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ON HOMOGENIZATION PROCEDURE FOR PERIODIC OPERATORS NEAR THE EDGE OF AN INTERNAL GAP

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§0. Introduction

1. The homogenization problem for periodic operators of mathematical physics is of significant interest for applications. At the same time, it is also of interest from the general-theoretical point of view. In this context, a typical and popular model is the family of operators

$$\mathcal{A}_\varepsilon = -\operatorname{div} g(x/\varepsilon) \operatorname{grad}, \quad x \in \mathbb{R}^d, \ \varepsilon > 0, \tag{0.1}$$

where $g(x)$ is a positive definite matrix-valued function periodic with respect to some lattice in $\mathbb{R}^d$. Usually, one deals with the equation

$$\mathcal{A}_\varepsilon u_\varepsilon = f, \quad x \in \Omega,$$

with some boundary conditions on $\partial\Omega$. Here $\Omega \subset \mathbb{R}^d$ is a fixed bounded domain. Another equation under consideration is

$$(\mathcal{A}_\varepsilon + \varkappa^2) u_\varepsilon = f, \quad x \in \mathbb{R}^d, \ \varkappa > 0. \tag{0.2}$$

Here we shall discuss only equation (0.2), treating it as an equation in $L_2(\mathbb{R}^d)$. Let A_ε be the operator in $L_2(\mathbb{R}^d)$ that corresponds to (0.1), and let $A = A_1$. The initial homogenization problem is to find a *constant matrix* g^0 such that the solution u^0 of the problem

$$(\mathcal{A}^0 + \varkappa^2) u^0 = f, \quad \mathcal{A}^0 = -\operatorname{div} g^0 \operatorname{grad}, \tag{0.3}$$

is the limit as $\varepsilon \to 0$ of the solutions u_ε of (0.2):

$$\lim_{\varepsilon \to 0} u_\varepsilon = u^0. \tag{0.4}$$

An extensive body of literature is devoted to homogenization problems. In the first place, the books [1]–[3] should be mentioned. Homogenization has different aspects (e.g., constructing the power series expansion in ε for $u_\varepsilon - u^0$, or estimating the error in (0.4)). The class of periodic problems of mathematical physics that are studied in the homogenization theory is quite extensive.

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2. In the field under discussion, application of the Floquet theory turns out to be useful. In particular, we mention the papers [4]–[6]. In [6], for a wide class of problems, it was shown that the possibility of homogenization is a *threshold effect* near the lower edge of the spectrum of the operator in question. From the abstract point of view, this corresponds to the study of the behavior of the resolvent $(A+\varepsilon^2\varkappa^2 I)^{-1}$ as $\varepsilon \to 0$. In [6], such a study was extended widely in abstract terms. When applied to the problem (0.2), this leads to the estimate

$$\|u_\varepsilon - u^0\|_{L_2(\mathbb{R}^d)} \le C\varepsilon \|f\|_{L_2(\mathbb{R}^d)} \tag{0.5}$$

with a constant C controlled explicitly.

3. The clear understanding of the threshold character of the homogenization problem gives rise to the following natural question. The spectrum of a periodic operator A has a band structure and may have gaps. Does it make sense to relate analogs of homogenization problems to the edges of internal gaps? Below, we discuss this question for the family (0.1) in the case where $d=1$. The possibility of broad generalizations is obvious. However, for simplicity, in the present exposition we lean entirely on the specific features of the one-dimensional case. The technique we apply is not based on the paper [6]. Rather, it is related to the methods used in [7] for treating another problem. In §1, we collect some well-known facts about expansion in eigenfunctions for the operator A in the one-dimensional case.

