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LIPSCHITZ PROPERTY OF THE FREE BOUNDARY IN THE PARABOLIC OBSTACLE PROBLEM

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Abstract. A parabolic obstacle problem with zero constraint is considered. It is proved, without any additional assumptions on a free boundary, that near the fixed boundary where the homogeneous Dirichlet condition is fulfilled, the boundary of the noncoincidence set is the graph of a Lipschitz function.

In this paper, the regularity properties of a free boundary in a neighborhood of the fixed boundary of a domain are studied for a parabolic obstacle problem with zero constraint.

For parabolic equations, the simplest obstacle problem can be formulated as follows. Suppose $\mathbb{D}$ is a domain in $\mathbb{R}^n$, $\mathbb{Q} = \mathbb{D} \times]0, T[$, and

$$\mathbb{K} = \{w \in H^1(\mathbb{Q}) : w \geq 0 \text{ a.e. in } \mathbb{Q}, w = \phi \text{ on } \partial'\mathbb{Q}\},$$

where ϕ is a nonnegative function defined on the parabolic boundary $\partial'\mathbb{Q}$ of the cylinder $\mathbb{Q}$. It is required to find a function $u \in \mathbb{K}$ such that

$$\int_{\mathbb{D}} \partial_t u(w-u)\,dx + \int_{\mathbb{D}} DuD(w-u)\,dx + \int_{\mathbb{D}} (w-u)\,dx \geq 0$$

for a.e. $t \in]0, T[$ and for all $w \in \mathbb{K}$.

It is known that if u is a solution of this problem, then, in the sense of distributions, u satisfies the equation

$$\Delta u - \partial_t u = \chi_\Omega \quad \text{in } \mathbb{Q}, \tag{0.1}$$

where $\Omega = \{(x,t) \in \mathbb{Q} : u(x,t) > 0\}$, and χ_Ω is the characteristic function of the set Ω. The set $\Omega = \Omega(u)$ is called the *noncoincidence set*, while the set $\Lambda(u) = \{(x,t) : u(x,t) = |Du(x,t)| = 0\}$ is the *coincidence set* for the solution u; $\Gamma(u) = \partial\Omega(u) \cap \Lambda(u)$ is the *free boundary.* The possibility must not be ruled out that the free boundary $\Gamma(u)$ and the fixed boundary $\partial'\mathbb{Q}$ meet at points where $\phi = 0$. Therefore, the points of contact may exist.

The regularity of the free boundary (far from $\partial'\mathbb{Q}$) was investigated only in the special case of the Stefan problem, where the boundary and initial conditions guarantee the additional property $\partial_t u \geq 0$; see [C1]. The nonnegativity of the time-derivative of the solution was used in [C1] to prove that $\partial_t u$ is continuous at the points of the free boundary.

This fact (i.e., the continuity of $\partial_t u$) is quite important for investigation of the regularity properties of the free boundary. For instance, I. Athanasopoulos and S. Salsa proved the following result.

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Theorem ([AtSa]). *Let $u(x,t) \geq 0$ in $\mathcal{C}_R := B_R(x^0) \times]t^0 - R^2, t^0 + R^2[$, where $(x^0, t^0) \in \partial\{u > 0\}$. Suppose that u is a solution of equation* (0.1) *in $\mathcal{C}_R$, and that $D_x^2 u \in L^\infty(\mathcal{C}_R)$ and the derivative $\partial_t u$ is continuous in $\mathcal{C}_R$.*

Suppose also that in some spatial direction, say e_1, the function u is monotone (*i.e., $D_{e_1} u \geq 0$*) *and that $\partial\{u > 0\}$ is the x_1-graph of a Lipschitz function f. Then f is a $C^{1+\alpha}$-function for some $0 < \alpha < 1$.*

It should be noted that the above theorem ensures the $C^{1+\alpha}$-regularity of the free boundary $\partial\{u > 0\}$ only at the interior points of $\mathcal{C}_R$. Unfortunately, $C^{1+\alpha}$-regularity may fail to occur at the points of contact between the free boundary and the fixed boundary. The following counterexample shows that in the t-direction the free boundary $\partial\{u > 0\}$ may intersect the fixed boundary transversally.

Counterexample. Let $n = 1$, and let $\mathcal{C}_r = \{(x,t) : 0 < x < r,\ -r^2 < t < r^2\}$. Suppose that a function u on $\mathcal{C}_1$ is a solution of the one-phase Stefan problem, i.e., u is a nonnegative solution of equation (0.1) with $\Omega = \{(x,t) \in \mathcal{C}_1 : u(x,t) > 0\}$ and $u_t \geq 0$ a.e. in $\mathcal{C}_1$. Assume that $u(0,t) = 0$ for $-1 < t < 1$ and

$$\operatorname*{ess\,sup}_{\mathcal{C}_1}\{|D_{xx}u| + |\partial_t u|\} \leq M.$$

Assume also that $(0,0)$ is a free boundary point, i.e., $D_x u(0,0) = 0$. From Theorems 2.9 and 3.3 of the present paper it follows that for some $r > 0$ the derivative $\partial_t u$ is continuous in the closure $\overline{\mathcal{C}}_r$ of the rectangle $\mathcal{C}_r$ and that the set $\partial\Omega \cap \overline{\mathcal{C}}_r$ is a graph $x = f(t)$ of some Lipschitz continuous function f. Under our assumptions it is evident that f is a monotone nonincreasing function, $f(t) = 0$ for $t \geq 0$, and

$$\begin{aligned} u(x,t) &= 0 \quad \text{if } 0 \leq x \leq f(t),\ (x,t) \in \overline{\mathcal{C}}_r, \\ u(x,t) &> 0 \quad \text{if } x > f(t),\ (x,t) \in \mathcal{C}_r. \end{aligned}$$

We exclude the case where $f \equiv 0$ for $-r^2 < t < r^2$ from our consideration; there is no loss of generality in assuming that $0 < f(t) < r/2$ for $-r^2 < t < 0$.

Now we set $v = \partial_t u$ and $y = x - f(t)$. Then, in the rectangle $\mathcal{C} = \{(y,t) : 0 < y < r/2,\ -r^2 < t < r^2\}$, the function v is a nonnegative solution of the equation $D_{yy}v - \partial_t v + f'(t) D_y v = 0$. Moreover, v is strictly positive inside the set $\mathcal{C}$. Together with the boundary condition $v|_{y=0} = 0$, this guarantees the estimate

$$(y,t) \geq \beta y \quad \text{in } \{(y,t) : 0 < y < \rho,\ -\rho^2 < t < \rho^2\}$$

with some positive constants β and ρ. Returning to the x-variable, we see that

$$\partial_t u \geq \beta\,(x - f(t)) \quad \text{in } \mathcal{C}_\rho \cap \Omega.$$

Since $D_x u = 0$ for $x = f(t)$ and $t \leq 0$, on the set $\mathcal{C}_r^- = \mathcal{C}_r \cap \{t \leq 0\}$ we have the estimate $|D_x u| \leq M\,(x - f(t))$. Therefore, if e_1 and e_0 are the standard basis vectors in $\mathbb{R}_x$ and $\mathbb{R}_t$, respectively, and if $e = a_0 e_0 + a_1 e_1$ with $a_1^2 + a_0^2 = 1$, $a_0 > 0$, $a_1 \leq 0$, then for such a direction e in $\mathcal{C}_\rho^- \cap \Omega$ we have

$$D_e u = a_0 \partial_t u + a_1 D_x u \geq (a_0\beta + a_1 M)(x - f(t)).$$

It follows that in $\Omega \cap \mathcal{C}_\rho^-$ the function u is monotone increasing in the directions e satisfying $a_0\beta > -a_1 M$. Since $u(0,0) = 0$, we obtain

$$u(x,t) = 0 \quad \text{in } \mathcal{C}_\rho^- \cap \Big\{(x,t) : 0 < x < -\frac{\beta}{M}t\Big\}.$$

Thus, we have shown that the free boundary $x = f(t)$ intersects the t-axis at the point $(0,0)$ transversally.

The main result of the present paper says that the boundary of the noncoincidence set Ω is Lipschitz continuous near the part of the lateral surface of $\mathbb{Q}$ where the solution is equal to zero. In particular, this implies that, locally, inside $\mathbb{Q}$ and near that part, the free boundary is the graph of a $C^{1+\alpha}$-function.

