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DEHORNOY'S ORDERING ON THE BRAID GROUP AND BRAID MOVES

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Abstract. In terms of Dehornoy's ordering on the braid group $\mathcal{B}_n$, restrictions are found that prevent us from performing the Markov destabilization and the Birman–Menasco braid moves. As a consequence, a sufficient condition is obtained for the link represented by a braid to be prime, and it is shown that all braids in $\mathcal{B}_n$ that are not minimal lie in a finite interval of Dehornoy's ordering.

Introduction

0.1. It is known that any classical link in $\mathbb{R}^3$ is representable by a closed braid, and the set of such closed braids is infinite. By a *braid move* we mean the passage from a closed braid $\widehat{\beta}$ to another closed braid representing the same link as $\widehat{\beta}$. Many essential results of the theory of braids and links are related to the moves of stabilization and destabilization introduced by Markov, as well as to the "exchange move" and the "flype" defined by Birman and Menasco. While stabilization can always be performed, the other moves mentioned above do not apply to some braids.

0.2. In the present paper, we establish some restrictions on the possibility of performing braid moves; the restrictions are formulated in terms of Dehornoy's ordering. Combining them with the known results concerning such transformations, we obtain conditions sufficient for the primeness of the link represented by a braid (these conditions are also formulated in terms of Dehornoy's ordering) and prove that *there exists* a similar criterion of the minimality of a braid.

0.3. The results of the paper can be roughly outlined as follows. All braids in the braid group $\mathcal{B}_n$ that lie outside a certain finite Dehornoy interval represent prime links, and their closures admit no nontrivial moves that do not increase the index of a braid.

0.4. Remarks.

0.4.1. All the restrictions obtained can be reformulated in terms of any ordering of Thurston type on the braid group. We prefer Dehornoy's ordering because it provides most compact proofs.

0.4.2. Assertions 1) and 2) of Theorem 5.1 (on the admissibility of destabilization, exchange move, and flype) were announced in [12]; they confirm Menasco's four conjectures formulated in Kirby's collection of problems [10] (with the exception of the conjecture on periodic braids: as shown in [12], the part of it that concerns flype is incorrect).

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0.4.3. It can be proved that for each nondegenerate random walk on the group $\mathcal{B}_n$ and for any finite Dehornoy interval $I \subset \mathcal{B}_n$ the probability to get into I at the kth step of the walk tends to zero as $k \to +\infty$. More precisely, there is a constant C (depending on the measure that determines the walk and on the interval I) such that the probability to get into I at the kth step of the walk is at most $C/\sqrt{k}$. The usual method for obtaining a "random" braid consists in generating a word $W = w_1 w_2 \dots w_k$ in the Artin generators, where each letter w_j is chosen equiprobably in the set $\{\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}\}$. This procedure models the right random walk on $\mathcal{B}_n$ with uniform distribution on $\{\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}\}$. Thus, the results obtained in the present paper imply, e.g., that a "random", "sufficiently long" braid represents a prime link with probability close to 1.

0.5. Structure of the paper. In §1, we define the braid group and introduce some related notions.

In §2, we introduce Dehornoy's ordering and related notions, and also prove several lemmas. Lemma 2.4 plays a key role in the proof of Theorem 5.1.

In §3, we make some remarks concerning the Dehornoy intervals.

In §4, we define destabilization, exchange move, flype, and also *template moves* of closed braids.

In §5, we prove Theorem 5.1 on the admissibility of braid moves.

In §6, we prove a criterion of the primeness of the link represented by a braid; this criterion is formulated in terms of Dehornoy's ordering (Theorem 6.2). The criterion is deduced from Theorem 5.1 and Proposition 6.1 (which is also proved in §6) saying that each closed braid of index > 2 that does not admit exchange move represents a prime link. (The proof of Proposition 6.1 involves results by Birman and Menasco [2], [3].) Corollary 6.3 gives a sufficient condition for the primeness of links representable by strongly D-positive, quasipositive, and positive braids.

In §7, we derive Theorem 7.3 on minimal braids from Theorem 5.1 and the "Markov theorem without stabilization" stated in [4]. We also formulate Conjecture 7.4, which indicates sharp bounds for the intervals the existence of which is claimed in Theorem 7.3; we present a general example of a nonminimal braid, which clarifies Conjecture 7.4. A special case of this example is a nonminimal braid that admits neither destabilization, nor exchange move, nor flype.

§1. The braid group $\mathcal{B}_n$

1.0. Definitions and notation. We denote by $\mathcal{B}_n$ the Artin group of braids of n strands:

$$\mathcal{B}_n := \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| \geq 2;\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle.$$

The elements of $\mathcal{B}_n$ are *braids*, and the generators $\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}$ are the *Artin generators*. Throughout, by a *word* we mean a word in the Artin generators. If no confusion is possible, we write $\beta = W$ if the word W represents the braid β. For $r \leq n$, we denote by $\overline{\mathcal{B}}_r$ the subgroup in $\mathcal{B}_n$ generated by $\sigma_1^{\pm 1}, \dots, \sigma_{r-1}^{\pm 1}$. Obviously, $\overline{\mathcal{B}}_r \cong \mathcal{B}_r$.

The conjugacy class $\widehat{\beta}$ of a braid $\beta \in \mathcal{B}_n$ is called a *closed braid*. The number n is the *index* of β and $\widehat{\beta}$.

Sometimes, we denote the braid $\alpha\beta\alpha^{-1}$ by β^α.