4. Let ν be an edge of an internal gap in the spectrum of A; for definiteness, we assume that ν is the right edge of the "periodic" gap. For A_ε, this edge moves to the point $\varepsilon^{-2}\nu$, i.e., to the high energy (high frequency) area. Accordingly, instead of (0.2), we need to consider the equation

$$(\mathcal{A}_\varepsilon - \varepsilon^{-2}\nu + \varkappa^2)u_\varepsilon = f. \tag{0.6}$$

As a result, the problem is reduced to the study of the resolvent $(A-(\nu-\varepsilon^2\varkappa^2)I)^{-1}$ for small ε. This resolvent is examined in §2. In §3, we discuss applications to analogs of homogenization problems. With each edge ν, we associate its own homogenized operator A^0_ν. However, in place of (0.5), now we obtain the estimate

$$\|u_\varepsilon - u^0_\varepsilon\|_{L_2(\mathbb{R}^d)} \le C\varepsilon \|f\|_{L_2(\mathbb{R}^d)}. \tag{0.7}$$

Here $u^0_\varepsilon = [\varphi_0^{(\varepsilon)}](A^0_\nu + \varkappa^2 I)^{-1}[\varphi_0^{(\varepsilon)}]f$, and $[\varphi_0^{(\varepsilon)}]$ is the operator of multiplication by the function $\varphi_0(x/\varepsilon)$, where φ_0 is a real-valued periodic solution of the equation $(\mathcal{A}_1-\nu)\varphi_0 = 0$. We can try to get rid of the factors $\varphi_0^{(\varepsilon)}$, passing to the *weak limit* for u_ε. However, it turns out that this limit is equal to zero. Therefore, admittedly, the replacement of u^0 in (0.5) by u^0_ε in (0.7) is in the heart of the matter. If $\nu=0$, then the "shift" $\varepsilon^{-2}\nu$ in (0.6) vanishes, and $\varphi_0 = 1$. Hence, (0.7) turns into (0.5).

The presence of the shift $\varepsilon^{-2}\nu$ in (0.6) shows that for $\nu \neq 0$ the threshold effect interacts with the high frequency effects. In this connection, we mention that nonstationary periodic problems in the high frequency limit were considered in [1]. However, the differences (compared with the present paper) in the statement of the problem, as well as in the technique and in the character of results are very large.

5. In the sequel, H^1 stands for the Sobolev space, and $\widetilde{H}^1(0,1)$ is the subspace of all functions in $H^1(0,1)$ the periodic extension of which belongs to $H^1_{\mathrm{loc}}(\mathbb{R})$. The symbol $\int$ denotes the integral over $\mathbb{R}$. We use the notation $D=-id/dx$.

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§1. Spectral properties of a periodic operator

1. In $L_2(\mathbb{R})$, we consider a selfadjoint operator A generated by a periodic differential expression

$$\mathcal{A} = Dg(x)D, \tag{1.1}$$

where g is a measurable function such that

$$0 < g_0 \le g(x) \le g_1 < \infty, \quad g(x+1) = g(x), \ \ x \in \mathbb{R}.$$

The precise definition of the operator A is given via the quadratic form

$$a[u,u] = \int g(x)|Du|^2\,dx, \quad u \in H^1(\mathbb{R}).$$

Along with A, we consider a family of selfadjoint operators A_ξ in $L_2(0,1)$. Here $\xi \in \mathbb{R}$ is a quasimomentum, and the operator A_ξ is determined by the quadratic form

$$a_\xi[u,u] = \int_{(0,1)} g(x)|Du|^2\,dx, \quad u \in H^1(0,1), \ \ u(1) = e^{i\xi}u(0). \tag{1.2}$$

Thus, A_ξ corresponds to the differential expression (1.1), and the boundary condition in (1.2) is equivalent to the fact that the function $\exp(-i\xi x)u(x)$ belongs to $\widetilde{H}^1(0,1)$. The spectrum of the operator A_ξ is discrete. Let $\lambda_n(\xi)$ and $\psi_n(\xi,x)$ be the consecutive eigenvalues and normalized eigenfunctions of A_ξ. The functions λ_n are (2π)-periodic. The eigenfunctions ψ_n can also be chosen to be (2π)-periodic in ξ. We put $\alpha_n = \lambda_n(0)$, $\beta_n = \lambda_n(\pi)$. Then

$$0 = \alpha_1 < \beta_1 \le \beta_2 < \alpha_2 \le \alpha_3 < \beta_3 \le \beta_4 < \cdots.$$

