Алгебра и анализ
Том. 15 (2003), вып. 4

St. Petersburg Math. J.
Vol. 15 (2004), No. 4, Pages 603–623
S 1061-0022(04)00825-8
Article electronically published on July 7, 2004

IRREDUCIBLE REPRESENTATIONS OF QUANTUM SOLVABLE ALGEBRAS AT ROOTS OF 1

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Abstract. The relationship between the irreducible representations of quantum solvable algebras at roots of 1 and the points of the variety of the center is studied. The quiver of the fiber algebra is characterized, and formulas for the dimension and for the number of the irreducible representations that lie over a point of the center variety are presented.

§1. Introduction

Quantum algebras have arisen in mathematical physics as deformations of the algebra $\mathbb{C}[G]$ of regular functions on a Lie group G, and of its universal enveloping algebra $U(\mathfrak{g})$. From the algebraic viewpoint, quantizing a $\mathbb{C}$-algebra R results in a $\mathbb{C}$-algebra R_q, which is a free module over the ring $\mathbb{C}[q, q^{-1}]$ of Laurent polynomials, and $R = R_q \bmod (q-1)$. If R is a Hopf algebra, it is natural to seek its quantizations in the class of Hopf algebras. The most familiar quantum algebras are the quantum universal enveloping algebra $U_q(\mathfrak{g})$ for a semisimple Lie algebra $\mathfrak{g}$, its dual Hopf algebra $\mathbb{C}_q[G]$, the algebra of quantum matrices, and the quantum Weyl algebra. The chain of examples can be extended by considering the multiparameter versions of these algebras and the quantum spaces of representations.

It is proposed to describe the space of primitive ideals. The tasks of interest are to construct a general theory in the spirit of the orbit method, as well as to classify the primitive ideals for specific quantum algebras. The problem reduces to the specializations $R_\varepsilon = R_q \bmod (q-\varepsilon)$ with $\varepsilon \in \mathbb{C}$. Two cases must be considered: ε is a root of 1 and ε is not a root of 1.

At the moment, the classification of primitive ideals is known for $\mathbb{C}_q[G]$ and the quantum universal enveloping algebra of the maximal solvable (respectively, nilpotent) subalgebra in $\mathfrak{g}$. The case where ε is not a root of unity was studied in the book [J]. The papers [DC-K], [DCKP1], [DCKP2], [DC-L], [DC-P1], [DC-P2] are devoted to the case where ε is a root of 1.

In a simplified form, classification of the primitive ideals in $\mathbb{C}_q[G]$ can be obtained as follows. The classification is based on the description of the symplectic leaves on G as orbits of dressing transformations (see [ST]). With any symplectic leaf Ω, we associate the ideal of functions vanishing on Ω. Its generators are some matrix elements of irreducible representations of the Lie group G. A quantum analog of this ideal can be constructed in the form of the ideal generated by the corresponding matrix elements of the irreducible representations of $U_q(\mathfrak{g})$. The resulting ideal is primitive if ε is not a root of 1, or it helps to stratify primitive ideals if ε is a root of 1.

2000 *Mathematics Subject Classification.* Primary 81R50.

Supported by RFBR (grant no. 02-01-00017).

The next example is the algebra of quantum matrices. This algebra is a bialgebra, but not a Hopf algebra, and the above methods do not apply to it. For the classification of the prime winding-invariant ideals, see [GLn1], [GLn2], [C2], [L]. One of the main goals is the case of $U_q(\mathfrak{g})$. This problem is far from being resolved (see [J], [DC-K]).

The above examples have led to some conjectures. The next goal is to prove these conjectures under the weakest possible assumptions imposed on R_q. Such assumptions must be easily verifiable, and the theory must cover the main examples.

This paper is devoted to the case of roots of 1. In what follows, we assume that ε is a primitive lth root of 1. In the above examples, the algebra R_ε is finite over its center, i.e., R_ε is an order. We note that this property is also fulfilled for the elliptic algebras [FO], for some new quantum groups that arise in the theory of special functions [IK], and for the reflection algebras [BG1].

The problem of description of the primitive ideals for orders is equivalent to the classification problem for irreducible representations. The restriction of an irreducible representation π of R_ε to the center is scalar, $\pi|_{Z_\varepsilon} = \chi \cdot \mathrm{id}$, and gives rise to a character χ (i.e., a point of the variety) of the center Z_ε. We deal with a usual problem for orders: to classify all irreducible representations of R_ε that lie over a given point χ of the variety of the center.

These orders have a common feature: the existence of the quantum adjoint action (see §2 and [DCKP2], [P3]). Acting on the center Z_ε, the quantum adjoint action determines a Poisson bracket. The variety of the center becomes a Poisson variety that splits into symplectic leaves. It is believed that the classification of the irreducible representations can be done in terms of the geometric and Poisson properties of the center variety.