The braid

$$\Delta := (\sigma_1\sigma_2 \dots \sigma_{n-1})(\sigma_1\sigma_2 \dots \sigma_{n-2}) \dots (\sigma_1\sigma_2)(\sigma_1) \in \mathcal{B}_n$$

is the *fundamental braid*. For $n > 2$, the braid $\Delta^2 \in \mathcal{B}_n$ generates the center of $\mathcal{B}_n$ (which is known to be isomorphic to $\mathbb{Z}$).

1.1. Lemma. *In the group $\mathcal{B}_n$, we have the following relations*:

$$(\sigma_i^{\pm 1})^\Delta = \sigma_{n-i}^{\pm 1}, \quad i = 1, \dots, n-1; \tag{1}$$

$$\Delta \in \overline{\mathcal{B}}_{n-1}\sigma_{n-1}\overline{\mathcal{B}}_{n-1}; \tag{2}$$

(3) $\Delta^2 = \sigma_1\rho\sigma_1\rho = \rho\sigma_1\rho\sigma_1$, *where ρ is a word in the generators $\sigma_2, \dots, \sigma_{n-1}$.*

Proof. (1) This fact is nearly obvious and well known (see, e.g., [1]).

(2) This follows directly from the definition of Δ.

(3) By (2), we have $\Delta = V\sigma_{n-1}W$, where V and W are words in the generators $\sigma_1^{\pm 1}, \dots, \sigma_{n-2}^{\pm 1}$. By (1), we see that $\Delta = \Delta^\Delta = X\sigma_1 Y$, where $X = V^\Delta$ and $Y = W^\Delta$ are words in the generators $\sigma_2^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}$. Since the braid Δ^2 commutes with any braid, we have

$$\Delta^2 = X\sigma_1 YX\sigma_1 Y = \sigma_1 YX\sigma_1 YX = YX\sigma_1 YX\sigma_1.$$

It remains to set $\rho := YX$. □

§2. The Dehornoy ordering

2.1. Definitions and notation. A braid $\beta \in \mathcal{B}_n$ is *D-positive* (or *Dehornoy-positive*) if for some $i \in \{1, \dots, n-1\}$ the braid β is represented by a word in the generators $\sigma_i, \sigma_{i+1}^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}$ and σ_i does occur in this word. A braid $\beta \in \mathcal{B}_n$ is *D-negative* if the braid β^{-1} is D-positive.

For $\beta_1, \beta_2 \in \mathcal{B}_n$, we write $\beta_1 \prec \beta_2$ if the braid $\beta_2\beta_1^{-1}$ is D-positive. The relation $\prec$ is a linear right-invariant ordering on the braid group (i.e., for any $\beta_1, \beta_2, \alpha \in \mathcal{B}_n$ we have $\beta_1 \prec \beta_2 \iff \beta_1\alpha \prec \beta_2\alpha$) (see [8]) and is called *Dehornoy's ordering*.

A set $I \subset \mathcal{B}_n$ is a *Dehornoy interval* if for any $x \prec z \prec y \in \mathcal{B}_n$ such that $x, y \in I$ we have $z \in I$. Let $a < b \in \mathbb{Z}$. We introduce the following natural notation for the Dehornoy intervals:

$$(\Delta^a, \Delta^b) := \{\beta \in \mathcal{B}_n \mid \Delta^a \prec \beta \prec \Delta^b\}.$$

2.2. Lemma. *Suppose a braid $\beta \in \mathcal{B}_n$ is represented by a word W containing r occurrences of σ_1 and s occurrences of σ_1^{-1} with $r + s > 0$. Then $\beta \in (\Delta^{-2s}, \Delta^{2r})$.*

Proof. We show that $\Delta^{-2s} \prec \beta$. (The inequality $\beta \prec \Delta^{2r}$ is proved in a similar way.) If $s = 0$, then by assumption we have $r > 0$ and the braid β is D-positive, i.e., $\Delta^{-2s} = e \prec \beta$. If $s \neq 0$, then by assumption we have

$$\beta = W = W_0\sigma_1^{-1}W_1\sigma_1^{-1}W_2 \dots W_{s-1}\sigma_1^{-1}W_s,$$

where the subwords $W_0, \dots, W_s$ (which may be empty) do not contain σ_1^{-1}. Since the braid $\Delta^2 = \rho\sigma_1\rho\sigma_1$ (see assertion (3) of Lemma 1.1) commutes with any braid, we have

$$\beta\Delta^{2s} = W_0(\Delta^2\sigma_1^{-1})W_1(\Delta^2\sigma_1^{-1})\dots(\Delta^2\sigma_1^{-1})W_s = W_0\rho\sigma_1\rho W_1\rho\sigma_1\rho \dots \rho\sigma_1\rho W_s.$$

Thus, the braid $\beta\Delta^{2s}$ is written without occurrences of σ_1^{-1} and with at least s ($s \geq 1$) occurrences of σ_1. This means that the braid $\beta\Delta^{2s}$ is D-positive, i.e., $\Delta^{-2s} \prec \beta$. □

2.3. Lemma. 1)$\mathcal{B}_n = \bigcup_{-\infty}^{\infty}[\Delta^{2y}, \Delta^{2(y+1)})$, *where* $[\Delta^{2y}, \Delta^{2(y+1)}) := \{\Delta^{2y}\} \cup (\Delta^{2y}, \Delta^{2(y+1)})$.