The spectrum of A is absolutely continuous and consists of the intervals (bands) $[\alpha_1, \beta_1]$, $[\beta_2, \alpha_2]$, $[\alpha_3, \beta_3]$, $\dots$. The intervals

$$(\beta_1, \beta_2), \ (\alpha_2, \alpha_3), \ (\beta_3, \beta_4), \ \dots$$

(if nonempty) represent the gaps in the spectrum. All eigenvalues $\lambda_n(\xi)$ are simple provided $\xi \ne 0, \pi \pmod{2\pi}$.

2. Let Λ be a fixed gap, and let ν be its edge. For definiteness, we assume that $\nu = \alpha_{2s+1}$ for some $s \in \mathbb{N}$. This means that ν is the *right* edge of the "*periodic*" gap. The arguments in the other three possible cases are quite similar, and for them (see Subsection 3.3) we shall give only final conclusions. We assume that $|\xi| \le \pi$ and denote $\lambda_{2s+1}(\xi) = \lambda(\xi)$, $\psi_{2s+1}(\xi,x) = \psi(\xi,x)$. Note that $\psi(\xi,x) = \exp(ix\xi)\varphi(\xi,x)$, where the function φ is periodic in x: $\varphi(\xi,x+1) = \varphi(\xi,x)$. The mapping $\xi \mapsto \lambda(\xi)$ covers (twice) the band M for which ν is the left edge. We mention the following facts. 1) The function λ is continuous and even for $|\xi| \le \pi$. It is real analytic for $|\xi| < \pi$. 2) We can choose the function $\psi(\xi,x)$ to be measurable in the pair of variables (ξ,x) and to be real analytic $H^1(0,1)$-valued (and then $C[0,1]$-valued) for $|\xi| < \pi$. The same can be said about the function $\varphi(\xi,x)$. 3) The function $\lambda(\xi)$ has a nondegenerate minimum point $\xi = 0$; for $0 \le \xi \le \pi$ this function is strictly monotone. Thus,

$$\lambda(\xi) - \nu = b\xi^2 + \xi^4\gamma(\xi), \quad |\xi| \le \pi, \ \ b = b_\nu > 0, \tag{1.3}$$

where $\gamma(\xi)$ is continuous, and for $|\xi| < \pi$ this function is real analytic. 4) We can choose the periodic function $\varphi_0(x) = \varphi(0,x) = \psi(0,x)$ to be real-valued. Then

$$\varphi(\xi,x) - \varphi_0(x) = \xi\theta(\xi,x), \quad |\xi| < \pi, \tag{1.4}$$

where θ is an $\widetilde{H}^1(0,1)$-valued real analytic function in ξ.

3. Let $F(A,\delta)$ denote the spectral projection for the operator A and the interval δ. We put

$$M_\sigma = [\nu, \lambda(\sigma)], \quad F_\sigma = F(A, M_\sigma), \quad 0 < \sigma \le \pi;$$

observe that $M = M_\pi$. We write the integral representation for the orthogonal projection F_σ. Let $X_\sigma : L_2(\mathbb{R}) \to L_2(-\sigma,\sigma)$ be the integral operator with the kernel

$$(2\pi)^{-1/2}\chi_\sigma(\xi)\overline{\psi(\xi,x)} = (2\pi)^{-1/2}\chi_\sigma(\xi)\exp(-i\xi x)\overline{\varphi(\xi,x)}; \tag{1.5}$$

here χ_σ is the indicator of the interval $(-\sigma,\sigma)$, and x is the integration variable. Then $\|X_\sigma\| = 1$ and

