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SOME CONVERGENCE PROBLEMS FOR WEAK NORMS

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Abstract. Let U be a normed space compactly embedded in a space V, let $\{U_n^*\}$ be a sequence of finite-dimensional subspaces of the dual space U^*, and let

$$U^{(n)} = \{u \in U \mid \chi(u) = 0, \chi \in U_n^*\}.$$

If the sequence $\{U_n^*\}$ is asymptotically dense in U^*, then $\|I_n\| \to 0$, where I_n denotes the operator that embeds $U^{(n)}$ in V. In particular, if $\{P_n\}$ is a sequence of finite-dimensional projections in U, and the sequence $\{\mathcal{R}(P_n^*)\}$ is asymptotically dense in U^*, then $\|u - P_n u\|_V / \|u - P_n u\|_U \to 0$. The norm $\|I_n\|$ is estimated in terms of the best approximation of the elements of the unit ball in V^* (this ball is compact in U^*) by elements of U_n^*. Usually, for projection methods of solving functional equations, the metric in which the convergence should be studied is dictated by the general convergence theorems (we mean, e.g., the energy metric for the Ritz method). The above arguments make it possible to establish a faster convergence of projection methods in weaker metrics. Some results of this type are obtained in the paper for the Ritz and Galerkin methods and for the method of moments.

In what follows, by "superconvergence" we shall mean the following. Suppose an approximation process is given in a Banach space U, i.e., an approximating sequence $\{u_n\}$ is assigned to every element $u \in U$. Also, let U be equipped with a seminorm $\nu(u)$ subordinate to the norm in U: $\nu(u) \le c\|u\|$. If $\nu(u - u_n)$ tends to zero faster than $\|u - u_n\|$, we talk about superconvergence.

Starting with the paper [1], superconvergence was studied in detail for the finite element method (see, e.g., [2] and the references therein). In the present paper we construct a certain abstract pattern of superconvergence in the case where $u_n = P_n u$, the P_n being projections. Any finite-dimensional projection has the form

$$P_n u = \sum_{k=1}^{n} \chi_k^n(u)\omega_k^n, \quad \chi_j^n(\omega_k^n) = \delta_{kj} \text{ (the Kronecker symbol)},$$

where $\{\omega_k^n\} \subset U$ is called the *coordinate* system, and $\{\chi_k^n\} \subset U^*$ is called the *projection* system. The main idea exploited in this paper is that, in superconvergence problems, the key role is played by the projection system rather than by the coordinate system. The general pattern is applied to the study of superconvergence for projection methods of solving boundary value problems for differential equations in the case of polynomial coordinate systems.

We introduce some notation and terminology. Let $U_n \subset U$ be a subspace. The best approximation of an element $u \in U$ by elements of U_n is defined to be

$$E_{U_n}(u)_U = \inf\{\|u - u_n\| \mid u_n \in U_n\}.$$

A sequence $\{U_n\}$ of subspaces of U is said to be *asymptotically dense* in U if for any $u \in U$ we have $E_{U_n}(u)_U \to 0$. If $u_1^n, \dots, u_n^n$ is a basis of U_n and the subspaces $\{U_n\}$ are

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asymptotically dense in U, then we say that the system $\{u_k^n\}$ $(n = 1, 2, \ldots, k = 1, \ldots, n)$ is *complete*. We denote by $\mathbb{P}_n^m$ the linear space of polynomials of degree at most n in m variables, and we put $\mathbb{P}_n = \mathbb{P}_n^1$.

Let U and F be Banach spaces, U^* the space dual to U, $\{U_n^*\}$ a sequence of subspaces in U^*, and $G \in \mathcal{L}(U, F)$ a compact linear operator. We introduce subspaces $U^{(n)} \subset U$ by the formula

$$U^{(n)} = \{u \in U \mid \chi_n u = 0, \chi_n \in U_n^*\},$$

and denote by $G_n = G|_{U^{(n)}}$ the restriction of G to $U^{(n)}$.

Theorem 1. *If the sequence U_n^* is asymptotically dense in U^*, then $\|G_n\| \to 0$.*

Proof. For each n, we construct an element $u_n \in U^{(n)}$ such that $\|u_n\| = 1$ and $\|Gu_n\| \geq \frac{1}{2}\|G_n\|$. For any $\chi \in U^*$ and $\varepsilon > 0$, we can find N such that for all $n > N$ we have $\|\chi - \chi_n\| < \varepsilon$ for some $\chi_n \in U_n^*$. Then, since $\chi_n(u_n) = 0$, for $n > N$ we obtain

$$|\chi(u_n)| = |\chi(u_n) - \chi_n(u_n)| \leq \varepsilon \|u_n\|_U = \varepsilon.$$

This proves that the sequence $\{u_n\}$ converges to zero weakly. By the compactness of G, it follows that $\|Gu_n\| \to 0$, and it remains to observe that $\|G_n\| \leq 2\|Gu_n\|$. $\square$

Now, suppose that U is embedded in a Banach space V $(U \subset V)$, and that the operator I that embeds U in V is compact.

