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OVERGROUPS OF ELEMENTARY SYMPLECTIC GROUPS

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ABSTRACT. Let R be a commutative ring, and let $l \geq 2$; for $l = 2$ it is assumed additionally that R has no residue fields of two elements. The subgroups of the general linear group $\mathrm{GL}(n, R)$ that contain the elementary symplectic group $\mathrm{Ep}(2l, R)$ are described. In the case where $R = K$ is a field, similar results were obtained earlier by Dye, King, and Shang Zhi Li.

In the present paper we consider a description of the subgroups in the general linear group $G = \mathrm{GL}(2l, R)$ over a commutative ring R that contain the elementary symplectic group $\mathrm{Ep}(2l, R)$. It turns out that for every such group H there exists a unique ideal A in R such that H lies between the group

$$\mathrm{EEp}(2l, R, A) = \mathrm{Ep}(2l, R)E(2l, R, A)$$

and its normalizer in $\mathrm{GL}(2l, R)$. More specifically, our main objective in the present paper is a proof of the following result.

Theorem 1. *Let R be a commutative ring. Suppose that either $l \geq 3$, or $l = 2$ and the ring R has no residue fields of two elements. Then for every subgroup H in $G = \mathrm{GL}(2l, R)$ that contains the elementary symplectic group $\mathrm{Ep}(2l, R)$, there exists a unique ideal $A \trianglelefteq R$ such that*

$$\mathrm{EEp}(2l, R, A) \leq H \leq N_G(\mathrm{EEp}(2l, R, A)).$$

An important supplement to Theorem 1 is the following result, in which we explicitly calculate the normalizer of $\mathrm{EEp}(2l, R, A)$. Namely, consider the reduction homomorphism $\rho_A : \mathrm{GL}(2l, R) \longrightarrow \mathrm{GL}(2l, R/A)$ and denote by $\mathrm{CGSp}(2l, R, A)$ the complete preimage of the group $\mathrm{GSp}(2l, R)$ with respect to ρ_A. Then the condition for a matrix to belong to $\mathrm{CGSp}(2l, R, A)$ is described by obvious quadratic congruences on its entries. Now we are in a position to state the second major result of the present paper.

Theorem 2. *Under the assumptions of Theorem 1, for any ideal $A \trianglelefteq R$ we have*

$$N_G(\mathrm{EEp}(2l, R, A)) = \mathrm{CGSp}(2l, R, A).$$

Thus, combining Theorems 1 and 2, we see that for any subgroup H with $\mathrm{Ep}(2l, R) \leq H \leq \mathrm{GL}(2l, R)$ there exists a unique ideal such that

$$\mathrm{EEp}(2l, R, A) \leq H \leq \mathrm{CGSp}(2l, R, A).$$

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In the case where $R = K$ is a field, there are only two possibilities for A, namely, $A = 0$ or $A = K$. We have

$$\mathrm{EEp}(2, K, 0) = \mathrm{Ep}(2l, K) = \mathrm{Sp}(2l, K), \quad \mathrm{CGSp}(2l, K, 0) = \mathrm{GSp}(2l, K).$$

On the other hand,

$$\mathrm{EEp}(2, K, K) = E(2l, K) = \mathrm{SL}(2l, K), \quad \mathrm{CGSp}(2l, K, K) = \mathrm{GL}(2l, K).$$

In this special case, Theorem 1 asserts that any overgroup of $\mathrm{Sp}(2l, K)$ in $\mathrm{GL}(2l, K)$ either is contained in $\mathrm{GSp}(2l, K)$, or contains $\mathrm{SL}(2l, K)$, so that this theorem boils down to a theorem of R. Dye [48]. Subsequently, R. Dye, O. King, and Shang Zhi Li generalized this result to overgroups of other classical groups, not necessarily split, but isotropic enough (see [46, 47, 54, 55], and [63]–[65]). In most cases, the normalizers of classical groups are maximal in $\mathrm{GL}(n, K)$ (or, in characteristic 2, in another classical group). From the viewpoint of the *Maximal Subgroup Classification Project*, they form the Aschbacher class $\mathcal{C}_8$; for an algebraically closed or a finite field they were studied in the context of classification of the maximal subgroups (see, e.g., [33, 58]).

On the other hand, isotropic classical groups in vector representations contain unipotent elements of small residue, such as transvections or products of two commuting transvections, so that in the field case a description of their overgroups follows also from MacLaughlin type theorems (description of the irreducible subgroups generated by small-dimensional elements). It seems that the most powerful specific results in this direction were obtained by E. L. Bashkirov in [3]–[7]; in fact, he described the subgroups of $\mathrm{GL}(n, K)$ that contain a classical group not over the field K itself, but over its subfield $L \leq K$ such that the extension K/L is algebraic. These results must follow also from Timmesfeld's general theorems on quadratic action subgroups [76].

The possibility to generalize these results to arbitrary commutative rings by using *decomposition of unipotents* was mentioned in [84, 73], but detailed proofs have never been published. In fact, it was even claimed there that by the same method one could describe the subgroups of $\mathrm{GL}(n, R)$ *normalized* by the elementary classical group. This would be a broad generalization not only of the above-mentioned results by R. Dye, J. King, and Shang Zhi Li on overgroups of classical groups, but also of the theorems on the normal structure of classical groups, such as the Wilson–Golubchik theorem for GL_n and the theorems of Golubchik, Abe–Suzuki, and Abe–Vaserstein for Sp_{2l} and SO_n (see the references in [84, 73, 39] and further discussion in §15 below).

Quite recently, some very special cases of our results were published in [88] and [66], where it was assumed that the ring R is local or Euclidean. Notice that in both cases the statements are obvious in essence, because for rings of stable rank at most 2 (in particular, for semilocal and Dedekind rings), any column of a matrix in $\mathrm{GL}(2l, R)$, $l \geq 2$, can be completed to an elementary symplectic matrix $x \in \mathrm{Ep}(2l, R)$ (see, e.g., [67, 79] or §§4 and 12 in the present paper). Thus, in these cases there is no need to apply either localization or decomposition of unipotents. As is explained in §12, this means that the results of [88] and [66] immediately follow from the auxiliary statements contained in §10 of the present paper, with $s = 1$.

A similar result for overgroups of the *even* split elementary orthogonal group $\mathrm{EO}(2l, R)$, $2 \in R^*$, $l \geq 3$, was proved by the authors in the preceding paper [16]. In a very general sense, the proof in the present paper follows the same lines as that in [16], but it substantially differs in one of the crucial steps as well as in many technical details. Here we mention the most important differences between the symplectic and the orthogonal cases. On the one hand:

- Decomposition of unipotents, which makes it possible to get unipotent elements in groups of smaller rank, is *overwhelmingly* more complicated in the symplectic case than in the orthogonal case, due to the presence of roots of different lengths; see [26, 9, 84, 73].

- As opposed to the orthogonal case, now the entries of an elementary matrix that stabilizes a column depend on the coordinates of this column quadratically rather than linearly, which makes summing an extremely painful procedure.

On the other hand:

- After we get an element in a maximal parabolic subgroup of type P_1, reduction to groups of smaller rank is considerably more straightforward, due to the fact that now the long root elements have residue 1 (in the orthogonal case we were forced to implement reduction to intersections of two maximal parabolics, which could be placed differently with respect to $\mathrm{EO}(2l, R)$).
- A column of a symplectic matrix is not subject to equations. Thus, for reduction to parabolic subgroups of type P_1, in the symplectic case we can use stability conditions rather than decomposition of unipotents.

The most significant difference between [16] and the present paper is that now for the key step in the proof of the main lemma (namely, for obtaining a nontrivial matrix in $H \cap P_1$) we are compelled to use localization. This obliges us to refer to fairly technical results of the *conjugation calculus* (see [36, 51, 52]). Moreover, this creates a considerable additional stress in all auxiliary statements, since now we are forced to calculate in a localization $\mathrm{GL}(2l, R_s)$, but for extraction of transvections we are still allowed to use only the images of matrices in $\mathrm{Ep}(2l, R)$, rather than all matrices in $\mathrm{Ep}(2l, R_s)$, as before.

In §§12 and 13 we present two further approaches to the proof of the main lemma, which are *easier* than localization, but unfortunately do not suffice to prove our main theorem in full generality, because they are applicable only under more restrictive conditions on l. Namely, in §12 we observe that for $2l > \mathrm{sr}(R)$ our theorem immediately follows from a theorem of Bass on surjective stability for the functor K_1. In §13 we show that, for $l \geq 3$, the version of decomposition of unipotents proposed in [40] can be used for reduction to parabolics of type P_1. We could not find a proof in this spirit that would work for $l = 2$, because in this case we would be forced to use the original version of the symplectic decomposition of unipotents, based on calculations with minors (see [26, 9, 84] and [73, Variations 1 and 7]). Not only is this method technically more demanding, but also it explicitly involves equations on the entries of the matrices in $\mathrm{GSp}(4, R)$.

Observe that, in contrast to [16], here we *do not* require that $2 \in R^*$. Perhaps, it is worth explaining what really goes on here. What we are really interested in is a description of the subgroups in $\mathrm{GL}(n, R)$ *normalized* by the elementary classical group, and we are going to present this result in a subsequent paper. Essentially, a result by Stepanov [27] reduces the description of the subgroups in G normalized by H to the description of the overgroups of H and of the normal subgroups in H. The cases of $\mathrm{EO}(2l, R)$ and $\mathrm{Ep}(2l, R)$ are complementary in this respect. From the viewpoint of Bak's theory, $\mathrm{EO}(2l, R)$ corresponds to the *minimum* value of the form parameter, whereas $\mathrm{Ep}(2l, R)$ corresponds to the *maximum* one; see [34, 35, 41, 50, 39]. This means that in the orthogonal case all complications related to the noninvertibility of 2 appear already at the level of description of its *overgroups*, whereas in the symplectic case they are all concealed in the description of the *normal subgroups* of the group $\mathrm{Sp}(2l, R)$ itself. In fact, it is known that, for $l \geq 3$, a description of the normal subgroups of $\mathrm{SO}(2l, R)$ is standard for an arbitrary commutative ring (this result was obtained by Golubchik [20, 22] in 1975, and by Vaserstein [81] and Fu An Li [62]) in 1988–89. On the other hand, the examples presented in [32] show that, without the invertibility of 2, a description of the normal subgroups in $\mathrm{Sp}(2l, R)$ fails to be standard. Note that for $l \geq 3$ this can be repaired by a relatively mild revision of the notion of standardness (see [30]–[32] and

[82, 61]), but for $l = 2$ the nonstandardness is of more malicious character (see [44]). Thus, a version of the condition $2 \in R^*$ is absolutely unavoidable in a description of the subgroups normalized by an elementary classical group, *both* in the orthogonal and the symplectic cases, but *for different reasons*.

As has already been mentioned, from the viewpoint of Aschbacher's *subgroup structure theorem* we are describing overgroups for the groups of class $\mathcal{C}_8$. Ideologically, the present paper is close to the work by Z. I. Borevich, the first author, A. V. Stepanov, and I. Z. Golubchik on overgroups of subsystem subgroups (the Aschbacher classes $\mathcal{C}_1 + \mathcal{C}_2$) in the classical groups over a commutative ring (see [8]–[14], [21, 26], and the references in [85, 73, 86]). Until quite recently, little was known about description of overgroups in the case of rings for other classes. Now the situation is changing. Recently, Stepanov [72] obtained striking results on subring subgroups (class $\mathcal{C}_5$). It turned out that, *as a rule*, such a description is not standard (roughly speaking, it is standard only for algebraic extensions and for rings of dimension 1). As another recent result, we mention the classification of the overgroups of tensored subgroups (classes $\mathcal{C}_4 + \mathcal{C}_7$). This classification was obtained by the first author and V. G. Khalin under the following assumptions: the ring R is commutative and the degrees of all factors are at least 3. Observe that for rings it does not make sense to consider the class $\mathcal{C}_6$, whereas a description of the overgroups for groups of class $\mathcal{C}_3$ is blocked by serious technical obstacles; it seems that such a description is possible only in some very special cases.

The present paper is organized as follows. The first three sections are of introductory nature. In §§1 and 2 we recall the main notation and the necessary facts about the groups $\mathrm{GL}(n, R)$ and $\mathrm{GSp}(2l, R)$, and in §3 we discuss the necessary results concerning the behavior of the functors of points of affine schemes and their elementary subfunctors under localization. The four sections that follow are devoted to the proof of Theorem 2. In §4 we prove that the highest weight orbits for the groups $E(n, R)$ and $\mathrm{Ep}(2l, R)$ coincide. In §5 we calculate the normalizer of $\mathrm{Ep}(2l, R)$ in $\mathrm{GL}(2l, R)$, and in §6 we start the study of transvections in subgroups containing $\mathrm{Ep}(2l, R)$. Finally, in §7 we characterize the matrices in $\mathrm{CGSp}(2l, R)$ and prove Theorem 2. From a technical viewpoint, §§8–11 form the core of the paper; these sections are directly devoted to the localization proof of Theorem 1. In §8 we recall the localization results we use in the sequel; in §§9 and 10 we extract transvections inside a parabolic subgroup of type P_1. The proof of Theorem 1 is finished in §11, where we show that if H contains $\mathrm{Ep}(2l, R)$ but is not contained in $\mathrm{GSp}(2l, R)$, then already $P_1 \cap H$ is not contained in $\mathrm{GSp}(2l, R)$. After this, in §§12 and 13 we sketch two further approaches to the proof of Theorem 1, based on stability conditions and decomposition of unipotents, respectively. In §14 we outline a broader context for our main results. Finally, in §15 we state some open problems.

