ &= (I_\Sigma f_Q, h_2)_H - (h_1, I_\Sigma g_Q)_H. \end{aligned} \tag{7.19}$$

But, clearly, (7.19) is equivalent to

$$\begin{aligned} &((t^2+1)^{1/2} f_Q, (1+t^2)^{-1/2}h_2)_{L^2(\Sigma,H)} - ((1+t^2)^{-1/2}h_1, (1+t^2)^{1/2}g_Q)_{L^2(\Sigma,H)} \\ &\quad = (I_\Sigma f_Q, h_2)_H - (h_1, I_\Sigma g_Q)_H. \end{aligned} \tag{7.20}$$

For the step functions $f_Q \in S_\Sigma$, identity (7.20) is obvious. Since S_Σ is dense in $\mathcal{D}(Q_\Sigma)$ in the graph norm (see (7.6)), identity (7.20) extends to the entire space $\mathcal{D}(Q_\Sigma) = L^2(\Sigma_1, H)$ by continuity (see Proposition 7.5).

We prove that the mapping $\Gamma = \{\Gamma_0, \Gamma_1\}$ is onto. For every pair $\{h_0, h_1\} \in H \oplus H$, we put $f(t) = (1+t^2)^{-1}K_\Sigma^{-1}h_1 + t(1+t^2)^{-1}h_0$. Since $I_\Sigma\big((1+t^2)^{-1}h\big) = K_\Sigma h$, the definition (7.16) shows that $\Gamma f = \{\Gamma_0 f, \Gamma_1 f\} = \{h_0, h_1\}$.

v) The obvious identity

$$f_\lambda := \frac{1}{t-\lambda}h = \frac{t}{1+t^2}h + f_Q, \quad f_Q = \frac{1+\lambda t}{(t-\lambda)(1+t^2)}h \in \mathcal{D}(Q_\Sigma), \tag{7.21}$$

yields $\Gamma_0(t-\lambda)^{-1}h = h$, and thus, $\gamma(\lambda)h = (t-\lambda)^{-1}h$. Now, the formula for $\gamma^*(\lambda)$ follows from Proposition 7.5.

vi) Applying (7.21) to both sides of (7.21), we see that

$$\Gamma_1 f_\lambda = Ch + I_\Sigma f_Q = Ch + \int_{\mathbb{R}} d\Sigma(t) \frac{1+\lambda t}{(t-\lambda)(1+t^2)} h = F(\lambda)h. \tag{7.22}$$

Now, (7.12), (7.22), and the relation $\Gamma_0 f_\lambda = h$ yield $M(\lambda) = F(\lambda)$. □

Corollary 7.10. *Let A be a simple symmetric operator in $\mathfrak{H}'$ with dense domain $\mathcal{D}(A)$, let $n_+(A) = n_-(A)$, and let $\widetilde{A} = \widetilde{A}^* \supset A$. Then there is a measure Σ in H $(\dim H = n_\pm(A))$ satisfying* (7.4) *and such that the pair $(A, \widetilde{A})$ is unitarily equivalent to the pair (A_Σ, Q_Σ) in $L^2(\Sigma, H)$, i.e., there exists a unitary mapping $U : \mathfrak{H} = L^2(\Sigma, H) \to \mathfrak{H}'$ such that $U^{-1}AU = A_\Sigma$ and $U^{-1}\widetilde{A}U = Q_\Sigma$.*

Proof. The operator A^* possesses a boundary triple $\widetilde{\Pi} = \{H, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ with $\mathcal{D}(\widetilde{A}) = \ker \widetilde{\Gamma}_0$; see [18]. The Weyl function $\widetilde{M}(\lambda)$ corresponding to $\widetilde{\Pi}$ admits a representation (7.10) with an operator measure Σ satisfying (7.4). Starting with Σ, we introduce the operators A_Σ and $Q_\Sigma = Q_\Sigma^*$ of the form (7.9) in $L^2(\Sigma, H)$ and consider the triple of the form (7.16) for A_Σ^*. Since $\overline{\mathcal{D}(A)} = \widetilde{\mathfrak{H}}$, we see that $\widetilde{M}(\lambda)$ satisfies (7.13). Proposition 7.9 shows that the Weyl function $M(\lambda)$ corresponding to Π coincides with $\widetilde{M}(\lambda) : M(\lambda) = \widetilde{M}(\lambda)$. Since the operator A_Σ is simple, the pairs $(A, \widetilde{A})$ and (A_Σ, Q_Σ) are unitarily equivalent. □

