Алгебра и анализ
Том. 15 (2003), вып. 4

St. Petersburg Math. J.
Vol. 15 (2004), No. 4, Pages 545–561
S 1061-0022(04)00821-0
Article electronically published on July 6, 2004

SOBOLEV SPACE ESTIMATES FOR SOLUTIONS OF EQUATIONS WITH DELAY, AND THE BASIS OF DIVIDED DIFFERENCES

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Abstract. Sharp Sobolev space estimates for solutions of neutral difference-differential equations with arbitrary index are obtained without the assumption that the roots of the characteristic quasipolynomial are separated. The proof is based on the fact that the system of divided differences of the exponential solutions forms a Riesz basis. Moreover, it is proved that, under more general conditions, the system of exponential solutions is minimal and complete.

§0. Introduction

In the present paper, we study the initial problem for the neutral difference-differential equation

$$a_{0m}u^{(m)}(t) + \cdots + a_{km}u^{(m)}(t-h_k) + \cdots + a_{nm}u^{(m)}(t-h_n) = 0, \quad t > 0,$$

with several delays and with the initial condition

$$u|_{(-h_n,0)} = g.$$

Here, the terms with integrals and the terms with lower derivatives are omitted. The solutions are studied in the scale of Sobolev spaces H^s for $s \geq m$. The basic assumption is that $a_{0m} \neq 0$ and $a_{nm} \neq 0$. Under this assumption, the zeros $\{\lambda_q\}$ of the characteristic quasipolynomial $L(\lambda)$ lie in the strip $\varkappa_- \leq \operatorname{Re} z \leq \varkappa_+$, where $\varkappa_+ := \sup \operatorname{Re} \lambda_q$ and $\varkappa_- := \inf \operatorname{Re} \lambda_q$. Using some properties of the family of exponential solutions, we prove the following sharp estimates for the solutions:

$$\|u\|_{H^s(T-h,T)} \leq d(T+1)^{M-1} e^{\varkappa_+ T} \|g\|_{H^s(-h,0)}, \quad T \geq 0, \tag{0.1}$$

$$\|u\|_{H^s(T-h,T)} \geq c(g) e^{\varkappa_- T}, \quad c > 0, \tag{0.2}$$

where the constant M is determined by the function $L(\lambda)$.

We note that the estimates obtained are valid also for all $s \geq m$ such that s is not a half-integer even if the zeros of $L(z)$ are not separated.

Under the separation assumption, the above estimates were obtained for all integral s in [7], [8]; for the nonintegral s, they were announced in [9], [27].

Estimates similar to (0.1) but with $\varkappa_+$ replaced by $\varkappa_+ + \varepsilon$, $\varepsilon > 0$, have long been known; see, e.g., [10], [14], [15]. Our estimate (0.1) is sharp in the following sense: the constants $\varkappa_\pm$ cannot be replaced by $\varkappa_+ - \varepsilon$ and $\varkappa_- + \varepsilon$, respectively, with any $\varepsilon > 0$. Moreover, the exponent M also cannot be reduced to fit for all g.

The proof is based on the properties of the family V of exponential solutions of the problem (in the case where the quasipolynomial $L(\lambda)$ has only simple roots λ_q, this family

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2000 *Mathematics Subject Classification.* Primary 34K40, 42B30, 46E35.
Key words and phrases. Equations with delay, exponential families, Riesz basis, Sobolev space.
Supported by RFFR (grants nos. 02-01-00790, 00-15-96100, 02-01-00554).

has the form $\{e^{\lambda_q t}\}$). The case where the roots are separated and s is an integer was studied by the first author in [7], [8]. In the present paper, we succeeded in lifting the above-mentioned restrictions by applying two recent results of the second author [1], [6].

For $m = 1$ (see (1.3)), estimates of the form (0.1) and the basis property of exponential solutions in the vector case (i.e., when the a_{kj} are square matrices) in the scale of Sobolev spaces of vector-valued functions with integral index were treated in [16], [17]. With a different notion of solutions, the basis property of the exponential solutions in the space $L^2((-h,0),\mathbb{C}^m)\oplus\mathbb{C}^m$ was studied in [20]. The completeness property of the exponential solutions was studied by many authors (starting, apparently, with N. Levinson [21]) both in the scalar and vector cases. We mention only the papers [18]–[20] (see also the references therein).

The outline of the present paper is as follows. After giving a definition of the solution of a problem in H^s and stating the compatibility conditions, we introduce some family of exponential solutions V and study its completeness and minimality. The study of decompositions with respect to V is split into several steps. Since we do not assume that the zeros $\{\lambda_q\}$ are separated, the family V may fail to constitute a Riesz basis in a subspace of H^s. Therefore, first we consider bases of subspaces generated by exponentials with close points λ_q. In each subspace of this form we take the basis formed by divided differences (functions of the form $e^{\mu t}$, $(e^{\mu t}-e^{\lambda t})/(\mu-\lambda)$, etc.), obtaining a family Φ.

If we remove m elements, then Φ becomes a basis in $L^2(-h,0)$ (see [1]). Using this fact and an idea of D. Russel [22], we see that, in the spaces H^m, $H^{m+1},\dots$, the initial family of divided differences forms a Riesz basis in the closure of its linear span, and its codimension is equal to $0,1,\dots$, respectively. To prove the basis property in a Sobolev space with nonintegral index, we use the results of the paper [6] on interpolation of subspaces and the basis property of exponentials in intermediate spaces. It turns out that Φ is a Riesz basis in the closure of its linear span in H^s for all s except the singular points $s = m+1/2$, $m+3/2,\dots$, and in the latter case Φ is a Riesz basis in a space with a stronger metric. Imposing the necessary compatibility conditions at $t = 0$ that ensure the solvability of the problem, we obtain a Riesz basis in the subspace $H^s_U \subset H^s$ of admissible initial functions.

The upper estimates are obtained by expanding the initial function g with respect to the resulting basis. The lower estimates are obtained in a similar way, but unlike the upper estimates, the lower ones are not uniform with respect to g: the constant $c(g)$ in (0.2) cannot be replaced with $c\|g\|$. At the singular points $s = m+1/2$, $m+3/2,\dots$ similar estimates are valid in spaces with metric stronger than that in H^s. The main results of the present paper were announced in [27].

§1. The initial problem for a neutral difference-differential equation

We consider the following initial problem:

$$\sum_{j=0}^{m}\sum_{k=0}^{n} a_{kj}u^{(j)}(t-h_k)+\sum_{j=0}^{m}\int_0^h B_j(\tau)u^{(j)}(t-\tau)\,d\tau = 0, \quad t>0, \tag{1.3}$$

$$u(t) = g(t), \quad t\in[-h,0]. \tag{1.4}$$

Here $0 = h_0 < h_1 < \cdots < h_n =: h$, the a_{kj} are complex coefficients, and the functions $B_j(\tau)$ are in $L^2(0,h)$. The function g is assumed to belong to the Sobolev space $H^s(-h,0)$ with $s \ge m$. For the definition and properties of Sobolev spaces, see the rest of this section and also [5, Chapter I].

Definition. By a solution of problem (1.3), (1.4) we mean a function u that belongs to $H^s(-h,T)$ for every $T>0$ and satisfies equation (1.3) in $H^{s-m}(0,T)$ and the initial condition (1.4) in $H^s(-h,0)$.

Let H^s_U denote the subspace of functions of class $H^s(-h,0)$ that satisfy the following $[s-m+1/2]$ compatibility conditions:

$$\sum_{j=0}^{m}\sum_{k=0}^{n} a_{kj} g^{(j+r)}(-h_k) + \sum_{j=0}^{m}\int_0^h B_j(\tau) g^{(j+r)}(-\tau)\,d\tau = 0, \tag{1.5}$$

$r=0,1,\dots,[s-m-1/2]$. By the trace theorem (see [5]), these conditions are necessary for the existence of smooth solutions if $s>m+1/2$.

We recall the definition of the Sobolev space for an arbitrary positive p. We define $H^p(\mathbb{R})$ to be the space of all $f\in L^2(\mathbb{R})$ such that

$$\|f\|^2_{H^p} := \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2(1+|\xi|^{2p})\,d\xi < +\infty,$$

where $\widehat{f}$ is the Fourier transform of f. For an interval $(-h,0)$, by definition, the Sobolev space $H^p=H^p(-h,0)$ consists of all restrictions of the functions in $H^p(\mathbb{R})$ to the interval $(-h,0)$ (with the corresponding quotient norm). It is well known that if $p=m$, then $H^m(-h,0)$ is the space of all functions $f\in L^2(-h,0)$ such that $f^{(m)}\in L^2(-h,0)$, with the following (equivalent) norm $\|f\|_{H_m}$:

$$\|f\|^2_{H^m} = \int_{-h}^{0} (|f(t)|^2 + |f^{(m)}(t)|^2)\,dt < \infty.$$

It is important that, for $0<p<1$, the space H^p is an interpolation space: $H^p = [L^2,H^1]_p = [L^2,H^1]_{p,2}$. For the details, see [5, Chapter I].