§1. Principal notation

Our notation is fairly standard for the most part and coincides with that in [8, 12, 73, 16]. All necessary definitions can be found in the monograph [50] by A. Hahn and O. T. O'Meara. Nevertheless, for the reader's convenience, in the present section we recall the notation and basic facts pertaining to subgroups of $\mathrm{GL}(n, R)$ to be used in the sequel.

First, let G be an arbitrary group. By the commutator of two elements $x, y \in G$ we always understand their left-normed commutator $[x, y] = xyx^{-1}y^{-1}$. By ${}^x y = xyx^{-1}$ and $y^x = x^{-1}yx$ we denote, respectively, the left and right conjugates of y by x. We write $H \leq G$ if H is a subgroup in G, while $H \trianglelefteq G$ means that H is a normal subgroup. For a subset $X \subseteq G$ we denote by $\langle X \rangle$ the subgroup it generates. For $H \leq G$, the symbol $\langle X \rangle^H$ denotes the smallest subgroup in G that contains X and is normalized by

H. For two subgroups $F, H \le G$, we denote by $[F, H]$ their mutual commutator subgroup generated by all commutators $[f, h]$ with $f \in F$, $h \in H$. The multiple commutators are also left-normed; in particular, $[E, F, H] = [[E, F], H]$.

Now, let R be an arbitrary associative ring with 1; except for the present section and §14, R is always assumed to be commutative. Let $M(m, n, R)$ be the R-bimodule of $(m \times n)$-matrices with entries in R, and let $M(n, R) = M(n, n, R)$ be the full matrix ring of degree n over R. Further, let R^* be the multiplicative group of R, and let $G = \mathrm{GL}(n, R) = M(n, R)^*$ be the general linear group of degree n over R. If R is commutative, $\mathrm{SL}(n, R)$ is the special linear group of degree n over R. As usual, a_{ij} denotes the entry of a matrix $a \in G$ at the position (i, j), i.e., $a = (a_{ij})$, $1 \le i, j \le n$. Next, $a^{-1} = (a'_{ij})$ is the inverse of a and a^t is its transpose. By $a_{*j} = (a_{1j}, \dots, a_{nj})^t$ we denote the jth column of the matrix a, and by $a_{i*} = (a_{i1}, \dots, a_{in})$ its ith row.

As usual, e is the identity matrix, and e_{ij} is a standard matrix unit, i.e., the matrix that has 1 at the position (i, j) and zeros elsewhere. For $\xi \in R$ and $1 \le i \ne j \le n$, we denote by $t_{ij}(\xi) = e + \xi e_{ij}$ an *elementary transvection*. The symbol $X_{ij} = \{t_{ij}(\xi), \xi \in R\}$, $i \ne j$, stands for the *root subgroup*. In the sequel, without any special reference we use standard relations among elementary transvections, such as the additivity formula $t_{ij}(\xi)t_{ij}(\zeta) = t_{ij}(\xi + \zeta)$ and the Chevalley commutator formula $[t_{ij}(\xi), t_{jh}(\zeta)] = t_{ih}(\xi\zeta)$; see, e.g., [1, 25, 42].

Now, let $I \trianglelefteq R$ be an ideal in R. We denote by $E(n, I)$ the subgroup in G generated by all elementary transvections of level I:

$$E(n, I) = \langle t_{ij}(\xi), \xi \in I, 1 \le i \ne j \le n \rangle .$$

In the most important case where $I = R$, the group $E(n, R)$ generated by all elementary transvections is called the (absolute) *elementary group*. In the sequel a major role is played by the *relative elementary group* $E(n, R, I)$. Recall that the group $E(n, R, I)$ is the normal closure of $E(n, I)$ in $E(n, R)$:

$$E(n, R, I) = \langle t_{ij}(\xi), \xi \in I, 1 \le i \ne j \le n \rangle^{E(n,R)}.$$

In the proofs, often without any special reference, we shall use a bunch of classical facts on elementary groups. The following fact, first proved in [28], is cited as the *Suslin theorem*.

Lemma 1. *If R is commutative and $n \ge 3$, then the group $E(n, R, I)$ is normal in* $\mathrm{GL}(n, R)$.

The following statement was proved by Vaserstein and Suslin [19] and, in the context of Chevalley groups, by Tits [77].

Lemma 2. *For $n \ge 3$, the relative elementary subgroup $E(n, R, I)$ is generated by all transvections of the form $z_{ij}(\xi, \zeta) = t_{ji}(\zeta)t_{ij}(\xi)t_{ji}(-\zeta)$, $\xi \in I$, $\zeta \in R$, $1 \le i \ne j \le n$.*

As above, let $I \trianglelefteq R$, and let R/I be the factor ring of R modulo I. Denote by $\rho_I : R \longrightarrow R/I$ the canonical projection sending $\lambda \in R$ to $\overline{\lambda} = \lambda + I \in R/I$. Applying this projection to all entries of a matrix, we get the reduction homomorphism $\rho_I : \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n, R/I)$ that sends a matrix $a = (a_{ij})$ to its class $\overline{a} = (\overline{a}_{ij})$ modulo I. The kernel of the homomorphism ρ_I is denoted by $\mathrm{GL}(n, R, I)$ and is called the *principal congruence subgroup* in G of level I.

Now, let $C(n, R)$ be the center of the group $\mathrm{GL}(n, R)$, which consists of the scalar matrices εe, $\varepsilon \in R^*$. The full preimage of the center of $\mathrm{GL}(n, R/I)$ is denoted by $C(n, R, I)$ and is called the *full congruence subgroup* of level I. The group $C(n, R, I)$ consists of all matrices congruent to a scalar matrix modulo I.

A key point in reduction modulo an ideal is the following *standard commutator formula*, proved by Vaserstein [78] and Borevich and Vavilov [8].

Lemma 3. *If R is commutative and $n \geq 3$, then*

$$[E(n,R), C(n,R,I)] = E(n,R,I).$$

Sometimes, instead of the elementary group $E(n,R)$ itself, it is more convenient to consider the *general elementary group* $\mathrm{GE}(n,R) = E(n,R)D(n,R)$, which is the product of the group $E(n,R)$ and the diagonal group $D(n,R)$. For $i \neq j$ and an invertible element $\varepsilon \in R^*$, we denote $d_i(\varepsilon) = e + (\varepsilon - 1)e_{ii}$ and $d_{ij}(\varepsilon) = d_i(\varepsilon)d_j(\varepsilon^{-1})$. From a geometric viewpoint, $d_i(\varepsilon)$ is an elementary *pseudoreflection*. In the theory of algebraic groups, the elements $d_{ij}(\varepsilon)$ are called *semisimple root elements* of the group SL_n and are denoted by $h_\alpha(\varepsilon)$. Since $d_{ij}(\varepsilon) \in E(n,R)$ (the Whitehead lemma), and $d_k(\varepsilon)$ normalizes every root subgroup X_{ij}, we have $\mathrm{GE}(n,R) = E(n,R)Q$, where $Q = \{d_n(\varepsilon), \varepsilon \in R^*\}$.

In the sequel we consider the usual left action of $\mathrm{GL}(n,R)$ on the *right* R-module $V = R^n$ of rank n. This module consists of all columns of height n over the ring R. Following P. Cohn, we denote by ${}^nR$ the *left* R-module V^* of rank n, which can be identified with the set of all rows of length n over R. The standard bases of R^n and ${}^nR$ will be denoted by $e_1, \dots, e_n$ and $e^1, \dots, e^n$, respectively. In other words, $e_i = e_{*i}$ is the ith column of the identity matrix, whereas $e^i = e_{i*}$ is its ith row. In this notation, $a_{*j} = ae_j$, $a_{i*} = e^i a$, and $a_{ij} = e^i a e_j$. These obvious formulas turn useful for cross-cultural communication: where in linear algebra one talks about the rows or columns of a matrix in the group G, the experts in K-theory and algebraic groups usually talk about the orbits of e^1 or e_1 under the action of G ("orbit of the highest weight vector").

A column $v = (v_1, \dots, v_n)^t \in R^n$ is said to be *unimodular* if $Rv_1 + \dots + Rv_n = R$, i.e., the *left* ideal generated by the elements $v_1, \dots, v_n$ coincides with R. Similarly, a row $u = (u_1, \dots, u_n) \in {}^nR$ is *unimodular* if $u_1R + \dots + u_nR = R$, i.e., the *right* ideal generated by the elements $u_1, \dots, u_n$ coincides with R. The unimodularity of rows/columns is a *necessary* condition for this column/row to be a column/row of an invertible matrix. However, except for some very special classes of rings, this condition is not sufficient in general: as we have just verified, actually, a column of an invertible matrix must lie in the orbit e_1.

We denote by P_i the ith standard *maximal parabolic subgroup* in $G = \mathrm{GL}(n,R)$. From a geometric viewpoint, the subgroup P_i, $i = 1, \dots, n-1$, is precisely the stabilizer of the submodule V_i in V generated by $e_1, \dots, e_i$. In the matrix form, P_i is realized as a group of upper block triangular matrices,

$$P_i = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}, x \in \mathrm{GL}(i,R), y \in M(i, n-i, R), z \in \mathrm{GL}(n-i,R) \right\}.$$

From the viewpoint of the dual module V^*, the group P_i is the stabilizer of the submodule $V^{i+1} = V_i^\perp$ and must be denoted by ${}_{i+1}P$. Any subgroup conjugate to P_i is called a *parabolic subgroup of type* P_i. We are most interested in the group P_1 (this is precisely the stabilizer of the submodule $e_1R \leq V$, which consists of all matrices whose *first* column is proportional to e_1) and in the group ${}_nP = P_{n-1}$ (which is the stabilizer of $Re^n \leq V^*$ and consists of all matrices whose *last* row is proportional to e^n). For reduction to smaller rank we need also the submaximal parabolic subgroups ${}_{j+1}P_i = P_{ij} = P_i \cap P_j$, where $1 \leq i < j \leq n-1$, stabilizing the flag $V_i < V_j$.

The group P_i can be decomposed as the semidirect product $P_i = L_i \leftthreetimes U_i$ of its *Levi subgroup* L_i and its *unipotent radical* U_i, where

$$L_i = \left\{ \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}, x \in \mathrm{GL}(i,R), z \in \mathrm{GL}(n-i,R) \right\},$$

$$U_i = \left\{ \begin{pmatrix} e & y \\ 0 & e \end{pmatrix}, y \in M(i,n-i,R) \right\}.$$

Alongside with the subgroup P_i, we consider its *opposite* subgroup P_i^- that stabilizes the submodule in V generated by $e_{i+1},\dots,e_n$ (thus, P_i^- is a subgroup of type P_{n-i} rather than of type P_i). In matrices, P_i^- is realized as

$$P_i^- = \left\{ \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}, x \in \mathrm{GL}(i,R), y \in M(n-i,i,R), z \in \mathrm{GL}(n-i,R) \right\}.$$

In other words, $P_i^- = P_i^t$. The unipotent radical U_i^- of the group P_i^- equals U_i^t.

§2. Symplectic groups

In the present section we briefly recall the principal notation pertaining directly to the symplectic case (see also [50, 24, 2, 23, 73, 15]). The symplectic groups are simply connected Chevalley groups of type C_l and are discussed in this quality in [25], [29]–[32], and [42, 70, 71, 80, 84]). On the other hand, the symplectic groups form a very special example of the Bak hyperbolic unitary groups $U^\lambda(2l,R,\Lambda)$ over a form ring (R,Λ); this example arises in the case where the involution on R is trivial, $\lambda=-1$, and $\Lambda=\Lambda_{\max}$. Thus, to these groups we can apply all results of the papers [34, 35, 41, 18, 50, 39, 40, 51]. In fact, usually, a proof for symplectic groups requires only a small part of the calculations necessary in the general case. Moreover, in [44, 66] symplectic groups were studied in the language of associative algebras with involution. In the following sections, when we are not aware of a convenient reference for a result we need directly for the symplectic case, without any further fuss we specialize relevant results obtained in any of these more general contexts.

In the present paper we always consider the even-dimensional case ($n=2l$). It will be convenient to index the rows and columns of matrices in G as in [42], namely, by $1,\dots,l,-l,\dots,-1$. We denote the sign of the index i by ε_i. Thus, $\varepsilon_i=+1$ if $i=1,\dots,l$, and $\varepsilon_i=-1$ otherwise. The same indexing is applied to all other objects. Thus, the base of the module $V=R^{2l}$ is indexed as $e_1,\dots,e_l,e_{-l},\dots,e_{-1}$, and the base of the module $V^*={}^{2l}R$ as $e^1,\dots,e^l,e^{-l},\dots,e^{-1}$.

Let $\mathrm{sdiag}(\lambda_1,\dots,\lambda_n)$ denote the matrix that has $\lambda_1,\dots,\lambda_n$ on the second diagonal (the one in the North-East to South-West direction) and zeroes elsewhere. We set $F=F_n=\mathrm{sdiag}(1,\dots,1,-1,\dots,-1)$, where the number of 1's as well as the number of -1's equals l. In other words, the entries of the matrix $F=(F_{ij})$ are determined by the condition $F_{ij}=\varepsilon_i\delta_{i,-j}$.