2) *If $\alpha \in (\Delta^{2a}, \Delta^{2b})$ and $\beta \in (\Delta^{2c}, \Delta^{2d})$, then $\alpha\beta \in (\Delta^{2(a+c)}, \Delta^{2(b+d)})$.*

3) *If $\beta \in (\Delta^{2a}, \Delta^{2b})$, then $\beta^{-1} \in (\Delta^{-2b}, \Delta^{-2a})$.*

4) *If $\beta \in (\Delta^{2a}, \Delta^{2b})$, then $\widehat{\beta} \subset (\Delta^{2(a-1)}, \Delta^{2(b+1)})$.*

5) *If $\widehat{\beta} \cap (\Delta^{2a}, \Delta^{2b}) \neq \varnothing$, then $\widehat{\beta} \subset (\Delta^{2(a-1)}, \Delta^{2(b+1)})$.*

Proof. 1) Obviously, this follows from Lemma 2.2.

2) Using the right invariance of Dehornoy's ordering and the fact that the braid Δ^2 commutes with any braid, we obtain

$$\Delta^{2(a+c)} = \Delta^{2c}\Delta^{2a} \prec \beta\Delta^{2a} = \Delta^{2a}\beta \prec \alpha\beta.$$

In a similar way, we show that $\alpha\beta \prec \Delta^{2(b+d)}$.

3) The inequality $\beta \prec \Delta^{2b}$ means that the braid $\beta\Delta^{-2b}$ is D-negative. This means that the braid $(\beta\Delta^{-2b})^{-1} = \Delta^{2b}\beta^{-1} = \beta^{-1}\Delta^{2b}$ is D-positive, i.e., $\Delta^{-2b} \prec \beta^{-1}$. The inequality $\beta^{-1} \prec \Delta^{-2a}$ is proved in a similar way.

4) We consider an arbitrary braid $\beta^\alpha \in \widehat{\beta}$. By assertion 1), we have $\alpha \in [\Delta^{2y}, \Delta^{2(y+1)})$ for some $y \in \mathbb{Z}$. If $\alpha = \Delta^{2y}$, then, obviously,

$$\beta^\alpha = \beta \in (\Delta^{2a}, \Delta^{2b}) \subset (\Delta^{2(a-1)}, \Delta^{2(b+1)}).$$

If $\alpha \in (\Delta^{2y}, \Delta^{2(y+1)})$, then, by assertion 3), we have $\alpha^{-1} \in (\Delta^{-2(y+1)}, \Delta^{-2y})$, and, by assertion 2),

$$\beta^\alpha = \alpha\beta\alpha^{-1} \in (\Delta^{2(y+a-(y+1))}, \Delta^{2(y+1+b-y)}) = (\Delta^{2(a-1)}, \Delta^{2(b+1)}).$$

5) This is a simple reformulation of property 4). □

2.4. Lemma. *Suppose a braid $\beta \in \mathcal{B}_n$ is represented by a word containing r occurrences of σ_{n-1} and s occurrences of σ_{n-1}^{-1} with $r+s>0$. Then*

$$\widehat{\beta} \cap (\Delta^{-2s}, \Delta^{2r}) \neq \varnothing,$$

and, in particular, by assertion 4) *of Lemma* 2.3, *we have*

$$\beta \in \widehat{\beta} \subset (\Delta^{-2(s+1)}, \Delta^{2(r+1)}).$$

Proof. Assertion (1) of Lemma 1.1 implies that $\beta^\Delta \in \widehat{\beta}$ is represented by a word containing exactly r occurrences of σ_1 and s occurrences of σ_1^{-1}; by Lemma 2.2, we have $\beta^\Delta \in (\Delta^{-2s}, \Delta^{2r})$. □

§3. The condition $\widehat{\beta} \cap (\Delta^{2a}, \Delta^{2b}) \neq \varnothing$

3.0. In the present paper, conditions of the form

$$(*) \qquad \widehat{\beta} \cap (\Delta^{2a}, \Delta^{2b}) \neq \varnothing$$

will be used repeatedly. In this connection, the following remarks are in order. (Statements 3.2–3.4 below are given without proof.)

3.1. "Positive" braids. We recall some definitions.

The braid $\beta \in \mathcal{B}_n$ is *strongly D-positive*[1] if $e \prec \beta^\alpha$ for all $\alpha \in \mathcal{B}_n$.

A braid β is *quasipositive* if $\beta = \sigma_1^{\beta_1} \dots \sigma_1^{\beta_k}$ for some $\beta_1, \dots, \beta_k \in \mathcal{B}_n$.

The braid β is *positive* if it is represented by a word containing only positive powers of the Artin generators.

It is known that any quasipositive braid is strongly D-positive (see [5], [11]). Obviously, any positive braid is quasipositive and, consequently, strongly D-positive.

The definitions imply directly that for each strongly D-positive (e.g., quasipositive or positive) braid β and for $a<b \in \mathbb{Z}$ the following relations are true:

$$\widehat{\Delta^{2b}\beta} \cap (\Delta^{2a}, \Delta^{2b}) = \widehat{\Delta^{2a}\beta^{-1}} \cap (\Delta^{2a}, \Delta^{2b}) = \varnothing.$$

[1]This notion was introduced by S. Yu. Orevkov.

3.2. Algorithms. There are effective algorithms for comparison of braids in Dehornoy's ordering (see, e.g., [8], [13]). *There is an effective algorithm allowing us to verify* $(*)$ *for* $a < b \in \mathbb{Z}$ *and* $\beta \in B_n$.