In this paper we study the quantum solvable algebras that are iterated skew polynomial extensions of $K[q, q^{-1}]$. As examples of such algebras, we mention the algebra of quantum matrices (see Example 2.14), the quantum Weyl algebra, the algebras $U_q(\mathfrak{b})$ and $U_q(\mathfrak{n})$ (see Example 2.15) and their numerous subalgebras. The algebra $\mathbb{C}_q[G]$ is not solvable, but localization reduces it to some solvable algebra. For the details concerning examples, see [P2]. The main goal is the construction of a quantum version of the Dixmier theory for $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a solvable Lie algebra (see [D]). Here are some problems around which the general theory is developed.

Problem 1. To prove that the symplectic leaves are algebraic (i.e., each of them is Zariski-open in its Zariski closure).

Problem 2. To prove that the dimension of an irreducible representation over χ is equal to $l^{\frac{d}{2}}$, where d is the dimension of the symplectic leaf of χ. This was conjectured in [DCKP1, 4.5] and [DC-P1, 25.1].

Problem 3. To describe the quiver of the algebra $R_{\varepsilon,\chi} := R_\varepsilon / m(\chi) R_\varepsilon$, where $m(\chi) = \mathrm{Ker}(\chi)$.

Problem 4. To find a formula for the number of irreducible representations over χ.

These problems are solved for $\mathbb{C}_q[G]$ and $U_q(\mathfrak{b})$. The solution of Problem 1 for these algebras can be obtained with the help of the method of dressing transformations. Formulas for dimension and the number of irreducible representations were found in [DC-P2]. The quivers were studied in [BG2].

In [P3], Problems 1 and 2 were solved for sufficiently large l (the point of good reduction of the stratification process). Our goal in this paper is to lift this undesirable restriction on l and to advance in describing the quiver and determining the number of irreducible representations over χ.

The main definition of the paper is that of a normal quantum solvable algebra (or NQS-algebra; see Definition 2.10). We require that this algebra satisfy certain conditions CN1, CN2. We present two examples (quantum matrices and $U(\mathfrak{n})$). Other examples can be found in [G], [P1], [P2]. Our definition of an admissible l (see Definition 2.18) is easily verifiable, and it is necessary for the solution of Problems 1–4.

We stratify the prime $\mathcal{D}$-stable spectrum of an NQS-algebra (see Theorem 3.2). It is proved that every prime $\mathcal{D}$-stable ideal is completely prime (Theorem 3.3). Problems 1 and 2 are solved in Theorem 4.2.

With any finite-dimensional algebra A we can associate a quiver (see [Pie, 6.4]). The vertices of the quiver are primitive idempotents $e_1, \dots, e_N$ such that their right ideals $e_1 A, \dots, e_N A$ represent nonisomorphic classes of principal indecomposable A-modules. Two vertices e_i, e_j are linked by the edge (e_i, e_j) if $e_i J e_j \neq 0$, where J is the radical of A. In this paper we prove (Theorem 4.3) that any two vertices e_i, e_j of the quiver of the finite-dimensional algebra $R_{\varepsilon,\chi}$ are linked by the edges (e_i, e_j) and (e_j, e_i). In particular, the quiver is connected.

In the last §5, we prove (Theorems 5.5 and 5.7) that the number of irreducible representations over χ is equal to l^t, where t is the dimension of a certain toric Lie subalgebra of the stabilizer $\mathfrak{g}(\chi)$ of χ (see Definition 5.6).

I am grateful to J. Cauchon for sending his new preprint [C1] to me. The method of stratification proposed in [C1] is used in this paper. Also, I would like to thank C. De Concini, C. Procesi, K. Brown, and I. Gordon for useful discussions.

§2. Quantum solvable algebras and FA-elements

We begin with general definitions and properties of skew extensions to be used throughout this paper.

Let R_F be a domain and an algebra over a field F.

Definition 2.1. We say that $x \in R_F$ is a *finite adjoint action element* (or x is an FA-element) if x is not a zero divisor and for every $a \in R_F$ there exists a polynomial $f_a(t) = c_0 t^N + c_1 t^{N-1} + \cdots + c_N$, $c_0 \neq 0$, $c_N \neq 0$, over F such that

$$c_0 x^N a + c_1 x^{N-1} a x + \cdots + c_N a x^N = 0. \tag{2.1}$$

An FA-element x gives rise to the denominator set $S_x := \{x^n\}_{n\in\mathbb{N}}$ (see [P1, Proposition 3.3]). Relation (2.1) can be written as

$$f_a(\mathrm{Ad}_x) a = 0, \tag{2.2}$$

where $\mathrm{Ad}_x(a) = x a x^{-1}$. If x is an FA-element in R, then it is an FA-element in $R S_x^{-1}$. The following statements are easy to prove.