Corollary 1. *Let $I_n = I|_{U^{(n)}}$, and suppose that the sequence $\{U_n^*\}$ is asymptotically dense in U^*. Then $\|I_n\| \to 0$.*

Let $\{P_n\}$ be a sequence of projections in the space U. Then the $\{P_n^*\}$ are projections in U^*, and

$$\mathcal{R}(P_n^*) = P_n^* U^* = \{\chi \in U^* \mid \chi(u - P_n u) = 0, u \in U\} = U_n^*.$$

Thus, we arrive at the following statement.

Corollary 2. *If $G \in \mathcal{L}(U, F)$ is a compact operator and the sequence $\{\mathcal{R}(P_n^*)\}$ is asymptotically dense in U^*, then there exists a numerical sequence $\alpha_n \to 0$ such that for all $u \in U$ we have*

$$\|Gu - GP_n u\| \leq \alpha_n \|u - P_n u\|.$$

Proof. It suffices to observe that $u - P_n u \in U^{(n)}$, so that we can put $\alpha_n = \|G_n\| \to 0$. $\square$

In the special case where the role of G is played by the compact operator that embeds U in V, we obtain the following.

Corollary 3. *If the sequence $\{\mathcal{R}(P_n^*)\}$ is asymptotically dense in U^*, then*

$$\|u - P_n u\|_V \leq \alpha_n \|u - P_n u\|_U, \quad \alpha_n \to 0,$$

for all $u \in U$.

Remark. In Corollaries 2 and 3 it is not assumed that $\|u - P_n u\|_U \to 0$.

In connection with Corollary 3, it should be noted that, in the obvious estimate

$$\|u - P_n u\|_V \leq \|I_n\|(1 + \|P_n\|) E_{U_n}(u)_U$$

for the discrepancy of the approximation of u by the projection $P_n u$, the last factor on the right depends only on the coordinate system and not on the projection system, the middle-term factor depends on both systems, and the first factor depends only on the projection system.

We describe at once how Theorem 1 applies to projection methods. Suppose we apply the projection method to a linear equation $Au = f$, where A is a bounded linear operator acting from a Banach space U to a Banach space F ($A \in \mathcal{L}(U, F)$). Let $u_n^* \in U_n$ be the corresponding approximate solution, i.e., the solution of the equation

$$A_n u_n = P_n f, \quad A_n = P_n A|_{U_n}.$$

Here P_n is a projection of F onto a finite-dimensional subspace F_n, and U_n is the linear hull of the coordinate system.

Theorem 2. *Suppose that the space U is compactly embedded in a space V, and that the following conditions are fulfilled*:

1) *the operator A admits a bounded inverse, $A^{-1} \in \mathcal{L}(F, U)$;*

2) *the operators A_n admit bounded inverses $A_n^{-1} \in \mathcal{L}(F_n, U_n)$ for all sufficiently large n;*

3) *the sequence $\{\mathcal{R}(P_n^*)\}$ of subspaces is asymptotically dense in F^*.*

Then the projection method is superconvergent: *there is a numerical sequence $\alpha_n \to 0$ such that*, *for any right-hand side $f \in F$, we have*

$$\|u^* - u_n^*\|_V \leq \alpha_n \|u^* - u_n^*\|_U,$$

where u^* is the exact solution.

Proof. The approximate solution is related to the exact one by the formula $u_n^* = Q_n u^*$, where $Q_n = A_n^{-1} P_n A$ is the projection of U onto U_n. It remains to show that the subspaces $\mathcal{R}(Q_n^*)$ form a sequence asymptotically dense in U^*. Let $P_n f = \sum_{j=1}^n \chi_j^n(f) g_j^n$, where $\chi_j^n(g_k^n) = \delta_{kj}$. Then $[A_n^{-1} P_n]^* \varphi = \sum_{j=1}^n \varphi(A_n^{-1} g_j^n) \chi_j^n$, whence $\mathcal{R}([A_n^{-1} P_n]^*) = F_n^* = \langle \chi_1^n, \dots, \chi_n^n \rangle$ is the linear hull of the elements χ_j^n. Since $Q_n^* = A^*[A^{-1} P_n]^*$, we have $\mathcal{R}(Q_n^*) = A^* F_n^*$. Let $\varphi \in U^*$ be arbitrary. Put $\psi = (A^*)^{-1} \varphi$. By condition 3) of the theorem, $E_{F_n^*}(\psi)_{F^*} \to 0$. Next,

$$\begin{aligned} E_{A^* F_n^*}(\varphi)_{U^*} &= \inf_{\varphi_n \in A^* F_n^*} \|\varphi - \varphi_n\| = \inf_{\psi_n \in F_n^*} \|A^* \psi - A^* \psi_n\| \\ &\leq \|A^*\| E_{F_n^*}(\psi)_{F^*} \to 0. \end{aligned}$$

We have proved the asymptotic density of the subspaces $\mathcal{R}(Q_n^*)$, and, with it, Theorem 2. $\square$

We turn to estimation of the norm of I_n. Since U is compactly embedded in V, the space V^* dual to V is compactly embedded in U^*.