3.3. Reformulation of condition $(*)$: Thurston-type orders. Dehornoy's ordering is a *Thurston-type* order on the braid group (see [14]). Let $\lessdot$ be a Thurston-type order on $\mathcal{B}_n$. We introduce the following natural notation:

$$(\Delta^a, \Delta^b)_\lessdot := \{\beta \in \mathcal{B}_n \mid \Delta^a \lessdot \beta \lessdot \Delta^b\}.$$

It is easily seen that for any $\beta \in \mathcal{B}_n$ and $a < b \in \mathbb{Z}$ we have

$$\widehat{\beta} \cap (\Delta^{2a}, \Delta^{2b}) \neq \varnothing \quad \Longleftrightarrow \quad \widehat{\beta} \cap (\Delta^{2a}, \Delta^{2b})_\lessdot \neq \varnothing \quad \Longrightarrow \quad \beta \in (\Delta^{2(a-1)}, \Delta^{2(b+1)})_\lessdot.$$

Thus, for example, we can reformulate Lemma 2.4 in terms of any Thurston-type order on $\mathcal{B}_n$.

We also observe that the sets of strongly D-positive and strongly $\lessdot$-positive braids coincide.

3.4. Modification of condition $(*)$: reducing the intervals. We denote by $\overline{\mathcal{B}}^\Delta_{n-1}$ the subgroup in $\mathcal{B}_n$ generated by $\sigma_2, \dots, \sigma_{n-1}$. The set $\overline{\mathcal{B}}^\Delta_{n-1}$ is a Dehornoy interval. We define

$$\langle \Delta^{2a}, \Delta^{2b} \rangle := (\Delta^{2a}, \Delta^{2b}) \setminus (\Delta^{2a}\overline{\mathcal{B}}^\Delta_{n-1} \cup \Delta^{2b}\overline{\mathcal{B}}^\Delta_{n-1}).$$

It can be shown that for any $\beta \in \mathcal{B}_n$ and $a < b \in \mathbb{Z}$ we have

$$\widehat{\beta} \cap (\Delta^{2a}, \Delta^{2b}) \neq \varnothing \quad \Longleftrightarrow \quad \widehat{\beta} \cap \langle \Delta^{2a}, \Delta^{2b} \rangle \neq \varnothing \quad \Longrightarrow \quad \beta \in \langle \Delta^{2(a-1)}, \Delta^{2(b+1)} \rangle.$$

Thus, replacing the parentheses in the notation for the intervals in Lemma 2.4 by the angular brackets once again we obtain a true assertion.

§4. Braid moves

4.0. It is known that any closed braid (and, respectively, any element of $\mathcal{B}_n$) represents an (oriented) link. The link represented by a closed braid $\widehat{\beta}$ will be denoted by $\mathcal{L}(\beta)$. By a *braid move* we mean the passage from the closed braid $\widehat{\beta}$ to a closed braid $\widehat{\alpha}$ representing the same link $\mathcal{L}(\beta)$. (One can say that a braid move is a partly determined many-valued mapping on the set of all closed braids.)

4.1. Destabilization. We say that a closed braid $\widehat{\alpha} \in \mathcal{B}_{n-1}$ of index $n-1$ is obtained from a closed braid $\widehat{\beta} \in \mathcal{B}_n$ of index n via *destabilization* if there is a braid $\gamma \in \widehat{\alpha}$ such that $\gamma\sigma_{n-1} \in \widehat{\beta}$ or $\gamma\sigma_{n-1}^{-1} \in \widehat{\beta}$:

$$\widehat{\beta} \ni \gamma\sigma_{n-1}^{\pm 1} \longmapsto \gamma \in \widehat{\alpha}.$$

Thus, *a closed braid* $\widehat{\beta}$ *admits destabilization if and only if*

$$\widehat{\beta} \cap \overline{\mathcal{B}}_{n-1}\sigma_{n-1}^{\pm 1} \neq \varnothing.$$

In other words, $\widehat{\beta}$ admits destabilization if $\widehat{\beta}$ contains a braid that can be written with a single occurrence of one of the generators σ_{n-1}, σ_{n-1}^{-1}.

4.2. Exchange move. Suppose $\widehat{\alpha}, \widehat{\beta} \subset \mathcal{B}_n$ are two closed braids of index n. We say that $\widehat{\alpha}$ is obtained from $\widehat{\beta}$ via an *exchange move* if there are $\gamma_1, \gamma_2 \in \overline{\mathcal{B}}_{n-1} \subset \mathcal{B}_n$ such that $\gamma_1\sigma_{n-1}\gamma_2\sigma_{n-1}^{-1} \in \widehat{\beta}$ and $\gamma_1\sigma_{n-1}^{-1}\gamma_2\sigma_{n-1} \in \widehat{\alpha}$:

$$\widehat{\beta} \ni \gamma_1\sigma_{n-1}\gamma_2\sigma_{n-1}^{-1} \longmapsto \gamma_1\sigma_{n-1}^{-1}\gamma_2\sigma_{n-1} \in \widehat{\alpha}.$$

Thus, *a closed braid $\widehat{\beta}$ admits an exchange move if and only if*

$$\widehat{\beta} \cap \overline{\mathcal{B}}_{n-1}\sigma_{n-1}\overline{\mathcal{B}}_{n-1}\sigma_{n-1}^{-1} \neq \varnothing.$$

4.3. Flype. A flype is usually defined with the help of the diagram shown in the upper part of Figure 1. The algebraic definition of this move is rather bulky, and we only define the *admissibility* of it.