§2. Characteristic function and exponential solutions

We define the characteristic function $L(\lambda)$ of equation (1.3) as follows:

$$L(\lambda) := \sum_{j=0}^{m}\sum_{k=0}^{n} a_{kj}\lambda^j e^{-\lambda h_k} + \sum_{j=0}^{m}\lambda^j\int_0^h B_j(\tau)e^{-\lambda\tau}\,d\tau.$$

Let $\Lambda=\{\lambda_q\}$ be the zeros of the characteristic function, and let ν_q be the multiplicity of λ_q; we arrange the λ_q in the ascending order of moduli (taking multiplicities into account). Consider the following family $V=\{v_{qj}\}$ of exponential solutions of (1.3):

$$v_{qj}(t) = t^j\exp(\lambda_q t), \quad j=0,1,\dots,\nu_q-1. \tag{2.6}$$

We note that the family

$$w_{qj}(t) = \frac{t^j\exp(\lambda_q t)}{|\lambda_q|^s+1}, \quad j=0,1,\dots,\nu_q-1,$$

is almost normalized in the space H^s. For nonintegral s this fact was proved in [6] in the case where the zeros of $L(\lambda)$ are simple. In the case of multiple zeros, the proof is similar.

If all terms involving integrals vanish, the function $L(\lambda)$ is called the characteristic quasipolynomial of equation (1.3) (see [10]).

To investigate the family of exponential solutions, we shall study the behavior of the characteristic function in the entire complex plane.

In what follows, we write

$$f(x)\asymp g(x), \quad x\in X,$$

if there exist positive constants c and C independent of x and such that

$$cg(x) \le f(x) \le Cg(x), \quad x \in X.$$

We use the symbols $\prec$ and $\succ$ for one-sided estimates of this type.

In the following lemma, we extract the principal parts of the characteristic function as $\operatorname{Re}\lambda \to +\infty$, $\operatorname{Re}\lambda \to -\infty$, and in the strip parallel to the real axis. To express the principal parts in terms of coefficients, we impose some subordination conditions on the integral terms.

Lemma 2.1. a) *Suppose that $a_{0j_+} \neq 0$ for some j_+, and that the coefficients a_{0j} and the integral terms $B_j(t)$ are zero for $j > j_+$. Then, as $\operatorname{Re}\lambda \to +\infty$, the characteristic function has the asymptotics*

$$L(\lambda) = a_{0j_+}\lambda^{j_+}(1 + o(1)) \tag{2.7}$$

uniformly with respect to $\operatorname{Im}\lambda$.

b) *Suppose that $a_{nj_-} \neq 0$ for some j_-, and that the coefficients a_{nj} and the integral terms $B_j(t)$ are zero for $j > j_-$. Then, as $\operatorname{Re}\lambda \to -\infty$, in the region $|\operatorname{Re}\lambda| > c\ln|\operatorname{Im}\lambda|$ with sufficiently large c, the characteristic function has the asymptotics*

$$L(\lambda) = a_{nj_-}\lambda^{j_-}e^{-\lambda h}(1 + o(1)) \tag{2.8}$$

uniformly with respect to $\operatorname{Im}\lambda$. If $a_{nm} \neq 0$, then this asymptotics remains valid as $\operatorname{Re}\lambda \to -\infty$ without any restrictions on the real and imaginary parts of λ.

c) *If at least one of the coefficients a_{km}, $k = 0, 1, \dots, n$, is nonzero, then, for sufficiently large c, C, and R, on the set $\{\lambda | c < \operatorname{Re}\lambda < C, |\lambda| > R\}$ the characteristic function admits the estimate*

$$|L(\lambda)| \asymp |\lambda|^m \tag{2.9}$$

uniformly with respect to $\operatorname{Im}\lambda$.

Proof. a) As $\operatorname{Re}\lambda \to +\infty$, the principal contribution of the nonintegral terms into the behavior of $L(\lambda)$ is given by the terms with minimal shift, and among them, by the terms with the largest power of λ. Under the conditions of statement a), the characteristic function can be represented in the form

$$L(\lambda) = a_{0j_+}\lambda^{j_+}\Bigg(1 + \sum_{j<j_+}\frac{a_{0j}}{a_{0j_+}}\lambda^{j-j_+} + \sum_{k=1}^{n}\sum_{j=0}^{m}\frac{a_{kj}}{a_{0j_+}}e^{-\lambda h_k}\lambda^{j-j_+} + \frac{1}{a_{0j_+}}\sum_{j=0}^{j_+}\lambda^{j-j_+}\int_0^h B_j(\tau)e^{-\lambda\tau}\,d\tau\Bigg).$$

By the Riemann–Lebesgue lemma, the integral terms tend to zero, which implies a).

b) As $\operatorname{Re}\lambda \to -\infty$, the principal contribution to the behavior of the function $L(\lambda)$ is given by the terms with the maximal shift. Therefore, the representation

$$L(\lambda) = a_{nj_-}\lambda^{j_-}e^{-\lambda h}\Bigg(1 + \sum_{j<j_-}\lambda^{j-j_-}\frac{a_{nj}}{a_{nj_-}} + \sum_{k<n}\sum_{j=0}^{m}\frac{a_{kj}}{a_{nj_-}}\lambda^{j-j_-}e^{\lambda(h-h_k)} + \frac{1}{a_{0j_-}}\sum_{j=0}^{j_-}\lambda^{j-j_-}\int_0^h B_j(\tau)e^{\lambda(h-\tau)}\,d\tau\Bigg)$$

implies b).

c) For $c < \operatorname{Re}\lambda < C$, if c is large, the principal contribution to the behavior of $L(\lambda)$ is given by the terms with the largest power, and among them, by the terms with the minimal shift. This yields c). Lemma 2.1 is proved. □

We note that many authors studied the behavior of the function $L(\lambda)$ obtaining estimates of $L(\lambda)$ in the case where $B_j(\tau) \equiv 0$, $j = 0, \dots, m$. Here, we only mention the book [10] (see also the references therein).

2.1. Completeness and minimality of the family of exponential solutions. Here, we use the characteristic function in order to study the completeness and minimality properties of the family V of exponential solutions (see (2.6)).

Theorem 2.1. a) *The family V is minimal in the space $H^p(-h,0)$ for $p > m - 1/2$.*

b) *Under any of the conditions occurring in statements* a)–c) *of Lemma 2.1, the family V is complete in the spaces $H^m(-h,0)$ and H^s_U.*

Proof. a) For simplicity, we only consider the case where the function L has simple zeros. Then $V = \{v_q\} = \{e^{\lambda_q t}\}$. It is also convenient to talk of a dual family rather than of a biorthogonal one. We show that, for any $\varepsilon > 0$, there is a family ψ_q of nonzero functionals over $H^p(-h,0)$, $p = m - 1/2 + \varepsilon$, such that

$$\psi_q(v_r) = 0 \quad \text{for } q \neq r, \qquad \psi_q(v_q) = 1. \tag{2.10}$$

The space dual to $H^p(-h,0)$, $p > 0$, is the set of distributions in $H^{-p}(\mathbb{R})$ with support in $[-h,0]$.

On the set $\mathcal{E}'$ of functionals over the space of infinitely differentiable functions defined on $[-h,0]$, we consider the Fourier–Laplace transformation

$$\widehat{\varphi}(\lambda) := \int_{-i\infty}^{i\infty} e^{-\xi\lambda}\varphi(\xi)\,d\xi.$$

We have

$$\widehat{\varphi}(\lambda) = \langle \varphi, e^{-\lambda \cdot} \rangle, \tag{2.11}$$

where $\langle \varphi, f(\cdot) \rangle$ is the value of the functional φ at f. For each q, the function $\widehat{\psi}_q(\lambda) := \frac{L(\lambda)}{L'(\lambda_q)(\lambda-\lambda_q)}$ is an entire function of exponential type 0 in the left half-plane and of type h in the right half-plane. The explicit form of the characteristic function $L(\lambda)$ shows that

$$\int_{-\infty}^{\infty} \frac{|L(iy)|^2}{(1+|y|^{p+1})^2}\,dy < \infty.$$

For $\widehat{\psi}_q$, we obtain

$$\int_{-\infty}^{\infty} \frac{|\widehat{\psi}_q(iy)|^2}{(1+|y|^p)^2}\,dy < \infty.$$

By the Paley–Wiener theorem, there exists a functional ψ_q in $[H^p(-h,0)]'$ that is the preimage of $\widehat{\psi}_q$ under the Fourier–Laplace transformation:

$$\langle \psi_q, e^{-\lambda \cdot} \rangle = \widehat{\psi}_q(\lambda) = \frac{L(\lambda)}{L'(\lambda_q)(\lambda - \lambda_q)}.$$

This yields the required family of functionals.

b) Assume that the family V of exponential solutions is not complete in $H^m(-h,0)$. Then there exists a nonzero function $f(t)$ in $H^m(-h,0)$ orthogonal to V. We show that this leads to a contradiction.