Proposition 2.2. *Suppose that $x, y \in \mathrm{Fract}(R_F)$ are FA-elements in a domain R_F and $xy = \gamma yx$ for some $\gamma \in F^*$. Then xy is also an FA-element.*

Proposition 2.3. *Suppose that the above domain R_F is generated by $x_1, \dots, x_n$ and $x \in \mathrm{Fract}(R_F)$. Also, suppose that for every j there exists a polynomial $f_j(t)$ satisfying (2.1) with $a = x_j$. Then x is an FA-element in R_F. If, moreover, $f_j(t)$ splits, $f_j(t) = (t - \gamma_j^{(1)}) \cdots (t - \gamma_j^{(n_j)})$, then, for any $a \in R_F$, the polynomial $f_a(t)$ also splits with roots in the semigroup generated by $\gamma_j^{(s)}$.*

Let τ be an endomorphism of R_F (τ is the identity on F), and let δ be a τ-derivation of R_F (i.e., $\delta(ab) = \delta(a)b + \tau(a)\delta(b)$ for all $a, b \in R_F$) that vanishes on F. The Ore extension (skew extension) $T_F = R_F[x; \tau, \delta]$ of R_F is generated by x and R_F with $xa = \tau(a)x + \delta(a)$ for all $a \in R_F$. Every element of T admits a unique presentation in the form $\sum x^i r_i$ (or $\sum r_i x^i$) with $r_i \in R$.

Proposition 2.4. *Let R_F and $T_F = R_F[x;\tau,\delta]$ be as before, with a diagonalizable automorphism τ. Suppose that $\tau\delta = \gamma\delta\tau$, $\gamma \neq 0$. The element x is an* FA-*element in T_F if and only if δ is locally nilpotent. Moreover, for a τ-eigenvector a, a polynomial $f_a(t)$ of degree N and satisfying* (2.1) *exists if and only if $\delta^N(a) = 0$.*

Proof. Let a be a τ-eigenvector, i.e., $\tau(a) = \beta a$. There exists a polynomial $f(t)$ obeying (2.1). Then

$$0 = c_0 x^N a + c_1 x^{N-1} a x + \cdots + c_N a x^N = f(\beta) a x^N + \{\text{terms of lower degree}\}.$$

This implies that $f(\beta) = 0$, $f(t) = f_1(t)(t-\beta)$, and $0 = f(\mathrm{Ad}_x)a = f_1(\mathrm{Ad}_x)(\mathrm{Ad}_x - \beta)a = f_1(\mathrm{Ad}_x)\delta(a)x^{-1}$. The element $\delta(a)$ is also a τ-eigenvector. After N steps we get $\delta^N(a) = 0$, where $N = \deg f(t)$. On the other hand, if $\delta^N(a) = 0$ and $\tau(a) = \beta a$, then the polynomial

$$f(t) := \prod_{i=1}^{N}(t - \beta\gamma^i)$$

satisfies (2.1). □

Let K be an algebraically closed field of zero characteristic, let q be an indeterminate, and let C be a localization of $K[q,q^{-1}]$ over some finitely generated denominator set. We denote $\Gamma = \{q^k : k \in \mathbb{Z}\}$ and put $F = \mathrm{Fract}(C) = K(q)$.

Definition 2.5. Let R be a unital domain, an algebra over C, and a free C-module. Let x be an element in R.

1) An element $x \in R$ is an FA-*element* if it is an FA-element in $R_F := R \otimes_C F$.

2) We say that x is an FA$_q$-*element* in R if it is an FA-element in $R_F := R \otimes_C F$ and for any $a \in R$ there exists a polynomial $f_a(t)$ satisfying (2.1) and such that it splits and all its roots belong to Γ.

Definition 2.6. We say that two elements a and b *q-commute* if $ab = q^k ba$ for some integer k.

Proposition 2.7 ([C1, Propositions 2.1–2.3])**.** *Let R be as in Definition* 2.5, *and let $T_F = R_F[x;\tau,\delta]$ be a skew extension, where τ is an automorphism, δ is a locally nilpotent τ-derivation, and $\tau\delta = q^s\delta\tau$ with $s \neq 0$. Denote*

$$\widehat{a} = \sum_{n=0}^{+\infty} \frac{(1-q^s)^{-n}}{(n)_{q^s}!}\delta^n\tau^{-n}(a)x^{-n}, \tag{2.3}$$

where $(n)_{q^s} = \frac{q^{sn}-1}{q^s-1}$. Then:

1) *the set $S_x = \{x^m\}_{m\in\mathbb{N}}$ is a denominator subset in T_F;*

2) *the map $a \mapsto \widehat{a}$ is an embedding of R in $T_F S_x^{-1}$;*

3) *$T_F S_x^{-1} = \widehat{R}_F[x^{\pm 1};\tau]$, where $\widehat{R}_F$ is the image of R under the map $a \mapsto \widehat{a}$.*

Throughout this paper, ε is a primitive lth root of 1 such that C admits specialization by $\varepsilon : C \to K$ with $q \mapsto \varepsilon$. For any ε, consider the specialization R_ε of R over K. In what follows we shall use two types of notation. If $a \in R$, we put $a_\varepsilon := a \bmod (q-\varepsilon)$. For $a \in R_\varepsilon$, we denote by $\underline{a} \in R$ an element of the preimage of a under the map $\pi_\varepsilon : R \to R_\varepsilon = R \bmod (q-\varepsilon)$. For any algebra A of R, we set $A_\varepsilon := (A + R(q-\varepsilon)) \bmod (q-\varepsilon) = \pi_\varepsilon(A)$.