Our arguments are based on the blow-up technique, in combination with various monotonicity formulas, and on the results of the paper [AUS2] concerning the global solutions of the parabolic obstacle problem with zero constraint (i.e., the solutions in the entire half-space $\{(x,t) \in \mathbb{R}^{n+1} : x_1 > 0\}$). It should be emphasized that our arguments do not require any additional assumptions on the free boundary.

Together with the monotonicity formula due to L. Caffarelli (see [C2], [CK] and [AUS2, Lemma 2.1]), we also use the functional introduced by G. Weiss for the study of some free boundary problems in the entire space $\mathbb{R}^{n+1}$. Changing Weiss's notation somewhat, we shall write this functional as follows:

$$W(r, x^*, t^*, u) := \frac{1}{r^4} \int_{t^*-4r^2}^{t^*-r^2} \int_{\mathbb{R}^n} \left(|Du|^2 + 2u + \frac{u^2}{t - t^*} \right) G(x - x^*, t^* - t)\, dx\, dt.$$

Here r is a positive parameter, u is a solution of (0.1) defined for $t \leq 0$ and all $x \in \mathbb{R}^n$ and having at most polynomial rate of growth at the infinity, (x^*, t^*) is a point of the free boundary, and

$$G(x,t) = \frac{\exp\left(-|x|^2/4t\right)}{(4\pi t)^{n/2}} \text{ for } t > 0 \quad \text{and} \quad G(x,t) = 0 \text{ for } t \leq 0.$$

In [W] it was shown that the functional W is monotone nondecreasing with respect to r and that the identity $\frac{\partial W}{\partial r} = 0$ for all $r > 0$ is equivalent to the degree 2 parabolic homogeneity of the function u.

For our purposes it was essential to introduce an appropriate local version of the Weiss functional. In particular, this permits us to make a conclusion about the homogeneity of the blow-up limits. For the "interior counterpart" of our problem, a local version of the Weiss functional W was introduced for the first time in [CPS]. Note that in [CPS] a more general free boundary problem was treated, without the assumption about the nonnegativity of the solution. In the present paper we introduce a modified local version of the Weiss functional W, in order to take a homogeneous Dirichlet condition on the fixed boundary into account. We observe also that we do not use the assumption $u \geq 0$ in the proofs of any statements concerning the functional W.

This paper is organized as follows. §1 is devoted to a local version of the Weiss monotonicity formula. In §2 we prove that $\partial_t u$ is continuous at the points of the free boundary that lie in a neighborhood of the fixed boundary. Finally, in §3 we analyze the properties of the free boundary near the fixed boundary.

Notation and definitions. Throughout the paper we use the following notation:

- $z = (x,t)$ are points in $\mathbb{R}^{n+1}$; here $x = (x_1, x') = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $n \geq 2$, and $t \in \mathbb{R}^1$;
- $\mathbb{R}^{n+1}_b = \{(x,t) \in \mathbb{R}^{n+1} : x_1 > b\}$, where $b \in \mathbb{R}$;
- $\mathbb{R}^{n+1}_+ = \mathbb{R}^{n+1}_0$;
- $\Pi_b = \{(x,t) \in \mathbb{R}^{n+1} : x_1 = b\}$;
- $\Pi = \Pi_0$;
- $e_1, \ldots, e_n$ is the standard basis in the x-space $\mathbb{R}^n$;
- e_0 is the standard basis vector in the t-space $\mathbb{R}^1$;
- χ_Ω denotes the characteristic function of the set $\Omega \subset \mathbb{R}^{n+1}$;
- $v_+ = \max\{v, 0\}$;

• $B_r(x^0)$ denotes the open ball with center x^0 and radius r in the x-space $\mathbb{R}^n$; $B_r^+(x^0) = B_r(x^0) \cap \mathbb{R}^{n+1}_+$; $B_r = B_r(0)$;
• $S_r(x^0) = \{x \in \mathbb{R}^n : |x - x^0| = r\}$, $S_r = S_r(0)$;
• $Q_r(z^0) = Q_r(x^0, t^0) = B_r(x^0) \times]t^0 - r^2, t^0]$ is a cylinder in $\mathbb{R}^{n+1}$;
• $Q_r^+(z^0) = Q_r^+(x^0, t^0) = Q_r(x^0, t^0) \cap \mathbb{R}^{n+1}_+$, $Q_r = Q_r(0,0)$, $Q_r^+ = Q_r^+(0,0)$.

We note that, unlike our previous publications, in this paper the top of the cylinder $Q_r(z^0)$ is included in the set $Q_r(z^0)$. If $Q = \mathbb{R}^{n+1}_b \cap Q_r(x^0, t^0)$, then $\partial' Q$ is the parabolic boundary of Q, i.e., $\partial' Q = \overline{Q} \setminus Q$.

D_i denotes the differentiation operator with respect to x_i; $\partial_t = \frac{\partial}{\partial t}$; $Du = (D_1 u, D' u) = (D_1 u, D_2 u, \dots, D_n u)$ is the spatial gradient of u; $D^2 u = D(Du)$ denotes the Hessian; D_ν stands for the operator of differentiation along the direction $\nu \in \mathbb{R}^{n+1}$, i.e., $|\nu| = 1$ and

$$D_\nu u = \sum_{i=1}^n \nu_i D_i u + \nu_0 \partial_t u;$$

$H = \Delta - \partial_t$ is the heat operator.

The index τ will always run from 2 to n. Also, we adopt the usual convention regarding summation with respect to repeated indices.

We use letters M, N, C (with or without indices) to denote various constants. To indicate that, say, N depends on some parameters, we list them in parentheses: $N(\dots)$.

Let M be a constant, $M \geq 1$.

We denote by $P_r(M, b)$ the *class of all local nonnegative solutions* of the parabolic obstacle problem, i.e., a function u belongs to $P_r(M, b)$ if u is continuous in $Q_r \cap \{(x,t) \in \mathbb{R}^{n+1} : x_1 \geq b\}$ and

(a) $H[u] = \chi_\Omega$ in $Q_r \cap \mathbb{R}^{n+1}_b$, where $\Omega = \Omega(u) := \{(x,t) \in Q_r \cap \mathbb{R}^{n+1}_b : u(x,t) > 0\}$;
(b) $u \geq 0$ in $Q_r \cap \mathbb{R}^{n+1}_b$, $u = 0$ on $\Pi_b \cap Q_r$;
(c) $\operatorname{ess\,sup}_{Q_r \cap \mathbb{R}^{n+1}_b} \{|D^2 u| + |\partial_t u|\} \leq M$

(the first equation in (a) is understood in the sense of distributions).

We also consider the *global nonnegative solutions* of the parabolic obstacle problem in the entire half-space $\mathbb{R}^{n+1}_b \cap \{t \leq 0\}$ that have at most quadratic growth in x and at most linear growth in t, i.e., the solutions for which

$$\operatorname*{ess\,sup}_{\mathbb{R}^{n+1}_b \cap \{t \leq 0\}} \{|D^2 u| + |\partial_t u|\} \leq M. \tag{0.2}$$

More precisely, we say that a continuous function u belongs to the class $P_\infty(M, b)$ if

(a′) $H[u] = \chi_\Omega$ in $\mathbb{R}^{n+1}_b \cap \{t \leq 0\}$, where $\Omega = \Omega(u) := \{(x,t) \in \mathbb{R}^{n+1}_b \cap \{t \leq 0\} : u(x,t) > 0\}$;
(b′) $u \geq 0$ in $\mathbb{R}^{n+1}_b \cap \{t \leq 0\}$, $u = 0$ on $\Pi_b \cap \{t \leq 0\}$;
(c′) inequality (0.2) is satisfied

(equation in (a′) is understood in the sense of distributions).

In both cases we shall use the following notation:
• $\Lambda(u) = \{(x,t) : u(x,t) = |Du(x,t)| = 0\}$;
• $\Gamma(u) = \partial\Omega(u) \cap \Lambda(u)$ is the free boundary;
• $\Gamma(u) \cap \Pi_b$ is the set of contact points.

It is assumed that $\Gamma(u) \neq \varnothing$.

We also define the class $P_\infty(M, -\infty)$ that corresponds formally to $b = -\infty$. In this case the half-space $\mathbb{R}^{n+1} \cap \{t \leq 0\}$ is considered instead of $\mathbb{R}^{n+1}_b \cap \{t \leq 0\}$, $\Pi_b = \varnothing$, and we omit the condition $u|_{\Pi_b} = 0$.