We consider the function

$$F(\lambda) = F_1(\lambda)/L(\lambda), \quad \text{where } F_1(\lambda) := \int_{-h}^{0} \lambda^m e^{\lambda t} \bar{f}^{(m)}(t)\,dt + \int_{-h}^{0} e^{\lambda t} \bar{f}(t)\,dt,$$

and prove that F is zero.

Since $f(t)$ is orthogonal to the exponential solutions $e^{\lambda_q t}$, $te^{\lambda_q t}, \dots,$ $t^{\nu_q-1}e^{\lambda_q t}$ in $H^m(-h,0)$, and the function $L(\lambda)$ has zero of order ν_q at λ_q, we see that the function $F(\lambda)$ is regular at all points λ_q; consequently, F is an is entire function.

First, we obtain an estimate for the numerator $F_1(\lambda)$ of $F(\lambda)$. In the entire complex plane, we have the following estimate for F_1:

$$|F_1(\lambda)| \prec |\lambda|^m(1+|e^{-\lambda h}|)/|\operatorname{Re}\lambda|^{1/2}. \tag{2.12}$$

In an arbitrary strip parallel to the imaginary axis, by the Riemann–Lebesgue lemma, we have the asymptotic formula

$$F_1(\lambda) = \lambda^m o(1), \quad |\lambda| \to +\infty. \tag{2.13}$$

Estimates (2.7), (2.8), and (2.12) imply that, for every $\varepsilon > 0$, there exists $R > 0$ such that in the region $\{\lambda \mid |\lambda| > R, |\arg\lambda - \pi/2| < \varepsilon, |\arg\lambda + \pi/2| < \varepsilon\}$ the function $F(\lambda)$ admits the estimate

$$|F(\lambda)| \le d_1|\lambda|^w \frac{1}{|\operatorname{Re}\lambda|^{1/2}}, \tag{2.14}$$

where $w := \max\{m - j_+, m - j_-\}$.

Using (2.14) and the fact that $F(\lambda)$ is an entire function of exponential type, and applying the standard argument based on the Phragmèn–Lindelöf theorem, we obtain the estimate

$$|F(\lambda)| \le d_2|\lambda|^w$$

in the entire complex plane. It follows that $F(\lambda)$ is a polynomial of degree at most w.

By statement c) of Lemma 2.1, only finitely many zeros lie in the strip $\{c < \operatorname{Re}\lambda < C\}$ for sufficiently large c.

Now, from (2.9) and (2.13) it follows that

$$|F(\lambda)| \to 0, \quad \operatorname{Im}\lambda \to \pm\infty, \quad \operatorname{Re}\lambda = \alpha,$$

whence $F(\lambda) \equiv 0$. Thus, for all λ we have

$$\int_{-h}^{0} \lambda^m e^{\lambda t} \bar{f}^{(m)}(t)\,dt + \int_{-h}^{0} e^{\lambda t}\bar{f}(t)\,dt \equiv 0.$$

To finish the proof of the fact that the family V is complete in $H^m(-h,0)$, it remains to observe that the above relation implies the identity $f(t) \equiv 0$, because the family $\{e^{\lambda t}\}_{\lambda\in\mathbb{C}}$ of exponentials is complete in $H^m(-h,0)$ (see, e.g., [22]).

Now, we study the completeness of V in $H^p(-h,0)$ for $m < p \le m+1$. For this, with the characteristic function $L(\lambda)$ we associate the functional ψ_L the Fourier–Laplace transform of which is $L(\lambda)$, $\widehat{\psi}_L = L$. By (2.11), ψ_L is vanishes at all exponential solutions, and we conclude that V is not complete in $H^p(-h,0)$ if ψ_L is bounded on $H^p(-h,0)$. Since $|L(\lambda)| \asymp |\lambda|^m$ on the line $\operatorname{Re}\lambda = \alpha$ by Lemma 2.1, we see that, on this line, the characteristic function is square-integrable with the weight $(1+|\lambda|^{2q})^{-1}$ if and only if $q > m + 1/2$. Therefore, the functional ψ_L is bounded on $H^p(-h,0)$ for $q > m+1/2$. Thus, we have proved that the family V is not complete in $H^p(-h,0)$ for $p > m+1/2$. Now, we check that V has defect 1 (the codimension of the closure of the linear span) if $m + 1/2 < p \le m+1$. Indeed, we take a point $\mu \notin \Lambda$ and add the element $e^{-\mu t}$ to V. The entire function $L(\lambda)(\lambda - \mu)$ corresponds to the new element. Using the same argument as in the proof of the completeness of V in H^m, we show that the new family is complete in H^{m+1}. Obviously, this family remains complete also in H^p for $p < m+1$.

Now, assume that V is not complete in H^p for $p \le m+1/2$. Then the defect of V is equal to 1, and there exists a functional ψ the kernel of which coincides with the closure of the linear span of V in H^p. The Fourier–Laplace transform $\widehat{\psi}$ of this functional is an entire function of exponential type, and the zeros of it coincide with Λ (with multiplicities taken into account). Indeed, should $\widehat{\psi}$ have an extra zero, we would have dealt with a complete family, as the following lemma shows.

Lemma 2.2. *Suppose the exponential family $\tilde{V} = \{t^j \exp(\mu_q t)\}$ with spectrum $\widetilde{\Lambda}$ is minimal in $H^p(-h,0)$, and its defect is 1. Then, adding to $\tilde{V}$ the element $e^{\mu t}$ with $\mu \notin \widetilde{\Lambda}$ or the element $t^r e^{\mu_q t}$ (if $e^{\mu_q t}$, $te^{\mu_q t}$, $\dots$, $t^{r-1}e^{\mu_q t}$ are already in $\tilde{V}$), we obtain a complete family.*

Proof. The arguments are the same as in the proof of a similar statement for the space of square-integrable functions (see [13]). For completeness, we outline the proof in the case where the spectrum Λ is simple. Let $e^{\mu t} \in \bigvee_{H^p(-h,0)} \widetilde{V}$. Then $\mathcal{H} := \bigvee_{H^p(-h,0)} e^{-\mu t}\widetilde{V}$ contains the function identically equal to 1. Since integration is a continuous operation and the indefinite integrals of the elements of $\mathcal{H}$ lie in $\mathcal{H}$, we have $t \in \mathcal{H}$, $t^2 \in \mathcal{H}$, $\dots$, so that the set of polynomials is dense in $\mathcal{H}$. Consequently, $\mathcal{H}$ coincides with H^p. Therefore, $\bigvee_{H^p(-h,0)} \widetilde{V}$ coincides with H^p, which is impossible. The lemma is proved. □

Now, we consider the function $\widehat{\psi}/L$. This function has no zeros, and its order and type do not exceed those of $\widehat{\psi}$ and L (see [23, p. 35]); $\widehat{\psi}/L = \exp(a\lambda + b)$. In this case, $\widehat{\psi}$ cannot be square-integrable with the weight $(1 + |\lambda|^p)^{-1}$ on the lines parallel to the imaginary axis. This contradicts the fact that the function $\widehat{\psi}$ is bounded on H^p. This proves the completeness in H^p for $p \le m + 1/2$.

To find the defect in $H^p(-h,0)$ for $p > m+1$, we need the following lemma.