On V, we introduce a symplectic inner product B with Gram matrix $F=F_{2l}$, by setting $B(e_i,e_j)=\varepsilon_i\delta_{i,-j}$. By definition, the *symplectic group* $\Gamma=\mathrm{Sp}(2l,R)$ consists of all matrices g in the general linear group $\mathrm{GL}(V)=\mathrm{GL}(2l,R)$ that preserve the inner product, $B(gu,gv)=B(u,v)$. In other words, a matrix $g=(g_{ij})\in\mathrm{SL}(2l,R)$ belongs to $\mathrm{Sp}(2l,R)$ if and only if $gFg^t=F$ or, what is the same,

$$g'_{ij}=\varepsilon_i\varepsilon_j g_{-j,-i} \quad \text{for all } i,j=1,\dots,-1.$$

From the viewpoint of the theory of algebraic groups, the group $\mathrm{Sp}(2l,R)$ is a Chevalley group of type C_l (the Ree theorem).

In the sequel we need an explicit form of the isomorphism between $V = R^{2l}$ and $V^* = {}^{2l}R$ determined by the inner product B. Usually, this isomorphism is referred to as (symplectic) *polarity*; to a column $v = (v_1, \dots, v_l, v_{-l}, \dots, v_{-1})^t \in R^{2l}$ it assigns the row $\widetilde{v} = (-v_{-1}, \dots, -v_{-l}, v_l, \dots, v_1) \in {}^{2l}R$, and, respectively, to a row $u = (u_1, \dots, v_l, v_{-l}, \dots, v_{-1}) \in {}^{2l}R$ it assigns the column $\widetilde{u} = (u_{-1}, \dots, u_{-l}, -u_l, \dots, -u_1) \in R^{2l}$. Clearly, the map $v \mapsto \widetilde{v}$ is linear, $\widetilde{\widetilde{v}} = v$, and for any $u, v \in R^{2l}$ we have $B(u, v) = \widetilde{u}v$. A straightforward calculation shows that $\widetilde{gv} = \widetilde{v}g^{-1}$ for all $v \in V$ and all $g \in \mathrm{Sp}(2l, R)$ (see, e.g., [39, Lemma 2.5]). In fact, this is merely another form of the equations determining the symplectic groups.

Along with the usual symplectic groups, we consider the *general symplectic groups* $\mathrm{GSp}(2l, R)$ that preserve the form B up to similarity, $B(gu, gv) = \lambda B(u, v)$. In other words, the matrices in $\mathrm{GSp}(2l, R)$ satisfy the relation $gFg^t = \lambda F$ for an appropriate *multiplier* $\lambda = \lambda(g) \in R^*$. Thus, the entries of a matrix in the group $\mathrm{GSp}(2l, R)$ satisfy the following condition:

$$\lambda(g)g'_{ij} = \varepsilon_i\varepsilon_j g_{-j,-i} \quad \text{for all } i, j = 1, \dots, -1.$$

As usual, by an (elementary) *symplectic transvection* we understand one of the matrices $T_{ij}(\xi)$, $\xi \in R$, $i \neq j$, of the form

$$T_{ij}(\xi) = T_{-j,-i}(-\xi) = e + \xi e_{ij} - \varepsilon_i\varepsilon_j\xi e_{-j,-i}, \quad i \neq \pm j,$$

or of the form

$$T_{i,-i}(\xi) = e + \xi e_{i,-i}, \quad j = -i.$$

From the viewpoint of the theory of Chevalley groups, these elements are short and long (elementary) *root elements* of the group Γ, respectively.

Relations among symplectic transvections are special cases of the *Chevalley commutator formula* (see, e.g., [42, 25, 70]); they are listed explicitly, e.g., in [15, 11, 39]. In the present paper the main role is played by the *elementary symplectic group* $\mathrm{Ep}(2l, R)$, i.e., the subgroup in Γ generated by all elementary symplectic transvections of the form $T_{ij}(\xi)$, $\xi \in R$, $i \neq j$. Many arguments in this paper depend critically on the following *Kopeiko–Taddei theorem* [23, 74] (see also [75, 82, 61, 84, 38, 39, 73, 51, 52] for other proofs and further references).

Lemma 4. *Let R be a commutative ring, and let $l \geq 2$. Then the elementary group* $\mathrm{Ep}(2l, R)$ *is normal in* $\mathrm{GSp}(2l, R)$.

The following easy but fundamental fact is well known (see, e.g., [70, Corollary 4.4]).

Lemma 5. *Let R be a commutative ring, and let $l \geq 2$; for $l = 2$ we assume additionally that R has no factor fields of* 2 *elements. Then the elementary group* $\mathrm{Ep}(2l, R)$ *is perfect.*

As was known to the XIX century classics, the proviso in the case of $l = 2$ is necessary: the group $\mathrm{Sp}(4, 2)$ over the field of 2 elements is isomorphic to the symmetric group S_6, and its commutator subgroup has index 2 (see, e.g., [24, 3.1.5 and 3.3.6]). Precisely this circumstance makes the analysis of the case of $\mathrm{Sp}(4, R)$ transcendentally more complicated than that of the case of $\mathrm{Sp}(2l, R)$, $l \geq 3$.

We denote by $W = W(C_l) \cong \mathrm{Oct}_l$ the Weyl group of the group Γ. As always, let $n = 2l$, and let the indices be numbered as $1, \dots, l, -l, \dots, -1$, in the usual way. Recall that the octahedral group Oct_l consists of all permutations in the symmetric group S_{2l} that commute with the change of signs, i.e., such that $w(-i) = -w(i)$ for all i. For a permutation $w \in W$, we denote by the same letter any symplectic monomial matrix of the form $\sum \lambda_i e_{w(i),j}$, where $\lambda_i \in R^*$.

In the localization process, we must explicitly use semisimple elements contained in $\mathrm{Ep}(2l, R)$. Namely, for any $i = 1, \dots, -1$, $\varepsilon \in R^*$, we denote by $D_i(\varepsilon)$ the *long semisimple root element*

$$D_i(\varepsilon) = d_{i,-i}(\varepsilon) = e + (\varepsilon - 1)e_{ii} + (\varepsilon^{-1} - 1)e_{-i,-i}.$$

If, moreover, $i \neq \pm j$, then $D_{ij}(\varepsilon)$ denotes the *short semisimple root element*

$$D_{ij}(\varepsilon) = e + (\varepsilon - 1)e_{ii} + (\varepsilon^{-1} - 1)e_{jj} + (\varepsilon^{-1} - 1)e_{-i,-i} + (\varepsilon - 1)e_{-j,-j}.$$

It is very well known (see, e.g., [25]) that the matrices $D_i(\varepsilon)$ and $D_{ij}(\varepsilon)$ are expressed as products of elementary symplectic transvections, i.e., they belong to $\mathrm{Ep}(2l, R)$.

The diagonal subgroup contained in $\mathrm{GSp}(2l, R)$ will be denoted by $T(2l, R)$.[1] It is generated by the long semisimple root elements $D_i(\varepsilon)$, where $i = 1, \dots, l$, $\varepsilon \in R^*$, and the matrices $h(\varepsilon) = \mathrm{diag}(\varepsilon, \dots, \varepsilon, 1 \dots, 1)$, where the number of ε's, as well as the number of 1's equals l (from the viewpoint of algebraic groups, these are weight elements of weight $\overline{\varpi}_l$). Since the group $T(2l, R)$ normalizes $\mathrm{Ep}(2l, R)$, we can consider their product $\mathrm{GEp}(2l, R) = \mathrm{Ep}(2l, R)T(2l, R)$, which is called the *general elementary symplectic group*.

The possibility to use localization in the proof of Theorem 1 of the present paper is based on the following straightforward observation. The fact stated below is a special case of the stabilization theorems mentioned in §11, but, obviously, for local rings it was known before and independently of algebraic K-theory. For the symplectic group, see, e.g., [53, 59, 57] or, in the context of Chevalley groups, [32].

Lemma 6. *If R is a semilocal ring, then* $\mathrm{SL}(n, R) = E(n, R)$, $\mathrm{GE}(n, R) = \mathrm{GL}(n, R)$, $\mathrm{Sp}(2l, R) = \mathrm{Ep}(2l, R)$, *and* $\mathrm{GSp}(2l, R) = \mathrm{GEp}(2l, R)$.

From a modern viewpoint, this lemma asserts that the functors SK_1 and $\mathrm{K}_1\mathrm{Sp}$, are trivial; this is true for all rings of stable rank 1 (see [1, 17, 18, 71]).

§3. Localization and the functor of points

Let R be a commutative ring with 1, S a multiplicative system in R, and $S^{-1}R$ a localization of R relative to S. In the present paper we use exclusively localizations with respect to the following two types of multiplicative systems.

- Localization in a maximal ideal, $S = R \setminus M$, where $M \in \mathrm{Max}(R)$ is a maximal ideal of the ring R; in this case we write $(R \setminus M)^{-1}R = R_M$.
- Principal localization, $S = \langle s \rangle = \{1, s, s^2, \dots\}$, where $s \in R$; in this case we write $\langle s \rangle^{-1}R = R_s$.

Denote by $F_S : R \longrightarrow S^{-1}R$ the localization homomorphism. In the special cases mentioned above, we write $F_M : R \longrightarrow R_M$ and $F_s : R \longrightarrow R_s$, respectively. By writing an element as a fraction a/s, we always mean that this element belongs to a localization $S^{-1}R$, where $s \in S$. If s is invertible in R and we wish to regard this fraction as an element of R, we write as^{-1}.

In the sequel, we view assignments $R \mapsto X(R)$, where $X = \mathrm{GL}_n$, SL_n, Sp_{2l}, GSp_{2l}, E_n, GE_n, Ep_{2l}, GEp_{2l}, as covariant functors from the category of commutative rings with 1 to the category of groups. The functors we consider belong to one of the following types.

- The *functors of points* of reductive group schemes: $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2l}, \mathrm{GSp}_{2l}$.
- The *elementary subfunctors* of the functors of points: E_n, Ep_{2l}.
- The products of elementary subfunctors by tori: $\mathrm{GE}_n, \mathrm{GEp}_{2l}$.

[1] In the papers devoted to overgroups of the maximal tori (see the references in [12, 85, 86]), this subgroup was usually denoted by $\overline{T}(2l, R)$, whereas $T(2l, R)$ was used for the diagonal subgroup of $\mathrm{Sp}(2l, R)$.

Ideologically, the proofs that involve localization are based on the interplay of three considerations: 1) the functors of points are compatible with localization, 2) the elementary subfunctors are compatible with factorization, 3) on local rings the values of semisimple groups and their elementary subfunctors coincide.

By the very definition, $\mathrm{GL}(n,-)$, $\mathrm{SL}(n,-)$, $\mathrm{Sp}(2l,-)$, and $\mathrm{GSp}(2l,-)$ are *affine group schemes* over $\mathbb{Z}$, i.e., *representable* functors. In particular, if X denotes any of the above functors, we have the corresponding localization homomorphism $F_S : X(R) \longrightarrow X(S^{-1}R)$. Clearly, under these homomorphisms the root elements go to root elements, $F_S(x_\alpha(\xi)) = x_\alpha(F_S(\xi))$, so that the homomorphism F_S maps $E(n,R)$ to $E(n,S^{-1}R)$, and $\mathrm{Ep}(2l,R)$ to $\mathrm{Ep}(2l,S^{-1}R)$. Since the tori are mapped to tori, the localization homomorphism maps $\mathrm{GSp}(2l,R)$ to $\mathrm{GSp}(2l,S^{-1}R)$ and $\mathrm{GEp}(2l,R)$ to $\mathrm{GEp}(2l,S^{-1}R)$. The most important property of these functors is that they commute with direct limits. The following statement is commonly known.

Lemma 7. *If R_i, $i \in I$, is an inductive system of rings, and X is one of the functors* $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2l}, \mathrm{GSp}_{2l}, E_n, \mathrm{GE}_n, \mathrm{Ep}_{2l}, \mathrm{GEp}_{2l}$, *then* $X(\varinjlim R_i) = \varinjlim X(R_i)$.

We apply this lemma to the following two situations.

- Consider the inductive system R_i of all finitely generated subrings in R with respect to inclusion; then $X(R) = \varinjlim X(R_i)$, which allows us to reduce all problems we consider to Noetherian rings.
- Let S be a multiplicative system. We view the system of rings R_s, $s \in S$, as an inductive system of rings with respect to the localization homomorphisms $F_t : R_s \longrightarrow R_{st}$. Then $X(S^{-1}R) = \varinjlim X(R_s)$.

One of the most important applications of localization is based on the fact that the functor of points of an affine scheme is fully determined by its values at local rings. For example, let X be a subscheme in the n-dimensional affine space R^n determined by an ideal $I(X) \le R[x_1,\dots,x_n]$ (this is the only case we are interested in), and let

$$X(A) = \mathrm{Hom}_{R\text{-Alg}}(R[x_1,\dots,x_n]/I(X), A)$$

be the set of its points in an R-algebra A. Then

$$X(R) = \{x \in R^n \mid \forall M \in \mathrm{Max}(R), F_M(x) \in X(R_M)\}.$$

Indeed, let $f \in I(X)$ and $(x_1,\dots,x_n) \in R^n$. Assume that

$$F_M(f(x_1,\dots,x_n) = 0$$

for any maximal ideal M. This means that there is $s_M \notin M$ such that $s_M f(x_1,\dots,x_n) = 0$. Since the ideal generated by all s_M, $M \in \mathrm{Max}(R)$, is not contained in any maximal ideal, it must coincide with the entire ring, so that $f(x_1,\dots,x_n) = 0$. For further reference, we state the following special case of the above principle.