For the global solutions $u \in P_\infty(M, b)$ we have

$$-1 \leq \partial_t u \leq 0. \tag{0.3}$$

For $b = -\infty$ inequalities (0.3) were proved in [CPS]. For $b > -\infty$ from the results of [AUS2] it follows that any global solution $u \in P_\infty(M, b)$ is independent of t and has the form $u = (x-a)_+^2/2$ with $a \geq b$.

Let $a > 0$ be some constant, let $u \in P_{2a}(M, 0)$, and let $z^0 = (x^0, t^0) \in \Gamma(u)$. For $r > 0$ we consider the functions

$$u_r(x,t) = \frac{u(rx + x^0, r^2 t + t^0)}{r^2}. \tag{0.4}$$

By the standard compactness arguments, we may pass to the limit along a subsequence $r_k \to 0$; as a result we obtain a global solution $u_0 \in P_\infty^+(M, -\infty)$. More precisely, this will be true if $x_1^0 > 0$. If $x_1^0 = 0$, then the function u_0 is defined only for $x_1 \geq 0$, and, in accordance with [AUS2], $u_0 = x_1^2/2$. In this case we extend u_0 by zero to the set $\{x_1 < 0, t \leq 0\}$, again obtaining a global solution $u_0 = (x_1)_+^2/2 \in P_\infty(M, -\infty)$. Usually, such a process is referred to as the blow-up limit passage. Any global solution u_0 obtained in this way is called a *blow-up limit* of the function u at the point z^0. In general, possibly different blow-up limits may be obtained at the same point if we choose different subsequences r_k.

§1. A monotonicity formula

Let $z^* = (x^*, t^*)$ be an arbitrary point in $\mathbb{R}^{n+1}$, let a and r be positive constants, and let v be a continuous function defined on $\mathcal{Q}_{a,r}(z^*) := B_a(x^*) \times]t^* - 4r^2, t^*[$ and satisfying $|Dv| \in L_2(\mathcal{Q}_{a,r}(z^*))$.

We define the local Weiss functional (cf. [W]) as follows:

$$W_a(r, x^*, t^*, v) := \frac{1}{r^4} \int_{t^* - 4r^2}^{t^* - r^2} \int_{B_a(x^*)} \left(|Dv|^2 + 2v + \frac{v^2}{t - t^*} \right) G(x - x^*, t^* - t)\, dx\, dt,$$

where

$$G(x,t) = \frac{\exp(-|x|^2/4t)}{(4\pi t)^{n/2}} \text{ for } t > 0 \quad \text{and} \quad G(x,t) = 0 \text{ for } t \leq 0.$$

Lemma 1.1. *Let v and z^* be as above.*

Then

$$W_a(\lambda r, x^*, t^*, v) = W_{a/r}(\lambda, 0, 0, v_r) \tag{1.1}$$

for any $\lambda \in]0, 1]$, where $v_r(x,t) = r^{-2} \cdot v(rx + x^*, r^2 t + t^*)$.

We omit the trivial proof.

Lemma 1.2. *Let $a > 0$ and $b \geq 0$ be given constants, let $u \in P_{2a}(M, -b)$, and let*

$$z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_a.$$

Suppose that the function u is extended by zero across the plane Π_{-b} to the set $Q_a(z^0) \cap \{x_1 < -b\}$; we preserve the notation u for this extension.

Then for $0 < r \leq a$ we have

$$\begin{aligned} &\frac{dW_a(r, x^0, t^0, u)}{dr} \\ &\quad = \frac{1}{r} \int_{-4}^{-1} \int_{B_{a/r}} \frac{|\partial' u_r|^2}{-t} G(x, -t)\, dx\, dt + J_a(r; u) \\ &\qquad + \frac{x_1^0 + b}{r^2} \int_{-4}^{-1} \int_{B_{a/r} \cap \{x_1 = \frac{-x_1^0 - b}{r}\}} |D_1 u_r|^2 G(x, -t)\, dx'\, dt, \end{aligned} \tag{1.2}$$

where u_r *is as in* (0.4),

$$\partial' u_r(x,t) := x \cdot Du_r(x,t) + 2t\partial_t u_r(x,t) - 2u_r(x,t), \tag{1.3}$$

$$J_a(r;u) := 2\int_{-4}^{-1}\int_{S_{a/r}} \frac{\partial' u_r}{r}(\vec{\gamma}\cdot Du_r)G(x,-t)\,dS_{a/r}\,dt - \frac{a}{r^2}\int_{-4}^{-1}\int_{S_{a/r}} \left(|Du_r|^2 + 2u_r + \frac{(u_r)^2}{t}\right) G(x,-t)\,dS_{a/r}\,dt,$$

and $\vec{\gamma}$ *is the unit vector of the outward normal to* $S_{a/r}$.

Proof. Using (1.1) and the relation

$$\frac{d}{dr}(D_i u_r) = D_i\left(\frac{du_r}{dr}\right),$$

we obtain

$$\frac{d}{dr}W_a(r,x^0,t^0,u) = \frac{d}{dr}W_{a/r}(1,0,0,u_r) = I_1 + I_2, \tag{1.4}$$

where

$$I_1 = 2\int_{-4}^{-1}\int_{B_{a/r}\cap\{x_1>\frac{-x_1^0-b}{r}\}} \left[Du_r\cdot D\left(\frac{du_r}{dr}\right) + \frac{du_r}{dr} + \frac{u_r}{t}\frac{du_r}{dr}\right] G(x,-t)\,dx\,dt,$$

$$I_2 = -\frac{a}{r^2}\int_{-4}^{-1}\int_{S_{a/r}\cap\{x_1>\frac{-x_1^0-b}{r}\}} \left(|Du_r|^2 + 2u_r + \frac{(u_r)^2}{t}\right) G(x,-t)\,dS_{a/r}\,dt - \frac{x_1^0+b}{r^2}\int_{-4}^{-1}\int_{B_{a/r}\cap\{x_1=\frac{-x_1^0-b}{r}\}} \left(|Du_r|^2 + 2u_r + \frac{(u_r)^2}{t}\right) G(x,-t)\,dx'\,dt.$$

Then, integrating the term $2Du_r\cdot D\left(\frac{d}{dr}u_r\right)G(x,-t)$ in I_1 by parts and using the identity

$$D_iG(x,-t) = \frac{x_i}{2t}G(x,-t),$$

we get

$$\begin{aligned} I_1 = 2\int_{-4}^{-1}\int_{B_{a/r}\cap\{x_1>\frac{-x_1^0-b}{r}\}} \frac{du_r}{dr}\left[-\Delta u_r - \frac{x_i}{2t}D_iu_r + 1 + \frac{u_r}{t}\right] G(x,-t)\,dx\,dt \\ + 2\int_{-4}^{-1}\int_{S_{a/r}\cap\{x_1>\frac{-x_1^0-b}{r}\}} \frac{du_r}{dr}\left(\vec{\gamma}\cdot Du_r\right) G(x,-t)\,dS_{a/r}\,dt \\ - 2\int_{-4}^{-1}\int_{B_{a/r}\cap\{x_1=\frac{-x_1^0-b}{r}\}} \frac{du_r}{dr}D_1u_rG(x,-t)\,dx'\,dt. \end{aligned} \tag{1.5}$$

The assumption $u(-b,x',t)=0$ implies that for $(x,t)\in\mathcal{E} := \left\{B_{a/r}\cap\{x_1 = \frac{-x_1^0-b}{r}\}\right\}\times\,]-4,-1[$ we have

$$u_r = |D'u_r| = \partial_t u_r = 0, \tag{1.6}$$

whence

$$\left(|Du_r|^2 + 2u_r + \frac{(u_r)^2}{t}\right)\Big|_{\mathcal{E}} = |D_1u_r|^2. \tag{1.7}$$

Moreover, since

$$\frac{du_r}{dr} = \frac{\partial' u_r}{r}, \tag{1.8}$$

from (1.6) and (1.3) it follows that

$$\left(-\frac{du_r}{dr}D_1u_r\right)\Big|_{\mathcal{E}} = \frac{x_1^0+b}{r^2}|D_1u_r|^2. \tag{1.9}$$

Substituting (1.7) and (1.9) in (1.5), and using (1.8), (1.3), and (1.4), we obtain the representation

$$\begin{aligned}&\frac{d}{dr}W_a(r,x^0,t^0,u)\\ &\quad = 2\int_{-4}^{-1}\int_{B_{a/r}\cap\{x_1>\frac{-x_1^0-b}{r}\}}\frac{\partial' u_r}{r}\left[1-H[u_r]-\frac{\partial' u_r}{2t}\right]G(x,-t)\,dx\,dt\\ &\qquad + J_a(r;u)+\frac{x_1^0+b}{r^2}\int_{-4}^{-1}\int_{B_{a/r}\cap\{x_1=\frac{-x_1^0-b}{r}\}}|D_1u_r|^2G(x,-t)\,dx\,dt.\end{aligned} \tag{1.10}$$

Strictly speaking, the formal calculations given above are correct if the function u has all derivatives up to the second order. Therefore, in the case of an arbitrary function $u \in P_{2a}(M,-b)$, we must regularize the function with respect to the t-variable. For instance, this can be done by using the Steklov average. For the smoothed function u, the representation (1.10) is proved as above. Now, letting the parameter of the averaging tend to zero, we easily show that (1.10) is true for the initial u.