Lemma 2.3. *Let ψ be a bounded functional on $H^p(-h,0)$, $p \ge 1$, and let $\psi(1) \ne 0$. Let B_ψ be the operator from $\operatorname{Dom} B_\psi = \operatorname{Ker}\psi$ to H^{p-1} defined by the relation $B_\psi f = \frac{d}{dt} f$. Then B_ψ yields an isomorphism between $\operatorname{Dom} B_\psi \subset H^p$ and H^{p-1}.*

Proof. Consider the integration operator J,

$$(Jf)(x) = \int_0^x f(t)\,dt.$$

Then J is bounded as an operator from H^{p-1} to H^p. The inverse to B_ψ can be represented in the form

$$(B_\psi^{-1}\varphi)(x) = (J\varphi)(x) - \frac{1}{\psi(1)}\psi(J\varphi).$$

Consequently, this operator is bounded as an operator from H^{p-1} to $\operatorname{Dom} B_\psi$. The lemma is proved. □

Now, we finish the proof of Theorem 2.1. From Lemma 2.3 and the properties of the family V in $H^p(-h,0)$ for $m < p \le m+1$ (see above), we conclude that the defect of V in $H^s(-h,0)$ is equal to $[s - m + 1/2]$. The subspace H^s_U is determined by this number of functionals (1.5), which are zero at all elements of V. Theorem 2.1 is proved. □

Remark 2.1. The proof of the completeness of the family of exponential solutions in the space $H^m = W^m_2$ can be extended to the case of the Banach space W^m_p, $p \ge 1$. This generalizes the well-known result of N. Levinson and C. McCalla [21].

Remark 2.2. From the proof of Lemma 2.2 it follows that, in the Sobolev spaces as well as in L^2, any nonminimal family of exponentials is complete.

Now, we state a result on the asymptotic behavior of the solutions of equation (1.3). We show that any solution of (1.3) that decays faster than any exponential is identically zero. Such statements are called theorems of Phragmèn–Lindelöf type (or statements about small solutions). Let U_α denote the subset of solutions $u(t)$ of (1.3) such that $e^{\alpha t}u(t) \in L^2(\mathbb{R}_+)$, $\alpha > 0$.

Theorem 2.2. *Under condition* b) *of Theorem* 2.1, *if* $u(t) \in \bigcap_{\alpha \geq 0} U_\alpha$, *then* $u(t) \equiv 0$.

Proof. This theorem can be proved in the same way as Theorem 2.1 (b) on the completeness of exponential solutions. We outline the arguments in the case where $B_j(\tau) \equiv 0$, $j = 1, 2, \dots, m$.

Consider the Laplace transform $\widehat{u}(\lambda)$ of a solution $u(t)$. Integrating by parts, we obtain

$$\widehat{u}(\lambda) = L^{-1}(\lambda) q(\lambda),$$

where

$$q(\lambda) = \sum_{k=0}^{n} \sum_{j=0}^{m} a_{kj} \left(\sum_{p=0}^{j-1} g^{(p)}(-h_k) \lambda^{j-p-1} - \lambda^j e^{-\lambda h_k} \int_{-h_k}^{0} e^{-\lambda t} g(t)\, dt \right).$$

The relation $u(t) \in U_\alpha$ implies that the function $\widehat{u}(\lambda)$ admits holomorphic extension to the half-plane $\{\lambda : \operatorname{Re} \lambda > -\alpha\}$. Now, since $u(t) \in \bigcap_{\alpha > 0} U_\alpha$, we see that $\widehat{u}(\lambda)$ is an entire function, and the form of $q(\lambda)$ shows that $\widehat{u}(\lambda)$ is an entire function of exponential type at most h.

As in the proof of Theorem 2.1, we can easily show that the function $\widehat{u}(\lambda)$ is bounded for $\lambda \in \{\lambda : |\arg \lambda| < \pi/2 - \varepsilon; |\arg \lambda| > \pi/2 + \varepsilon\}$, $\varepsilon > 0$. The Phragmèn–Lindelöf theorem shows that $\widehat{u}(\lambda)$ is an entire function bounded in the entire complex plane. Since $\widehat{u}(\lambda) \to 0$ as $\operatorname{Re} \lambda \to +\infty$, we obtain $\widehat{u}(\lambda) \equiv 0$, whence $u(t) \equiv 0$ by the inversion theorem. □

Remark 2.3. Some conditions ensuring the absence of small solutions of a first order vector system were obtained in [18] (see also [15], [19]). However, using the standard passage from (1.3) to such a system, we obtain a characteristic function the determinant of which fails to have the maximal exponential type mh, and this does not allow us to apply the results of [18].

2.2. A basis of subspaces. If we deal with several delays and these delays h_j are incommensurable, then, as a rule, the set Λ is not separated, and the set V does not form an unconditional basis in its span in $H^p(0, T)$, no matter what T and p may be. However, it turns out that if we combine (unite) close points of the spectrum, then the subspaces spanned by the corresponding exponentials constitute an unconditional basis.

In what follows we assume that the conditions $a_{0m} \neq 0$ and $a_{nm} \neq 0$ are satisfied. Then, by Lemma 2.1, the spectrum Λ of V lies in a strip parallel to the imaginary axis,

$$\varkappa_- := \inf_q \operatorname{Re} \lambda_q \leq \operatorname{Re} \lambda \leq \sup_q \operatorname{Re} \lambda_q =: \varkappa_+.$$

As in [2], [3], [25], we construct certain subspaces of exponentials. For every $\lambda \in \Lambda$, we denote by $D_\lambda(r)$ the disk of radius r with center λ. Let $G^{(q)}(r)$, $q = 1, 2, \dots$, be the connected components of $\bigcup_{\lambda \in \Lambda} D_\lambda(r)$. Let $\Lambda^{(q)}(r)$ be the sequence of points of Λ lying in $G^{(q)}$, let $\Lambda^{(q)}(r) := \Lambda \cap G^{(q)}(r)$, and let $\mathcal{L}^{(q)}(r)$ be the subspaces formed by the corresponding exponentials $t^n e^{\lambda t}$, $\lambda \in \Lambda^{(q)}(r)$, $n = 0, \dots, \nu_\lambda - 1$ (ν_λ is the multiplicity of λ). We denote by $\mathcal{L} := \mathcal{L}(r)$ the family $\{\mathcal{L}^{(q)}(r)\}$ and by $M_q = M_q(r)$ the number of points in $\Lambda^{(q)}$ (with multiplicities taken into account).

Proposition 2.1. *If* $a_{0m} \neq 0$ *and* $a_{nm} \neq 0$, *then, for sufficiently small* r, *the numbers* $M_q(r)$ *are bounded uniformly in* q.

We consider arbitrary m zeros $\lambda_1, \lambda_2 \dots, \lambda_m$ of L. The asymptotic relations obtained before imply that the function

$$L_0(\lambda) = L(\lambda) / \prod_1^m (\lambda - \lambda_j)$$

is of sine type (more precisely, it becomes a function of this type after the substitution $\lambda \mapsto i\lambda$). The required result follows from [24].

Theorem 2.3. *Let $a_{0m} \neq 0$ and $a_{nm} \neq 0$. Assume that $s \geq m$ is not a half-integer ($s \neq m+1/2, m+3/2, \dots$). Then the family $\mathcal{L}$ of subspaces is a Riesz basis in H^m and in H^s_U.*

Proof of Theorem 2.3. a) First, we study the properties of $\mathcal{L}$ in L^2. For this, we construct the subspaces $\{\mathcal{L}_0^{(q)}\}$ corresponding to the function $L_0\ (\lambda)$. The difference between the families $\mathcal{L}$ and $\mathcal{L}_0 := \{\mathcal{L}_0^{(q)}(r)\}$ is as follows: if λ_j is a simple zero of $L(\lambda)$, then we remove $e^{\lambda_j t}$ from the basis of the corresponding subspace $\{\mathcal{L}^{(q)}\}$; if λ_j is a zero of multiplicity $n+1$, then we remove the element $t^n e^{\lambda_j t}$ with the highest exponent n.

The operator of multiplication by $e^{\alpha t}$ is an isomorphism of the space $H^n(-h,0)$ for each n, and it shifts the spectrum Λ by α (in $\mathbb{C}$). Therefore, when studying the basis property, we assume that Λ lies in the half-plane $\operatorname{Re} z < 0$. All elements of V lie in $H^n(-h,\infty)$ for every n.

The following result guarantees that the family $\mathcal{L}_0$ is a basis in the case of a semiaxis.

Proposition 2.2 ([2], [3])**.** *If Λ lies in a strip located in the left half-plane and parallel to the real axis, and Λ can be represented as a finite union of separated sets, then for every $r > 0$ the family $\mathcal{L}_0$ of subspaces forms an unconditional basis in the closure of its linear span in $L^2(0,\infty)$.*

Since $L_0\ (\lambda)$ is a function of sine type and of exponential type 0 in the right half-plane and type h in the left half-plane, the projection P_h from the closure of the linear span of $\mathcal{L}_0$ in $L^2(0,\infty)$ onto $L^2(-h,0)$ is an isomorphism (see [11], [4]). Thus, the family $\mathcal{L}_0$ is a basis in $L^2(-h,0)$. Observe that the same family is a basis in L^2 on every interval of length h. Indeed, the shift operator $f(t) \longmapsto f(t+R)$ is an isomorphism between the spaces $H^n(-h,0)$ and $H^n(R-h,R)$ for every R, and it preserves the subspaces $\mathcal{L}_0^{(q)}$.

b) Now, we proceed to the study of the basis property in Sobolev spaces with integral index.