$$F_\sigma = X_\sigma^* X_\sigma, \quad 0 < \sigma \le \pi. \tag{1.6}$$

Now, let $\nu - \varkappa^2 \in \Lambda$, and let $\varepsilon \in (0,1]$ be a parameter. Then the point $\nu - \varepsilon^2\varkappa^2$ is regular for A. In accordance with (1.6), we have

$$(A - (\nu - \varepsilon^2\varkappa^2)I)^{-1}F_\sigma = X_\sigma^*[(\lambda(\xi) - \nu + \varepsilon^2\varkappa^2)^{-1}]X_\sigma. \tag{1.7}$$

Here and in what follows, we denote by $[f]$ the operator of multiplication by the function f.

§2. Approximation of the operator $(A - (\nu - \varepsilon^2\varkappa^2)I)^{-1}$

1. A homogenization problem is reduced (see §3) to the construction of an appropriate approximation for the operator

$$S_\nu(\varepsilon) = (A - (\nu - \varepsilon^2\varkappa^2)I)^{-1}. \tag{2.1}$$

A crucial point is to study the operator (1.7), i.e., the operator $S_\nu(\varepsilon)F_\sigma$ for small $\varepsilon > 0$. Let X_σ^0 be the integral operator with the kernel

$$(2\pi)^{-1/2}\chi_\sigma(\xi)\exp(-ix\xi)\varphi_0(x), \tag{2.2}$$

which differs from the kernel (1.5) by the replacement of the function $\varphi(\xi,x)$ with the function $\varphi_0(x)$. Our nearest goal is to estimate the difference

$$G_{\sigma,\varepsilon} = X_\sigma^*[(\lambda(\xi) - \nu + \varepsilon^2\varkappa^2)^{-1}]X_\sigma - (X_\sigma^0)^*[(b\xi^2 + \varepsilon^2\varkappa^2)^{-1}]X_\sigma^0 \tag{2.3}$$

in terms of ε. In what follows, by C and c (possibly, with indices) we denote various constants independent of ε. By (1.3), $\lambda(\xi) - \nu + \varepsilon^2\varkappa^2 = b\xi^2 + \varepsilon^2\varkappa^2 + \xi^4\gamma(\xi)$. We fix $\sigma < \pi$ so that $2\xi^4|\gamma(\xi)| \le b\xi^2$ for $|\xi| \le \sigma$. Then, clearly,

$$|(\lambda(\xi) - \nu + \varepsilon^2\varkappa^2)^{-1} - (b\xi^2 + \varepsilon^2\varkappa^2)^{-1}| \le c_1.$$

It follows that for the norm of the operator

$$G_{\sigma,\varepsilon}^{(1)} = X_\sigma^*[(\lambda(\xi) - \nu + \varepsilon^2\varkappa^2)^{-1} - (b\xi^2 + \varepsilon^2\varkappa^2)^{-1}]X_\sigma, \tag{2.4}$$

which acts in $L_2(\mathbb{R})$, we have

$$\|G_{\sigma,\varepsilon}^{(1)}\| \le c_1. \tag{2.5}$$

2. Now we consider the operator

$$X_\sigma^*[(b\xi^2+\varepsilon^2\varkappa^2)^{-1}]X_\sigma = W_{\sigma,\varepsilon}^* W_{\sigma,\varepsilon}, \tag{2.6}$$

where the operator $W_{\sigma,\varepsilon} : L_2(\mathbb{R}) \to L_2(-\sigma,\sigma)$ is defined by the relation

$$W_{\sigma,\varepsilon} = [(b\xi^2+\varepsilon^2\varkappa^2)^{-1/2}\chi_\sigma(\xi)]X_\sigma. \tag{2.7}$$