We say that a closed braid $\widehat{\beta}$ of index n *admits a flype* if

$$\widehat{\beta} \cap \overline{\mathcal{B}}_{n-1}\sigma_{n-1}\overline{\mathcal{B}}_{n-1}\sigma_{n-1}^{\pm 1}\overline{\mathcal{B}}_{n-1}\sigma_{n-1}^{-1} \neq \varnothing.$$

In Figure 1, the diagram involved in the definition of a flype is presented in such a way that the possibility of writing the corresponding braid in the required form is obvious.

4.4. Remark. Each closed braid of index greater than 2 that admits destabilization also admits an exchange move. Each closed braid of index greater than 2 that admits an exchange move also admits a flype.

These facts are immediate consequences of the definitions and the following obvious identities:

$$\begin{aligned} \sigma_1 &= \sigma_1(\sigma_2\sigma_1\sigma_1^{-1}\sigma_2^{-1}) = \sigma_2\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}, \\ si_1^{-1} &= \sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}. \end{aligned}$$

4.5. Template moves. A *form* of index n and length d is a collection of braids and indices of the type

$$(\xi_1, \dots, \xi_d; k_1, \dots, k_d) \in (\mathcal{B}_n)^d \times \{1, \dots, n\}^d.$$

Suppose π is a permutation of d elements and

$$F = (\xi_1, \dots, \xi_d; k_1, \dots, k_d) \quad \text{and} \quad F' = (\xi'_1, \dots, \xi'_d; k_{\pi(1)}, \dots, k_{\pi(d)})$$

are two forms of indices n and n', respectively. The triple (F, F', π) is a *template* if for every collection $\{\gamma_j \in \overline{\mathcal{B}}_{k_j}\}_{j=1}^d$ the braids $\gamma_1\xi_1 \dots \gamma_d\xi_d \in \mathcal{B}_n$ and $\gamma_{\pi(1)}\xi'_1 \dots \gamma_{\pi(d)}\xi'_d \in \mathcal{B}_{n'}$ represent one and the same link:

$$\mathcal{L}(\gamma_1\xi_1 \dots \gamma_d\xi_d) = \mathcal{L}(\gamma_{\pi(1)}\xi'_1 \dots \gamma_{\pi(d)}\xi'_d).$$

Suppose $\widehat{\alpha}$ and $\widehat{\beta}$ are closed braids of indices n' and n, respectively. We say that $\widehat{\alpha}$ is obtained from $\widehat{\beta}$ via a *template move of type* (F, F', π) if there are braids $\gamma_j \in \mathcal{B}_{k_j}$ such that $\gamma_1\xi_1 \dots \gamma_d\xi_d \in \widehat{\beta}$ and $\gamma_{\pi(1)}\xi'_1 \dots \gamma_{\pi(d)}\xi'_d \in \widehat{\alpha}$:

$$\widehat{\beta} \ni \gamma_1\xi_1 \dots \gamma_d\xi_d \longmapsto \gamma_{\pi(1)}\xi'_1 \dots \gamma_{\pi(d)}\xi'_d \in \widehat{\alpha}.$$

A template move is said to be *nonincreasing* if $n \geq n'$. A template move T is said to be *trivial* if each closed braid to which T can be applied is taken by T to the same braid.

The following statement is obvious.

4.6. Assertion. *A closed braid $\widehat{\beta}$ admits a template move of type (F, F', π) if and only if*

$$\widehat{\beta} \cap \overline{\mathcal{B}}_{k_1}\xi_1\overline{\mathcal{B}}_{k_2}\xi_2 \dots \overline{\mathcal{B}}_{k_d}\xi_d \neq \varnothing.$$

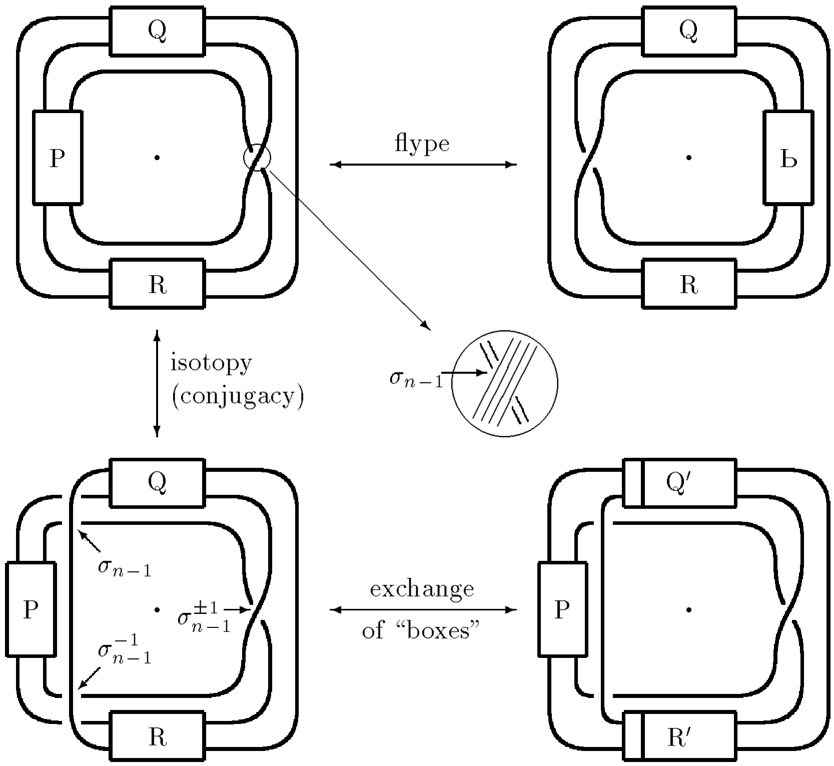

FIGURE 1.