From the assumption $u \in P_{2a}(M,-b)$ it follows that $\Gamma(u_r)$ has zero Lebesgue measure, and

$$H[u_r] = \chi_{\{u_r>0\}} \quad \text{in } \mathcal{Q} := \left\{B_{a/r}\cap\{x_1 > (-x_1^0-b)/r\}\right\}\times\,]-4,-1[. \tag{1.11}$$

Therefore, for $(x,t)\in\mathcal{Q}$ we have

$$\frac{\partial' u_r}{r}\left[1-H[u_r]-\frac{\partial' u_r}{2t}\right] = -\frac{|\partial' u_r|^2}{2rt}. \tag{1.12}$$

Combining (1.10) and (1.12), we complete the proof. □

Remark. Under the conditions of Lemma 1.2, the functional $W_a(r,x^0,t^0,u)$ is uniformly bounded for $0<r\le a$. Moreover, there exists a universal constant $C_0 = C_0(n,M)$ such that

$$|J_a(r;u)| \le C_0\left(1+\frac{1}{r}\right)\left(\frac{a}{r}\right)^{n+4}\exp\left(-a^2/16r^2\right) \tag{1.13}$$

for all functions of class $P_{2a}(M,-b)$, for all values of the parameters r and a indicated, and for an arbitrary $b\ge 0$. In particular,

$$\lim_{r\to 0^+}|J_a(r;u)| = 0, \quad a>0. \tag{1.14}$$

Corollary 1.3. *Let $a>0$ and $b\ge 0$ be given constants, and let $u\in P_{2a}(M,-b)$. Then for any point $z^0=(x^0,t^0)\in\Gamma(u)\cap Q_a$ the function $W_a(r,x^0,t^0,u)$ has a limit as $r\to 0^+$.*

For $a=1$ and $b=0$, the corresponding limit

$$\omega(x^0,t^0,u) = \lim_{r\to 0^+}W_1(r,x^0,t^0,u) \tag{1.15}$$

will be called the *balanced energy* of the function u at the point (x^0,t^0) of the free boundary. From (1.1) it follows that

$$\omega(x^0,t^0,u) = \int_{-4}^{-1}\int_{\mathbb{R}^n}\left(|Du_0|^2+2u_0+\frac{(u_0)^2}{t}\right)G(x,-t)\,dx\,dt, \tag{1.16}$$

where u_0 is an arbitrary blow-up limit of the solution u at the point (x^0,t^0).

§2. Regularity properties of solutions

Lemma 2.1. *Let* $u \in P_2(M,0)$, *let* $z^0 = (x^0,t^0) \in \Gamma(u) \cap Q_1$, *and let* u_0 *be a blow-up limit of* u *at* z^0. *Then* u_0 *is a homogeneous function of degree* 2 *on the set* $\mathbb{R}^{n+1} \cap \{t \leq 0\}$, *i.e.,*

$$u_0(\varkappa x, \varkappa^2 t) = \varkappa^2 u_0(x,t), \quad \varkappa > 0, \ (x,t) \in \mathbb{R}^{n+1} \cap \{t \leq 0\}.$$

Remark. Observe that the statement of Lemma 2.1 concerns only the blow-up limits of u at some fixed point $z^0 \in \Gamma(u)$.

Proof. It suffices to consider the case where $x_1^0 > 0$. We take a subsequence r_k that tends to 0^+ as $k \to \infty$ and is such that the functions

$$u_k(x,t) = \frac{u(r_k x + x^0, r_k^2 t + t^0)}{r_k^2}$$

tend to u_0, i.e., $u_0(x,t) = \lim_{k\to\infty} u_k(x,t)$. Obviously, $u_k \in P_{1/r_k}(M, -x_1^0/r_k)$ and $(0,0) \in \Gamma(u_k)$. From (1.15) and (1.1) it follows that for arbitrary numbers $\lambda > \mu > 0$ we have

$$\begin{aligned} 0 \longleftarrow_{k\to\infty} & W_1(\lambda r_k, x^0, t^0, u) - W_1(\mu r_k, x^0, t^0, u) \\ &= W_{1/r_k}(\lambda, 0, 0, u_k) - W_{1/r_k}(\mu, 0, 0, u_k) \\ &= \int_\mu^\lambda \frac{dW_{1/r_k}(\theta, 0, 0, u_k)}{d\theta}\, d\theta. \end{aligned} \tag{2.1}$$

On the other hand, by (1.2), we have

$$\begin{aligned} &\frac{dW_{1/r_k}(\theta,0,0,u_k)}{d\theta} \\ &\quad \geq \frac{1}{\theta}\int_{-4}^{-1}\int_{B^+_{1/r_k\theta}} \frac{|\partial'(u_k)_\theta|^2}{-t} G(x,-t)\,dx\,dt + J_{1/r_k}(\theta; u_k) \\ &\quad = \frac{1}{\theta^5}\int_{-4\theta^2}^{-\theta^2}\int_{B^+_{1/r_k}} \frac{|\partial' u_k|^2}{-t} G(x,-t)\,dx\,dt + J_{1/r_k}(\theta; u_k). \end{aligned} \tag{2.2}$$

Now, combining (2.1) and (2.2), recalling estimate (1.13), and letting $k \to \infty$, we get the identity

$$\partial' u_0 = x \cdot Du_0 + 2t\partial_t u_0 - 2u_0 \equiv 0, \quad t \in [-\lambda^2, -\mu^2].$$

Therefore, u_0 is a homogeneous function of degree 2 for all t in the interval $[-\lambda^2, -\mu^2]$. Since λ and μ are arbitrary positive constants with $\lambda > \mu$, this completes the proof. □

Lemma 2.2. *Let* $u_0 \in P_\infty(M, -\infty)$, *let* $(0,0) \in \Gamma(u_0)$, *and let*

$$W_\infty(r,0,0,u_0) := \frac{1}{r^4}\int_{-4r^2}^{-r^2}\int_{\mathbb{R}^n} \left(|Du_0|^2 + 2u_0 + \frac{(u_0)^2}{t}\right) G(x,-t)\,dx\,dt, \quad r > 0.$$

Then the relation

$$\frac{dW_\infty(r,0,0,u_0)}{dr} = 0 \quad \textit{for all } r > 0$$

implies that the function u_0 *is homogeneous of degree* 2, *i.e.,*

$$u_0(\varkappa x, \varkappa^2 t) = \varkappa^2 u_0(x,t), \quad \varkappa > 0, \ (x,t) \in \mathbb{R}^{n+1} \cap \{t < 0\}.$$

Proof. This was proved in [W]. □

Lemma 2.3. *Let $u \in P_2(M,0)$. For any $\varepsilon > 0$ there exists $\rho^* = \rho^*(\varepsilon, n, M) > 0$ such that if $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_1$ and $0 \le x_1^0 < \rho^*$, then for $\rho \in [x_1^0, \rho^*]$ we have*

$$\sup_{Q_\rho^+(z^0)} \left| u(x,t) - \frac{\big((x_1 - x_1^0)_+\big)^2}{2} \right| \le \varepsilon \rho^2, \tag{2.3}$$

$$\sup_{Q_\rho^+(z^0)} \left| D_1 u(x,t) - \big(x_1 - x_1^0\big)_+ \right| \le \varepsilon \rho, \tag{2.4}$$

$$\sup_{Q_\rho^+(z^0)} \left| D_\tau u(x,t) \right| \le \varepsilon \rho, \quad \tau = 2, \dots, n. \tag{2.5}$$