In order to apply Lemma 2.3, we consider the functional ψ_0 equal to the Fourier–Laplace transform of $L_0(\lambda)$. The same argument as in the proof of Theorem 2.1 shows that the functional ψ_0 is bounded on $H^1(-h,0)$. Now, Lemma 2.3 implies that the family $\mathcal{L}_0$ of subspaces forms a Riesz basis in the closure of its linear span in the subspace $\operatorname{Ker}\psi_0 \subset H^1(-h,0)$ of codimension 1. Indeed, the subspaces $\mathcal{L}_0^{(q)}$ lie in $\operatorname{Ker}\psi_0$ and are invariant under differentiation. Therefore, the basis property is preserved under the mapping B_ψ. Moreover, the space $L^2(-h,0)$, which coincides with $\bigvee_{L^2}\mathcal{L}_0$, is the preimage of the kernel $\operatorname{Ker}\psi_0$.

Applying Lemma 2.3 m times, we see that the family $\{\mathcal{L}_0^{(q)}\}$ of subspaces is a Riesz basis in the closure of its linear span in the space $H^m(-h,0)$, and that the defect of the family $\mathcal{L}_0$ in $H^m(-h,0)$ is equal to m. To the family $\mathcal{L}_0$, we add the subspace formed by the elements of V that do not belong to the family $\{\mathcal{L}_0^{(q)}\}$. As a result, by Lemma 2.2, we obtain a basis $\{\mathcal{L}(r)\}$ in $H^m(-h,0)$.

Suppose $s = m+n$ is an integer. Then V has defect n in $H^s(-h,0)$, and all elements of V, which are solutions of (1), satisfy n conditions (1.5). This proves the theorem for the integral values of s.

c) Now, we proceed to the study of the basis property of $\{\mathcal{L}(r)\}$ in the space H^s with nonintegral values of s. First, we consider the family $\mathcal{L}_0$ in $H^p(-h,0)$ for $0 < p \leq 1$. This family is a Riesz basis in $L^2(-h,0)$ and also in the subspace $\mathcal{H} = \operatorname{Ker}\psi_0$ of codimension 1 in $H^1(-h,0)$. The operator interpolation theory says that this family is a Riesz basis in the interpolation spaces $\mathcal{H}^p := [L^2(-h,0), \mathcal{H}]_p$. The problem of determining the metric in such subspaces was solved in [6]. In the simplest case where the Fourier–Laplace transform $L_0(\lambda)$ of ψ_0 is bounded from above and bounded away from zero as $\lambda \to \pm i\infty$, the following statement is valid.

Proposition 2.3. *Suppose that the relation*

$$|L_0(\alpha + iy)| \asymp 1$$

is valid on the line $\operatorname{Re}\lambda = \alpha$. *Then, for* $0 \le p < 1/2$, *the space* $[L^2, \mathcal{H}]_p$ *coincides with* H^p *and has an equivalent metric*; *if* $1/2 < p \le 1$, *then* $[L^2, \mathcal{H}]_p$ *is a proper subspace of* H^p *of codimension* 1 *and has an equivalent metric*; *if* $p = 1/2$, *then the space* $[L^2, \mathcal{H}]_{1/2}$ *is dense in* $H^{1/2}$ *and has a stronger metric.*

This allows us to conclude that the family $\mathcal{L}_0$ is a Riesz basis in $H^p(-h, 0)$ for $p < 1/2$ and is a Riesz basis in the closure of its linear span of codimension 1 in the space $H^p(-h, 0)$ for $1/2 < p \le 1$.

Applying Lemmas 2.3 and 2.2 once again, we see that, if s is not a half-integer, in the space H^s the family $\mathcal{L}_0$ is an unconditional basis in the closure of its linear span, and that the defect is equal to $[s + 1/2]$. By Lemma 2.2, the defect of V is $[s - m + 1/2]$; consequently, $\mathcal{L}$ is an unconditional basis in H^s_U, because the space H^s_U is determined by the functionals at which the exponential solutions vanish. Theorem 2.3 is proved completely. □

2.3. The basis of divided differences. If the set Λ is not separated, then V is not uniformly minimal. To obtain a basis, we need to choose appropriate linear combinations of exponentials in each subspace $\mathcal{L}^{(q)}$. It turns out (see [1]) that it is convenient to take the so-called *divided differences*.

Let μ_k, $k = 1, \dots, m$, be arbitrary complex numbers, not necessarily distinct.

Definition 2.1. The divided difference of order zero corresponding to a point μ is $[\mu](t) := e^{\mu t}$. The divided difference of order $m-1$ corresponding to points $\mu_1, \mu_2, \dots, \mu_r$ is defined as follows:

$$[\mu_1, \dots, \mu_r](t) = \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{m-2}} d\tau_{r-1} t^{r-1} \exp\big(tZ(\mu_1, \dots, \mu_r)\big),$$

where

$$Z(\mu_1, \dots, \mu_r) := [\mu_1 + \tau_1\delta_1 + \dots + \tau_{r-1}\delta_{r-1}], \quad \delta_j := \mu_{j+1} - \mu_j.$$

If the points μ_k are distinct, then it is easy to obtain explicit formulas for the divided differences:

$$[\mu_1, \dots, \mu_r](t) = \sum_{k=1}^{r} \frac{e^{\mu_k t}}{\prod_{j \ne k}(\mu_k - \mu_j)}.$$

In particular, $[\mu_1, \mu_2] = \frac{e^{\mu_1 t} - e^{\mu_2 t}}{\mu_1 - \mu_2}$. For more details about the divided differences, see [1] and [12, p. 228].

Let $\Lambda^{(q)}(r) = \{\lambda_{j,q}\}$, $j = 1, \dots, M_q(r)$, be the sets introduced in Subsection 2.2. We denote by $\Phi := \{\Phi^{(q)}(r)\}$ the family of divided differences corresponding to the points of $\Lambda^{(q)}(r)$,

$$\Phi^{(q)}(r) = \{[\lambda_{1,q}], [\lambda_{1,q}, \lambda_{2,q}], \dots, [\lambda_{1,q}, \dots, \lambda_{M_q,q}]\} = \{\varphi_{q,1}, \varphi_{q,2}, \dots, \varphi_{q,M_q}\}.$$

The family $\Phi^{(q)}(r)$ depends on the enumeration of the points in $\Lambda^{(q)}(r)$, though each divided difference depends on $\lambda_{j,q}$ in a symmetric way. Observe that the functions

$$\widetilde{\varphi}_{q,j} := \frac{\varphi_{q,j}}{(|\mu_q| + 1)^s}$$

are almost normalized in the space $H^s(-h, 0)$; here $\mu_q := \max_{j=1,\dots,M_q} |\lambda_{q,j}|$.

Theorem 2.4. *Suppose that $a_{0m} \neq 0$ and $a_{nm} \neq 0$, and that $s \geq m$ is not a half-integer ($s \neq m+1/2, m+3/2, \dots$). Then the family $\{\varphi_{q,j}\}_{q,j}$ is an unconditional basis in the spaces $H^m(-h,0)$ and H^s_U.*

Proof. Since $\{\mathcal{L}^{(q)}(r)\}$ is a Riesz basis in $H^m(-h,0)$ by Theorem 2.3, it suffices to check that, for all q, in the finite-dimensional space $\mathcal{L}^{(q)}(r)$ the divided differences $\Phi^{(q)}$ form a basis *uniform with respect to* q in the sense that the operators that take $\Phi^{(q)}$ to orthonormal bases are bounded together with their inverses uniformly with respect to q. For the space L^2, this was done in [1]; it is easily seen that for the Sobolev spaces H^n with integral n the proof is similar. Therefore, the statement is also true for the intermediate spaces. $\square$

§3. Estimates of solutions

Using Theorems 2.1 a) and 2.3 and the approach suggested in [7], [8], we can obtain the following upper bound for the solutions of problem (1.3).

Theorem 3.1. *Suppose that $a_{0m} \neq 0$, $a_{nm} \neq 0$, and that s, $s \geq m$, is not a half-integer ($s \neq m+1/2, m+3/2, \dots$). Let the initial function g be in H^s_U. Then problem* (1.3), (1.4) *has a unique solution $u(t)$ satisfying the estimate*

$$\|u\|_{H^s(T-h,T)} \leq d(T+1)^{M-1} e^{\varkappa_+ T} \|g\|_{H^s(-h,0)}, \quad T \geq 0, \tag{3.15}$$

where the constant d is independent of g and T, and M is defined as follows:

$$M := \lim_{r\to 0} \lim_{\varepsilon \to 0} \max_q \{M_q(r) \mid \varkappa_q > \varkappa_+ - \varepsilon\}.$$

Remark 3.1. 1. Roughly speaking, M is the maximal multiplicity of the points of the spectrum near its right boundary $\operatorname{Re}\lambda = \varkappa_+$.