Next, we put

$$W_{\sigma,\varepsilon}^0 = [(b\xi^2+\varepsilon^2\varkappa^2)^{-1/2}\chi_\sigma(\xi)]X_\sigma^0 \tag{2.8}$$

and estimate the difference

$$W_{\sigma,\varepsilon} - W_{\sigma,\varepsilon}^0 = [(b\xi^2+\varepsilon^2\varkappa^2)^{-1/2}\xi\chi_\sigma(\xi)]U_\sigma.$$

Here U_σ is the operator with the kernel (see (1.4))

$$(2\pi)^{-1/2}\chi_\sigma(\xi)\exp(-ix\xi)\overline{\theta(\xi,x)}.$$

First, we check that the operator U_σ is bounded. Obviously, the operator with the kernel $(2\pi)^{-1/2}\chi_\sigma(\xi)\exp(-ix\xi)$ is bounded. It remains to use the fact that the function $\overline{\theta(\xi,x)}$ is a *multiplier* on the set of kernels of the bounded integral operators that take $L_2(\mathbb{R})$ to $L_2(-\sigma,\sigma)$. The function $\overline{\theta(\xi,x)}$ is analytic in $\xi \in [-\sigma,\sigma]$ and continuous and periodic in x. Moreover,

$$\max_{x\in\mathbb{R}} \|\theta(\cdot,x)\|_{C^1[-\sigma,\sigma]} < \infty.$$

Therefore, $\overline{\theta(\xi,x)}$ is a multiplier. For the details about multipliers for integral kernels see, e.g., [8, §§8, 9].

Now we observe that $|\xi|(b\xi^2+\varepsilon^2\varkappa^2)^{-1/2} \le b^{-1/2}$, whence

$$\|W_{\sigma,\varepsilon} - W_{\sigma,\varepsilon}^0\| \le c_2. \tag{2.9}$$

3. We write down the identity

$$\begin{aligned} G_{\sigma,\varepsilon} &= G_{\sigma,\varepsilon}^{(1)} + W_{\sigma,\varepsilon}^* W_{\sigma,\varepsilon} - (W_{\sigma,\varepsilon}^0)^* W_{\sigma,\varepsilon}^0 \\ &= G_{\sigma,\varepsilon}^{(1)} + W_{\sigma,\varepsilon}^*(W_{\sigma,\varepsilon} - W_{\sigma,\varepsilon}^0) + (W_{\sigma,\varepsilon} - W_{\sigma,\varepsilon}^0)^* W_{\sigma,\varepsilon}^0 \end{aligned} \tag{2.10}$$

(cf. (2.2), (2.3), and also (2.4), (2.6), (2.8)). Since $\varepsilon(b\xi^2+\varepsilon^2\varkappa^2)^{-1/2} \le \varkappa^{-1}$, from (2.7) and (2.8) it follows that

$$\varepsilon\|W_{\sigma,\varepsilon}\| \le \varkappa^{-1}, \quad \varepsilon\|W_{\sigma,\varepsilon}^0\| \le c_3. \tag{2.11}$$

Now, using (2.10) and also estimates (2.5), (2.9), and (2.11), we obtain the following statement.

Proposition 2.1. *For the operator* (2.3) *we have*

$$\varepsilon\|G_{\sigma,\varepsilon}\| \le C_0, \quad 0 < \varepsilon \le 1. \tag{2.12}$$

4. Now it is easy to obtain a convenient approximation of the full resolvent (2.1). Obviously,

$$\|S_\nu(\varepsilon)(I - F_\sigma)\| \le c_4, \quad 0 < \varepsilon \le 1. \tag{2.13}$$

Next, we consider the selfadjoint operator A_ν^0 in $L_2(\mathbb{R})$ generated by the differential expression $\mathcal{A}_\nu^0 = b_\nu D^2$. By (2.2), we have