Proof. We begin with the proof of the first inequality. Arguing by contradiction, suppose that (2.3) fails. Then there exists a number $\varepsilon_0 > 0$ and sequences $u^j \in P_2(M,0)$, $\rho_j \searrow 0$, and $z^j = (x^j, t^j) \in \Gamma(u^j) \cap Q_1$ such that $\rho_j > x_1^j \ge 0$ and

$$\sup_{Q_{\rho_j}^+(z^j)} \left| u_j(x,t) - \frac{((x_1 - x_1^j)_+)^2}{2} \right| > \varepsilon_0 \rho_j^2. \tag{2.6}$$

We define v_j by the formula

$$v_j(x,t) = \frac{u_j(\rho_j x + x^j, \rho_j^2 t + t^j)}{\rho_j^2}$$

for $(x,t) \in Q_{1/\rho_j} \cap \mathbb{R}^{n+1}_{-b_j}$, where $b_j = x_1^j/\rho_j$, $b_j \in [0,1]$. Observe that $(0,0) \in \Gamma(v_j)$ and $v_j|_{x_1 = -b_j} = 0$. Moreover, in an appropriate function space the functions v_j converge (along a subsequence) to a global solution $v_0 \in P_\infty^+(M, -b)$, where $b = \lim_{j\to\infty} b_j$, $b \in [0,1]$.

Since $(0,0) \in \Gamma(v_0)$ and $v_0|_{x_1=-b} = 0$, from [AUS2] we deduce that $v_0 = \left((x_1)_+\right)^2/2$. Therefore, for all sufficiently large j we have the inequality

$$\sup_{Q_1 \cap \{x_1 > -b_j\}} \left| v_j(x,t) - \frac{\left((x_1)_+\right)^2}{2} \right| \le \frac{\varepsilon_0}{2}. \tag{2.7}$$

On the other hand, (2.6) implies that

$$\begin{aligned}
&\sup_{Q_1 \cap \{x_1 > -b_j\}} \left| v_j(x,t) - \frac{\left((x_1)_+\right)^2}{2} \right| \\
&\qquad = \sup_{Q_1 \cap \{x_1 > -b_j\}} \left| \frac{u_j(\rho_j x + x^j, \rho_j^2 t + t^j)}{\rho_j^2} - \frac{\left((x_1)_+\right)^2}{2} \right| \\
&\qquad = \sup_{Q_{\rho_j}^+(z^j)} \left| \frac{u_j(y,s)}{\rho_j^2} - \frac{((y_1 - x_1^j)_+)^2}{2\rho_j^2} \right| > \varepsilon_0.
\end{aligned}$$

This contradiction with (2.7) completes the proof of (2.3).

It only remains to observe that estimates (2.4) and (2.5) are proved in the same way as (2.3). □

Lemma 2.4. *Let $u \in P_2(M,0)$, let $\varepsilon \in \left]0, \frac{1}{4(2n+1)}\right[$, and let $N_1 \ge 0$ and N_τ (with $\tau = 2, \dots, n$) be some constants. Suppose that, for a point $z^0 = (x^0, t^0) \in Q_1^+$ and some $\rho < 1$, in $Q_\rho^+(z^0)$ we have the inequality*

$$\rho \bigg(\sum_{j=1}^n N_j D_j u \bigg) - u \ge -\varepsilon \rho^2.$$

Then

$$\rho\Bigg(\sum_{j=1}^{n} N_j D_j u\Bigg) - u \geq 0 \quad \textit{in } Q^+_{\rho/2}(z^0).$$

Proof. Suppose the conclusion of the lemma fails. Then there is a function $u \in P_2(M, 0)$ and a point $z^0 \in Q^+_1$ such that for some $\rho < 1$ the assumptions of the lemma are satisfied, but there is a point $z^* = (x^*, t^*) \in Q^+_{\rho/2}(z^0)$ with

$$\rho\Bigg(\sum_{j=1}^{n} N_j D_j u(x^*, t^*)\Bigg) - u(x^*, t^*) < 0. \tag{2.8}$$

Let

$$w(x,t) = \rho\Bigg(\sum_{j=1}^{n} N_j D_j u(x,t)\Bigg) - u(x,t) + \frac{1}{2n+1}\left(|x - x^*|^2 - (t - t^*)\right).$$

Then w is caloric in $Q_{\rho/2}(z^*) \cap \Omega(u)$, and, by (2.8), $w(x^*, t^*) < 0$. Observe also that the condition $u \geq 0$ implies the inequality $D_1 u \geq 0$ on Π, so that $w \geq 0$ on the set $\partial\Omega(u) \cap Q_{\rho/2}(z^*)$. By the maximum principle, the negative infimum of w is attained on $\partial' Q_{\rho/2}(z^*) \cap \Omega(u)$. Thus, we obtain

$$-\frac{\rho^2}{4(2n+1)} \geq \inf_{\partial' Q_{\rho/2}(z^*) \cap \Omega(u)} \left\{\rho\Bigg(\sum_{j=1}^{n} N_j D_j u\Bigg) - u\right\} \geq -\varepsilon\rho^2,$$

a contradiction. This proves the lemma. □

Lemma 2.5. *Let $u \in P_2(M, 0)$. There exists $\rho_0 = \rho_0(n, M) > 0$ such that if $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_1$ and $x^0_1 \leq \rho_0$, then for any $\rho \in [x^0_1, \rho_0]$ we have*

$$\rho \cdot D_1 u - u \geq 0 \quad \textit{in } Q^+_{\rho/2}(z^0).$$

Proof. We fix $\varepsilon = \frac{1}{16(2n+1)}$. Successive application of Lemmas 2.3 and 2.4 finishes the proof. □

Lemma 2.6. *Let $u_0 \in P_\infty(M, -\infty)$ be a homogeneous function of degree 2 in $\mathbb{R}^{n+1} \cap \{t \leq 0\}$. Also, suppose that $(0,0) \in \Gamma(u)$. Then either*

$$u_0(x,t) = \frac{\left((x \cdot e)_+\right)^2}{2} \tag{2.9}$$

for some direction $e \in \mathbb{R}^n$ such that $e \cdot e_1 \geq 0$, or, in some rotated coordinate system in the x-space, we have

$$u_0(x,t) = \sum_{i=1}^{n} \frac{a_i}{2} x_i^2 - ct, \tag{2.10}$$

where $\sum_{i=1}^{n} a_i = 1 - c$, $a_i \geq 0$, $c \geq 0$.

Proof. First we consider the case where the interior of $\Lambda(u_0)$ is empty. Then the function v_0 defined by the formula

$$v_0(x,t) = u_0(x,t) - \frac{x_1^2}{2}$$

is caloric in $\mathbb{R}^{n+1} \cap \{t \leq 0\}$ and has quadratic growth with respect to x and at most linear growth with respect to t. By the Liouville theorem (see Lemma 2.1 in [AUS1]), the function v_0, and, consequently, the function u_0, is a polynomial of degree 2, i.e., there exist constants $a_i \geq 0$ and $c \geq 0$ such that the exact representation (2.10) is valid.

The case where the interior of $\Lambda(u_0)$ is not empty requires a more detailed analysis. Since u_0 is homogeneous, from the existence of at least one interior point of the set $\Lambda(u_0)$ it follows that for every $t_1 < 0$ the set $\Lambda(u_0) \cap \{t = t_1\}$ has nonempty interior in $\mathbb{R}^n$.

Next, arguing in the same way as in the proof of Theorem II in [AUS2], we can show that the function u_0 is one-dimensional in the variables x, i.e., $u_0 = u_0(y,t)$, where $y = (x \cdot e)$ for some $e \in \mathbb{R}^n$, $y \in \mathbb{R}$, and $t \leq 0$. Moreover, the function $D_e u$ does not change its sign. This result follows by the dimensional reduction based on a version of the monotonicity formula due to L. Caffarelli [C2], [CK].

Since u_0 is nonnegative, homogeneous, and one-dimensional in the space variables, for all $t \leq 0$ we have

$$u_0(0,t) = -mt, \quad m \geq 0. \tag{2.11}$$

If $m = 0$, then Theorem I in [AUS2] immediately gives the desired inequality (2.9). We show that $m = 0$ is the only possibility.