Lemma 8. *Let $g \in \mathrm{GL}(2l,R)$ be such that $F_M(g) \in \mathrm{GSp}(2l,R_M)$ for all $M \in \mathrm{Max}(R)$. Then $g \in \mathrm{GSp}(2l,R)$.*

It should be mentioned that for functors that are not affine schemes, a similar statement fails completely. Thus, $E(n,R) = \mathrm{SL}(n,R)$ and $\mathrm{Ep}(2l,R) = \mathrm{Sp}(2l,R)$ for any commutative local ring, but it is well known that there exist rings for which $E(n,R) \ne \mathrm{SL}(n,R)$ and $\mathrm{Ep}(2l,R) \ne \mathrm{Sp}(2l,R)$. Nevertheless, for some special types of rings/matrices, $F_M(g) \in \mathrm{Ep}(2l,R_M)$ still implies $g \in \mathrm{Ep}(2l,R)$; see, in particular, the Quillen theorem in §8.

On the other hand, elementary subfunctors of group schemes have an important technical advantage over the functor of points, namely, they carry the epimorphisms to epimorphisms. The point is that if $A \longrightarrow B$ is a surjective homomorphism of rings, and G is a reductive group, then $G(A) \longrightarrow G(B)$ is not necessarily an epimorphism. On the other hand, if E is the elementary subfunctor in G, then the homomorphism $E(A) \longrightarrow E(B)$ is always surjective. In the sequel we repeatedly use the fact that for any ideal $I \trianglelefteq R$ the image of the group $\mathrm{Ep}(2l, R)$ in $\mathrm{GL}(n, R/I)$ with respect to reduction modulo I coincides with $\mathrm{Ep}(2l, R/I)$.

§4. The orbit of the highest weight vector

A key role in the proof of our theorem is played by the following easy observation.

Proposition 1. *If R is a commutative ring, then any column of a matrix in* $\mathrm{GE}(2l, R)$ *can be completed to a matrix in* $\mathrm{GEp}(2l, R)$.

For fields, this statement is commonly known and means precisely that any nonzero column can be a column of a symplectic matrix. In this case the statement follows immediately from the Bruhat decomposition and the fact that the difference of any two weights of the vector representation of a symplectic group is a root. For rings this fact follows, e.g., from [19, Lemma 5.5]; see remark a) on page 1010 therein. Of course, in [19] this result is stated as the coincidence of the orbits, $\mathrm{GE}(2l, R)e_1 = \mathrm{GEp}(2l, R)e_1$. However, since we need the same line of arguments in a subsequent paper, as a pattern for generalization to Chevalley groups, and since we could not find a proof in the existing literature, we reproduce a detailed proof below. First, observe that for $n \geq 2$ any column of a matrix in $\mathrm{GE}(n, R)$ is already a column of a matrix in $E(n, R)$, and for even $n = 2l$ any column of a matrix in $\mathrm{GEp}(2l, R)$ is a column of a matrix in $\mathrm{Ep}(2l, R)$. Consequently, it suffices to verify that $E(2l, R)e_1 = \mathrm{Ep}(2l, R)e_1$.

The following obvious lemma shows that the elementary group is generated by the unipotent radicals of two opposite parabolic subgroups. This immediately follows from the Chevalley commutator formula.

Lemma 9. *For any $i = 1, \dots, n-1$, the group $E(n, R)$ is generated by the subgroups U_i and U_i^-.*

We apply this statement in the cases where $n = 2l$ and $i = 1$ or $i = l$.

The inner product B is obtained by the antisymmetrization of the inner product C with Gram matrix $\mathrm{sdiag}(1, \dots, 1, 0, \dots, 0)$, namely, $C(e_i, e_j) = \delta_{i,-j}$ if $\varepsilon_i = 1$ and $C(e_i, e_j) = 0$ otherwise. Clearly, $C(u, v) = u_1 v_{-1} + \dots + u_l v_{-l}$ for all $u, v \in R^{2l}$. The following statement is obvious (see, e.g., [23] or [39, Lemmas 7.1 and 7.2]).

Lemma 10. *For any $v \in R^{2l-2}$, the matrices*

$$Y^+(v) = \begin{pmatrix} 1 & \widetilde{v} & 0 \\ 0 & e & v \\ 0 & 0 & 1 \end{pmatrix}, \quad Y^-(v) = \begin{pmatrix} 1 & 0 & 0 \\ v & e & 0 \\ 0 & -\widetilde{v} & 1 \end{pmatrix}$$

belong to the group $\mathrm{Ep}(2l, R)$.

Proof. Indeed,

$$Y^+(v) = T_{1,-1}(-C(v,v)) \prod T_{i,-1}(v_i), \quad Y^-(v) = Y_{-1,1}(C(v,v)) \prod T_{i1}(v_i),$$

where both products are taken over $i = 2, \dots, -2$ in the natural order. $\square$

Lemma 11. *For any $u \in {}^{2l-1}R$ and $v \in R^{2l-1}$, there exist $x, y \in E(2l-1, R)$ such that*

$$\begin{pmatrix} 1 & ux \\ 0 & x \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ v & y \end{pmatrix} \in \mathrm{Ep}(2l, R).$$

Proof. Put $u = (w, a)$, where $w \in {}^{2l-2}R$ and $a \in R$, and $v = (z, b)^t$, where $z \in R^{2l-2}$ and $b \in R$. By the preceding lemma, the products

$$\begin{pmatrix}1 & w & a\\ 0 & e & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & e & \widetilde{w}\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & w & a\\ 0 & e & \widetilde{w}\\ 0 & 0 & 1\end{pmatrix} = Y^+(w)T_{1,-1}(a),$$

$$\begin{pmatrix}1 & 0 & 0\\ z & e & 0\\ b & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & e & 0\\ 0 & -\widetilde{z} & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ z & e & 0\\ b & -\widetilde{z} & 1\end{pmatrix} = Y^-(z)T_{-1,1}(b)$$

belong to $\mathrm{Ep}(2l, R)$, so that we can set, for instance,

$$x = \begin{pmatrix}e & \widetilde{w}\\ 0 & 1\end{pmatrix}, \quad y = \begin{pmatrix}e & 0\\ -\widetilde{z} & 1\end{pmatrix}. \quad \square$$

Now we are ready to finish the proof of the proposition.

Proof of Proposition 1. Consider a vector v belonging to the orbit of the vector e_1 under the action of $E(2l, R)$. By Lemma 7, such a vector can be presented in the form $x_1 \cdots x_t e_1$, where all factors $x_1, \dots, x_t$ belong to U_1 or to U_1^-. We wish to show that this vector already belongs to $\mathrm{Ep}(2l, R)e_1$. We argue by induction on t. First, observe that for $t = 0$ our statement is obvious, while for $t \geq 1$ by Lemma 9 there exists $y \in L_1$ such that $x_1 y \in \mathrm{Ep}(2l, R)$. We rewrite $x_1 \cdots x_t e_1$ in the form

$$(x_1 y)x_2^y \cdots x_t^y y^{-1} e_1 = (x_1 y)x_2^y \cdots x_t^y e_1.$$

It remains to show that $z_2 \cdots z_t e_1 \in \mathrm{Ep}(2l, R)e_1$, where (since L_1 normalizes U_1 and U_1^-) all factors $z_i = x_i^y$ still belong to U_1 or to U_1^-, but this is precisely our induction hypothesis. $\square$

An answer to the following problem seems to be unknown, apart from some trivial cases where it follows from stability conditions (see [67, 79] or §12 below).

Problem 1. *Does there exist a column of a matrix in* $\mathrm{GL}(2l, R)$ *that cannot be completed to a matrix belonging to* $\mathrm{GSp}(2l, R)$?

§5. The normalizer of $\mathrm{Ep}(2l, R)$

The Kopeiko–Taddei theorem tells us that the normalizer of $\mathrm{Ep}(2l, R)$ in $\mathrm{GL}(2l, R)$ *contains* $\mathrm{GSp}(2l, R)$. In this section we show that in fact it *coincides* with $\mathrm{GSp}(2l, R)$. It seems incredible that such a result could be unknown to the experts, but we could not find its proof in the literature *even for the case of fields*. Moreover, although for fields this result follows from the work of R. Dye, O. King, Shang Zhi Li, and Bashkirov, we could not find even an explicit statement of this result in the available literature. The first author offers a bottle of champagne to anybody who can supply a reference to a statement of this results prior to [58] for finite fields or prior to [45, Lemma 2] for the general case.

Let us start with a more convenient form of the equations that determine whether a matrix belongs to the group $\mathrm{GSp}(2l, R)$. Observe that the multiplier does not arise in these equations.

Proposition 2. *A matrix* $g = (g_{ij}) \in \mathrm{GL}(2l, R)$ *belongs to the group* $\mathrm{GSp}(2l, R)$ *if and only if*

$$g_{ir} g'_{sj} = \varepsilon_i \varepsilon_j \varepsilon_r \varepsilon_s g_{-j,-s} g'_{-r,-i}$$

for all $i, j, r, s = 1, \dots, -1$.

Proof. Let G be an affine scheme over $\mathbb{Z}$ determined by the above equations. Clearly, $\mathrm{GSp}(2l,R) \subseteq G(R)$. By Lemma 8, it suffices to establish the inverse inclusion for the case of a local ring R. Let M be a maximal ideal of R.

First, we show that if $g = (g_{ij}) \in G(R)$, then there exists a pair (i,r) such that $g_{ir}g'_{-r,-i} \in R^*$. Indeed, suppose that, contrary to the expectations, $g_{ir}g'_{-r,-i} \in M$ for all i,r. Since the matrix g is invertible for any i, there exists r such that $g_{ir} \notin M$, and for any j there exists s such that $g'_{sj} \notin M$. Then $g_{ir}g'_{sj} \in R^*$, but $g_{-j,-s}g'_{-r,-i} \in M$, so that the matrix g cannot belong to $G(R)$.

Now, we fix a pair (i,r) such that $g_{ir}g'_{-r,-i} \in R^*$. Set $\lambda = \varepsilon_i\varepsilon_r g_{ir}(g'_{-r,-i})^{-1} \in R^*$. Then the equations on g take the form $\lambda g'_{sj} = \varepsilon_s\varepsilon_j g_{-j,-s}$ for all j,s. But this means precisely that the matrix g belongs to $\mathrm{GSp}(2l,R)$. □

Let E, F be two subgroups of the group G. Recall that the *transporter* of the subgroup E into the subgroup F is the set

$$\mathrm{Tran}_G(E,F) = \{g \in G \mid E^g \le F\}.$$

Mostly we use this notation when $E \le F$, and then

$$\mathrm{Tran}_G(E,F) = \{g \in G \mid [g,E] \le F\}.$$

The proof of the following theorem mimics that of Proposition 2 in [13] or, in the orthogonal case, of Proposition 2 in [11].

Theorem 3. *Suppose R is a commutative ring, $l \ge 2$, $E = \mathrm{Ep}(2l,R)$, $\Gamma = \mathrm{Sp}(2l,R)$, and $G = \mathrm{GL}(2l,R)$. Then*

$$N_G(E) = N_G(\Gamma) = \mathrm{Tran}_G(E,\Gamma) = \mathrm{GSp}(2l,R).$$

Proof. Obviously, $\mathrm{GSp}(2l,R) \le N_G(\Gamma)$. This immediately follows from Proposition 2. Indeed, let $x \in \mathrm{Sp}(2l,R)$, $g \in \mathrm{GSp}(2l,R)$. We show that $y = gxg^{-1} \in \Gamma$. For this, it suffices to verify the equations $y'_{ij} = \varepsilon_i\varepsilon_j y_{-j,-i}$ for all i,j. Indeed, $y'_{ij} = \sum g_{ih}x'_{hk}g'_{kj}$, where the sum is taken over all h,k. Since the matrix x is symplectic, using the equations in the preceding proposition for the matrix g, we see that $y'_{ij} = \varepsilon_i\varepsilon_j \sum g_{-j,-k}x_{-k,-h}g'_{-h,-j} = \varepsilon_i\varepsilon_j y_{-j,-i}$, as expected. The Kopeiko–Taddei theorem asserts that $\mathrm{GSp}(2l,R) \le N_G(E)$.

Conversely, we have $N_G(E), N_G(\Gamma) \le \mathrm{Tran}_G(E,\Gamma)$. Thus, to prove the theorem it suffices to verify the inclusion $\mathrm{Tran}_G(E,\Gamma) \le \mathrm{GSp}(2l,R)$. Take any matrix $g = (g_{ij}) \in \mathrm{Tran}_G(E,\Gamma)$.

1) Writing the conditions that the matrix $y = gT_{r,-r}(1)g^{-1}$ is symplectic, we immediately get the equations $g_{ir}g'_{-r,j} = -\varepsilon_i\varepsilon_j g_{-j,r}g'_{-r,-i}$.