If $u_\varepsilon = u \bmod (q-\varepsilon)$ is in the center Z_ε of R_ε, then $\mathcal{D}_u(a) = \frac{u\underline{a} - \underline{a}u}{q-\varepsilon} \bmod (q-\varepsilon)$ determines a derivation of R_ε. We call $\mathcal{D}_u$ the *quantum adjoint action* of u (see [DCKP1], [DCKP2], and [P3]). An ideal is stable with respect to the quantum adjoint action ($\mathcal{D}$-stable) if it is stable with respect to all $\mathcal{D}_u$. The formula $\{a,b\} = \mathcal{D}_{\underline{a}}(b)$ for $a,b \in Z_\varepsilon$ yields a Poisson bracket on $\mathcal{M} = \mathrm{Maxspec}\ Z_\varepsilon$.

Next we present two versions of reduction of Proposition 2.7 modulo $q-\varepsilon$.

Corollary 2.8. *Let T, R, τ, δ, and q^s be as in Proposition 2.7. Suppose that R is generated by elements $x_1, \dots, x_n$ and that τ is a diagonal automorphism with eigenvalues in Γ. Let N be such that $\delta^N(x_i) = 0$ for all $1 \le i \le n$. If l is relatively prime to s and $l \ge N$, then*
1) $T_\varepsilon S_{x_\varepsilon}^{-1} \cong R_\varepsilon[x_\varepsilon^{\pm 1}; \tau]$, *and*
2) x_ε^l *lies in the center* $Z(T_\varepsilon)$.

Proof. Let $\mathfrak{N}_1$ denote the denominator subset in C generated by $q^{sn}-1$, $1 \le n \le d$. The elements $x^{\pm 1}, \widehat{x}_1, \dots, \widehat{x}_M$ generate $\widehat{T} := TS_x^{-1}\mathfrak{N}_1^{-1}$. We denote by $\widehat{R}$ the subalgebra generated by $\widehat{x}_1, \dots, \widehat{x}_M$ over $C\mathfrak{N}_1^{-1}$. By Proposition 2.7, the map $a \mapsto \widehat{a}$ provides an isomorphism of $R\mathfrak{N}_1^{-1}$ onto $\widehat{R}$. We have $\widehat{T} = \widehat{R}[x; \tau, \delta]$. After reduction modulo $q-\varepsilon$, we obtain assertion 1).

Since $x\widehat{x}_j = q^{n_j}\widehat{x}_j x$ for some n_j, the element x_ε^l lies in the center $Z(\widehat{T}_\varepsilon)$. This proves assertion 2). □

Corollary 2.9. *Let T, R, τ, δ, and q^s be as before, and let l be relatively prime to s. Suppose that $x_\varepsilon^l \in Z(T_\varepsilon)$. Then $T_\varepsilon S_{x_\varepsilon}^{-1} \cong R_\varepsilon[x_\varepsilon^{\pm 1}; \tau]$.*

Proof. Taking

$$x^l a = \tau^l(a)x^l + \sum_{i=1}^{l-1} \binom{l}{i}_{q^s} \tau^{l-i}\delta^i(a)x^{l-i} + \delta^l(a)$$

modulo $q-\varepsilon$, we obtain $x^l a = ax^l + \delta^l(a) \bmod (q-\varepsilon)$ and $\delta^l(a) \in (q-\varepsilon)R$ for any $a \in R$. If $n = lm+r$, $0 \le r < l$, then $\delta^n(a) \in (q-\varepsilon)^m R$. On the other hand, $(n)_{q^s}! = (q-\varepsilon)^m c(q)$, where $c(\varepsilon) \ne 0$. Hence

$$\frac{\delta^n(a)}{(n)_{q^s}!} \in Rc^{-1}(q).$$

Consider the denominator subset $\mathfrak{N}_x$ in C generated by $q^n - 1$, where l does not divide n and $\frac{q^{lm}-1}{q-\varepsilon}$, $m \in \mathbb{N}$. For any $a \in R$, the element $\widehat{a}$ (see Proposition 2.3) lies in the localization of T over S_x and $\mathfrak{N}_x$, and $TS_x^{-1}\mathfrak{N}_x^{-1} = R\mathfrak{N}_x^{-1}[x; \tau]$. Reducing modulo $q-\varepsilon$ yields the claim.

□

Let $\mathbb{S} = (s_{ij})$ be an integral skew-symmetric matrix of size $M \times M$. We denote $q_{ij} = q^{s_{ij}}$ and form the matrix $\mathbb{Q} = (q_{ij})$. We choose a subset $\mathfrak{k} := \{t_1, \dots, t_m\}$, where $1 \le t_1 < \dots < t_m \le M$, and call it the distinguished subset.