2. If s is a half-integer ($s = m+n+1/2$ and n is a positive integer), then estimates similar to (3.15) are valid with a stronger norm, $\|\cdot\|_{*,m+n+1/2}$. We have

$$\|u\|_{H^{m+n+1/2}(T-h,T)} \leq \|u\|_{*,m+n+1/2} \leq d(T+1)^{M-1} e^{\varkappa_+ T} \|g\|_{*,m+n+1/2}.$$

3. If the set Λ is separated, the constant M can be estimated as follows:

$$M \leq N := \max_{\lambda_q \in \Lambda} \nu_q.$$

Proof of Theorem 3.1. We enumerate the points in $\Lambda^{(q)} = \{\lambda_{q,1}, \dots, \lambda_{j,q}, \dots, \lambda_{q,M_q}\}$ in such a way that $\operatorname{Re}\lambda_{q,j}$ be monotone nondecreasing in j and put $\varkappa_q := \max_j \operatorname{Re}\lambda_{q,j}$. Then $\varkappa_q = \operatorname{Re}\lambda_{q,M_q}$.

Next, we expand the initial function $g(t) \in H^s_U$ with respect to the basis of subspaces:

$$g(t) = \sum_q \psi_q(t),$$

where the functions $\psi_q(t)$ are in $\mathcal{L}^{(q)}$; they in turn can be expanded with respect to the divided differences,

$$\psi_q(t) = \sum_{j=0}^{M_q-1} c_{q,j} \varphi_{q,j}(t).$$

We have

$$\|g\|^2_{H^s(-h,0)} \asymp \sum_q \|\psi_q\|^2_{H^s(-h,0)}, \tag{3.16}$$

$$\|\psi_q\|^2_{H^s(-h,0)} \asymp \sum_j |c_{q,j}|^2 (1+|\lambda_{q,j}|)^{2s} \tag{3.17}$$

with constants independent of g and q. Since each component ψ_q is a linear combination of exponential solutions, the formal solution can be written as

$$u(t) = \sum_q \psi_q(t),$$

and it satisfies the initial condition for $t \in [-h, 0]$ and converges in the space $H^s(T-h, T)$. Consequently, this series is a solution of the problem in the sense of our definition.

The following two lemmas allow us to estimate the divided differences $\varphi_{q,j}$ and the functions ψ_q.

Lemma 3.1. *For all n, q, and j, we have*

$$|(d/dt)^n \varphi_{q,j}(t)| \le C(1+t)^{M_q-1} e^{\varkappa_q t}(1+\mu_q^n), \tag{3.18}$$

$$\|\varphi_{q,j}\|_{H^n(T-h,T)} \le C_1(1+T)^{M_q-1} e^{\varkappa_q T}(1+\mu_q^n), \tag{3.19}$$

where the constants C and C_1 depend only on n and M_q.

Proof. For $Z = Z(\lambda_{q,1}, \lambda_{q,2}, \dots, \lambda_{q,j})$, Definition 2.1 implies the relation

$$\frac{d^n}{dt^n}\varphi_{q,j}(t) = \sum_{k=0}^{n} d_k \int_0^1 \dots \int_0^{\tau_{j-2}} t^{\max\{j-1-k,0\}} Z^{n-k} e^{tZ}\, d\tau_1 \cdots d\tau_{j-1}.$$

Putting $\delta_{q,j} = \lambda_{q,j+1} - \lambda_{q,j}$ and using the relations

$$\begin{aligned} \operatorname{Re} Z &= \operatorname{Re}(\lambda_{q,1} + \delta_{q,1}\tau_1 + \dots + \delta_{q,j-1}\tau_{j-1}) \\ &\le \operatorname{Re}(\lambda_{q,1} + \delta_{q,1} + \dots + \delta_{q,j-1}) = \operatorname{Re}\lambda_{q,M_q} = \varkappa_q, \\ |Z| &\le |\lambda_{q,j}| + |\delta_1| + \dots + |\delta_{j-1}| \le 2j \max|\lambda_{q,j}| = 2j\mu_q, \end{aligned}$$

we obtain the estimate

$$\left|\frac{d^n}{dt^n}\varphi_{q,j}(t)\right| \le C_2 \sum_{k=0}^{n} \int_0^1 \dots \int_0^{\tau_{j-2}} t^{\max\{j-1-k,0\}} \mu_q^{n-k} e^{t\varkappa_q}\, d\tau_1 \cdots d\tau_{j-1}.$$

This yields (3.18). Next, the estimates

$$\begin{aligned} \|\varphi_{q,j}\|^2_{H^n(T-h,T)} &\prec (1+\mu_q^{2n}) \int_{T-h}^{T} |t^{j-1} e^{t\varkappa_q}|^2\, dt \prec (1+\mu_q^{2n}) e^{2\varkappa_q T} \int_{T-h}^{T} t^{2(j-1)}\, dt \\ &\prec (1+\mu_q^{2n}) e^{2\varkappa_q T}(1+T)^{2(j-1)} h \end{aligned}$$

imply (3.19). Lemma 3.1 is proved. ∎

Lemma 3.2. *For all q and p, the inequality*

$$\|\psi_q\|_{H^p(T-h,T)} \prec (1+T)^{M_q-1} e^{\varkappa_q T}(1+\mu_q^p)\|c_q\|_{l^2}$$

is valid with $c_q := \{c_{q,j}\}$ and $\|c_q\|_{l^2} = (\sum_{j=0}^{M_q-1} |c_{qj}|^2)^{1/2}$.

For the integral values of p, this statement follows from the preceding lemma and the inequalities

$$\begin{aligned} \|\psi_q\|_{H^p(T-h,T)} &\le \sum_{j=0}^{M_q-1} |c_{qj}|\, \|\varphi_{q,j}\|_{H^p(T-h,T)} \le \|c_q\|_{l^2} \left(\sum_{j=0}^{M_q-1} \|\varphi_{q,j}\|^2_{H^p(T-h,T)} \right)^{1/2} \\ &\prec (1+T)^{M_q-1} e^{\varkappa_q T}(1+\mu_q^p)\|c_q\|_{l^2}. \end{aligned}$$

Now, Lemma 3.2 is a consequence of the following well-known inequality (see [5, (2.43)]) for the norms in the interpolation spaces $[X, Y]_\theta$:

$$\|y\|_{[X,Y]_\theta} \le C\|y\|_X^{1-\theta}\|y\|_Y^{\theta}.$$

We return to the proof of Theorem 3.1. It is easily seen that the shift $t \to t+T$ maps $\mathcal{L}^{(q)}$ to $\mathcal{L}^{(q)}$. Using the notation $\widetilde{\psi}_q(\tau) := \psi_q(\tau+T)$, $\tau \in [-h,0]$, we can represent the solution on the interval $[T-h,T]$ in the form

$$u(t) = u(\tau+T) = \sum_q \widetilde{\psi}_q(\tau).$$

Since $\mathcal{L}$ is a basis, we obtain the estimates

$$\|u\|^2_{H^s(T-h,T)} \asymp \sum_q \|\widetilde{\psi}_q\|^2_{H^s(-h,0)}$$

with constants independent of T. Returning to the variable t and using Lemma 3.1, we see that

$$\|u\|^2_{H^s(T-h,T)} \prec \sum_q (1+T)^{2M_q-2} e^{2\varkappa_q T}(1+\mu_q^{2s})\|c_q\|^2_{l^2}. \tag{3.20}$$

We put

$$M(r,\varepsilon) := \max\{M_q(r) \mid \varkappa_q > \varkappa_+ - \varepsilon\}.$$

This is a monotone nondecreasing function of ε; it has a limit $M(r)$ as $\varepsilon \to +0$, and if ε is sufficiently small, then $M(r,\varepsilon) = M(r)$. Therefore, there exists T such that, for all q, we have

$$(1+T)^{2(M_q(r)-1)} e^{\varkappa_q T} \le 2(1+T)^{2(M(r)-1)} e^{\varkappa_+ T}. \tag{3.21}$$

Now from (3.20) and (3.21) we deduce the inequality

$$\|u\|^2_{H^s(T-h,T)} \prec (1+T)^{2M(r)-2} e^{2\varkappa_+ T} \sum_q (1+\mu_q^{2s})\|c_q\|^2_{l^2}. \tag{3.22}$$

Since for all $\lambda_{qj} \in \Lambda^{(q)}(r)$ we have $1+\lambda_{qj} \le 2(1+\mu_q)$ if r is sufficiently small, inequality (3.22) implies that

$$\|u\|^2_{H^s(T-h,T)} \prec (1+T)^{2M(r)-2} e^{2\varkappa_+ T} \sum_q (1+\lambda_{qj}^{2s})\|c_q\|^2_{l^2}.$$

By (3.17) and (3.16),

$$\begin{aligned}\|u\|^2_{H^s(T-h,T)} &\prec (1+T)^{2M(r)-2} e^{2\varkappa_+ T} \textstyle\sum_q \|\widetilde{\psi}_q\|^2_{H^s(-h,0)} \\ &\prec (1+T)^{2M(r)-2} e^{2\varkappa_+ T}\|g\|^2_{H^s(-h,0)}.\end{aligned}$$

To finish the proof of Theorem 3.1, it remains to observe that the function $M(r)$ is monotone nonincreasing as $r \to 0$.