Theorem 3. *Let U_n^* is a finite-dimensional subspace in U^*, let I be the embedding operator from U to V, and let I_n be the restriction of I to the subspace*

$$U^{(n)} = \{u_n \in U \mid \chi_n(u_n) = 0, \chi_n \in U_n^*\}.$$

Then

$$\|I_n\| = \sup\{E_{U_n^*}(\psi)_{U^*} \mid \psi \in V^*, \|\psi\|_{V^*} \leq 1\}.$$

Proof. By the duality criterion of the best approximation element in the dual space (see [3, p. 22]), for $\psi \in V^* \subset U^*$ we have

$$E_{U_n^*}(\psi)_{U^*} = \sup\{\psi(u) \mid u \in U^{(n)}, \|u\| \leq 1\}.$$

Therefore,

$$\sup_{\|\psi\|_{V^*} \leq 1} E_{U_n^*}(\psi)_{U^*} = \sup_{u \in U^{(n)}, \|u\| \leq 1} \sup_{\|\psi\|_{V^*} \leq 1} \psi(u) = \sup_{u \in U^{(n)}, \|u\| \leq 1} \|u\|_V = \|I_n\|. \qquad \square$$

Now we consider applications of the theorems just proved to the study of projection methods for solving functional equations.

The Ritz method. Let H be a separable real Hilbert space, and let A be a selfadjoint and positive definite operator in H. We assume that the operator A^{-1} is compact. With the operator A, we associate its energy space H_A, which embeds in H compactly. The scalar product and the norm in H_A will be denoted by $[\cdot,\cdot]$ and $\|\cdot\|_A$. In order to apply Theorem 3, we must consider the embedding of the space H^* dual to H in the space dual to H_A; the latter dual space is identified with H_A.

Lemma. *The embedding of H^* in H_A is given by the formulas*

$$H^* = \mathcal{D}(A), \quad \|u\|_{H^*} = \|Au\|.$$

Proof. The claim means that, for $u \in H_A$, the functional $F(v) = [u, v]$ is bounded on H (i.e., for all $v \in H_A$ we have $|[u, v]| \le c\|v\|$) if and only if $u \in \mathcal{D}(A)$, and then $\|F\| = \|Au\|$. We prove this. If $u \in \mathcal{D}(A)$, then $F(v) = (Au, v)$, whence $|F(v)| \le \|Au\|\,\|v\|$ and $\|F\| = \|Au\|$. Conversely, if $|[u, v]| \le c\|v\|$ for all $v \in H_A$, then, in particular, for all $v \in \mathcal{D}(A)$ we have $|[u, v]| = |(Av, u)| \le c\|v\|$. But this means that $u \in \mathcal{D}(A^*) = \mathcal{D}(A)$. □

Suppose the Ritz method with a coordinate system $\{\omega_k^n\} \subset H_A$, $n = 1, 2, \dots$, $k = 1, \dots, n$, complete in H_A is applied to finding an approximate solution of the equation $Au = f$. In accordance with the Ritz method, the approximate solution is the H_A-orthogonal projection of the exact solution to the linear hull U_n of the coordinate system. Therefore, Theorem 3 and the lemma imply the following statement.

Theorem 4. *For the exact and the Ritz approximate solutions of the equation in question, we have*

$$\|u^* - u_n^*\| \le \alpha_n \|u^* - u_n^*\|_A,$$

where

$$\alpha_n = \sup_{u \in \mathcal{D}(A): \|Au\| \le 1} E_{U_n}(u)_{H_A}.$$

We show how this theorem can be applied to elliptic equations. In a bounded domain $\Omega \in \mathbb{R}^m$ with sufficiently smooth boundary $\partial\Omega$, consider the elliptic equation

$$(1) \qquad Au = -\sum_{k,j=1}^{m} \frac{\partial}{\partial x_k}\left(a_{kj}\frac{\partial u}{\partial x_j}\right) + ru = f$$

with one of the boundary conditions

$$(2) \qquad u|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial N}\Big|_{\partial\Omega} = 0, \quad \text{or} \quad \frac{\partial u}{\partial N} + \sigma u|_{\partial\Omega} = 0.$$

Here $\frac{\partial}{\partial N}$ is the derivative along the conormal. Under the first boundary condition it is assumed that $r(x) \ge 0$, under the second it is assumed that $r(x) \ge r_0 > 0$, and under the third it is assumed that either $r \ge r_0 > 0$, $\sigma \ge 0$, or $r \ge 0$, $\sigma \ge \sigma_0 > 0$. The coefficients of the equation and the function σ are assumed to be sufficiently smooth so as to ensure that for $f \in L_2(\Omega)$ the exact solution u^* belong to the Sobolev space W_2^2 and $\|u^*\|_{W_2^2} \le C\|f\|_{L_2}$.