(2.14)

$$\begin{aligned} (X_\sigma^0)^*[(b_\nu\xi^2+\varepsilon^2\varkappa^2)^{-1}]X_\sigma^0 &= [\varphi_0]\Phi^*[(b_\nu\xi^2+\varepsilon^2\varkappa^2)^{-1}\chi_\sigma(\xi)]\Phi[\varphi_0] \\ &= [\varphi_0]\Phi^*[(b_\nu\xi^2+\varepsilon^2\varkappa^2)^{-1}]\Phi[\varphi_0] - [\varphi_0]\Phi^*[(b_\nu\xi^2+\varepsilon^2\varkappa^2)^{-1}(1-\chi_\sigma(\xi))]\Phi[\varphi_0]. \end{aligned}$$

Here Φ is the Fourier operator in $L_2(\mathbb{R})$. Clearly, the first term on the right-hand side of (2.14) coincides with the operator $[\varphi_0](A_\nu^0 + \varepsilon^2\varkappa^2 I)^{-1}[\varphi_0]$, and the norm of the second term is estimated by a constant independent of $\varepsilon \in (0,1]$. Combining this with (2.3), (2.12), and (2.13), we arrive at the following statement.

Proposition 2.2. *We have*

$$\varepsilon\|S_\nu(\varepsilon) - [\varphi_0](A_\nu^0 + \varepsilon^2\varkappa^2 I)^{-1}[\varphi_0]\| \le C, \quad 0 < \varepsilon \le 1. \tag{2.15}$$

§3. Homogenization near the edge of an internal gap

1. In $L_2(\mathbb{R})$, we consider the selfadjoint operator $A(\varepsilon)$ generated by the differential expression

$$\mathcal{A}(\varepsilon) = Dg(x/\varepsilon)D, \quad 0 < \varepsilon \le 1.$$

Clearly, $A(1) = A$ corresponds to (1.1). Obviously, the operator $A(\varepsilon)$ has a gap $\Lambda(\varepsilon) = \varepsilon^{-2}\Lambda$ the right edge of which is the point $\varepsilon^{-2}\nu$. A homogenization for $A(\varepsilon)$ near the edge of the internal gap requires that we take account of the movement of that edge to the point $\varepsilon^{-2}\nu$. Accordingly, we must study approximation for the operator

$$R_\nu(\varepsilon) := (A(\varepsilon) - (\varepsilon^{-2}\nu - \varkappa^2)I)^{-1}, \quad 0 < \varepsilon \le 1. \tag{3.1}$$

The scale transformation reduces this question to estimate (2.15). Indeed, let T_ε, $\varepsilon > 0$, be the family of unitary operators in $L_2(\mathbb{R})$ defined by the rule $T_\varepsilon : u(x) \mapsto \varepsilon^{1/2}u(\varepsilon x)$. Next, we put $\varphi_0^{(\varepsilon)}(x) = \varphi_0(x/\varepsilon)$ and

$$R_\nu^0(\varepsilon) := [\varphi_0^{(\varepsilon)}](A_\nu^0 + \varkappa^2 I)^{-1}[\varphi_0^{(\varepsilon)}]. \tag{3.2}$$

Then, obviously, for the operators (2.1) and (3.1) we have

$$R_\nu(\varepsilon) = \varepsilon^2 T_\varepsilon^* S_\nu(\varepsilon) T_\varepsilon \tag{3.3}$$

and, similarly,

$$R_\nu^0(\varepsilon) = \varepsilon^2 T_\varepsilon^*[\varphi_0](A_\nu^0 + \varepsilon^2\varkappa^2 I)^{-1}[\varphi_0]T_\varepsilon. \tag{3.4}$$

Now, (2.15) directly implies our main result.