2) Now, let $r \ne \pm s$. Writing the conditions that the matrix $y = gT_{rs}(1)g^{-1}$ is symplectic, we get

$$g_{ir}g'_{s,j} - \varepsilon_r\varepsilon_s g_{i,-s}g'_{-r,j} = \varepsilon_i\varepsilon_j(\varepsilon_r\varepsilon_s g_{-j,r}g'_{s,-i} - g_{-j,-s}g'_{-r,-i}).$$

3) If $l \ge 3$, we get the equations of Proposition 2 for $r \ne s$. Indeed, let q be an index such that all six indices $\pm r, \pm s, \pm q$ are pairwise distinct. We write the condition that the matrix $y = gT_{rq}(1)T_{qs}(1)g^{-1}$ is symplectic and compare the resulting equations with those in item 2) (for pairs r,q and q,s) to obtain $g_{ir}g'_{s,j} = \varepsilon_i\varepsilon_j\varepsilon_r\varepsilon_s g_{-j,-s}g'_{-r,-i}$.

4) Now, suppose that $l = 2$. We write the conditions that the matrix

$$y = gT_{r,-s}(1)T_{-s,s}(1)g^{-1}$$

is symplectic and compare the resulting equations with those obtained in 2) (for pairs $r,-s$) and in 1) (for pairs $-s,s$); again, this yields the same equation as in 3).

5) It only remains to verify the equations for $r = s$. This can be done exactly as above, by considering the matrix $y = gT_{r,-r}(1)T_{-r,r}(1)g^{-1}$, but it is easier to use the

same standard trick as in [8, §3]. Namely, recall that $l \geq 2$ and pick $s \neq \pm r$. Then items 3) and 4) imply that

$$g_{ir}g'_{rj} = \sum g_{ir}g'_{sh}g_{hs}g'_{rj} = \varepsilon_i\varepsilon_j \sum g_{-h,-s}g'_{-r,-i}g_{-j,-r}g'_{-s,-h} = \varepsilon_i\varepsilon_j g_{-j,-r}g'_{-r,-i}$$

(the sums are taken over $h = 1, \dots, -1$). This finishes the proof of the inclusion $\mathrm{Tran}_G(E, \Gamma) \leq \mathrm{GSp}(2l, R)$ and of the theorem. □

Usually, we shall apply this result in the following form.

Corollary. *Under the conditions of Lemma* 5,

$$\mathrm{Tran}_G(\mathrm{Ep}(2l, R), \mathrm{GSp}(2l, R)) = \mathrm{GSp}(2l, R).$$

Proof. We show that for any $g \in \mathrm{GL}(2l, R)$ the inclusion $[g, \mathrm{Ep}(2l, R)] \leq \mathrm{GSp}(2l, R)$ implies that $g \in \mathrm{GSp}(2l, R)$. Indeed, the Kopeiko–Taddei theorem implies

$$[g, \mathrm{Ep}(2l, R), \mathrm{Ep}(2l, R)] \leq \mathrm{Ep}(2l, R).$$

Since the group $\mathrm{Ep}(2l, R)$ is perfect, it follows from the three subgroup lemma that $g \in N_G(\mathrm{Ep}(2l, R)) = \mathrm{GSp}(2l, R)$. □

§6. Elementary transvections in subgroups normalized by $\mathrm{Ep}(2l, R)$

In this section we assume that $l \geq 2$. Observe that the proof of the following two lemmas is easier than in the orthogonal case, because now all transvections $t_{i,-i}(\xi)$ belong to the elementary symplectic group $\mathrm{Ep}(2l, R)$ itself. As a result, there is no need to consider the case of $i = -j$ in Lemma 12 separately.

Lemma 12. *For any ideal $A \trianglelefteq R$ we have*

$$E(2l, A)^{\mathrm{Ep}(2l,R)} = E(2l, R, A).$$

Proof. Clearly, the left-hand side is contained in the right-hand side. To prove the reverse inclusion, by Lemma 2 it suffices to check that the $z_{ij}(\xi, \zeta) = {}^{t_{ji}(\zeta)}t_{ij}(\xi)$ belong to $H = E(2l, A)^{\mathrm{Ep}(2l,R)}$. For $i = -j$ this is obvious, since $t_{-i,i}(\zeta) = T_{-i,i}(\zeta) \in \mathrm{Ep}(2l, R)$ in this case. On the other hand, for $i \neq \pm j$ we have ${}^{t_{ji}(\zeta)}t_{ij}(\xi) = {}^{T_{ji}(\zeta)}t_{ij}(\xi)$. □

Lemma 13. *Let H be a subgroup in* $\mathrm{GL}(2l, R)$ *that contains* $\mathrm{Ep}(2l, R)$. *For $i \neq \pm j$, we put $A_{ij} = \{\xi \in R \mid t_{ij}(\xi) \in H\}$. Then for any $h \neq \pm k$ we have $A_{hk} = A_{ij} = A$ for an ideal $A \trianglelefteq R$.*

Proof. Clearly, all subsets A_{ij}, $i \neq j$, are additive subgroups. Now we show that in fact they are ideals. Suppose $i \neq \pm j$ and $\xi \in A_{ij}$. If $l \geq 3$, we can use the same argument as in the proof of Lemma 4 in [16]. Namely, taking any $\zeta \in R$ and any index h such that the indices $\pm i, \pm j, \pm h$ are pairwise distinct, we consider the commutator

$$t_{ih}(\xi\zeta) = [t_{ij}(\xi), t_{jh}(\zeta)] = [t_{ij}(\xi), T_{jh}(\zeta)] \in H.$$

This means that $A_{ij}R \subseteq A_{ih}$. In particular, $A_{ij} \subseteq A_{ih}$. Similarly,

$$t_{hj}(\zeta\xi) = [t_{hi}(\zeta), t_{ij}(\xi)] = [T_{hi}(\zeta), t_{ij}(\xi)] \in H,$$

so that $RA_{ij} \subseteq A_{hj}$. In particular, $A_{ij} \subseteq A_{hj}$. This shows that for $l \geq 3$ all subgroups A_{ij} with $i \neq \pm j$ coincide and are ideals. We denote their common value by A.

It only remains to consider the case where $l = 2$. To this end, observe that, since H contains $\mathrm{Ep}(2l, R)$, we have $A_{ij} = A_{-j,-i}$ for any $i \neq \pm j$. Now,

$$t_{i,-j}(\xi\zeta) = [t_{ij}(\xi), t_{j,-j}(\zeta)] = [t_{ij}(\xi), T_{j,-j}(\zeta)] \in H.$$

This means that $A_{ij}R \subseteq A_{i,-j}$. In particular, $A_{ij} \subseteq A_{i,-j}$. Similarly,

$$t_{-i,j}(\zeta\xi) = [t_{-i,i}(\zeta), t_{ij}(\xi)] = [T_{-i,i}(\zeta), t_{ij}(\xi)] \in H,$$

whence $RA_{ij} \subseteq A_{-i,j}$. In particular, $A_{ij} \subseteq A_{-i,j}$. Combining these inclusions with the fact that $A_{ij} = A_{-j,-i}$, we conclude that our lemma holds also for $l = 2$. $\square$

Summarizing two preceding lemmas, we get the following result.

Proposition 3. *Let H be a subgroup in $\mathrm{GL}(2l, R)$ that contains $\mathrm{Ep}(2l, R)$. Then there exists a unique largest ideal $A \trianglelefteq R$ such that*

$$\mathrm{EEp}(2l, R, A) = \mathrm{Ep}(2l, R)E(2l, R, A) \leq H.$$

Namely, if $t_{ij}(\xi) \in H$ for some $i \neq \pm j$, then $\xi \in A$.

§7. Proof of Theorem 2

Proposition 3 focuses our attention on the subgroups $\mathrm{EEp}(2l, R, A)$ for ideals $A \trianglelefteq R$. Lemma 12 asserts precisely that $\mathrm{EEp}(2l, R, A)$ is generated (as a subgroup) by all elementary symplectic transvections $T_{ij}(\xi)$, $i \neq j$, $\xi \in R$, and all elementary transvections $t_{ij}(\zeta)$, $i \neq \pm j$, $\zeta \in A$, of level A.

Lemma 14. *Under the conditions of Lemma 5, the group $\mathrm{EEp}(2l, R, A)$ is perfect for any ideal $A \trianglelefteq R$.*

Proof. It suffices to show that all generators of the group $\mathrm{EEp}(2l, R, A)$ lie in its commutator subgroup. For the symplectic transvections $T_{ij}(\xi)$, this follows from Lemma 5. On the other hand, for the transvections $t_{ij}(\zeta)$ with $i \neq \pm j$ and $\zeta \in A$ this is obvious, because $t_{ij}(\zeta) = [t_{i,-j}(\zeta), t_{-j,j}(1)]$, where both transvections on the right-hand side belong to $\mathrm{EEp}(2l, R, A)$. $\square$

In the remaining part of the section we calculate the normalizer of $\mathrm{EEp}(2l, R, A)$ in $\mathrm{GL}(2l, R, A)$. Namely, we return to the reduction homomorphism $\rho_A : \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n, R/A)$ and concentrate on the full preimage of the group $\mathrm{GSp}(2l, R/A)$ relative to this reduction:

$$\mathrm{CGSp}(2l, R, A) = \rho_A^{-1}(\mathrm{GSp}(2l, R/A)).$$

Clearly, $\mathrm{CGSp}(2l, R, A) \geq \mathrm{GSp}(2l, R)\,\mathrm{GL}(2l, R, A)$. In general, $\mathrm{CGSp}(2l, R, A)$ is *strictly* larger than $\mathrm{GSp}(2l, R)\,\mathrm{GL}(2l, R, A)$. In fact, it is easily seen that the multiplier of any matrix in $\rho_A(\mathrm{GSp}(2l, R))$ belongs to $\rho_A(R^*)$, where, as usual, $\rho_A : R \longrightarrow R/A$ is the reduction homomorphism modulo A. At the same time, any element of $(R/A)^*$ can be the multiplier of a matrix in $\mathrm{GSp}(2l, R/A)$. It is well known that, in general, for rings of dimension ≥ 1 the homomorphism $\rho_A : R^* \longrightarrow (R/A)^*$ is not surjective.

From Proposition 2 it follows that the group $\mathrm{CGSp}(2l, R, A)$ is determined by quadratic congruences in matrix entries.

Proposition 4. *Let A be an ideal in R. A matrix $g = (g_{ij}) \in \mathrm{GL}(2l, R)$ belongs to $\mathrm{CGSp}(2l, R, A)$ if and only if*

$$g_{ir}g'_{sj} \equiv \varepsilon_i\varepsilon_j\varepsilon_r\varepsilon_s g_{-j,-s}g'_{-r,-i} \pmod{A}$$

for all i, j, r, s.

This proposition explains the abbreviation $\mathrm{CGSp}(2l, R, A)$: the congruence general symplectic group. Now we are in a position to finish the proof of Theorem 2.

Proof of Theorem 2. Since $E(2l, R, A)$ and $\mathrm{GL}(n, R, A)$ are normal in $\mathrm{GL}(2l, R)$, the homomorphism theorem shows that Theorem 2 gets the following form:

$$N_G(\mathrm{EEp}(2l, R, A)) \leq N_G(\mathrm{EEp}(2l, R, A)\,\mathrm{GL}(2l, R, A)) = \mathrm{CGSp}(2l, R, A).$$

In particular,

$$[\mathrm{CGSp}(2l, R, A), \mathrm{EEp}(2l, R, A)] \leq \mathrm{EEp}(2l, R, A)\,\mathrm{GL}(2l, R, A).$$

On the other hand, it is completely clear that $\mathrm{EEp}(2l,R,A)$ is normal in the group on the right-hand side of this inclusion. Indeed, it is easily seen that even

$$[\mathrm{GSp}(2l,R,A)\,\mathrm{GL}(2l,R,A),\mathrm{EEp}(2l,R,A)]\leq \mathrm{EEp}(2l,R,A).$$

To verify this, we consider a commutator of the form

$$[xy,hg],\quad x\in \mathrm{GSp}(2l,R),\ \ y\in \mathrm{GL}(n,R,A),\ \ h\in \mathrm{Ep}(2l,R),\ \ g\in \mathrm{E}(2l,R,A).$$

Then $[xy,hg]={}^x[y,h]\cdot[x,h]\cdot{}^h[xy,g]$, and at this point the Suslin normality theorem, the standard commutator formula, and the Kopeiko–Taddei theorem immediately imply that all factors on the right-hand side belong to $\mathrm{EEp}(2l,R,A)$. Summarizing the above, we see that

$$[\mathrm{CGSp}(2l,R,A),\mathrm{EEp}(2l,R,A),\mathrm{EEp}(2l,R,A)]\leq \mathrm{EEp}(2l,R,A).$$

Now we cannot use the three subgroup lemma, and we must refer directly to the Hall–Witt identity. For this, we need a slightly more precise version of the last inclusion:

$$[[\mathrm{CGSp}(2l,R,A),\mathrm{EEp}(2l,R,A)],[\mathrm{CGSp}(2l,R,A),\mathrm{EEp}(2l,R,A)]]\leq \mathrm{EEp}(2l,R,A).$$

Observe that we have already checked that the left-hand side is generated by the commutators $[uv,[z,y]]$ with $u,y\in \mathrm{EEp}(2l,R,A)$, $v\in \mathrm{GL}(2l,R,A)$, $z\in \mathrm{CGSp}(2l,R,A)$. However,

$$[uv,[z,y]]={}^u[v,[z,y]]\cdot[u,[z,y]].$$

The second commutator is an element of $\mathrm{EEp}(2l,R,A)$, and the first is an element of $[\mathrm{GL}(2l,R,A),\mathrm{E}(2l,R)]\leq \mathrm{E}(2l,R,A)$.