Suppose $m > 0$. For definiteness, we assume that $D_e u_0 \geq 0$ (otherwise we replace e by $-e$). Now, combining (2.11) with the inequality $D_e u_0 \geq 0$, we see that $u_0(y,t) > 0$ on the set $\mathcal{D} := \{(y,t) : y \in \mathbb{R}_+,\ t < 0\}$. Therefore, $H[u_0] = 1$ and $H[\partial_t u_0] = 0$ on $\mathcal{D}$.

Now we introduce the function

$$v(y,t) = \begin{cases} \partial_t u_0(y,t) + m & \text{if } y \geq 0,\ t \leq 0, \\ -\partial_t u_0(-y,t) - m & \text{if } y < 0,\ t \leq 0. \end{cases}$$

Obviously, v is bounded and caloric in $\mathbb{R}^2 \cap \{t < 0\}$, and it vanishes for $y = 0$. By Liouville's theorem, $v \equiv 0$ in $\mathbb{R}^2 \cap \{t < 0\}$, and elementary integration yields the exact representation for u_0 on the set $\mathcal{D}$:

$$u_0(y,t) = -mt + \frac{1-m}{2} y^2. \tag{2.12}$$

Relation (2.12) implies immediately that

$$D_e u_0 = 0 \quad \text{if } y = 0,\ t \leq 0. \tag{2.13}$$

On the other hand, from inequalities (0.3) it follows that

$$D_{ee} u_0 \geq 0 \quad \text{in } \mathbb{R}^2 \cap \{t \leq 0\}. \tag{2.14}$$

Combining (2.13), (2.14), and the assumption that $D_e u_0$ is nonnegative yields the identity $D_e u_0 \equiv 0$ on the set $\{y < 0, t \leq 0\}$. The latter means that on the set $\{y < 0, t \leq 0\}$ we have the exact representation

$$u_0(y,t) = -mt. \tag{2.15}$$

The representation (2.15) and the equation $H[u_0] = 1$ show that $m = 1$. However, this is impossible because for $u_0 = -t$ the set of interior points of $\Lambda(u_0)$ is empty. $\square$

Corollary 2.7. *Let $u \in P_2(M,0)$, let $z^0 = (x^0,t^0) \in \Gamma(u) \cap Q_1$, and let $x_1^0 \leq \rho_0$, where ρ_0 is the constant occurring in Lemma* 2.5*. Suppose that u_0 is a blow-up limit of the solution u at the point z^0.*

Then, for $x \in \mathbb{R}^n$ and $t \leq 0$, either

$$u_0(x,t) = \frac{(x \cdot e)_+^2}{2} \quad \textit{and} \quad \omega(x^0,t^0,u) = W_\infty(1,0,0,u_0) = \frac{15}{4} \tag{2.16}$$

for some direction $e \in \mathbb{R}^n$ such that $e \cdot e_1 \geq 0$, or in some rotated coordinate system in the x-space $\mathbb{R}^n$ we have

$$u_0(x,t) = \sum_{i=1}^n \frac{a_i}{2} x_i^2 - ct \quad \textit{and} \quad \omega(x^0,t^0,u) = W_\infty(1,0,0,u_0) = \frac{15}{2}. \tag{2.17}$$

Here a_i and c are the constants occurring in Lemma 2.6, *and $\omega(x^0, t^0, u)$ is the balanced energy at the point z^0 (see* (1.15)*).*

Proof. If $x_1^0 = 0$, from the results of [AUS2] it follows that $u_0 = (x_1)_+^2/2$. Let $x_1^0 > 0$. Then Lemma 2.5 guarantees the nonnegativity of $D_1 u$ in $Q_{\rho_0/2}(z^0)$; consequently, $D_1 u_0 \geq 0$ in $\mathbb{R}^n \times \{t \leq 0\}$. Therefore, Lemmas 2.1 and 2.6 imply the first identities in (2.16) and (2.17).

Next, by (1.16), for the case where $u_0(x,t) = (x \cdot e)_+^2/2 = (y_1)_+^2/2$ we have

$$\omega(x^0, t^0, u) = \int_{-4}^{-1} dt \int_{\mathbb{R}^{n-1}} dy_2 \cdots dy_n \int_0^\infty \left(2y_1^2 + \frac{y_1^4}{4t} \right) G(y, -t)\, dy_1.$$

Similarly, by (1.16), if $u_0(x,t) = \sum_{i=1}^n \frac{a_i}{2} x_i^2 - ct$, then

$$\begin{aligned}
&\omega(x^0, t^0, u) \\
&\quad = \int_{-4}^{-1} dt \int_{\mathbb{R}^n} \Bigg(\sum_{i=1}^n a_i^2 x_i^2 + \sum_{i=1}^n a_i x_i^2 - 2ct \\
&\qquad\qquad + c^2 t - c \sum_{i=1}^n a_i x_i^2 + (4t)^{-1} \sum_{i,j=1}^n a_i a_j x_i^2 x_j^2 \Bigg) G(x, -t)\, dx.
\end{aligned}$$

Now the second identities in (2.16) and (2.17) follow immediately from the direct calculations of the integrals written above. □

Lemma 2.8. *There exists $\delta = \delta(n, M) > 0$ such that if $u \in P_2(M, 0)$ and*

$$z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_1 \cap \{x_1 \leq \delta\},$$

then

$$\omega(x^0, t^0, u) = \frac{15}{4}. \tag{2.18}$$

Here $\omega(x^0, t^0, u)$ is the balanced energy at the point z^0 (see (1.15)*).*

Proof. We set $r := x_1^0$ and consider the function

$$u_r(x,t) = \frac{u(rx + x^0, r^2 t + t^0)}{r^2}.$$

If r is sufficiently small, then, by Lemma 2.3, the function u_r is close to a global solution $u_0(x) = (x_1)_+^2/2$. More precisely, for any $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that for $r < \rho \leq R$ we have the inequality

$$|u_r - u_0| + |Du_r - Du_0|^2 \leq \varepsilon \frac{\rho^2}{r^2} \quad \text{in } Q_{\rho/r} \cap \{x_1 > -1\}.$$

In particular,

$$|u_r(x,t) - u_0(x)| + |Du_r(x,t) - Du_0(x)|^2 \leq \varepsilon \left(|x|^2 + |t| + 1 \right) \quad \text{in } Q_{R/r} \cap \{x_1 > -1\}.$$

Using (1.1), we deduce the inequality

$$\begin{aligned}
W_1(r, x^0, t^0, u) &= W_{1/r}(1, 0, 0, u_r) \\
&\leq C\varepsilon \int_{-4}^{-1} dt \int_{B_{R/r}} \left(1 + |x|^4\right) G(x, -t)\, dx \\
&\quad + \int_{-4}^{-1} dt \int_{B_{R/r}} \left(|Du_0|^2 + 2u_0 + \frac{u_0^2}{t} \right) G(x, -t)\, dx \\
&\quad + \int_{-4}^{-1} dt \int_{R/r < |x| < 1/r} \left(|Du_r|^2 + 2u_r + \frac{u_r^2}{t} \right) G(x, -t)\, dx.
\end{aligned}$$

Now, choosing ε and, after that, r small, and invoking (0.2) and the second identity in (2.16), we obtain

$$W_1(r, x^0, t^0, u) \leq \frac{15}{4} + C\varepsilon + C\int_{-4}^{-1} dt \int_{|x|>\frac{R}{r}} (1+|x|^4)G(x,-t)\,dx \leq 4.$$

Finally, taking (1.2) and (1.13) into account, for sufficiently small δ and $\tilde{r} < r < \delta$ we conclude that

$$W_1(\tilde{r}, x^0, t^0, u) \leq W_1(r, x^0, t^0, u) + \frac{N_1(n,M)}{r^{n+5}} \exp\left(-\frac{1}{16r^2}\right) \leq 5. \tag{2.19}$$

Now (2.18) follows from (2.19) and Corollary 2.7. □

Remark. Let $u \in P_2(M, 0)$ and let $z = (x,t) \in \Gamma(u) \cap Q_1 \cap \{x_1 \leq \delta\}$, where δ is the constant occurring in Lemma 2.8. Then the convergence of $W_1(\rho, x, t, u)$ to $15/4$ as $\rho \searrow 0$ is uniform with respect to $z = (x,t)$. This can be proved easily by an argument similar to the proof of the Dini theorem, because for $\rho > 0$ the functions $W_1(\rho, x, t, u)$ are continuous with respect to (x,t), the limit function is a constant, and the convergence in question is monotone up to exponentially small terms.