4.7. Remarks.

4.7.1. Birman and Menasco [4] introduced "isotopic" templates and the corresponding braid moves. Comparison of the definitions of templates in [4] and in the present paper readily shows that each isotopic template determines an algebraic template (in the sense of the above definition). Thus, the set of template moves described in the present paper includes the set of moves described in [4]. The question whether the two sets coincide remains open.

4.7.2. By definition, the index of the braids admitting a given template move is fixed. Destabilization, an exchange move, and a flype are "unions" of the corresponding template moves "over all indices".

§5. CONDITIONS OF THE ADMISSIBILITY OF TRANSFORMATIONS

5.1. Theorem. *Let $\beta \in \mathcal{B}_n$.*

1) *If a closed braid $\widehat{\beta}$ admits destabilization or an exchange move, then $\widehat{\beta} \cap (\Delta^{-2}, \Delta^2) \neq \varnothing$; in particular, $\beta \in (\Delta^{-4}, \Delta^4)$.*
2) *If a closed braid $\widehat{\beta}$ admits a flype, then $\widehat{\beta} \cap (\Delta^{-4}, \Delta^4) \neq \varnothing$; in particular, $\beta \in (\Delta^{-6}, \Delta^6)$.*
3) *Suppose $\mathcal{T}$ is a nontrivial nonincreasing template move. Then there is $r \in \mathbb{N}$ such that if a closed braid $\widehat{\beta}$ admits the move $\mathcal{T}$, then $\beta \in (\Delta^{-2r}, \Delta^{2r})$. (In other words, all closed braids admitting the move $\mathcal{T}$ lie in a finite Dehornoy interval.)*

The remarks in §3 enable us to reformulate Theorem 5.1 in many ways.

Proof. Assertions 1) and 2) follow directly from Lemma 2.4 and the definitions of the admissibility of moves (see §4).

3) We denote by (F, F', π) the template of the move $\mathcal{T}$, where

$$F = (\xi_1, \dots, \xi_d;\, k_1, \dots, k_d) \quad \text{and} \quad F' = (\xi_1', \dots, \xi_d';\, k_{\pi(1)}, \dots, k_{\pi(d)})$$

are forms of indices n and n', respectively. (Here we have $n \geq n'$ because by assumption the move $\mathcal{T}$ is nonincreasing.)

a) Suppose that $k_j < n$ for all j. We denote by r_j the number of occurrences of the generators $\sigma_{n-1}^{\pm 1}$ in a certain word representing ξ_j. The definitions imply that each closed braid admitting the move $\mathcal{T}$ contains a braid represented by a word with exactly $r_1 + \dots + r_d$ occurrences of the generators $\sigma_{n-1}^{\pm 1}$. (A similar assertion was used in [4, Proof of Proposition 1.1].) It remains to apply Lemma 2.4 and set

$$r = r_1 + \dots + r_d + 1.$$

b) Suppose that for some $i \in \{1, \dots, d\}$ we have $k_i = n$. Then $n' = n$ because $n \geq n' \geq k_i = n$. We show that in this case the move $\mathcal{T}$ is trivial. Suppose $\widehat{\alpha}$ is obtained from $\widehat{\beta}$ via the move $\mathcal{T}$. This means that there exist braids $\gamma_j \in \overline{\mathcal{B}}_{k_j}$ satisfying

$$\beta_1 := \gamma_1 \xi_1 \dots \gamma_d \xi_d \in \widehat{\beta} \quad \text{and} \quad \alpha_1 := \gamma_{\pi(1)} \xi_1' \dots \gamma_{\pi(d)} \xi_d' \in \widehat{\alpha}.$$

Using conjugation, we can assume that $i = 1$ and that $\pi(1) = 1$, i.e., $k_{\pi(1)} = k_1 = n$. Under the above conditions, the definition of a template implies that for each braid $\gamma \in \mathcal{B}_n$ we have

$$\mathcal{L}(\gamma \beta_1) = \mathcal{L}(\gamma \alpha_1).$$

In particular, setting $\gamma = \alpha_1^{-1}$, we obtain $\mathcal{L}(\alpha_1^{-1}\beta_1) = \mathcal{L}(e)$, i.e., the braid $\alpha_1^{-1}\beta_1 \in \mathcal{B}_n$ represents a trivial n-component link. It is easily seen (see, e.g., [6, Lemma 7]) that the latter is possible only if $\alpha_1^{-1}\beta_1 = e$, whence $\alpha_1 = \beta_1$ and $\widehat{\alpha} = \widehat{\beta}$. Consequently, the transformation $\mathcal{T}$ is trivial. $\square$

§6. Condition sufficient for the primeness of a link

6.0. Definitions. We recall some notions from the link theory.

A link $L \subset S^3$ is *trivial* if there is a sphere $S^2 \subset S^3$ such that $L \subset S^2$.

A link $L \subset S^3$ is *split* if there is a sphere $S^2 \subset S^3 \setminus L$ bounding no ball (in $S^3 \setminus L$).

A link $L \subset S^3$ is *composite* if there is a sphere $S^2 \subset S^3$ that intersects the link L at two points and splits L into two links ("tangles") none of which is an unknotted arc.

A link is *prime* if it is neither composite, nor split, nor trivial.

6.1. Proposition. *Suppose that $\beta \in \mathcal{B}_n$ with $n > 2$, and that the link $\mathcal{L}(\beta)$ is not prime (i.e., $\mathcal{L}(\beta)$ is composite, split, or trivial). Then the closed braid $\widehat{\beta}$ admits an exchange move.*

Proposition 6.1 is proved below (see Sections 6.5–6.9).