Now we are ready to finish the proof. Since the group $\mathrm{EEp}(2l,R)$ is perfect by the preceding lemma, it suffices to show that $[z,[x,y]]\in \mathrm{EEp}(2l,R,A)$ for all $x,y\in \mathrm{EEp}(2l,R,A)$, $z\in \mathrm{CGSp}(2l,R,A)$. Indeed, the Hall–Witt identity yields

$$[z,[x,y]]={}^{xz}[[z^{-1},x^{-1}],y]\cdot{}^{xy}[[y^{-1},z],x^{-1}],$$

where the second commutator belongs to $\mathrm{EEp}(2l,R,A)$ by the above. Removing the conjugation by x in the first commutator and carrying the conjugation by z inside the commutator, we see that it only remains to prove that $[[x^{-1},z],[z,y]y]\in \mathrm{EEp}(2l,R,A)$. Indeed,

$$[[x^{-1},z],[z,y]y]=[[x^{-1},z],[z,y]]\cdot{}^{[z,y]}[[x^{-1},z],y],$$

where both commutators on the right-hand side belong to $\mathrm{EEp}(2l,R,A)$, while the conjugating element in the second commutator belongs to $\mathrm{EEp}(2l,R,A)\,\mathrm{GL}(2l,R,A)$ and thus normalizes $\mathrm{EEp}(2l,R,A)$. $\square$

§8. Localization methods

If R has zero divisors, the localization homomorphism $F_s:\mathrm{GSp}(2l,R)\mapsto \mathrm{GSp}(2l,R_s)$ is not necessarily injective. There exist two principal methods to fight zero divisors: localization and patching and localization-completion.

Quillen–Suslin's method. The most famous is the method introduced by Quillen and Suslin [28], which was used later by Abe [29]–[31], Kopeiko [23], Tulenbaev and Taddei [75], Vaserstein [78], [80]–[82], Fu An Li [61, 62], and others. This method consists in throwing in independent polynomial variables with subsequent specialization (for further references, see [50, 84, 39]). Vaserstein called this method localization and patching. This method is based on the following remarkable local-global principle, usually called the *Quillen theorem*, which we state only for the symplectic case we are interested in.

Lemma 15. *Let $g\in \mathrm{GSp}(2l,R[t],tR[t])$, $l\geq 2$. For $g\in \mathrm{Ep}(2l,R)$ it is necessary and sufficient that $F_M(g)\in \mathrm{Ep}(2l,R_M[t])$ for all maximal ideals $M\in \mathrm{Max}(R)$.*

For the first time, a similar result at the level of K_0 was proved by Quillen. A generalization to K_1 was proposed by Suslin; it was based on roughly the same ideas but contained several important and new technical elements. For $\mathrm{GL}(n, R[t])$ this theorem was proved by Suslin (see [28]), who himself called it the Quillen theorem, but it seems fair to call it the Quillen–Suslin theorem. In the symplectic case this is Theorem 3.6 of the paper [23] by Kopeiko, but in the published text (unlike Kopeiko's Thesis, where complete proofs were given) only the main steps were listed, with an observation that the minutiae of the proof are *analogous* to those in the linear case [28]. In the context of Chevalley groups, all details of the calculations were published by Abe in [29, Theorem 1.15]; see also [49].

Now it is completely clear how to proceed. We pass from the ring R to the polynomial ring $R[t]$, where t is not a zero divisor, use the Quillen theorem, and finally specialize $t \mapsto s$.

Bak's method. The following easy observation ([36, Lemma 4.10] or [52, Lemma 5.1]) allows us to control the effects produced by the noninjectivity of the localization homomorphism in the Noetherian case.

Lemma 16. *Assume that the ring R is Noetherian, $s \in R$. Then there exists a natural m such that* $\mathrm{Ker}(F_s) = \mathrm{GL}(n, R, \mathrm{Ann}(s^m))$. *In other words, the restriction of the localization homomorphism F_s to the principal congruence subgroup of level s^m is injective: if $F_s(h) = F_s(g)$ for some $h, g \in \mathrm{GL}(n, R, s^m R)$, then $h = g$.*

Proof. The localization homomorphism

$$F_s : \mathrm{GL}(2l, R, s^m R) \longrightarrow \mathrm{GL}(2l, R_s)$$

is injective if and only if the homomorphism $F_s : s^m R \longrightarrow R_s$ is injective. For $i \geq 0$, consider the annihilator $\mathrm{Ann}(s^i)$ of the element s^i in R. Since the ring R is Noetherian, there exists m such that $\mathrm{Ann}(s^m) = \mathrm{Ann}(s^{m+1}) = \cdots$. If $F_s(s^m a) = 0 \in R_s$, then $s^k s^m a = 0 \in R$ for some $k \geq 0$. But since $\mathrm{Ann}(s^{m+k}) = \mathrm{Ann}(s^m)$, we have already $s^m a = 0 \in R$, as stated. $\square$

In fact, the above observation is one of the *four* major ideas underlying Bak's localization-completion method. The three others are second localization, patching, and completion; see [36]–[38], [51], and particularly [52], where a detailed and *accessible* exposition of all ingredients of the method was given. The choice between the Quillen–Suslin method and reduction to Noetherian rings is a matter of taste, since everything that can be proved by one of these methods can also be proved by the other. However, it seems to us that Bak's method is somewhat easier.

The following lemma, which is a special case of Theorem 5.3 in [52], allows us to get rid of the denominators.

Lemma 17. *Let $l \geq 2$. For any finite collection of elements $g_1, \dots, g_n \in \mathrm{GEp}(2l, K)$ and any $k \geq 0$ there exist $m \geq 0$ such that*

$$[g_i, F_s(\mathrm{GSp}(2l, R, s^m R))] \leq \mathrm{Ep}(2l, F_s(s^k R)).$$

Results of this sort constitute "conjugation calculus". Observe that in the above form (with GSp, rather than GEp on the right-hand side) this is a fairly difficult and extremely powerful result. In particular, the normality of the elementary subgroup is a trivial special case, obtained by setting $s = 1$ and $n = 1$.

§9. Behavior of transvections under localization

We continue to assume that H is a subgroup of $\mathrm{GL}(2l, R)$, $l \geq 2$, that contains the elementary symplectic group $\mathrm{Ep}(2l, R)$, but is not contained in $\mathrm{GSp}(2l, R)$. We approach the core of the proof: reduction to proper parabolic subgroups. We wish to show that if H contains a matrix $y \notin \mathrm{GSp}(2l, R)$ whose first column coincides with the first column of the identity matrix, then H contains a nontrivial transvection. In this section we set stage for this reduction; the reduction itself is carried through in the following section.

This and the following sections play the same role in our proof as do §§6–8 in the paper [16], but since in the orthogonal group there are no elements of residue 1, there we had to assume that *two* among the columns of the matrix y coincided with the respective columns of the identity matrix, and the cases where these were the 1st and the (-1)st columns or the 1st and the 2nd columns required quite different calculations. In this sense, reduction to groups of smaller rank in the symplectic case is considerably easier than that in the orthogonal case.

On the other hand, for the localization proof an additional tension arises due to the fact that now we assume that a matrix with the required property can be found not in the group H itself, but in one of the groups $F_s(H)\,\mathrm{GSp}(2l, R_s)$. However, for extraction of transvections we can use not all elements of $\mathrm{Ep}(2l, R_s)$, but only elements of $F_s(\mathrm{Ep}(2l, R))$. This forces us to perform additional commutations and refer to standard localization arguments as well as to a rather tough Theorem 5.3 in [52], to find a required matrix already in $F_s(H)$. Observe that to prove the theorem in the case where we can find a matrix with the required properties in the group H itself, all these supplementary efforts are superfluous,[2] so that the content of the present section can be ignored completely, whereas in the next section one could set $s = 1$ in all lemmas.

Our goal in the present section is the proof of the following important auxiliary result.

Proposition 5. *Let H be a subgroup in $\mathrm{GL}(2l, R)$ that contains $\mathrm{Ep}(2l, R)$. Assume that there exists $s \in R$ such that $F_s(H)\,\mathrm{GSp}(2l, R_s)$ contains a nontrivial transvection. Then H contains a nontrivial transvection $t_{ij}(\xi)$, $i \neq \pm j$, $\xi \in R$.*

The proofs of this proposition and of the lemmas used therein illustrate *standard* tricks underlying application of localization methods (see, e.g., [36, 51] and the proof of Lemma 5.2 in [52]).

Lemma 18. *Let H be a subgroup in $\mathrm{GL}(2l, R)$ that contains $\mathrm{Ep}(2l, R)$. Next, let $X \leq \mathrm{GL}_n$ be a group subscheme. Assume that, for some $s \in R$,*

$$F_s(H)\,\mathrm{GSp}(2l, R_s) \cap X(R_s) \not\leq \mathrm{GSp}(2l, R_s).$$

Then there exists $t \in R$ such that

$$F_t(H)\,\mathrm{GEp}(2l, R_t) \cap X(R_t) \not\leq \mathrm{GSp}(2l, R_t).$$

Proof. Let $F_s(g)x$ be such an element, where $g \in H$ and $x \in \mathrm{GSp}(2l, R_s)$. By Lemma 11, there exists a maximal ideal $M \in \mathrm{Max}(R)$ such that $s \notin M$ and $F_M(g) \notin \mathrm{GSp}(2l, R_M)$. Since the ring R_M is local, we see that $\mathrm{GSp}(2l, R_M) = \mathrm{GEp}(2l, R_M)$. On the other hand, since $\mathrm{GEp}(2l, R_M) = \varinjlim \mathrm{GEp}(2l, R_t)$, where the limit is taken over all $t \notin M$, there exists $t = sq \notin M$ such that $F_q(x) \in \mathrm{GEp}(2l, R_t)$. Then

$$F_q(F_s(g)x) = F_t(g)F_q(x) \in F_t(H)\,\mathrm{GEp}(2l, R_t) \cap X(R_t),$$

and by our choice of M, still $F_t(g) \notin \mathrm{GSp}(2l, R_t)$. $\square$

[2]This is so, in particular, at the stable level, or when symplectic decomposition of unipotents works for any matrix in $\mathrm{GL}(2l, R)$, for example, under the assumption that $l \geq 3$, $2 \in R^*$.

Lemma 19. *Under the conditions of the above lemma, if* $y \in F_s(H)\,\mathrm{GEp}(2l, R_s)$, *then there exists* $n \in \mathbb{N}_0$ *such that*

$$[y, T_{ij}(s^n/1)] \in F_s(H)$$

for all $i \neq j$.

Proof. We express y as $y = gx$, where $g \in F_s(H)$ and $x \in \mathrm{GEp}(2l, R_s)$. Then for all n we have

$$[y, T_{ij}(s^n/1)] = {}^g[x, T_{ij}(s^n/1)][g, T_{ij}(s^n/1)].$$

By Lemma 17, we can choose n in such a way that $[x, T_{ij}(s^n/1)] \in F_s(\mathrm{Ep}(2l, R)) \in F_s(H)$ for all $i \neq j$. All other factors on the right-hand side belong to $F_s(H)$ from the very beginning. □

Proof of Proposition 5. Lemma 18 allows us to assume that

$$t_{ij}(a/s^k) \in F_s(H)\,\mathrm{GEp}(2l, R_s)$$

for some $i \neq \pm j$, $a \in R$, $k \geq 0$, where $a/s^k \neq 0$. Consider the commutator $[t_{ij}(a/s^k), t_{j,-j}(s^{n+k}/1)] = t_{i,-j}(s^n a/1)$. Choosing n as in Lemma 19, we see that $t_{i,-j}(s^n a/1) \in F_s(H)$; therefore, there exists $g \in H$ such that $F_s(g) = t_{i,-j}(s^n a/1)$. Since $F_s(t_{i,-j}(s^n a)) = t_{i,-j}(s^n a/1)$, we get $g = t_{i,-j}(s^n a)y$ for some $y \in \mathrm{Ker}(F_s)$. This means that there exists m such that $y \in \mathrm{GL}(2l, R, \mathrm{Ann}(s^m))$. Consider the commutator $z = [g, t_{-j,j}(s^m)] \in H$. Since $[y, t_{-j,j}(s^m)] = e$, we have $z = [t_{i,-j}(s^n a), t_{-j,j}(s^m)] = t_{i,j}(s^{n+m}a)$. If $s^{m+n}a = 0$, then $a \in \mathrm{Ker}(F_s)$, which is impossible, because we have assumed that $a/s^k \neq 0 \in R_s$. This shows that $z = t_{i,j}(s^{m+n}a) \in H$ is the desired nontrivial transvection. □

§10. Extraction of transvections in the parabolic subgroup P_1

Our aim in this section is to prove the following result. Recall that the maximal parabolic subgroup P_1 consists of all matrices whose first column is proportional to the first column of the identity matrix. We continue to tacitly assume that $l \geq 2$.