In particular, if $\rho_k \searrow 0$ as $k \to \infty$, $z^k = (x^k, t^k) \in \Gamma(u) \cap Q_1 \cap \{x_1 \leq \delta\}$, and $z^k = (x^k, t^k) \longrightarrow z^0 = (x^0, t^0)$, then

$$\lim_{k\to\infty} W_1(\rho_k, x^k, t^k, u) = \omega(x^0, t^0, u) = \frac{15}{4}. \tag{2.20}$$

Theorem 2.9. *Let $u \in P_2(M, 0)$. Then $\partial_t u$ is continuous on the set $Q_{1/2} \cap \{0 \leq x_1 < \delta\}$, where δ is the same constant as in Lemma* 2.8.

Proof. We consider a point $z^0 = (x^0, t^0) \in \Gamma(u) \cap Q_{1/2} \cap \{0 \leq x_1 < \delta\}$ and prove that

$$\lim_{\Omega(u)\ni z\to z^0} \partial_t u(z) = 0. \tag{2.21}$$

It suffices to show that the lower limit

$$m := \liminf_{\Omega(u)\ni z\to z^0} \partial_t u(z) \tag{2.22}$$

is nonnegative, whereas the corresponding upper limit is nonpositive.

First, we assume that $t^0 < 0$. Let m be defined by (2.22), and let $z^k = (x^k, t^k) \in \Omega(u)$ be a sequence such that $z^k \to z^0 = (x^0, t^0)$ as $k \to \infty$ and

$$\lim_{k\to\infty} \partial_t u(z^k) = m.$$

Denoting by $K_r(z) = K_r(x,t)$, $r > 0$, the cylinder $K_r(z) = B_r(x) \times]t - r^2, t + r^2[$, for each point z^k we define the corresponding distance to the free boundary as follows:

$$r_k = \sup\{r > 0 : K_r(z^k) \cap \Gamma(u) = \varnothing\}.$$

Clearly, $r_k \to 0$ as $k \to \infty$.

Consider the functions

$$u_k(x,t) = \frac{u(r_k x + x^k, r_k^2 t + r_k^2 + t^k)}{r_k^2}.$$

We observe that $K_1(0,-1) \cap \{x_1 > -x_1^k/r_k\} \subset \Omega(u_k)$, and that $\partial_t u_k(0,-1) = \partial_t u(x^k, t^k)$ tends to m as $k \to \infty$.

Therefore, the u_k converge (along a subsequence) to a global solution $u_0 \in P_\infty(M, -b)$ with the following properties:

$$\partial_t u_0(0,-1) = m, \quad \text{and} \quad H[u_0] = 1 \text{ in } K_1(0,-1) \cap \{x_1 > -b\}, \tag{2.23}$$

where $b = \lim_{k\to\infty} -x_1^k/r_k$. If $b < \infty$, from the results of [AUS2] it follows that u_0 does not depend on t, so that in (2.23) we have $m = 0$.

If $b = \infty$, we need a more detailed analysis. Observe that in this case the limit function u_0 is a global solution defined in the entire space $\mathbb{R}^{n+1}$, and

$$\partial_t u_0(x,t) \geq m, \quad (x,t) \in K_1(0,-1).$$

The latter inequality follows from the fact that in $K_1(0,-1)$ the functions $\partial_t u_k$ converge pointwise to $\partial_t u_0$, and from assumption (2.22).

Thus, the function $\partial_t u_0$ is caloric in $K_1(0,-1)$ and has a local minimum at the point $(0,-1)$. Consequently, by the maximum principle,

$$\partial_t u_0(x,t) \equiv m \quad \text{in } Q_1(0,-1). \tag{2.24}$$

In accordance with our definition of r_k, for each k there exists a point $(y^k, s^k) \in \Gamma(u_k) \cap \partial K_1(0,-1)$ and a corresponding point $(r_k y^k + x^k, r_k^2 s^k + r_k^2 + t^k) \in \Gamma(u) \cap \partial K_{r_k}(z^k)$ such that

$$(r_k y^k + x^k, r_k^2 s^k + r_k^2 + t^k) \to (x^0, t^0) \quad \text{as } k \to \infty. \tag{2.25}$$

Let (y^0, s^0) denote the limit of a subsequence of points (y^k, s^k) as $k \to \infty$. Obviously, $(y^0, s^0) \in \Gamma(u_0) \cap \partial K_1(0,-1)$. Next, using (1.1), (2.25), and (2.20), we see that

$$\begin{aligned} W_\infty(\rho, y^0, s^0, u_0) &= \lim_{k\to\infty} W_{\frac{1}{r_k}}(\rho, y^k, s^k, u_k) \\ &= \lim_{k\to\infty} W_1(\rho r_k, r_k y^k + x^k, r_k^2 s^k + r_k^2 + t^k, u) = \omega(x^0, t^0, u) \\ &= \frac{15}{4} \end{aligned} \tag{2.26}$$

for any $\rho > 0$.

From Lemma 2.2 it follows that u_0 is a homogeneous function of degree 2 for $t \leq s^0$. More precisely, for any $x \in \mathbb{R}^n$, $t \leq 0$, and $\lambda > 0$ we have

$$u_0(y^0 + \lambda x, s^0 + \lambda^2 t) = \lambda^2 u_0(y^0 + x, s^0 + t).$$

Moreover, Lemma 2.5 shows that for any $x \in \mathbb{R}^n$ and $t \leq s^0$ we have

$$D_1 u_0 \geq 0 \quad \text{in } \mathbb{R}^{n+1} \cap \{t \leq s^0\}.$$

Now, using Lemma 2.6, Corollary 2.7, and relation (2.26), for $t \leq s^0$ we obtain the representation

$$u_0(x,t) = \frac{\big((x - y^0)\cdot e\big)_+^2}{2}, \tag{2.27}$$

where e is a direction in the x-space $\mathbb{R}^n$.

If $s^0 > -2$, then for $m \neq 0$ the representation (2.27) contradicts formula (2.24) in the cylinder $B_1 \times]-2, s^0[$. If $s^0 = -2$, then, by (2.24), we have

$$u_0(x,t) = \frac{\big((x - y^0)\cdot e\big)_+^2}{2} + m(t+2) \quad \text{in } Q_1(0,-1),$$

whence $H[u_0] = 1 - m$ in $Q_1(0,-1)$. Assuming that $m \neq 0$, we arrive at a contradiction with (2.23).

Thus, we have shown that for $t^0 < 0$ the lower limit of $\partial_t u$ vanishes at the point $z^0 = (x^0, t^0) \in \Gamma(u)$. The above arguments (with small changes) remain valid for $t^0 = 0$. In this case we consider the sets $K_1(0,-1) \cap \{t \leq b_0\}$ in place of $K_1(0,-1)$, where $b_0 = -\lim_{k\to\infty} t^k/r_k^2 - 1$.

The proof of the claim that the upper limits of $\partial_t u(z)$ as $z \to z^0 \in \Gamma(u)$ near the fixed boundary $x_1 = 0$ are nonpositive is even simpler. Assuming that this is not the case and arguing in the same way as before we get a global solution u_0 for which (2.24) is true with $m > 0$. But this is impossible by (0.3). The proof is complete. $\square$

§3. Regularity properties of the free boundary

Lemma 3.1. *Let $u \in P_2(M, 0)$, let δ be the constant as in Lemma 2.8, and let $0 < \varepsilon_1 < \frac{1}{16(2n+1)}$, $N_1 \geq 0$, N_0, and N_τ (with $\tau = 2, \dots, n$) be some constants. Then for an arbitrary point $z^0 = (x^0, t^0) \in Q_{1/2} \cap \{0 \leq x_1^0 \leq \delta/2\}$ and $\rho < \delta/2$ the inequality*

$$\rho\left(\sum_{i=1}^{n} N_i D_i u\right) + \rho^2 N_0 \partial_t u - u \geq -\varepsilon_1 \rho^2 \quad \text{in } Q^+_{\rho/2}(z^0), \tag{3.1}$$

implies that

$$\rho\left(\sum_{i=1}^{n} N_i D_i u\right) + \rho^2 N_0 \partial_t u - u \geq 0 \quad \text{in } Q^+_{\rho/4}(z^0).$$

Proof. Suppose the conclusion of the lemma fails.