Assertion 1) of Theorem 5.1 and Proposition 6.1 immediately imply the following result (the case of $n = 2$, which is not covered by Proposition 6.1, is trivial):

6.2. Theorem (condition sufficient for the primeness of a link). *Suppose $\beta \in \mathcal{B}_n$ $(n > 1)$. If $\widehat{\beta} \cap (\Delta^{-2}, \Delta^2) = \varnothing$ (e.g., $\beta \notin (\Delta^{-4}, \Delta^4)$), then the link $\mathcal{L}(\beta)$ is prime (i.e., noncomposite, nonsplit, and nontrivial).*

Theorem 6.2, as well as Theorem 6.1, admits a number of reformulations (see §3).

Theorem 6.2 and the definitions given in §3 imply the following result.

6.3. Corollary. *If a braid β is strongly D-positive (e.g., quasipositive or positive), then the link $\mathcal{L}(\Delta^2\beta)$ is prime.*

6.4. Remark. The assertion of Corollary 6.3 is known in the case of positive braids (see [7]).

6.5. The remaining part of the section is devoted to the proof of Proposition 6.1.

We recall that a closed braid $\widehat{\beta}$ of index n is said to be *split* if for some $r \in \{1, \dots, n-1\}$ there is a braid $\alpha \in \widehat{\beta}$ written in the generators $\{\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}\} \setminus \{\sigma_r^{\pm 1}\}$.

A closed braid $\widehat{\beta}$ is *composite* if for some $r \in \{2, \dots, n-1\}$ we have

$$\widehat{\beta} \cap \overline{\mathcal{B}}_r \overline{\mathcal{B}}_{n-r+1}^{\Delta} \neq \varnothing.$$

6.6. Remark. We easily see that each split braid is composite.

We need the following result.

6.7. Theorem (Birman–Menasco [2], [3]; cf. [9])**.** *Suppose that a closed braid $\widehat{\beta}$ represents an unknot, a split link, or a composite link. Then there is a finite sequence of closed braids*

$$\widehat{\beta} = \widehat{\beta}_0 \mapsto \widehat{\beta}_1 \mapsto \widehat{\beta}_2 \mapsto \dots \mapsto \widehat{\beta}_k = \widehat{\alpha}$$

such that $\widehat{\alpha}$ is a braid of index 1 (*i.e.*, $\alpha = e \in \mathcal{B}_1$), *a split braid, or a composite braid, respectively, and for each $j = 1, \dots, k$ the closed braid $\widehat{\beta}_j$ is obtained from $\widehat{\beta}_{j-1}$ via destabilization or an exchange move.*

6.8. Remarks.

6.8.1. The original formulations of Birman and Menasco involve the transformation of isotopy of a braid in the complement of the axis. We have defined a closed braid as a conjugacy class, which allows us to avoid mentioning isotopies.

6.8.2. The definition of a composite braid used by Birman and Menasco in [2] differs from ours. Each braid that is composite in the sense of [2] is also composite by our definition.

Proof of Proposition 6.1. Suppose that $\beta \in \mathcal{B}_n$ $(n > 2)$ and that the link $\mathcal{L}(\beta)$ is nonprime, i.e., $\mathcal{L}(\beta)$ is an unknot, a split link, or a composite link.

If the closed braid $\widehat{\beta}$ is not composite (split), then the admissibility of an exchange move for $\widehat{\beta}$ follows from the Birman–Menasco theorem (Theorem 6.7). (We recall that if $n > 2$, then a braid admitting destabilization also admits an exchange move.)

We show that if $\widehat{\beta}$ is composite (e.g., split), then $\widehat{\beta}$ admits an exchange move. By the definition of a composite braid, for some $r \in \{2, \dots, n-1\}$ we have:

$$\exists \beta_1 \in \overline{\mathcal{B}}_r \subset \overline{\mathcal{B}}_{n-1},\ \beta_2 \in \overline{\mathcal{B}}_{n-r+1}^{\Delta} \subset \overline{\mathcal{B}}_{n-1}^{\Delta} : \beta_1\beta_2 \in \widehat{\beta}.$$

We observe that $\beta_2^{\Delta} \in \overline{\mathcal{B}}_{n-r+1} \subset \overline{\mathcal{B}}_{n-1}$ and

$$\beta_2 = \beta_2^{\Delta^2} = \Delta \beta_2^{\Delta} \Delta^{-1}.$$

By assertion (2) of Lemma 1.1,

$$\Delta = V \sigma_{n-1} W,$$

where $V, W \in \overline{\mathcal{B}}_{n-1}$. It follows that

$$V^{-1}\beta_1(\beta_2 V) = V^{-1}\beta_1(V\sigma_{n-1}W\beta_2^{\Delta}W^{-1}\sigma_{n-1}^{-1}) \in \overline{\mathcal{B}}_{n-1}\sigma_{n-1}\overline{\mathcal{B}}_{n-1}\sigma_{n-1}^{-1}.$$

Since $V^{-1}\beta_1\beta_2 V \in \widehat{\beta}$, this means that $\widehat{\beta}$ admits an exchange move. $\square$

§7. Minimal braids

7.1. A braid β (as well as the closed braid $\widehat{\beta}$) is *minimal* if the index of β is minimal among the indices of all possible braids representing the link $\mathcal{L}(\beta)$.