Suppose we want to solve problem (1), (2) under the second or the third boundary condition and apply the Ritz method with polynomial coordinate functions, i.e., we seek the solution u_n in the form $u_n(x) = p_n(x)$, where $p_n \in \mathbb{P}_n^m$ in a polynomial of degree n with coefficients to be determined.[1] The well-known multidimensional analog of the Jackson theorem for the case of L_2 implies that the number α_n occurring in Theorem 4 satisfies $\alpha_n \le C_1/n$. The same estimate is obtained also in the case of the first boundary condition if we seek an approximate solution in the form[2] $u_n(x) = \omega(x)p_n(x)$, where

[1] The parameter n has changed its meaning: now this is the degree of the polynomial, rather than the number of coordinate functions; the latter number is considerably larger.

[2] The corresponding coordinate functions will also be called polynomial functions.

$p_n \in \mathbb{P}_n^m$ and $\omega(x)$ is a sufficiently smooth function vanishing on the boundary of Ω and satisfying $\omega(x) > 0$ for $x \in \Omega$ and $\operatorname{grad} \omega(x) \neq 0$ for $x \in \partial\Omega$. Instead of an analog of the Jackson theorem, in this case we should apply an analog of Kharrik's approximation theorem (see [4]). For the L_2-generalization of that theorem, see [5]. So, the following statement is true.

Theorem 5. *Under the above assumptions, for the discrepancy of the approximate solution of* (1), (2) *obtained by the Ritz method with polynomial coordinate functions we have*

$$\|u^* - u_n^*\|_{L_2} \leq \frac{C}{n} \|u^* - u_n^*\|_{W_2^{(1)}}.$$

Here we have used the fact that the norm in the energy space H_A is equivalent to the norm in $W_2^{(1)}(\Omega)$. The analog of Jackson's or Kharrik's theorem mentioned above allows us to deduce the following corollary to Theorem 5.

Corollary. *If the exact solution of problem* (1), (2) *belongs to* $W_2^l(\Omega)$, *then the rate of convergence for the Ritz method is characterized by the relation*

$$\|u^* - u_n^*\|_{L_2} = \mathcal{O}(n^{-l}).$$

This statement is not new (see [6]).

Similar results can be obtained for differential equations of order higher than two. Here we restrict ourselves to the case of the simplest boundary value problem for an ordinary differential equation. The L_2-generalization of Kharrik's theorem that we need in this case is elementary; below we formulate it in the form to be applied in what follows, and give a proof.

So, let $W_2^{(l)}$ ($l \in \mathbb{N}$) be the space of functions f on $[-1, 1]$ that possess the following properties: f is continuously differentiable $l-1$ times, $f^{(l-1)}$ is absolutely continuous, and $f^{(l)} \in L_2(-1, 1)$. Let $\dot{W}_2^{(m)}$ denote the subspace of $W_2^{(m)}$ consisting of all f subject to the conditions

$$f(-1) = \cdots = f^{(m-1)}(-1) = 0, \quad f(1) = \cdots = f^{(m-1)}(1) = 0.$$

For $f \in \dot{W}_2^{(m)}$ we put $\|f\|_m = \|f^{(m)}\|_{L_2}$.

Lemma. *Suppose $l > m$. There is a constant c such that for every $f \in W_2^{(l)} \cap \dot{W}_2^{(m)}$ we can find a sequence $\{p_n\}$ of polynomials with the following property: for the polynomials $Q_n(x) = (1 - x^2)^m p_n(x)$ we have*

$$\|f - Q_n\|_m \leq \frac{c}{n^{l-m}} \|f^{(l)}\|_{L_2}.$$

Proof. The set of derivatives of order m of all functions in $W_2^{(l)} \cap \dot{W}_2^{(m)}$ coincides with the set of functions in $W_2^{(l-m)}$ that are orthogonal (in L_2) to all polynomials of degree $m-1$. At the same time, the set of derivatives of order m of all polynomials of the form $Q_n(x) = (1 - x^2)^m p_n(x)$ coincides with the set of all polynomials in $\mathbb{P}_{n+m}$ that are orthogonal to the polynomials of degree $m-1$. Therefore, the best approximation relative to the norm in $\dot{W}_2^{(m)}$ of a function $u \in W_2^{(l)} \cap \dot{W}_2^{(m)}$ by polynomials of the form Q_n coincides with the best approximation $u^{(m)}$ in L_2 by polynomials of degree $n+m$. It remains to employ the well-known estimates for the best L_1-approximations of functions in $W_2^{(l-m)}$ by polynomials (see, e.g., [7, p. 148]). $\square$