Definition 2.10. We say that R is a *normal quantum solvable algebra* (or an NQS-algebra) over C if R is generated by elements x_i, $1 \le i \le M$, and x_j^{-1}, $j \in \mathfrak{k}$, such that the monomials $x_1^{t_1} \cdots x_M^{t_M}$ with $t_j \in \mathbb{Z}$, $j \in \mathfrak{k}$, and $t_j \in \mathbb{N}$, $1 \le j \le M$, $j \notin \mathfrak{k}$, form a free C-basis, the algebra C lies in the center of R, and the following conditions are satisfied:
1) $x_i x_j = q_{ij} x_j x_i$ for all i and $j \in \mathfrak{k}$;
2) for $1 \le i < j \le M$, we have

$$x_i x_j = q_{ij} x_j x_i + r_{ij}, \tag{2.4}$$

where r_{ij} is a sum of monomials $cx_{i+1}^{t_{i+1}} \cdots x_{j-1}^{t_{j-1}}$ with $c \in C$. The definition of a quantum solvable algebra is given in Remark 2.12.

The subalgebra $Y_\mathfrak{k}$ generated by C and the $x_i^{\pm 1}$ with $i \in \mathfrak{k}$ is an algebra of twisted Laurent polynomials. The subalgebras R_i generated by C, the x_j with $j \ge i$, and their inverses for the distinguished subscripts, form a chain $R = R_1 \supset R_2 \supset \cdots \supset R_M$ (we call this chain the *right filtration*). It can be proved that each R_i is a skew extension of R_{i+1} (see [GL1, 1.2]). This means that the map $\tau_i : x_j \mapsto q_{ij}x_j, i < j$, extends to an automorphism of R_{i+1}, and the map $\delta_i : x_j \mapsto r_{ij}$ extends to a τ_i-derivation of R_{i+1}. All automorphisms τ_i are the identity on C, and all τ_i-derivations δ_i are equal to zero on C. Formula (2.4) implies that $R_i = R_{i+1}[x_i; \tau_i, \delta_i]$ for $i \notin \mathfrak{k}$ and $R_i = R_{i+1}[x_i^{\pm 1}, \tau_i]$ for $i \in \mathfrak{k}$. An NQS-algebra is a Noetherian domain (see [MC-R, 1.2.9]), a C-algebra, and a free C-module.

The NQS-algebra R admits another filtration (we call it the *left filtration*):

$$R'_1 \subset R'_2 \subset \cdots \subset R'_M = R,$$

where R'_i is generated by C, the elements $x_1, \ldots, x_i$, and their inverses for the distinguished subscripts. Again, we have $R'_i = R'_{i-1}[x_i; \tau'_i, \delta'_i]$ (respectively, $R'_i = R'_{i-1}[x_i^{\pm 1}, \tau'_i]$ for a distinguished i), where τ'_i (respectively, δ'_i) is an automorphism (respectively, τ'_i-derivation) of R_{i-1}. We put $\delta_i = \delta'_i = 0$ if i is distinguished.

Furthermore, for any $1 \leq \alpha < \beta \leq M$ we denote by $R_{[\alpha,\beta]}$ the subalgebra generated by C and the elements x_i and x_j^{-1} such that $\alpha \leq i, j \leq \beta$ and $j \in \mathfrak{k}$.

Observe that $R_{[\alpha,\beta]} = R_{[\alpha-1,\beta]}[x_\alpha; \tau_\alpha, \delta_\alpha]$ for $\alpha \notin \mathfrak{k}$, and $R_{[\alpha,\beta]} = R_{[\alpha-1,\beta]}[x_\alpha^{\pm 1}; \tau_\alpha]$ for $\alpha \in \mathfrak{k}$. Similarly, $R_{[\alpha,\beta]} = R_{[\alpha,\beta-1]}[x_\beta; \tau'_\beta, \delta'_\beta]$ for $\beta \notin \mathfrak{k}$, and $R_{[\alpha,\beta]} = R_{[\alpha,\beta-1]}[x_\beta^{\pm 1}; \tau'_\beta]$ for $\beta \in \mathfrak{k}$. We impose the following conditions on an NQS-algebra.

Condition CN1. We require that R be an iterated q-skew extension for the left and the right filtration. This means that $\tau_i\delta_i = q_i\delta_i\tau_i$ for some $q_i = q^{s_i}$, $s_i \in \mathbb{Z}$, and $\tau'_i\delta'_i = q'_i\delta'_i\tau'_i$ for some $q'_i = q^{s'_i}$, $s'_i \in \mathbb{Z}$. We require that $s_i \neq 0$ (respectively, $s'_i \neq 0$) if $\delta_i \neq 0$ (respectively, $\delta'_i \neq 0$). We call $\{s_i\}, \{s'_i\}$ the systems of exponents of R.