We pass to lower estimates for the solutions of problem (1.3), (1.4).

Theorem 3.2. *Suppose that $a_{0m} \neq 0$ and $a_{nm} \neq 0$, and that $s \ge m$ is not a half-integer ($s \neq m+1/2, m+3/2, \dots$). Then, for every nonzero function $g \in H^s_U(-h,0)$, the solution u of problem* (1.3), (1.4) *satisfies the estimate*

$$\|u\|_{H^s(T-h,T)} \ge c e^{\varkappa_- T}, \quad c = c(g) > 0, \ \ T > 0, \tag{3.23}$$

where $c(g)$ depends on g.

Proof. By (3.16) and (3.17), for every q we have

$$\|g\|_{H^s(-h,0)} \ge d_0 \|\psi_q\|_{H^s(-h,0)}.$$

Let q be such that $\psi_q \neq 0$, and let $\xi_1, \xi_2, \dots, \xi_N$ be points belonging to Λ^q with multiplicities $\nu_1, \dots, \nu_N$. We expand ψ_q in the exponential solutions,

$$\psi_q(t) = \sum_{j=1}^{N} \sum_{r=1}^{\nu_j} c_{jr} t^{r-1} e^{\xi_j t}, \quad t \in [-h,0]. \tag{3.24}$$

We keep only nonzero summands in (3.24) and enumerate the real parts $\operatorname{Re}\xi_j$ in ascending order,

$$\alpha_1 < \alpha_2 < \cdots < \alpha_R.$$

Let β_{jr}, $r = 1, 2, \ldots, \gamma_j$, be the imaginary parts of the points in $\Lambda^{(q)}$ that have real part α_j, and let ν_{jr} be the multiplicity of $\alpha_j + i\beta_{jr}$. In (3.24) we collect the terms with the same real part $\operatorname{Re}\lambda = \alpha_j$, obtaining

$$\psi_q(t) = \sum_{j=1}^{R}\sum_{r=1}^{\gamma_j}\sum_{n=1}^{\nu_{jr}} c_{jrn} t^{n-1} e^{i\beta_{jr}} e^{\alpha_j t}.$$

In this expression, we distinguish the principle part (as $t \to +\infty$) with $\sigma := \max \alpha_j = \alpha_R$, and, among the terms obtained, we find the terms with the highest exponent $l = \max_j \nu_{j,R}$. We have

$$u(t) = t^l e^{\sigma t} \sum_{r=1}^{\gamma_R} c_{jr} e^{i\beta_{Rr} t} + u_1(t),$$

where $u_1(t)$ consists of all terms with smaller real parts α_j or with the same real part α_R but with a smaller power of t. For $u_1(t)$ we have the obvious estimate

$$|u_1(t)| \prec t^{l-1} e^{\sigma t}.$$

Now, the claim of the theorem can easily be obtained with the help of the following lemma.

Lemma 3.3. *Let $\beta_j \in \mathbb{R}$, $j = 1, \ldots, I$, be different constants. Then the inequality*

$$\Big\| \sum_{j=1}^{I} c_j e^{j\beta_j t} \Big\|_{H^s(T-h,T)} \geq c(s, q, \beta_1, \ldots, \beta_I) \Big(\sum_{j=1}^{I} |c_j|^2 \Big)^{1/2}$$

is valid uniformly with respect to T.

Proof. We use a compactness argument. We put

$$Q(d_1, d_2, \ldots, d_I) := \Big\| \sum_{j=1}^{I} d_j e^{i\beta_j t} \Big\|_{H^s(-h,0)}$$

and consider the function Q on the sphere $d_1^2 + \cdots + d_I^2 = 1$. This function is continuous and has a positive minimum $Q_{\min}$ on this compact set, because the functions $e^{i\beta_j t}$ are linearly independent. Then, for all R, we have

$$\Big\| \sum_{j=1}^{I} d_j e^{i\beta_j t} \Big\|_{H^s(R-h,R)} \geq Q_{\min},$$

because

$$\sum_{j=1}^{I} d_j e^{i\beta_j (t+R)} = \sum_{1}^{I} e^{i\beta_j R} d_j e^{j\beta_j t} = \sum_{1}^{I} \widetilde{d}_j e^{j\beta_j t}$$

and $|\widetilde{d}_j| = |d_j|$. Lemma 3.3 is proved. □

In general, the quantity $c(g)$ in (3.23) cannot be replaced by $c\|g\|$:

$$\inf_g \lim_{T\to\infty} e^{\varkappa_- T} \frac{\|u\|_{H^s(T-h,T)}}{\|g\|_{H^s(T-h,T)}} = 0.$$

We give an example for the space $L^2(-h, 0)$. The case of the space H^s is similar.

Let Λ consist of the pairs

$$\{in, in + \delta_n\}_{n\in\mathbb{Z}},$$

where $\delta_n \to 0$. As the initial functions, we take $g_m(t) := e^{imt} + e^{i(m+\delta_n)t}$. Then

$$\|g_m\|_{L^2(-h,0)} \xrightarrow[m\to\infty]{} 2\sqrt{h},$$

and, for $T = T_m = \pi/\delta_m$, we obtain

$$\|u_m\|^2_{L^2(T_m-h,T_m)} = \int_{-h}^{0} |1 - e^{i\delta_m t}|^2\, dt \xrightarrow[m\to\infty]{} 0.$$

Consequently,

$$\inf_m \frac{\|u_m\|_{L^2(T_m-h,T_m)}}{\|g_m\|_{L^2(-h,0)}} = 0.$$

Proposition 3.1. *Let $a_{0m} \neq 0$ and $a_{nm} \neq 0$. Assume that $s \geq m$ is not a half-integer ($s \neq m + 1/2, m + 3/2, \dots$) and that the zeros of L are simple and separated. Then the quantity $c(g)$ in (3.23) can be replaced by $c\|g\|$.*

Indeed, in this case, we can write

$$g(t) = \sum_{\lambda_q \in \Lambda} c_q e^{\lambda_q t}$$

and

$$u(t) = \sum_{\lambda_q \in \Lambda} c_q e^{\lambda_q t} = \sum_{\lambda_q \in \Lambda} c_q e^{\lambda_q T} e^{\lambda_q (t-T)},$$

whence

$$\|u\|^2_{H^s(T-h,T)} \asymp \sum_{\lambda_q \in \Lambda} |c_q e^{\lambda_q T}|^2 \geq e^{2\varkappa_- T} \sum_{\lambda_q \in \Lambda} |c_q|^2 \asymp e^{2\varkappa_- T} \|g\|^2_{H^s(-h,0)}.$$

Remark 3.2. Estimates (3.15) and (3.23) are sharp. Indeed, in [28] there is an example of an equation with the quasipolynomial $l(\lambda) = \lambda + a - e^{-\lambda}(\lambda - a)$ where l has pure imaginary simple roots. From (3.15) and (3.23) it follows that the quantity $\|u\|_{H^s(T-h,T)}$ is uniformly bounded from above and from below for all $T > 0$. Thus, in (3.15) and (3.23) we cannot replace $\varkappa_+$ and $\varkappa_-$ by $\varkappa_+ - \varepsilon$ and $\varkappa_- + \varepsilon$, respectively, for any $\varepsilon > 0$.

In conclusion, we indicate how the properties of V can be related to spectral problems for differential operators.