Now, consider the boundary value problem

$$L(u) = \sum_{k=0}^{m} (-1)^k (a_k u^{(k)})^{(k)} = f, \tag{3}$$

$$u(-1) = \cdots = u^{m-1}(-1) = 0, \quad u(1) = \cdots = u^{m-1}(1) = 0, \tag{4}$$

where the functions a_k are k times continuously differentiable, $a_m(x) \geq d > 0$, and $a_k(x) \geq 0$ for $k = 0, \dots, m-1$. The operator A defined by $Au = L(u)$ is considered on $\mathcal{D}(A) = W_2^{(2m)} \cap \dot{W}_2^{(m)}$; it is positive definite, its energy space coincides with $\dot{W}_2^{(m)}$, and the norm in the energy space is equivalent to the norm $\|\cdot\|_m$. Moreover, there exist constants $0 < c_1 < c_2$ such that for all $u \in \mathcal{D}(A)$ we have $c_1\|u^{(2m)}\|_{L_2} \leq \|Au\|_{L_2} \leq c_2\|u^{(2m)}\|_{L_2}$.

Suppose that, to find an approximate solution of problem (3), (4), we apply the Ritz method with coordinate functions $\omega_k^n(x) = \omega_k(x) = (1-x^2)^m p_{k-1}(x)$, where $p_{k-1} \in \mathbb{P}_{k-1}$. The next statement follows immediately from Theorem 4 and the previous lemma.

Theorem 6. *For the discrepancy of the approximate solution obtained by the Ritz method, we have the estimate*

$$\|u^* - u_n^*\|_{L_2} \leq \frac{c}{n^m}\|u^* - u_n^*\|_m.$$

Applying the same lemma once again, we arrive at the following statement.

Corollary. *If the exact solution of the problem under consideration belongs to* $W_2^{(l)}$ $(l > 2m)$, *then*

$$\|u^* - u_n^*\|_{L_2} \leq \frac{c}{n^l}\|(u^*)^{(l)}\|_{L_2}.$$

The Galerkin method. In a real Hilbert space H, we consider the linear equation

$$Au + Bu = f \tag{5}$$

under the following assumptions. The operator A is selfadjoint and positive definite, and its inverse is compact. With A we associate the corresponding energy space H_A. The operator B is defined on H_A and is bounded as an operator from H_A to H, i.e., $B \in \mathcal{L}(H_A, H)$. Equation (5) is uniquely solvable. Under these assumptions, (5) is equivalent to the equation

$$u + Tu = A^{-1}f, \tag{6}$$

which will be considered in the space H_A. Here $T = A^{-1}B \in \mathcal{L}(H_A, H_A)$; it is easy to check that T is compact. For the solution of (5) we apply the Galerkin method, i.e., we seek an approximate solution in the form $u_n = \sum_{k=1}^n c_k\omega_k^n$, where $\{\omega_k^n\}$ is a coordinate system complete in H_A. The coefficients c_k are found from the system of equations

$$\sum_{k=1}^{n} \big([\omega_k^n, \omega_j^n] + (B\omega_k^n, \omega_j^n)\big)c_k = (f, \omega_j^n),$$

where $(\cdot,\cdot)$ and $[\cdot,\cdot]$ are the scalar products in H and H_A, respectively. This system can be rewritten as

$$\sum_{k=1}^{n} \big[(I+T)\omega_k^n, \omega_j^n\big]c_k = [A^{-1}f, \omega_j^n],$$

which is the usual system of the Galerkin method for equation (6) viewed as an equation in the Hilbert space H_A. Under our assumptions, this system is uniquely solvable for n

sufficiently large, and the approximate solutions u_n^* converge to the exact solution u^* at the rate of the best approximations:

$$\|u^* - u_n^*\|_A \le cE_{U_n}(u^*)_{H_A},$$

where U_n is the linear hull of the coordinate elements (see [8]). For any $u_n \in U_n$ we have $\big[(I+T)(u^* - u_n^*), u_n\big] = 0$, or, what is the same, $[u^* - u_n^*, v_n] = 0$ for all $v_n \in V_n = (I+T^*)U_n$. Therefore, $u^* - u_n^*$ belongs to the subspace

$$V^{(n)} = \{v \in H_A \mid [v, v_n] = 0, v_n \in V_n\} \subset H_A,$$

and we see that

$$\|u^* - u_n^*\|_H \le \|I_n\| \, \|u^* - u_n^*\|_A,$$

where I_n is the operator that embeds $V^{(n)}$ in H. By Theorem 3 and the lemma before Theorem 4, we have

$$\|I_n\| = \sup_{v \in M_1} E_{V_n}(v)_{H_A}, \quad M_1 = \{v \in \mathcal{D}(A) \mid \|Av\| \le 1\}.$$

For $v \in M_1$ we put $u = (I+T^*)^{-1}v$. Let $u_n \in U_n$ be the element yielding the best approximation of u in U_n: $\|u - u_n\|_A = E_{U_n}(u)_{H_A}$. Then $v_n = (I+T^*)u_n \in V_n$ and $\|v - v_n\|_A \le \|I+T\| E_{U_n}(u)_{H_A}$. Thus, $E_{V_n}(v)_{H_A} \le \|I+T\| E_{U_n}(u)_{H_A}$, so that

$$\|I_n\| \le \|I+T\| \sup_{u \in M_2} E_{U_n}(u)_{H_A}, \quad M_2 = (I+T^*)^{-1}M_1.$$