Condition CN2. All τ_i and τ'_i extend to diagonal automorphisms of R and generate commuting diagonal subgroups H and H'.

Proposition 2.11. *Let R be an* NQS-*algebra over C. Put $n = M - m$. Let $\tilde{R}_i$, $i \notin \mathfrak{k}$, be a subalgebra generated by R_i and $Y_\mathfrak{k}$. The chain $R = \tilde{R}_1 \supset \tilde{R}_2 \supset \cdots \supset \tilde{R}_n \supset \tilde{R}_{n+1} = Y_\mathfrak{k}$ is a chain of skew extensions $\tilde{R}_i \cong \tilde{R}_{i+1}[x; \tilde{\tau}_i, \tilde{\delta}_i]$. If, moreover, R satisfies Condition* CN1, *then $\tilde{\tau}_i\tilde{\delta}_i = q_i\tilde{\delta}_i\tilde{\tau}_i$ with the same $q_i = q^{s_i}$ as in* CN1.

Proof. We put $\tilde{\tau}_i(a) = \tau_i(a)$ (respectively, $\tilde{\delta}(a) = \delta(a)$) for $a \in R_{i-1}$, and $\tau_i(x_j) = q_{ij}x_j$ (respectively, $\tilde{\delta}(x_j) = 0$) for $j < i$, $j \in \mathfrak{k}$. A direct calculation completes the proof. $\square$

Remark 2.12. A quantum solvable algebra defined in [P1]–[P3] as an algebra R generated by elements $x_1, x_2, \ldots, x_n, x_{n+1}^{\pm 1}, \ldots, x_{n+m}^{\pm 1}$ with $M = n+m$ and such that the monomials $x_1^{t_1} \cdots x_n^{t_n} x_{n+1}^{t_{n+1}} \cdots x_{n+m}^{t_{n+m}}$ with $t_1, \ldots, t_n \in \mathbb{N}$ and $t_{n+1}, \ldots, t_{n+m} \in \mathbb{Z}$ form a free C-basis and the following conditions are satisfied:

1) $x_ix_j = q_{ij}x_jx_i$, for all i and $n + 1 \leq j \leq M$;

2) $x_ix_j = q_{ij}x_jx_i + r_{ij}$, $1 \leq i < j \leq n$, where r_{ij} is an element of the subalgebra R_{i+1} generated by $x_{i+1}, \ldots, x_n, x_{n+1}^{\pm 1}, \ldots, x_{n+m}^{\pm 1}$.

Proposition 2.9 claims that an NQS-algebra is a quantum solvable algebra. Conditions CN1 and CN2 are comparable with more general Conditions Q1–Q4 in [P2] and Conditions 3.2–3.4 in [P3].

Proposition 2.13. *Any* FA-*element in a quantum solvable algebra* (*in particular*, *an* NQS-*algebra*) *R is an* FA_q-*element.*

Proof. Let R be a quantum solvable algebra (see the previous remark). For a monomial $w = x_1^{t_1} \cdots x_M^{t_M}$, we denote $\deg(w) = (t_1, \ldots, t_M)$. The lexicographic ordering provides a filtration in R. The algebra $A_\mathbb{Q} := \mathrm{gr}(R)$ is generated by $a_i = \mathrm{gr}(x_i)$, $1 \leq i \leq M$, and

a_j^{-1}, $j \in \mathfrak{k}$. The relations are $a_i a_j = q_{ij} a_j a_i$. The algebra $A_{\mathbb{Q}}$ is the localization of the algebra of twisted polynomials. As usual, $(\cdot,\cdot)$ denotes the standard scalar product in $\mathbb{Z}^M$. For two monomials $a, b \in A_{\mathbb{Q}}$ with $\deg(a) = \underline{m}$, $\deg(b) = \underline{n}$, we have $ab = q^{(\mathbb{S}\underline{m},\underline{n})}ba$. For every $u, v \in R$ with $\deg(u) = \underline{m}$, $\deg(v) = \underline{n}$, we have

$$uv = q^{(\mathbb{S}\underline{m},\underline{n})}vu + \{\text{terms of lower degree}\}. \tag{2.5}$$

Let $u, v \in R$, and let u be an FA-element. Let $f(t)$ be the corresponding polynomial that satisfies (2.1) with $x := u$ and $a := v$. If $\gamma := q^{(\mathbb{S}\underline{m},\underline{n})}$, then, by (2.5),

$$0 = c_0 u^N v + c_1 u^{N-1} vu + \cdots + c_N vu^N = f(\gamma)vu^N + \{\text{terms of lower degree}\}.$$

Consequently, $f(\gamma) = 0$ and $f(t) = (t - \gamma)f_1(t)$. The element $v_1 := uv - \gamma vu$ is annihilated by $f_1(\mathrm{Ad}_u)$. The proof can be finished by induction on the degree of the polynomial $f(t)$. □

Here are two most familiar examples of NQS-algebras.