Proposition 6. *Let H be a subgroup in* $\mathrm{GL}(2l, R)$ *that contains* $\mathrm{Ep}(2l, R)$. *Suppose that*

$$F_s(H)\,\mathrm{GSp}(2l, R_s) \cap P_1(R_s) \nsubseteq \mathrm{GSp}(2l, R_s)$$

for some $s \in R$. *Then H contains a nontrivial transvection.*

The assumption of this proposition says that there is a matrix y in $F_s(H)\,\mathrm{GSp}(2l, R_s)$ that belongs to $P_1(R_s)$ but not to $\mathrm{GSp}(2l, R_s)$. We start with two auxiliary assertions. We show that the desired conclusion is true if we impose stronger conditions on y, namely, require that y belongs already to the unipotent radical $Y = {}_{-1}U_1$ of the submaximal parabolic subgroup ${}_{-1}P_1$, or to the subgroup ${}_{-1}P_1 \leq P_1$ itself. We recall that ${}_{-1}P_1$ consists of all matrices whose first column is proportional to the first column of the identity matrix, and the last row is proportional to the last row of the identity matrix.

Let $Y = Y(R)$ denote the *Heisenberg subgroup* in $\mathrm{GL}(2l, R)$, which is generated by all elementary transvections $t_{1j}(\xi)$ and $t_{i,-1}(\xi)$ with $j \neq 1$, $i \neq -1$, $\xi \in R$. Clearly, Y consists of all matrices y such that $y_{ij} = \delta_{ij}$ for all pairs (i, j) except for the pairs of the form $(1, j)$, $j \neq 1$, and $(i, -1)$, $i \neq -1$. As has been observed, from the viewpoint of algebraic groups, Y is the unipotent radical of ${}_{-1}P_1$, so that, in particular, $[{}_{-1}P_1, Y] \leq Y$.

Lemma 20. *Let H be a subgroup in* $\mathrm{GL}(2l, R)$ *that contains* $\mathrm{Ep}(2l, R)$. *If*

$$F_s(H)\,\mathrm{GSp}(2l, R_s) \cap Y(R_s) \nsubseteq \mathrm{GSp}(2l, R_s)$$

for some $s \in R$, *then H contains a nontrivial transvection.*

Proof. The group Y is algebraic: it is defined by the equations $x_{ij} = \delta_{ij}$ for all pairs (i, j) except for the pairs $(1, j)$, $j \neq 1$ and $(i, -1)$, $i \neq -1$. Now, referring to Lemma 18 and changing s if necessary, we may assume that $y \in F_s(H)\operatorname{GEp}(2l, R_s)$. As in Lemma 19, we pick an n such that $[y, T_{ij}(s^n R/1)] \subseteq F_s(H)$ for all $i \neq j$. If necessary, we can increase n so that it become larger than the largest exponent with which s appears in the denominators of the entries y_{ij}. Then, moreover, $[y, T_{ij}(s^n R/1)] \subseteq F_s(E(2l, R))$. On the other hand, a straightforward calculation shows that

$$z = [y, t_{i,-i}(s^n/1)] = t_{1,-i}(s^n y_{1i}/1) t_{i,-1}(-s^n y_{-i,-1}) t_{1,-1}(a),$$

where $s^n y_{1i}/1, s^n y_{-i,-1}, a \in F_s(R)$. Multiplying z by

$$t_{1,-1}(-a) T_{i,-1}(s^n y_{-i,-1}) \in F_s(\operatorname{Ep}(2l, R)) \leq F_s(H),$$

we can conclude that $t_{1,-i}(s^n(y_{1i} + \varepsilon_i y_{-i,-1})) \in F_s(H)$. Since $y \notin \operatorname{GSp}(2l, R_s)$, there exists an index $i = 2, \ldots, -2$ such that $y_{1i} + \varepsilon_i y_{-i,-1} \neq 0 \in R_s$. But this means precisely that $F_s(H)$ contains a nontrivial transvection. It only remains to invoke Proposition 4. $\square$

Lemma 21. *Let H be a subgroup in $\operatorname{GL}(2l, R)$ that contains $\operatorname{Ep}(2l, R)$. Assume that*

$$F_s(H)\operatorname{GSp}(2l, R_s) \cap {}_{-1}P_1(R_s) \nsubseteq \operatorname{GSp}(2l, R_s)$$

for some $s \in R$. Then H contains a nontrivial transvection.

Proof. The group ${}_{-1}P_1$ is algebraic as well: it is defined by the equations $y_{ij} = 0$ for the pairs (i, j) equal to $(i, 1)$, $i \neq 1$, or $(-1, j)$, $j \neq -1$. Thus, as usual, Lemma 18 allows us to assume that $y \in F_s(H)\operatorname{GEp}(2l, R_s)$. We take the same n as in the proof of the preceding lemma and for each $j \neq \pm 1$ consider the matrix

$$z_j = y T_{1j}(s^n/1) y^{-1} = e + \sum y_{11} s^n y'_{jk} e_{1k} - \varepsilon_j \sum y_{-k,-j} s^n y'_{-1,-1} e_{-k,-1},$$

where both sums are taken over $k = 2, \ldots, -1$. Clearly, $y \in H \cap Y$. It follows that $z_j \in Y(R_s) \cap F_s(H)$, and we fall into the setting of the preceding lemma and can conclude that either all matrices z_j lie in $\operatorname{GSp}(2l, R_s)$, or H contains a nontrivial transvection.

It remains to consider the case where $z_j \in \operatorname{GSp}(2l, R_s)$ for all $j \neq \pm 1$. Since, moreover, $z_j \in Y(R_s)$, we have $z_j \in \operatorname{Sp}(2l, R_s)$ automatically. This means that $y_{11} y'_{jk} = \varepsilon_j \varepsilon_k y_{-k,-j} y'_{-1,-1}$ for all $j, k \neq \pm 1$. Thus, the matrix obtained from y by deleting the rows and columns with indices ± 1, which we denote by $\overline{y}$, belongs to $\operatorname{GSp}(2l - 2, R_s)$. Now, look at the matrix $h^{-1} = \operatorname{diag}(1, \overline{y}, \lambda(y)) \in \operatorname{GSp}(2l, R_s)$. The matrix $x = yh \in F_s(H)\operatorname{GSp}(2l, R_s)$ does not belong to $\operatorname{GSp}(2l, R_s)$ together with y. Clearly, $x_{ij} = \delta_{ij}$ for all pairs (i, j), except, possibly, for the pairs $(1, j)$ and $(i, -1)$, so that x belongs to $Y(R_s) D_1(R_s^*)$. If $x_{-1,-1}^{-1} \neq x_{11}$, then, since $[h, T_{12}(s^n)] \in F_s(\operatorname{Ep}(2l, R))$, the above argument yields

$$z = [x, T_{12}(s^n)] = t_{12}(s^n(x_{11} - 1)) t_{-2,-1}(s^n(1 - x_{-1,-1}^{-1})) t_{1,-1}(a) \in F_s(H),$$

where $s^n(x_{11} - 1), s^n(1 - x_{-1,-1}^{-1}), a \in F_s(R)$. This shows that $t_{12}(s^n(x_{11} - x_{-1,-1}^{-1})) \in F_s(H)$ is a nontrivial transvection in $F_s(H)$. On the other hand, if $x_{-1,-1}^{-1} = x_{11}$, then, multiplying x by $D_1(x_{11}^{-1}) \in \operatorname{Ep}(2l, R_s)$, we get the matrix $x D_1(x_{11}^{-1}) \in F_s(H)\operatorname{GSp}(2l, R_s) \cap Y(R_s)$, which does not fall into $\operatorname{GSp}(2l, R_s)$. To finish the proof, it suffices to invoke Proposition 4 once again. $\square$

Proof of Proposition 6. Again P_1 is algebraic: it is defined by the equations $y_{i1} = 0$, $i \neq 1$. As always, Lemma 18 allows us to assume that $y \in F_s(H)\operatorname{GEp}(2l, R_s)$. As usual, there exists a sufficiently large n such that

$$z = y T_{1,-1}(s^n/1) y^{-1} = e + \sum s^n y'_{-1,j} e_{1j} \in F_s(H),$$

where the sum is taken over all $j = 1, \dots, -1$. Clearly, z satisfies the conditions of the preceding lemma, and if $z \notin \mathrm{GSp}(2l, R_s)$, we can finish here: the matrix z yields a nontrivial transvection. However, if $z \in \mathrm{GSp}(2l, R_s)$, then $y'_{-1,j} = 0$ for all $j \neq -1$, so that in this case the matrix y itself satisfies the conditions of the preceding lemma. $\square$

§11. Proof of Theorem 1

The core of the proof of Theorem 1 is the following alternative, which we verify by using localization.

Main Lemma. *Assume that $l \geq 2$. For $l = 2$ we stipulate additionally that R has no residue fields of two elements. Next, let H be a subgroup in* $\mathrm{GL}(2l, R)$ *that contains* $\mathrm{Ep}(2l, R)$. *Then either $H \leq \mathrm{GSp}(2l, R)$, or H contains a nontrivial elementary transvection of the form $t_{ij}(\xi)$, $i \neq \pm j$, $\xi \in R$.*

Proof. Let $g \in H$ be a matrix that does not belong to $\mathrm{GSp}(2l, R)$. By Lemma 11, there exists a maximal ideal $M \in \mathrm{Max}(R)$ such that $F_M(g) \notin \mathrm{GSp}(2l, R_M)$. Since the ring R_M is local, Lemma 6 and Proposition 1 jointly assert that $\mathrm{Ep}(2l, R_M)$ acts transitively on the columns of invertible matrices (this is a very special case of the corollary to Lemma 22 of the next section). This means that there exists a matrix $x \in \mathrm{GSp}(2l, R_M)$ such that the first column of the matrix $xF_M(g)$ coincides with the first column of the identity matrix. Since $\mathrm{Ep}(2l, R_M) = \varinjlim \mathrm{Ep}(2l, R_s)$, where the limit is taken over all $s \notin M$, there exists $s \notin M$ and a matrix $x \in \mathrm{Ep}(2l, R_s)$ such that already the first column of the matrix $y = xF_s(g)$ coincides with the first column of the identity matrix and, of course, $y \notin \mathrm{GSp}(2l, R_s)$. Now Proposition 6 implies that H must contain a nontrivial transvection, as claimed. $\square$

At this point, the proof of the theorem can be completed by reduction modulo the largest ideal $A \trianglelefteq R$ such that $E(2l, R, A) \leq H$.

Proof of the theorem. As above, let A be the largest ideal such that $E(2l, R, A) \leq H$. The existence of such an ideal was established in Proposition 3. Let $\overline{H} = \rho_A(H)$ be the image of the group H under the reduction homomorphism $\rho_A : \mathrm{GL}(2l, R) \longrightarrow \mathrm{GL}(2l, R/A)$. Clearly, the group $\overline{H}$ contains $\mathrm{Ep}(2l, R/A)$, and application of the main lemma shows that we have the following alternative: either $\overline{H} \leq \mathrm{GSp}(2l, R/A)$, or $\overline{H}$ contains a nontrivial elementary transvection $t_{ij}(\xi + A)$, $i \neq \pm j$, $\xi \in R \setminus A$. We show that the second possibility cannot occur. Indeed, presenting $t_{ij}(\xi) \in H\,\mathrm{GL}(2l, R, A)$ in the form $t_{ij}(\xi) = ab$, $a \in H$, $b \in \mathrm{GL}(2l, R, A)$, we consider the commutator $[t_{ij}(\xi), T_{j,-j}(1)] = t_{i,-j}(\xi)$. Substituting the expression for $t_{ij}(\xi)$, we get $t_{i,-j}(\xi) = [ab, T_{j,-j}(1)] = {}^a[b, T_{j,-j}(1)][a, T_{j,-j}(1)]$. The first of the commutators on the right-hand side belongs to $E(2l, R, A)$ by the standard commutator formula, while the second lies in H. This means that $t_{i,-j}(\xi) \in H$, where $\xi \notin A$, which contradicts the maximality of A. This shows that always $\overline{H} \leq \mathrm{GSp}(2l, R/A)$, but then Theorem 2 implies the desired inclusion

$$H \leq \mathrm{CGSp}(2l, R, A) = N_G(\mathrm{EEp}(2l, R, A)).$$

This finishes the proof of Theorem 1. $\square$

Thus, once again we get a "fan" description of the intermediate subgroups in the sense of Z. I. Borevich; see the references in [8, 85, 86].

§12. The use of stability conditions

We recall that a column $v = (v_1, \dots, v_{d+1})^t \in R^{d+1}$ with coefficients in R is said to be *stable* if there exist $b_1, \dots, b_d$ such that the *left* ideal generated by the elements $v_1 + b_1 v_{d+1}, \dots, v_1 + b_n v_{d+1}$ coincides with the left ideal generated by $v_1, \dots, v_{d+1}$. One says that the *stable rank* of the ring R does not exceed d, and one writes $\mathrm{sr}(R) \leq d$ if any *unimodular* column of height $d+1$ is stable. In its simplest version, the Bass theorem (see [1, Theorem 3.5 on page 196]) asserts that if R is a commutative ring and $\dim(\mathrm{Max}(R)) = d$, then $\mathrm{sr}(R) \leq d+1$. We mention a few immediate consequences of this theorem:

- $\mathrm{sr}(R) = 1$ if R is semilocal;
- $\mathrm{sr}(R) \leq 2$ if R is Dedekind;
- if K is a field, then $\mathrm{sr}(K[x_1, \dots, x_n]) \leq m+1$.