Then there is a function $u \in P_2(M, 0)$ and some points $z^0 \in Q_{1/2} \cap \{0 \leq x_1^0 \leq \delta/2\}$ and $z^* = (x^*, t^*) \in Q^+_{\rho/4}(z^0)$ such that inequality (3.1) is fulfilled and

$$\rho\left(\sum_{i=1}^{n} N_i D_i u(x^*, t^*)\right) + \rho^2 N_0 \partial_t u(x^*, t^*) - u(x^*, t^*) < 0. \tag{3.2}$$

Consider the function

$$\begin{aligned} w(x,t) = \rho\left(\sum_{i=1}^{n} N_i D_i u(x,t)\right) &+ \rho^2 N_0 \partial_t u(x,t) \\ &- u(x,t) + \frac{1}{2n+1}\left(|x - x^*|^2 - (t - t^*)\right). \end{aligned}$$

This function is caloric in $Q_{\rho/4}(z^*) \cap \Omega(u)$, and $w(x^*, t^*) < 0$ by (3.2).

Also, we observe that Theorem 2.9 and the condition $u \geq 0$ imply the inequalities

$$w \geq 0 \text{ on } \Pi, \quad w \geq 0 \text{ on } \Gamma(u) \cap Q_{\rho/4}(z^*).$$

By the maximum principle, the negative infimum of w is attained on $\partial' Q_{\rho/4}(z^*) \cap \Omega(u)$. Thus,

$$-\frac{\rho^2}{16(2n+1)} \geq \inf_{\partial' Q_{\rho/4}(z^*) \cap \Omega(u)} \left\{\rho\left(\sum_{i=1}^{n} N_i D_i u\right) + \rho^2 N_0 \partial_t u - u\right\} \geq -\varepsilon_1 \rho^2,$$

a contradiction. This proves the lemma. $\square$

Lemma 3.2. *Let $u \in P_2(M, 0)$. There exists $\rho = \rho(n, M) > 0$ and a cone*

$$\mathcal{K} := \left\{e \in \mathbb{R}^{n+1} : \frac{e}{|e|} \cdot e_1 > (1 + \rho^2)^{-1/2}\right\}$$

such that for any point $z^0 = (x^0, t^0) \in Q_{1/2} \cap \Lambda(u) \cap \{x_1 \leq \rho\}$ and any direction $e \in \mathcal{K}$ we have

$$D_e u \geq 0 \quad \text{in } Q^+_{\rho/8}(z^0).$$

Proof. We fix the constant ε_1 occurring in Lemma 3.1, take $\varepsilon = \varepsilon_1/n$, and set

$$\rho_1 = \min\{\rho^*, \rho_0, \delta/2\}, \quad \rho = \varepsilon \rho_1 / M, \tag{3.3}$$

where $\rho^* = \rho^*(\varepsilon_1, n, M)$, ρ_0, and δ are the constants defined in Lemmas 2.3, 2.5, and 2.8, respectively.

First, suppose that $z^0 \in \Gamma(u)$. Applying Lemmas 2.5 and 2.3, we obtain

$$\rho_1 D_1 u - u \geq 0 \quad \text{in } Q^+_{\rho_1/2}(z^0), \tag{3.4}$$

$$\sup_{Q^+_{\rho_1}(z^0)} |D_\tau u| \leq \varepsilon\rho_1, \quad \tau = 2, \dots, n. \tag{3.5}$$

Combining (3.3), (3.4), and (3.5), we get the estimate

$$\rho_1\Big(D_1 u + \sum_{i=2}^n N_\tau D_\tau u\Big) + \rho_1^2 N_0 \partial_t u - u \geq -\varepsilon_1 \rho_1^2 \quad \text{in } Q^+_{\rho_1/2}(z^0),$$

where N_τ and N_0 are some constants satisfying $|N_\tau| \leq 1$, $|N_0| \leq \varepsilon/M$.

Since $u \geq 0$, we can apply Lemma 3.1 to obtain the inequality $D_e u \geq 0$ in $Q^+_{\rho/4}(z^0)$ for any $e \in \mathcal{K}$.

Now we assume that $z^0 \in \Lambda(u) \setminus \Gamma(u)$. If the cylinder $Q^+_{\rho/8}(z^0)$ is not contained in $\Lambda(u)$, we take $z^* = (x^*, t^*) \in \Gamma(u) \cap Q_{\rho/8}(z^0)$ with the maximum possible value of t^*. From the previous arguments it follows that $D_e u \geq 0$ in $Q^+_{\rho/4}(z^*)$ for any $e \in \mathcal{K}$. It only remains to observe that $Q_{\rho/8}(z^0) \cap \{t \leq t^*\} \subset Q_{\rho/4}(z^*)$, and for $t^0 > t^*$ the function u vanishes in $Q^+_{\rho/8}(z^0) \cap \{t \geq t^*\}$. The proof is complete. □

Theorem 3.3. *Let $u \in P_2(M, 0)$, let ρ be the same constant as in Lemma* 3.2, *and let $Q = Q_{1/4} \cap \{0 \leq x_1 \leq \rho/64\}$. There exists a Lipschitz continuous nonnegative function f defined on $\Pi \cap Q_{1/4}$ and such that*

$$\Omega(u) \cap Q = \{(x,t) \in Q : x_1 > f(x_2, \dots, x_n, t)\}.$$

Proof. We define f as follows:

$$f(x', t) = \sup\{x_1 \in [0, \rho/64] : u(x_1, x', t) = 0\}. \tag{3.6}$$

We must prove that f is a Lipschitz function.

For an arbitrary point $z^0 \in Q$ we introduce the cones

$$\mathcal{K}_-(z^0) = Q \cap \{z^0 - \mathcal{K}\}, \quad \mathcal{K}_+(z^0) = Q \cap \{z^0 + \mathcal{K}\}$$

and the sets

$$\Sigma_s(z^0) = \{(x,t) \in \mathcal{K}_-(z^0) : t = s\},$$

where $\mathcal{K}$ is the cone described in Lemma 3.2. Observe that the tangent of the opening angle of the cone $\mathcal{K}$ is equal to ρ, so that $\Sigma_s(z^0)$ is nonempty only if $s \in\,]t^0 - \rho x_1^0,$ $(t^0 + \rho x_1^0)_-[$, where $(t^0 + \rho x_1^0)_- = \min\{0; t^0 + \rho x_1^0\}$.

First, we prove that

$$\mathcal{K}_-(z^0) \subset \Lambda(u), \quad z^0 \in \Lambda(u) \cap Q. \tag{3.7}$$

The inclusion (3.7) is equivalent to the following statement: for $z^0 \in \Lambda(u) \cap Q$ we have

$$\Sigma_s(z^0) \subset \Lambda(u) \tag{3.8}$$

for all $t \in\,]t^0 - \rho x_1^0, (t^0 + \rho x_1^0)_-[$. Lemma 3.2 shows that (3.8) is true at least for $t \in\,]t^0 - \rho x_1^0, t^0]$. If $t^0 = 0$, then (3.7) is proved.

Consider a point $z^0 \in \Lambda(u) \cap Q$ with $t^0 < 0$ and suppose that for this point (3.7) fails. Then there exists $s \in\,]t^0, (t^0 + \rho x_1^0)_-[$ such that

$$\mathcal{K}_-(z^0) \cap \{t \leq s\} \subset \Lambda(u) \tag{3.9}$$

and (3.8) fails for $t > s$ that are close to s. Using Lemma 3.2 once again, we see that this can happen only if $\mathcal{K}_-(z_0) \cap \{t > s\} \subset \Omega(u)$. By (3.9), we conclude that $\Sigma_s(z^0) \subset \Gamma(u)$, while the set $\Sigma_s(z^0) \cap \Pi$ consists of contact points. However, from Theorem III in [AUS2] it follows that such a behavior of the free boundary near the contact points is impossible. This contradiction proves (3.7).

By using (3.7), it can easily be shown that

$$\mathcal{K}_+(z) \subset \Omega(u), \quad z \in \overline{\Omega}(u) \cap Q. \tag{3.10}$$

From (3.7) and (3.10) it follows that the function f defined by (3.6) satisfies the Lipschitz condition with the constant ρ^{-1}. □

Corollary 3.4. *Let f be the same function as in Theorem* 3.3. *Then in a neighborhood of every point (x', t) satisfying $0 < f(x', t) < \rho/64$, the function f belongs to the class $C^{1+\alpha}$ with some $0 < \alpha < 1$.*

Proof. This statement is an obvious consequence of Theorem 3.3, Lemma 3.2, and the result of Athanasopoulos and Salsa proved in [AtSa]. □

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