Example 2.14. Quantum matrices. The algebra $M_q(n, K)$ of regular functions on quantum matrices is generated by $C := K[q, q^{-1}]$ and the quantum matrix entries $\{a_{ti}\}_{t,i=1}^n$ that satisfy the relations $a_{ti}a_{sj} - a_{sj}a_{ti} = (q - q^{-1})a_{si}a_{tj}$ for $i < j$, $t < s$ and $a_{ti}a_{sj} = qa_{sj}a_{ti}$ for $t < s, i = j$ and $t = s, i < j$.

The algebra $M_q(n, K)$ is an NQS-algebra with respect to the generators $x_{(i-1)n+j} = a_{ij}$. It satisfies Conditions CN1 (see [G], [P2]) and CN2 (the map $\tau_{ij} : a_{ij} \mapsto qa_{ij}$, i.e., the multiplication of the ith row by q, is an automorphism of R). Other examples can be obtained by considering subalgebras (like quantum triangular matrices), some generalizations, and multiparameter versions of this algebra.

Example 2.15. Consider $U_q(\mathfrak{n})$, where $\mathfrak{n}$ is the upper nilpotent subalgebra of a semisimple Lie algebra. The algebra $U_q(\mathfrak{n})$ is generated over $C = K[q, q^{-1}, (q^{d_i} - q^{-d_i})^{-1}]$ by E_i, $i = 1, \dots, n$, with the quantum Chevalley–Serre relations. We fix a reduced expression $w_0 = s_{i_1} \cdots s_{i_N}$ of the longest element in the Weyl group W. We consider the convex ordering $\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_2), \dots, \beta_N = s_{i_1} \cdots s_{i_{N-1}}(\alpha_N)$ in the set Δ^+ of positive roots and the quantum root vectors $E_{\beta_s} = T_{i_1} \cdots T_{i_{s-1}} E_{i_s}$, $1 \le s \le N$ (see [Lu]). The relations for the E_{β_i} are as follows (see [LS]):

$$E_{\beta_i}E_{\beta_j} - q^{-(\beta_i,\beta_j)}E_{\beta_j}E_{\beta_i} = \sum_{m \in \mathbb{Z}_+^N} c_m E^m,$$

where $i < j$, $c_m \in K[q, q^{-1}]$, and $c_m \neq 0$ only if $m = (m_1, \dots, m_N)$ is such that $m_s = 0$ for $s \le i$ and for $s \ge j$. The algebras $U_q(\mathfrak{n})$ and $U_q(\mathfrak{b})$ and the subalgebras $U_q^w(\mathfrak{n})$ (see [C1], [DC-P1]) are NQS-algebras. They satisfy Conditions CN1 (see [G], [P2]) and CN2 (the map $\tau_\alpha : E_\beta \mapsto q^{(\alpha,\beta)}E_\beta$ is an automorphism of R).

Proposition 2.16. *Let R be an* NQS-*algebra satisfying Condition* CN1, *with the systems of exponents $\{s_i\}, \{s'_i\}$.*

1) *All x_α, $1 \le \alpha \le M$, are* FA-*elements in R. Let N_α (see Proposition* 2.4) *be chosen in such a way that $\delta_\alpha^{N_\alpha}(x_j) = 0$, $\alpha < j$, and $(\delta'_\alpha)^{N_\alpha}(x_j) = 0$, $j < \alpha$.*

2) *For any $1 \le \alpha \le M$, $\alpha \notin \mathfrak{k}$, consider two denominator subsets: $\mathfrak{N}_\alpha$ is generated by $q^{s_\alpha t} - 1$ and $q^{s'_\alpha t} - 1$, $1 \le t < N_\alpha$, and S_α is generated by x_α. The algebra $RS_\alpha^{-1}\mathfrak{N}_\alpha^{-1}$ is an* NQS-*algebra with the distinguished subset $\mathfrak{k} \cup \{\alpha\}$ over $C\mathfrak{N}_\alpha^{-1}$ and with the same matrix $\mathbb{Q}$ and the systems of exponents as for the algebra R.*

Proof. Statement 1) is proved as in [P1, Lemma 4.3]. To prove 2), we apply Proposition 2.5 to two extensions $R_\alpha = R_{\alpha+1}[x_\alpha; \tau_\alpha, \delta_\alpha]$ and $R'_\alpha = R'_{\alpha-1}[x_\alpha; \tau'_\alpha, \delta'_\alpha]$, and consider the new system $\widehat{x}_1, \dots, \widehat{x}_{\alpha-1}, x_\alpha^{\pm 1}, \widehat{x}_{\alpha+1}, \dots, \widehat{x}_M$ of generators of RS_α^{-1}. □

Corollary 2.17. *Let R be as in Proposition* 2.16. *Suppose that l is relatively prime to s_α, s'_α, and that $x^l_{\alpha\varepsilon}$ lies in the center of R_ε; then $R_\varepsilon S^{-1}_{\alpha\varepsilon}$ is isomorphic to the specialization of some* NQS-*algebra modulo $q-\varepsilon$.*