7.2. The Markov theorem without stabilization, formulated in [4], asserts, in particular, the following: *For each $n \in \mathbb{N}$, there exists a finite collection $\mathcal{BM}(n)$ of nonincreasing template braid moves with the following property: if $\widehat{\beta}$ is a closed braid of index n, and $\widehat{\alpha}$ is any minimal braid with $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$, then there is a finite sequence of closed braids*

$$\widehat{\beta} = \widehat{\beta}_0 \mapsto \widehat{\beta}_1 \mapsto \widehat{\beta}_2 \mapsto \cdots \mapsto \widehat{\beta}_k = \widehat{\alpha}$$

such that for each $j = 1, \dots, k$ the closed braid $\widehat{\beta}_j$ is obtained from $\widehat{\beta}_{j-1}$ via a move in $\mathcal{BM}(n)$.

The part of the "Markov theorem without stabilization" cited above implies that if a closed braid $\widehat{\beta} \in \mathcal{B}_n$ of index n admits no braid moves in $\mathcal{BM}(n)$, then $\widehat{\beta}$ is a unique minimal closed braid representing the link $\mathcal{L}(\beta)$. Thus, the "Markov theorem without stabilization" and assertion 3) of Theorem 5.1 imply the following result.

7.3. Theorem (on minimal braids). *For each $n \in \mathbb{N}$, there is $r = r(n) \in \mathbb{N}$ such that every braid $\beta \in \mathcal{B}_n \setminus (\Delta^{-2r}, \Delta^{2r})$ is minimal. Furthermore, $\widehat{\beta}$ is a unique closed braid of index n representing $\mathcal{L}(\beta)$.*

There are reasons to believe that the following statement is true.

7.4. Conjecture. Under the assumptions of Theorem 3, we can set $r(n) := n$.

7.5. A nonminimal braid outside the interval $(\Delta^{-2(n-2)}, \Delta^{2(n-2)})$. For $n > 2$, we present an example of a nonminimal braid of index n lying outside the interval $(\Delta^{-2(n-2)}, \Delta^{2(n-2)})$. Thus, we show that $r(n) \geq n - 1$. Put

$$\delta := \sigma_1\sigma_2 \dots \sigma_{n-1} \in \mathcal{B}_n; \quad \delta^\Delta = \sigma_{n-1} \dots \sigma_2\sigma_1 \in \mathcal{B}_n.$$

It is known that

$$\delta^n = (\delta^\Delta)^n = (\delta\delta^\Delta)(\sigma_2\sigma_3 \dots \sigma_{n-1})^{n-1} = \Delta^2.$$

For $n, m \in \mathbb{N} \setminus \{1\}$, we define

$$\beta(n,m) := (\delta\delta^\Delta)^{m-1}\delta \in \mathcal{B}_n.$$

7.6. Assertion. 1) *The braids $\beta(n,m)$ and $\beta(m,n)$ represent one and the same link*:

$$\mathcal{L}(\beta(n,m)) = \mathcal{L}(\beta(m,n)).$$

Thus, for $n > m$ the braid $\beta(n,m)$ is not minimal.

2) $\Delta^{2(m-1)} \prec \beta(n,m)$.

Proof. 1) It suffices to look at Figure 2, which depicts a link corresponding to the braid $\widehat{\beta}(n,m)$ with respect to the axis A_1 and to the braid $\widehat{\beta}(m,n)$ with respect to the axis A_2 (we have taken $(n,m) = (5,4)$ in that figure).

2) Indeed,

$$\beta(n,m)\Delta^{-2(m-1)} = (\delta\delta^\Delta\Delta^{-2})^{m-1}\delta = (\sigma_2\sigma_3 \dots \sigma_{n-1})^{-(n-1)(m-1)}\delta \succ e.$$

□

The required example is given by the braid $\beta(n, n-1)$. Indeed, by Assertion 7.6, the braid $\beta(n, n-1)$ is not minimal and $\Delta^{2(n-2)} \prec \beta(n, n-1)$.

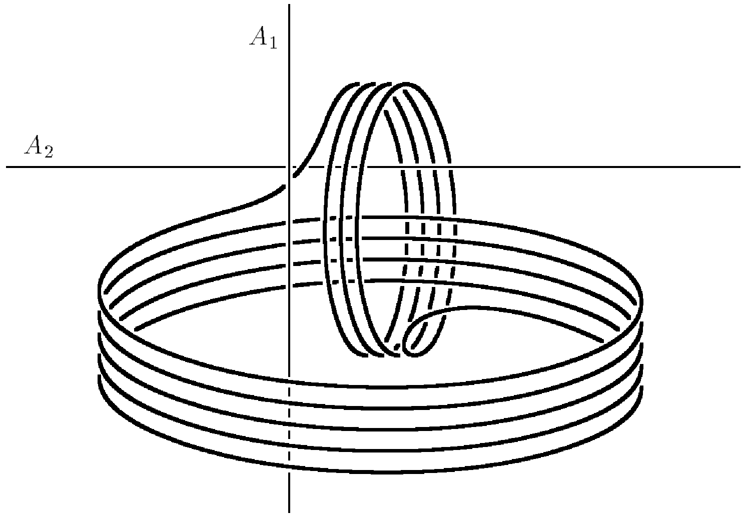

FIGURE 2.

7.7. Remark. For $n > 4$, we have $\beta(n, n-1) \succ \Delta^6$, and, by, Theorem 5.1, the closed braid $\beta(n, n-1)$ does not admit destabilization, an exchange move, and a flype. Consequently, for $n > 4$ these three braid moves do not suffice for reducing an arbitrary closed braid of index n to a minimal one. This partially answers the question on the sufficiency of braid moves that was posed in [4].

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