The following easy result, which is implied immediately by the *definition* of the stable rank, is known as the *Bass transitivity theorem* (see [1, Theorem 3.3 on page 192]).

Lemma 22. *If $n > \mathrm{sr}(R)$, then the group $E(n,R)$ acts transitively on the set of all unimodular columns of height n.*

In fact, for our purposes it suffices that, under the conditions of the theorem, the group $E(n,R)$ act transitively on the columns of invertible matrices. In other words, any column of a matrix in $\mathrm{GL}(n,R)$ can be completed to a matrix in $E(n,R)$. This statement is called the *surjective stability* theorem for the functor K_1 (see [1, Theorem 4.1 on page 197] or *any* paper on stability for K_1, e.g., [17] or [71]). As we already know, this result can also be stated as follows: $\mathrm{GL}(n,R)e_1 = \mathrm{GE}(n,R)e_1$. Proposition 1 and the transitivity theorem imply the following statement.

Corollary. *If R is a commutative ring and $n = 2l > \mathrm{sr}(R)$, then any unimodular column can be completed to a matrix in* $\mathrm{GEp}(2l,R)$.

In particular, this corollary *immediately* implies that the principal theorem of the present paper is valid for all rings of dimension not exceeding 2, because in this case $\mathrm{sr}(R) \leq 3$, so that all groups $\mathrm{Ep}(2l,R)$, $l \geq 2$, act transitively on the set of unimodular columns of height $2l$. Thus, if $g \in H$ is any matrix that does not belong to $\mathrm{GSp}(2l,R)$, then there exists a matrix $g \in \mathrm{Ep}(2l,R)$ such that $y = xg \in P_1$ and, of course, still $y \notin \mathrm{GSp}(2l,R)$. Applying Proposition 6 to the matrix y, we can conclude that H contains a transvection. This finishes the proof of the main lemma, and at the same time of Theorem 1 in the case in question.

This argument works, *for instance*, for the semilocal and Dedekind rings, thus covering all results of the papers [88] and [66] completely. Recall that most of the tricks we had to recourse to in the previous sections are superfluous in these cases. For example, this proof never appeals to any of the results of §§8 and 9, and in all lemmas of §10 we could set $s = 1$.

§13. Decomposition of unipotents

The following result is yet another variation of the theme of "decomposition of unipotents" (see [26, 9, 84, 73]). We use the straightforward version of the decomposition of unipotents in the symplectic group, proposed in [40]; in the case where $l \geq 3$ this version allows us to reduce all necessary calculations to calculations in the maximal parabolic subgroup P_1 of the group $\mathrm{GL}(2l,R)$. In this section we give yet another proof of the main lemma, which does not involve localization, but, instead, makes use of the following fundamental result, which is a special case of the "Theme" in [40] (=Variation 10 in [73, §15]).

Decomposition of unipotents. *Suppose R is an arbitrary commutative ring and $l \geq 3$, $2 \in R^*$. Then for any $x \in \mathrm{GL}(2l,R)$ the matrices $g \in \mathrm{Ep}(2l,R)$ such that $(gx)_{*i} = x_{*i}$ for some i, $1 \leq i \leq -1$, generate* $\mathrm{Ep}(2l,R)$.

Notice that all the proofs of this decomposition we are aware of invoke one of the following results:

- Description of the overgroups $\mathrm{Ep}(2l-2,R)$ in $\mathrm{GSp}(2l,R)$; this is a special case of the results of [10, 12, 13].
- Calculation of $H^1(\mathrm{Sp}(2l-2,R),R^{2l-2})$; this is a special case of the main result of [68].

Both calculations depend on the condition $2 \in R^*$.

In a slightly more fanciful version, decomposition of unipotents in the symplectic group was carried through in [9]; see, e.g., [85, §12] and [73, §14]. The method works starting with $l \geq 2$, and provides new proofs of various structure results, such as the Kopeiko–Taddei theorem, the structure of normal subgroups of $\mathrm{GSp}(2l,R)$, etc. Nevertheless, it is hard and, perhaps, impossible to use this method for the proof of our principal theorem, for the following reasons:

- It is based on the use of minors (see [26, 9] or [73, §6]) and is considerably more demanding technically.
- It requires reduction to *two* different types of maximal parabolic subgroups, namely, P_1 and P_l, as well as to reductive groups of smaller rank.
- Most importantly, it explicitly resorts to the equations on matrices in $\mathrm{GSp}(2l,R)$.

Proof of the main lemma under the conditions $l \geq 3$, $2 \in R^*$. By assumption, there exists $x \in H \setminus \mathrm{GSp}(2l,R)$. By the corollary to Theorem 2, there exists an elementary symplectic matrix $g \in \mathrm{Ep}(2l,R)$ such that $[g,x] \notin \mathrm{GSp}(2l,R)$. Decomposition of unipotents implies that g can be expressed in the form $g = g_1 \cdots g_s$, where $g_i \in \mathrm{Ep}(2l,R)$, and each g_i, $1 \leq i \leq s$, stabilizes at least one column of the matrix x^{-1}, i.e., $(g_i x^{-1})_{*j} = (x^{-1})_{*j}$ for a suitable j, $1 \leq j \leq -1$. Since $[g,x] \notin \mathrm{GSp}(2l,R)$, we have $[g_i,x] \notin \mathrm{GSp}(2l,R)$ for at least one i, so that H contains a matrix $y = x g_i x^{-1} \notin \mathrm{GSp}(2l,R)$ for which $y_{*j} = (x g_i x^{-1})_{*j} = (x x^{-1})_{*j} = e_{*j}$ by our choice of j. Conjugating by an element of the Weyl group W, without loss of generality we can assume that H contains a matrix $y \notin \mathrm{GSp}(2l,R)$ whose *first* column coincides with the first column of the identity matrix. It only remains to invoke Proposition 5. □

§14. The groups $\mathrm{SL}(2,R)$ and $\mathrm{Ep}(2,R)$ over a noncommutative ring

The most remarkable advance in our understanding of the structure of symplectic groups over rings was the paper [44] by Costa and Keller. The most important conceptual (as well as technical!) aspect of that paper consists in the observation that the major difference between the symplectic group and all other Chevalley groups, and all specific complications arising in the analysis of the symplectic case, are explained by the fact that in reality the group Sp_{2l} of *any* rank is isomorphic to the group SL_2 over another ring and behaves as a group of rank 1.

Namely, let A be a ring with involution, i.e., an antiautomorphism $a \mapsto \overline{a}$ of order 2. In general, there is no analog of the determinant in the group $\mathrm{GL}(2,A)$. Nevertheless, it is more or less clear what should be meant by $\det(x) = 1$. As in the commutative case, it is natural to define the *special linear group* $\mathrm{SL}(2,A) = \mathrm{Sp}(2,A)$ by the condition

$$\mathrm{SL}(2,A) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in \mathrm{GL}(2,A) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \overline{d} & -\overline{b} \\ -\overline{c} & \overline{a} \end{pmatrix} \right\}.$$

Involution on the right-hand side serves to take into account the noncommutativity of the ring A and guarantee that $\mathrm{SL}(2,A)$ is indeed a group. When the involution is trivial (and, consequently, the ring A is commutative!) we recover the usual group $\mathrm{SL}(2,A)$. Another most important example of a ring with involution is $A = M(l,R)$, where R is commutative, while the involution is given by the transposition $a \mapsto a^t$ (for this operation to be an involution, the ring R must be commutative!). In this case the equations that determine $\mathrm{SL}(2, A)$ coincide with the following:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix}.$$

Thus, $\mathrm{SL}(2, M(l, R)) = \mathrm{Sp}(2l, R)$ (see [2, §12] or [44, §1]). Precisely this realization, which makes it possible to imitate all calculations of the paper [43], constitutes the crux of the approach taken in [44]. All calculations of the present paper can easily be translated into this language. Even more, some of them would take a shorter and more natural shape. The reason why we, nevertheless, have not taken this approach is as follows: this would severely obscure analogy with other Chevalley groups over commutative rings, so that our calculations would lose their value as a pattern for analysis of such situations as, say, embedding of F_4 in E_6.

We return to the general case. It is easily seen that the elementary transvections

$$t_{12}(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad t_{21}(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

belong to $\mathrm{SL}(2, A)$ if and only if the element a is *Hermitian*, i.e., $\overline{a} = a$. Let $H(A)$ denote the set of all Hermitian elements of the ring A. Then it is natural to define the *elementary symplectic group* $\mathrm{Ep}(2, A)$ by

$$\mathrm{Ep}(2, A) = \langle t_{12}(a), t_{21}(a), a \in H(A) \rangle.$$

It is easy to check that $\mathrm{Ep}(2, M(l, R)) = \mathrm{Ep}(2l, R)$, so that this definition agrees with the usual definition (observe that in [2, §12] the group $\mathrm{Ep}(2l, R)$ was *defined* precisely like that).

From this viewpoint, in the present paper we talk about description of the subgroups in $\mathrm{GL}(2, A)$ that contain $\mathrm{Ep}(2, A)$, and we completely[3] solve this problem for the case of $A = M(l, R)$, where the ring R is commutative, involution on A is transposition, and $l \geq 2$. It is natural to pose a similar question in the general case.

Problem 2. *Describe the subgroups in* $\mathrm{GL}(2, A)$ *that contain* $\mathrm{Ep}(2, A)$.

In this setting, conditions on the rank could be stated in terms of the existence of nontrivial orthogonal idempotents in the ring A.

Let us mention a natural variation on this theme, in the spirit of the early papers by Z. I. Borevich and the first author. Clearly, the subgroup $\mathrm{SD}(2, A)$ of diagonal matrices contained in $\mathrm{SL}(2, R)$ has the form

$$\mathrm{SD}(2, A) = \left\{ d_{12}(a) = \begin{pmatrix} a & 0 \\ 0 & \overline{a}^{-1} \end{pmatrix},\ a \in A^* \right\}.$$

In the classical case, this is precisely the group $\mathrm{GL}(l, R)$ in the hyperbolic embedding in $\mathrm{Sp}(2l, R)$. However, in accordance with our general philosophy, it is natural to consider not all diagonal matrices, but only the "elementary" ones, i.e., the matrices that fall into $\mathrm{Ep}(2, R)$. Set $T(2, A) = \mathrm{SD}(2, A) \cap \mathrm{Ep}(2, A)$. In the classical case $T(2, M(2, l))$ contains *at least* $\mathrm{GE}(l, R)$ in the hyperbolic embedding (see [2, Lemma 13.1]).

Problem 3. *Describe the subgroups in* $\mathrm{GL}(2, A)$ *that contain* $T(2, A)$.

Notice that a description of the overgroups of $T(2, K)$ in $\mathrm{GL}(2, A)$ is known in the following cases:

[3] Apart from the exceptional case where $l = 2$ and the ring R has a residue field $\mathbb{F}_2$, but an exhaustive analysis of this case would require a modification of the answer and the full force of the methods developed in [44]!

- $A = K$ is a field (for the trivial involution, this is a result of J. King [56]; in the special cases where $-1 \in K^{*2}$ or $|K^*| > |K^*/K^{*2}|$ it was established earlier by the first author and E. V. Dybkova in [15]; for a nontrivial involution this is a recent result of Dybkova [89]).
- $A = T$ is a skew-field with an involution of orthogonal type (this is a very recent result of Dybkova [90]).
- $A = M(n, R)$, R is commutative, and $l \geq 3$ (this is a *very* special case of the results of the first author; see [9, 10, 12, 15]).

§15. Concluding remarks

We list some further problems that have already been solved or can be easily solved by the methods of [16] and the present paper.

- Describe the overgroups of $\mathrm{EO}(2l+1, R)$ in $\mathrm{GL}(2l+1, R)$.
- Describe the overgroups of $\mathrm{EU}(2l, R, \Lambda)$ in $\mathrm{GL}(2l, R)$ (see [69]).
- Generalize the results of [16] to overgroups of nonsplit, but sufficiently isotropic orthogonal groups $\mathrm{EO}(n, R, f)$.
- Describe the overgroups of the elementary Chevalley group of type F_4 in the Chevalley group of type E_6.

The last problem is closely related to the content of the present paper, since in both cases we consider simply a Chevalley group $G(\Phi, R)$ and its twisted form ${}^2G(\Phi, R)$ (see [42, 25]). Here the involution on R is trivial, so that twisting occurs exclusively as a result of folding roots. It would be of interest to consider the following generalization: take two groups that have a representation where the orbits of the highest weight vectors under the action of elementary groups coincide.

Let us mention another problem whose complete solution would be significant, since it would serve as a *common* generalization of a huge number of preceding results, *in particular*, [87], [20]–[22], [78, 8], [81]–[83], [61, 62, 46, 47, 48, 39, 40, 54, 55, 63, 64], and many others.

Problem 4. *Describe the subgroups in* $\mathrm{GL}(n, R)$ *normalized by an elementary classical group.*

Under the assumption that $2 \in R^*$, it is clear that the *standard* answer to this problem can be stated in terms of pairs $A \leq B$ of ideals of the ring R, and the authors intend to return to this theme in a subsequent paper. The simplest examples show that for the general case the standard description fails drastically, and this answer requires modification in the spirit of form ideals [34, 35, 50, 39, 40], admissible pairs [30, 31, 32], quasiideals [80, 81, 82], Jordan ideals, or radices [43, 44]. At the present stage it is not completely clear what the answer will look like without the invertibility of 2.

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