The operator T^* is related to A and B as follows. For $w \in H_A$, we have $g = T^*w$ if $[g, u] = [w, Tu]$ for all $u \in H_A$, i.e., T^* is defined by the relation

$$[T^*w, u] = (w, Bu), \quad u \in H_A. \tag{7}$$

We apply the aforesaid to elliptic equations. In a bounded domain $\Omega \in \mathbb{R}^m$ with a sufficiently smooth boundary, consider the elliptic equation

$$Lu = -\sum_{k,j=1}^{m} \frac{\partial}{\partial x_k}\left(a_{kj}\frac{\partial u}{\partial x_j}\right) + \sum_{k=1}^{m} b_k \frac{\partial u}{\partial x_k} + cu = f \tag{8}$$

with one of the boundary conditions

$$u|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial N}\Big|_{\partial\Omega} = 0, \quad \text{or} \quad \frac{\partial u}{\partial N} + \sigma u|_{\partial\Omega} = 0,$$

where $\frac{\partial}{\partial N}$ denotes the derivative along the conormal. All coefficients a_{kj}, b_k and c, and also the function $\sigma > 0$ are assumed to be sufficiently smooth. We want to find out what the operator T^* is and, accordingly, what the norm $\|I_n\|$ is, provided that the operator L is split naturally into the sum $A + B$. We only consider the third boundary condition; the other cases are treated similarly. In the case of the third boundary condition, the operator A is represented by the sum in (8) involving the second order derivatives, and the product (w, Bu) reshapes as follows:

$$(w, Bu) = -\int_\Omega \left(\sum_{k=1}^{m} \frac{\partial}{\partial x_k}(b_k w)u + cuw\right) dx + \int_{\partial\Omega} \sum_{k=1}^{m} b_k wu \cos(\overline{n}, x_k)\, dS.$$

For $T^*w = v$, identity (7) takes the form

$$\begin{aligned}[T^*w, u] &= \int_\Omega \sum_{k,j=1}^{m} a_{kj} \frac{\partial v}{\partial x_k}\frac{\partial u}{\partial x_j}\, dx + \int_{\partial\Omega} \sigma uv\, dS \\ &= -\int_\Omega zu\, dx + \int_{\partial\Omega} gu\, dS,\end{aligned}$$

where

$$z = -\sum_{k=1}^{m} \frac{\partial}{\partial x_k}(b_k w) + cw, \quad g = \sum_{k=1}^{m} b_k w \cos(\overline{n}, x_k).$$

Therefore, $v = T^* w$ solves the problem

$$-\sum_{k,j=1}^{m} \frac{\partial}{\partial x_k}\left(a_{kj}\frac{\partial v}{\partial x_j}\right) = z, \quad \frac{\partial v}{\partial N} + \sigma v|_{\partial\Omega} = g.$$

Since $\|z\| \le c_5\|w\|_A$, and g is the trace of some function $G \in W_2^{(1)}$ such that $\|G\|_{W_2^{(1)}} \le c_1\|w\|_A$, we can write $\|T^*w\|_{W_2^{(2)}} \le c_7\|w\|_A$ (see [9]). So, for any $w \in H_A$ we have $T^*w \in W_2^{(2)}$ and $\|T^*w\|_{W_2^{(2)}} \le c_1\|w\|_A$. Now, let $u \in M_2$, and let $u + T^*u = v \in M_1$, i.e., $v \in \mathcal{D}(A)$ and $\|Av\| \le 1$. The latter means that $\|v\|_{W_2^{(2)}} \le c_2$, which implies that $\|v\|_{W_2^{(1)}} \le c_3$. Next, $\|u\|_A \le \|(I+T^*)^{-1}\|\,\|v\|_A$, $\|T^*u\|_{W_2^{(2)}} \le c_1\|(I+T^*)^{-1}\|\,\|v\|_A$, and $\|u\|_{W_2^{(2)}} \le c_4$, where $c_4 = c_1\|(I+T^*)^{-1}\|c_3 + c_2$. Thus, for the problem in question we have

$$\|I_n\| \le C \sup_{\|u\|_{W_2^{(2)}} \le 1} E_{U_n}(u)_{W_2^{(1)}} \tag{9}$$

(we have used the fact that the norm in H_A is equivalent to the norm in $W_2^{(1)}$).

We also note that in the case of the second boundary condition the role of A can be played, for instance, by the operator

$$Au = -\sum_{k,j=1}^{m} \frac{\partial}{\partial x_k}\left(a_{kj}\frac{\partial u}{\partial x_j}\right) + u,$$

which is positive definite.

This proves the following statement.

Theorem 7. *For the embedding operator I_n arising when we solve equation* (8) *with the boundary conditions as indicated by the Galerkin method, we have estimate* (9).