References

[1] N. I. Akhiezer and I. M. Glazman, *Theory of linear operators in Hilbert space*, "Nauka", Moscow, 1966; English transl., Pitman, Boston–London, 1981. MR **34:**6527

[2] Yu. M. Berezanskiĭ, *On expansion according to eigenfunctions of general selfadjoint differential operators*, Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), no. 3, 379–382. (Russian) MR **18:**323b

[3] ———, *Expansions in eigenfunctions of selfadjoint operators*, "Naukova Dumka", Kiev, 1965; English transl., Transl. Math. Monogr., vol. 17, Amer. Math. Soc., Providence, RI, 1968. MR **36:**5769

[4] Yu. M. Berezanskiĭ, G. F. Us, and Z. G. Sheftel′, *Functional analysis*, "Vyshcha Shkola", Kiev, 1990; English transl., Oper. Theory Adv. Appl., vols. 85, 86, Birkhäuser Verlag, Basel, 1996. MR **97i:**46001a; MR **97i:**46001b

[5] C. A. Berger, *A strange dilation theorem*, Notices Amer. Math. Soc. **12** (1965), no. 5, 590. (Abstract 625–152).

[6] M. Sh. Birman and M. Z. Solomyak, *Spectral theory of selfadjoint operators in Hilbert space*, Leningrad. Univ., Leningrad, 1980; English transl., Math. Appl. (Soviet Ser.), Reidel, Dordrecht, 1987. MR **82k:**47001

[7] M. Sh. Birman and S. B. Èntina, *Stationary approach in abstract scattering theory*, Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), no. 2, 401–430; English transl., Math. USSR-Izv. **1** (1967), no. 1, 391–420. MR **35:**790

[8] M. S. Brodskiĭ, *Triangular and Jordan representations of linear operators*, "Nauka", Moscow, 1969; English transl., Transl. Math. Monogr., vol. 32, Amer. Math. Soc., Providence, RI, 1971. MR **41:**4283

[9] V. I. Vasyunin and N. K. Nikol′skiĭ, *Control subspaces of minimal dimension. Elementary introduction*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **113** (1981), 41–75; English transl., J. Soviet Math. **22** (1983), no. 6, 1719–1742. MR **84e:**93016

[10] ———, *Quasi-orthogonal decompositions with respect to complementary metrics, and estimates for univalent functions*, Algebra i Analiz **2** (1990), no. 4, 1–81; English transl., Leningrad Math. J. **2** (1991), no. 4, 691–764. MR **93b:**46003b

[11] J. B. Garnett, *Bounded analytic functions*, Pure Appl. Math., vol. 96, Academic Press, Inc., New York–London, 1981. MR **83g:**30037

[12] I. M. Gel′fand and I. Ya. Vilenkin, *Generalized functions*. Vol. 4. *Applications of harmonic analysis*, Fizmatgiz, Moscow, 1961; English transl., Academic Press, New York–London, 1964 [1977]. MR **30:**4152

[13] I. M. Gel′fand and A. G. Kostyuchenko, *Expansion in eigenfunctions of differential and other operators*, Dokl. Akad. Nauk SSSR (N. S.) **103** (1955), no. 3, 349–352. (Russian) MR **17:**388

[14] I. M. Gel′fand and G. E. Shilov, *Generalized functions*. Vol. 3. *Theory of differential equations*, Fizmatgiz, Moscow, 1958; English transl., Academic Press, New York–London, 1967 [1977]. MR **36:**506

[15] F. Gesztesy, N. Kalton, K. Makarov, and E. Tsekanovskii, *Some applications of operator-valued Herglotz functions*, Operator Theory, System Theory and Related Topics (Beer-Sheva/Rehovot, 1997), Oper. Theory Adv. Appl., vol. 123, Birkhäuser, Basel, 2001, pp. 271–321. MR **2002f:**47049

[16] V. I. Gorbachuk and M. L. Gorbachuk, *Boundary value problems for operator-differential equations*, "Naukova Dumka", Kiev, 1984; English transl., Math. Appl. (Soviet Ser.), vol. 48, Kluwer Academic Publishers Group, Dordrecht, 1991. MR **92m:**34133