The completeness, minimality, and basis properties of the family V of exponential solutions are closely related to the study of the operator of differentiation with multipoint conditions. We explain this in the case where s is an integer. Consider the operator $\mathbb{D}g = \frac{d}{dt} g(t)$, $t \in (-h, 0)$, the domain $\mathrm{Dom}(\mathbb{D})$ of which consists of all functions that belong to $H^s[-h, 0]$ and satisfy the compatibility conditions (1.5) for $s \geq m + 1$. Direct inspection shows that the family of exponential solutions of equation (1.3) coincides with the family of eigenfunctions and adjoined functions of $\mathbb{D}$ that correspond to the eigenvalues λ_q. Thus, the results on completeness, minimality, and the Riesz basis property of the family of exponential solutions obtained in the present paper can be restated as results on completeness, minimality, and the Riesz basis property of the family of eigenfunctions and the adjoined functions of the differentiation operator $\mathbb{D}$ with multipoint conditions of the form (1.5).

It should be noted that the proof of the Riesz basis property for subspaces, as given by the first author in [16], [17] (in the vector case for $m = 1$), was based on the study of the resolvent of $\mathbb{D}$.

Many authors studied the spectral properties of the differentiation operator $\mathbb{D}$ in other spaces (largely, in L^p and C). We only mention the survey by A. M. Sedletskiĭ [26] (see also the references therein).

In turn, the investigation of the differentiation operator in the scale of Sobolev spaces with integral index s can be related to the study of the differentiation operator $\mathbb{D}$ with a spectral parameter in the boundary conditions. The case of nth order differential operators with a spectral parameter in the boundary conditions was treated in [25]. The methods of this paper can also be applied to the study of the unconditional basis property of $\mathcal{L}^{(p)}$ in the space H^m.

In our opinion, the study of the basis property, together with finding estimates for solutions, in the vector case in the scale of Sobolev spaces with an arbitrary real index is a topical problem that deserves attention.

References

1. S. A. Avdonin and S. A. Ivanov, *Riesz bases of exponentials and divided differences*, Algebra i Analiz **13** (2001), no. 3, 1–17; English transl., St. Petersburg Math. J. **13** (2002), no. 3, 339–351. MR 1850184 (2002g:42009)
2. V. I. Vasyunin, *Unconditionally convergent spectral decompositions and interpolation problems*, Trudy Mat. Inst. Steklov. **130** (1978), 5–49; English transl. in Proc. Steklov Inst. Math. **1979**. MR 0505683 (80b:47041)
3. N. K. Nikol′skiĭ, *Treatise of the shift operator*, "Nauka", Moscow, 1980; English transl., Grundlehren Math. Wiss., vol. 273, Springer-Verlag, Berlin, 1986. MR 0827223 (87i:47042)
4. S. V. Khrushchev, N. K. Nikol′skiĭ, and B. S. Pavlov, *Unconditional bases of exponentials and of reproducing kernels*, Complex Analysis and Spectral Theory (Leningrad, 1979/1980), Lecture Notes in Math., vol. 864, Springer-Verlag, Berlin–New York, 1981, pp. 214–335. MR 0643384 (84k:46019)
5. J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*. Vol. 1, Dunod, Paris, 1968. MR 0247243 (40:512)
6. S. A. Ivanov and N. Kalton, *Interpolation of subspaces and applications to exponential bases*, Algebra i Analiz **13** (2001), no. 2, 93–115; English transl., St. Petersburg Math. J. **13** (2002), no. 2, 221–239. MR 1834861 (2002f:46147)
7. V. V. Vlasov, *On a class of differential-difference equations of neutral type*, Izv. Vyssh. Uchebn. Zaved. Mat. **1999**, no. 2, 20–29; English transl., Russian Math. (Iz. VUZ) **43** (1999), no. 2, 17–26. MR 1685756 (2000b:34136)
8. ______, *On the properties of a system of exponential solutions of differential-difference equations in Sobolev spaces*, Izv. Vyssh. Uchebn. Zaved. Mat. **2001**, no. 6, 23–29; English transl., Russian Math. (Iz. VUZ) **45** (2001), no. 6, 21–27. MR 1858506 (2002d:34103)
9. V. V. Vlasov and S. A. Ivanov, *The basis property and estimates for solutions of equations with aftereffect in a scale of Sobolev spaces*, Uspekhi Mat. Nauk **56** (2001), no. 3, 151–152; English transl., Russian Math. Surveys **56** (2001), no. 3, 595–596. MR 1859727
10. R. Bellman and K. Cooke, *Differential-difference equations*, Academic Press, New York–London, 1963. MR 0147745 (26:5259)
11. B. S. Pavlov, *The basis property of a system of exponentials and the condition of Muckenhoupt*, Dokl. Akad. Nauk SSSR **247** (1979), no. 1, 37–40; English transl., Soviet Math. Dokl. **20** (1979), no. 4, 655–659. MR 0545940 (84j:42042)
12. G. E. Shilov, *Mathematical analysis. Second special course*, "Nauka", Moscow, 1965. (Russian)
13. N. Levinson, *Gap and density theorems*, Amer. Math. Soc. Colloq. Publ., vol. 26, Amer. Math. Soc., New York, 1940. MR 0003208 (2:180d)
14. D. Henry, *Linear autonomous neutral functional differential equations*, J. Differential Equations **15** (1974), 106–128. MR 0338520 (49:3284)
15. J. Hale, *Theory of functional differential equations*, 2nd ed., Appl. Math. Sci., vol. 3, Springer-Verlag, New York–Heidelberg, 1977. MR 0508721 (58:22904)
16. V. V. Vlasov, *Estimates for solutions of differential-difference equations of neutral type*, Izv. Vyssh. Uchebn. Zaved. Mat. **2000**, no. 4, 14–22; English transl., Russian Math. (Iz. VUZ) **44** (2000), no. 4, 12–20. MR 1782521 (2001i:34137)
17. ______, *On the basis property of exponential solutions of functional-differential equations in Sobolev spaces*, Dokl. Akad. Nauk **381** (2001), no. 3, 302–304. (Russian) MR 1892528 (2002j:34131)
18. S. M. Verduyn Lunel, *Series expansions and small solutions for Volterra equations of convolution type*, J. Differential Equations **85** (1990), no. 1, 17–53. MR 1052326 (91c:45004)
19. M. C. Delfour and A. Manitius, *The structural operator F and its role in the theory of retarded systems*. II, J. Math. Anal. Appl. **74** (1980), 359–381. MR 0572658 (84h:34111)

20. S. M. Verduyn Lunel and D. V. Yakubovich, *A functional model approach to linear neutral functional-differential equations*, Integral Equations Operator Theory **27** (1997), no. 3, 347–378. MR 1433007 (98g:34127)
21. N. Levinson and C. McCalla, *Completeness and independence of the exponential solutions of some functional differential equations*, Stud. Appl. Math. **53** (1974), 1–15. MR 0342813 (49:7557)
22. D. Russell, *On exponential bases for the Sobolev spaces over an interval*, J. Math. Anal. Appl. **87** (1982), no. 2, 528–550. MR 0658032 (83g:46035)
23. B. Ya. Levin, *Distribution of zeros of entire functions*, Gostekhizdat, Moscow, 1956; English transl., Amer. Math. Soc., Providence, RI, 1964. MR 0156975 (28:217) MR 0087740 (19:402c)
24. ______, *On bases for exponential functions in L^2*, Khar′kov. Gos. Univ. Uchen. Zap. **115** (1961) (Zap. Mat. Otdel. Fiz.-Mat. Fak. Khar′kov. Gos. Univ. i Khar′kov. Mat. Obshch. (4) **27** (1961)), 39–48. (Russian).
25. A. A. Shkalikov, *Boundary value problems for ordinary differential equations with a parameter in the boundary conditions*, Trudy Sem. Petrovsk. No. 9 (1983), 190–229; English transl. in J. Soviet Math. **33** (1986), no. 5–6. MR 0731903 (85m:34028)
26. A. M. Sedletskiĭ, *Biorthogonal expansions of functions in exponential series on intervals of the real axis*, Uspekhi Mat. Nauk **37** (1982), no. 5, 51–95; English transl., Russian Math. Surveys **37** (1983), no. 5, 57–108. MR 0676613 (84g:42025)
27. V. V. Vlasov and S. A. Ivanov, *Estimates for solutions of equations with aftereffect in Sobolev spaces and a basis of divided differences*, Mat. Zametki **72** (2002), no. 2, 303–306; English transl., Math. Notes **72** (2002), no. 1–2, 271–274. MR 1942555 (2003i:34151)
28. P. S. Gromova and A. M. Zverkin, *Trigonometric series whose sum is a continuous unbounded function on the real axis and is a solution of an equation with deviating argument*, Differentsial′nye Uravneniya **4** (1968), no. 10, 1774–1784. (Russian) MR 0241778 (39:3117)

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Received 18/FEB/2003

Translated by B. M. BEKKER