Again, applying an analog of Jackson's or Kharrik's theorem, we obtain the following consequence of Theorem 7.

Corollary. *If the boundary value problem indicated is solved by the Galerkin method with polynomial coordinate functions, then*

$$\|I_n\| = \mathcal{O}\left(\frac{1}{n}\right),$$

and if the exact solution u^* belongs to $W_2^{(l)}$, then

$$\|u_n^* - u^*\|_{L_2} = \mathcal{O}\left(\frac{1}{n^l}\right).$$

Like the corollary to Theorem 5, the second statement in this corollary is not new (see [6]).

The moment method. Let H be a separable real Hilbert space, and let A be a linear operator in H with dense domain and with $\mathcal{R}(A) = H$. We assume that the inverse A^{-1} exists and is bounded, $A^{-1} \in \mathcal{L}(H,H)$, and that, moreover, A^{-1} is a compact operator. We introduce the new Hilbert space $H_0 = \mathcal{D}(A)$ with the scalar product $[u,v] = (Au, Av)$. The norm in H_0 will be denoted by $\|\cdot\|_0$. The embedding of H_0 in H is compact. Let

$A_0 \in \mathcal{L}(H_0, H)$ be defined by[3] $A_0 u = Au$. Obviously, $\|A_0\| = \|A_0^{-1}\| = 1$. Consider the embedding of the dual space H^* in $H_0^* = H_0$. Let $F_f \in H^*$ be given by $F_f u = (f, u)$. If $u \in H_0$, then $u = A^{-1}g$ and

$$F_f u = (f, A^{-1}g) = \big((A^{-1})^* f, Au\big) = [A^{-1}(A^{-1})^* f, u].$$

Thus, $F_f u = [w, u]$, where $w = A^{-1}(A^{-1})^* f$, and

$$\|F_f\|_0 = \|A^{-1}(A^{-1})^* f\|_0 = \|(A^{-1})^* f\|.$$

Now, suppose that the equation $Au = f$ is solved by the moment method: the approximate solution U_n is sought in an n-dimensional subspace[4] $U_n \subset \mathcal{D}(A)$ and must satisfy the conditions

$$(Au_n, \chi_j^n) = (f, \chi_j^n), \quad j = 1, \dots, n,$$

where $\{\chi_j^n\}$ $(n = 1, 2, \dots,\ j = 1, \dots, n)$ is a system of elements in H, called the projection system. We denote by V_n^* the linear hull of the elements $A^{-1}\chi_j^n$: $V_n^* = \langle A^{-1}\chi_1^n, \dots, A^{-1}\chi_n^n \rangle$.

The conditions that determine the approximate solution can be expressed as the following requirement: $[u_n, \varphi_n] = [u^*, \varphi_n]$ for all $\varphi_n \in V_n^*$. Here u^* is the exact solution. Therefore,

$$\|u_n^* - u^*\| \le \|I_n\| \cdot \|u_n^* - u^*\|_0 = \|I_n\| \cdot \|Au_n^* - f\|, \tag{10}$$

where I_n is the operator that embeds the subspace

$$V^{(n)} = \{v \in H_0 \mid [v, \varphi_n] = 0, \varphi_n \in V_n^*\} \subset H_0$$

in H. Relation (10) means, in particular, that if the system $\{\varphi_j^n = A^{-1}\chi_j^n\}$ is complete in H_0 (this is equivalent to the completeness of the system $\{\chi_j^n\}$ in H), then the approximate solutions tend to the exact one faster than the discrepancies of the approximate solutions tend to zero.

By Theorem 3,

$$\begin{aligned}\|I_n\| &= \sup\{E_{V_n^*}(\psi)_{H_0} \mid \psi = A^{-1}(A^{-1})^* g,\ g \in H,\ \|g\| \le 1\} \\ &= \sup_{\|g\|=1} \min_{c_j} \Big\| A^{-1}(A^{-1})^* g - \sum_{j=1}^n c_j A^{-1}\chi_j^n \Big\|_0 \\ &= \sup_{\|g\|=1} \min_{c_j} \Big\| (A^{-1})^* g - \sum_{j=1}^n c_j \chi_j^n \Big\|.\end{aligned}$$

Thus, we have proved the following statement.

Theorem 8. *For the moment method under consideration, we have the estimate*

$$\|u_n^* - u^*\| \le \alpha_n \|u_n^* - u^*\|_0 = \alpha_n \|Au_n^* - f\|, \quad \alpha_n = \sup_{\|g\| \le 1} E_{F_n^*}((A^{-1})^* g)_H,$$

where $F_n^* = \langle \chi_1^n, \dots, \chi_n^n \rangle$ is the linear hull of the projection system.

[3] The operators A and A_0 should be distinguished at least because they have different adjoints. It is easy to check that $A_0^* = A_0^{-1}$.

[4] So far, we do not need to specify this subspace. The only requirement is that it must contain an approximate solution.