[17] N. Dunford and J. T. Schwartz, *Linear operators*. Part II. *Spectral theory. Selfadjoint operators in Hilbert space*, John Wiley and Sons, Inc., New York, 1988. MR **90g:**47001b

[18] V. A. Derkach and M. M. Malamud, *The extension theory of Hermitian operators and the moment problem*, Analysis-3, Itogi Nauki i Tekhniki Ser. Sovrem. Mat. Prilozh., vol. 5, VINITI, Moscow, 1993; English transl., J. Math. Sci. **73** (1995), no. 2, 141–242. MR **95m:**47009

[19] T. Kato, *Perturbation theory for linear operators*, 2nd ed., Grundlehren Math. Wiss., vol. 132, Springer-Verlag, Berlin–New York, 1976. MR **53:**11389

[20] I. S. Kats, *On Hilbert spaces generated by monotone Hermitian matrix-functions*, Khar′kov. Gos. Univ. Uchen. Zap. **34** (1950), 95–113 = Zap. Mat. Otdel. Fiz.-Mat. Fak. i Khar′kov. Mat. Obshch. (4) **22** (1950), 95–113. (Russian) MR **18:**222b

[21] A. A. Kirillov, *Elements of the theory of representations*, "Nauka", Moscow, 1972; English transl., Grundlehren Math. Wiss., vol. 220, Springer-Verlag, Berlin–New York, 1976. MR **53:**10985

[22] M. G. Kreĭn, *On the Hermitian operators with directed functionals*, Akad. Nauk Ukrain. RSR. Zbirnik Prac′ Inst. Mat. **10** (1948), 83–106. (Ukrainian) MR **14:**56c

[23] M. M. Malamud and S. M. Malamud, *On the spectral theory of operator measures*, Funktsional. Anal. i Prilozhen. **36** (2002), no. 2, 83–89; English transl., Funct. Anal. Appl. **36** (2002), no. 2, 154–158. MR **2003e:**47044

[24] S. M. Malamud and L. L. Oridoroga, *On the principal vector of an operator measure*, Spectral and Evolution Problems, Vol. 2 (Sevastopol, 2000), Nat. Taurida Univ., Simferopol, 2001, pp. 124–127. MR **2002f:**47044

[25] M. A. Naĭmark, *Spectral functions of a symmetric operator*, Izv. Akad. Nauk SSSR Ser. Mat. **4** (1940), no. 3, 277–318. (Russian) MR **2:**105a

[26] N. K. Nikol′skiĭ, *Treatise on the shift operator*, "Nauka", Moscow, 1980; English transl., Grundlehren Math. Wiss., vol. 273, Springer-Verlag, Berlin, 1986. MR **87i:**47042

[27] A. I. Plesner, *Spectral theory of linear operators*, "Nauka", Moscow, 1965; English transl., Frederick Ungar Publishing Co., New York, 1969. MR **33:**3106

[28] M. Reed and B. Simon, *Methods of modern mathematical physics*. IV. *Analysis of operators*, Academic Press, New York–London, 1978. MR **58:**12429c

[29] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, North-Holland Publishing Co., Amsterdam–London; American Elsevier Publishing Co., Inc., New York, 1970. MR **43:**947

[30] ———, *On certain classes of power-bounded operators in Hilbert space*, Acta Sci. Math. (Szeged) **27** (1966), 17–25. MR **33:**6413

[31] A. V. Skorokhod, *Integration in Hilbert space*, "Nauka", Moscow, 1975; English transl., Ergeb. Math. Grenzgeb., vol. 79, Springer-Verlag, New York–Heidelberg, 1974. MR **57:**6360

[32] A. S. Kholevo, *Probabilistic and statistical aspects of quantum theory*, "Nauka", Moscow, 1980; English transl., North-Holland Ser. Statist. Probab., vol. 1, North-Holland Publishing Co., Amsterdam, 1982. MR **85i:**81038a

[33] D. V. Yakubovich, *Local spectral multiplicity of a linear operator with respect to a measure*, Investigations on Linear Operators and Function Theory. 23, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **222** (1995), 293–306; English transl., J. Math. Sci. **87** (1997), no. 5, 3971–3979. MR **96j:**47004

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