Proposition 3.1. *For the difference of the operators* (3.1) *and* (3.2) *we have*

$$\|R_\nu(\varepsilon) - R_\nu^0(\varepsilon)\| \le C\varepsilon, \quad 0 < \varepsilon \le 1. \tag{3.5}$$

2. Estimate (3.5) allows us to talk about *homogenization* for the operator $A(\varepsilon)$. Indeed, the resolvent of the operator $A_\nu^0 = b_\nu D^2$ is involved in (3.2). This operator is independent of ε and has constant coefficient b_ν. However, we have not succeeded in avoiding the dependence of $R_\nu^0(\varepsilon)$ on ε, because of the factors $[\varphi_0^{(\varepsilon)}]$. Obviously, if $\nu = 0$, then $\varphi_0 = 1$, which corresponds to averaging in the usual sense: for $\nu = 0$ the operator (3.2) reduces to $(A^0 + \varkappa^2 I)^{-1}$ with $A^0 = b_0 D^2$, and in (3.1) the shift by $\varepsilon^{-2}\nu$ disappears.

The presence of the factors $[\varphi_0^{(\varepsilon)}]$ in (3.2) for $\nu \ne 0$ lies in the essence of the problem. It seems that dependence on ε can be avoided by passage to the weak limit. Indeed, let $f \in L_\infty(\mathbb{R})$, $f(x) = f(x+h)$, and let $f^{(\varepsilon)}(x) = f(x/\varepsilon)$. It is well known that the $f^{(\varepsilon)}$ converge weakly in $L_{2,\mathrm{loc}}(\mathbb{R})$ to the mean value $\langle f\rangle$ over the period. This easily implies the existence of the weak $L_2(\mathbb{R})$-limit

$$\operatorname*{w-lim}_{\varepsilon\to 0}[f^{(\varepsilon)}](A_\nu^0 + \varkappa^2 I)^{-1}[f^{(\varepsilon)}] = \langle f\rangle^2(A_\nu^0 + \varkappa^2 I)^{-1}.$$

Combined with (3.5), this yields the relation

$$\operatorname*{w-lim}_{\varepsilon\to 0} R_\nu(\varepsilon) = \langle\varphi_0\rangle^2(A_\nu^0 + \varkappa^2 I)^{-1}.$$

However, for $\nu \neq 0$ we have $\langle \varphi_0 \rangle = 0$, because the eigenfunctions of the periodic problem are orthogonal to constants. Thus,

$$\operatorname*{w-lim}_{\varepsilon \to 0} R_\nu(\varepsilon) = 0, \tag{3.6}$$

which is much less informative than (3.5).

3. We briefly discuss the case where $\nu \neq \alpha_{2s+1}$. First, let $\nu = \beta_{2s}$, i.e., ν is the right edge of the "antiperiodic" gap. Then estimate (3.5) survives if in (3.2) we replace φ_0 by ψ_0, where $\psi_0(x) = \psi(\pi, x)$ is the normalized real-valued eigenfunction of the antiperiodic problem. Relation (3.6) also remains true. Indeed, ψ_0 is an eigenfunction of the periodic problem on the interval $[0, 2]$, and its mean value over the period is equal to zero.

Now, let $\nu = \alpha_{2s}$, i.e., ν is the left edge of a "periodic" gap. Then $\lambda_{2s}(\xi)$ has a nondegenerate maximum point $\xi = 0$: $\lambda_{2s}(\xi) = \nu - b_\nu \xi^2 + O(\xi^4)$, $b_\nu > 0$. The operator (3.2) keeps its form, but in (3.5) the operator $R_\nu(\varepsilon)$ should be replaced by

$$\widetilde{R}_\nu(\varepsilon) := (A(\varepsilon) - (\varepsilon^{-2}\nu + \varkappa^2)I)^{-1},$$

and the difference should be replaced by the sum:

$$\|\widetilde{R}_\nu(\varepsilon) + R^0_\nu(\varepsilon)\| \leq C\varepsilon, \quad 0 < \varepsilon \leq 1. \tag{3.7}$$

Finally, in the case where $\nu = \beta_{2s+1}$, estimate (3.7) remains true if in (3.2) we replace φ_0 by ψ_0. For the left edge of the gap, we have

$$\operatorname*{w-lim}_{\varepsilon \to 0} \widetilde{R}_\nu(\varepsilon) = 0.$$

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