We present an application of this theorem to the moment method for ordinary differential equations. Consider the boundary value problem

$$
\begin{aligned}
&Au = u^{(m)} + a_1 u^{(m-1)} + \cdots + a_m u = f, \\
&l_\nu(u) = \sum_{j=0}^{m-1} \big(\alpha_{\nu j} u^{(j)}(-1) + \beta_{\nu j} u^{(j)}(1)\big) = 0, \quad \nu = 1, \ldots, m.
\end{aligned}
\tag{11}
$$

The coefficients a_j are assumed to be $m-j$ times continuously differentiable. As to the boundary conditions, we assume that if a polynomial p of degree at most $m-1$ ($p \in \mathbb{P}_{m-1}$) satisfies these conditions ($l_\nu(p) = 0$ for $\nu = 1, \ldots, m$), then p is identically zero.[5] Also, we assume that problem (11) is uniquely solvable.

For solving problem (11), we use the following version of the moment method. As the approximate solution, we take a polynomial of degree at most $n+m$ that satisfies the boundary conditions and is such that

$$\int_{-1}^{1} \big(Au_n^*(x) - f(x)\big) q_n(x)\, dx = 0$$

for all polynomials $q_n \in \mathbb{P}_n$. The convergence of this method can easily be established with the help of Kantorovich's theorem on projection methods (see [8, p. 84]). This theorem says that, under the above assumptions, the approximate solutions u_n^* exist for all sufficiently large n, and

$$\|u_n^* - u^*\|_{W_2^{(m)}} \le c_3 \|u_n^* - u^*\|_0 \le c_4 E_{U_n}(u^*)_{H_0} \le c_5 E_{\mathbb{P}_n}(u^{*(m)})_{L_2},$$

where $\|u\|_0 = \|Au\|_{L_2}$. In particular, if $u^* \in W_2^{(l)}(-1,1)$ with $l > m$, then

$$\|u_n^* - u^*\|_{W_2^{(m)}} = \mathcal{O}(1/n^{l-m}).$$

We now turn to problems concerning superconvergence. In our case, the coefficient α_n in Theorem 8 has the form

$$\alpha_n = \sup_{\|g\|_{L_2} \le 1} E_{\mathbb{P}_n}\big((A^{-1})^* g\big)_{L_2}.$$

But $(A^{-1})^* = (A^*)^{-1}$, and A^* is an invertible differential operator of order m with continuous coefficients. Therefore, for the function $v = (A^{-1})^* g$ we have $\|v^{(m)}\|_{L_2} \le c_6 \|g\|_{L_2}$. Consequently, by the Jackson theorem, $\alpha_m \le c_7 n^{-m}$.

So, the following statement is true.

Theorem 9. *Under the above assumptions on the boundary value problem, the moment method converges, and*

$$
\begin{aligned}
\|u_n^* - u^*\|_{W_2^{(m)}} &\le C_1 E_{\mathbb{P}_n}(u^{*(m)})_{L_2}, \\
\|u_n^* - u^*\|_{L_2} &\le \frac{C_2}{n^m} \|u_n^* - u^*\|_{W_2^{(m)}}.
\end{aligned}
$$

If $u^* \in W_2^{(l)}$ $(l > m)$, *then* $\|u_n^* - u^*\|_{L_2} = \mathcal{O}(1/n^l)$.

Remark 1. A close result on the convergence of the moment method for the same problem in the space $C[-1,1]$ was obtained in [10].

Remark 2. By the same means, a similar result on superconvergence can be obtained for the moment method also in the case of the first boundary value problem for the elliptic equation. Then the approximate solution is sought on the basis of the requirement that

[5] This requirement can easily be lifted.

the discrepancy be orthogonal to all polynomials of degree n (the coordinate system may be arbitrary in this case). This leads to the estimate

$$\|u_n^* - u^*\|_{L_2} \leq \frac{c}{n^2} \|u_n^* - u^*\|_{W_2^2}.$$

However, for a polynomial coordinate system, it remains unclear how the rate of convergence of u_n^* to u^* depends on the smoothness properties of the solution.

As a special case of the moment method for problem (11), we mention the Galerkin method: as before, we seek an approximate solution in the form of a polynomial of degree $n+m$ satisfying the boundary conditions, but the coefficients of this polynomial must obey the requirement that the discrepancy should be L_2-orthogonal to all such polynomials (the projection system coincides with the coordinate one). However, in general, in this case the quantity $E_{F_n^*}\big((A^{-1})^* f\big)_H$ is the best approximation of a function satisfying some boundary conditions (these are determined by the domain of A) by polynomials satisfying *some other* boundary conditions. Therefore, we cannot expect that $\|I_n\| \leq c/n^m$ in this case. Nevertheless, for $m = 2k$ and for the simplest boundary value problem ($u^{(j)}(\pm 1) = 0$, $j = 0, \dots, k-1$), a result similar to Theorem 9 can be obtained also for the Galerkin method, and by the same means.

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