Proof. Consider the multiplicatively closed subset $\mathfrak{N}_{\alpha,l}$ generated by $q^{s_\alpha n}-1$, $q^{s'_\alpha n}-1$ (where $1 \le n < N_\alpha$ and l does not divide n) and $\frac{q^{s_\alpha lm}-1}{q-\varepsilon}$, $\frac{q^{s'_\alpha lm}-1}{q-\varepsilon}$ for $1 \le lm < N_\alpha$. Since l is relatively prime to s_α and s'_α, the polynomials in $\mathfrak{N}_{\alpha,l}$ do not vanish at $q=\varepsilon$. The element $x^l_{\alpha\varepsilon}$ belongs to the center of R_ε; the proof of Corollary 2.9 shows that $\widehat{x}_1,\dots,\widehat{x}_{\alpha-1},x^{\pm1}_\alpha,\widehat{x}_{\alpha+1},\dots,\widehat{x}_M \in RS^{-1}_\alpha\mathfrak{N}^{-1}_{\alpha,l}$. Reducing the generators modulo $q-\varepsilon$, we get the system $\widehat{x}_{1\varepsilon},\dots,\widehat{x}_{\alpha-1,\varepsilon},x^{\pm1}_{\alpha\varepsilon},\widehat{x}_{\alpha+1,\varepsilon},\dots,\widehat{x}_{M\varepsilon}$ of generators of $R_\varepsilon S^{-1}_{\alpha\varepsilon}$. □

For an NQS-algebra R, let $N := N_R = \max\{N_\alpha\}$. For $1 \le i_1 < \cdots < i_k \le M$ and $\mu := \{i_1,\dots,i_k\} \supset \mathfrak{k}$, we denote by $\mathbb{S}_\mu$ the submatrix (s_{ij}), $i,j\in\mu$, of $\mathbb{S}$.

Definition 2.18. We say that a positive integer l (respectively, a primitive lth root ε of unity) is admissible for an NQS-algebra R if the following conditions are satisfied.

1) l is relatively prime to all elementary divisors of all submatrices $\mathbb{S}_\mu$, $\mu\supset\mathfrak{k}$;
2) l is relatively prime to s_i and s'_i, $1\le i\le M$;
3) $l \ge N$.

Lemma 2.19. *Suppose ε satisfies conditions* 2) *and* 3) *of Definition* 2.18, *and R is an* NQS-*algebra satisfying Condition* CN1. *Then the elements $\{x^l_{i\varepsilon}\}$ lie in the center Z_ε of R_ε.*

Proof. It suffices to apply Corollary 2.8. □

Proposition 2.20. *Let R and ε be as in Lemma* 2.19.

1) *If x is an* FA-*element of R_F (respectively, R_ε), then the linear operator Ad_x is diagonalizable in $R_FS^{-1}_x$ (respectively, $R_\varepsilon S^{-1}_x$).*

2) *For any* FA-*element x in R the element x^l_ε belongs to Z_ε.*

Proof. Lemma 2.19 implies that for an admissible ε, the algebra R_ε is finite over its center. The set of admissible roots of unity is infinite. Statement 1) is a consequence of [P1, Corollary 2.5 and Proposition 3.4].

We prove 2). For an FA-element x and for any a in R there exists a minimal polynomial $f(t)$ satisfying (2.1). By Proposition 2.13, the roots of $f(t)$ belong to Γ. Suppose that $q^{\alpha_1},\dots,q^{\alpha_N}$ are the roots of $f(t)$. The element $u=x^l$ is also an FA-element of R. The operators Ad_x and Ad_{x^l} are simultaneously diagonalizable. The roots of the corresponding polynomial $f_*(t)$ for x^l are $\lambda_i := q^{\alpha_i l}$, $1\le i\le N$. This implies that $f_*(t)=(t-1)^N \bmod (q-\varepsilon)$ and $(\mathrm{Ad}_{x^l_\varepsilon}-\mathrm{id})a=0$. On the other hand, by 1), x_ε and x^l_ε are FA-elements in R_ε. Thus, $\mathrm{Ad}_{x^l_\varepsilon}$ is diagonalizable. It follows that $x^l_\varepsilon\in Z_\varepsilon$, and we obtain 2). □

Definition 2.21. Let R and ε be as before. We say that an ideal is $\mathcal{D}_0$-*stable* if it is stable with respect to all derivations $\mathcal{D}_{x^l_i}$, $1\le i\le M$.

Notation 2.22. For any automorphism $\tau\in H$, let θ be the following diagonal derivation of R_ε:

$$\theta(a)=\frac{\tau^l-\mathrm{id}}{q-\varepsilon} \bmod (q-\varepsilon).$$

In a similar way we introduce θ' for $\tau'\in H'$. We denote by Θ the commutative subalgebra spanned by $\theta_1,\dots,\theta_M$, and similarly for Θ'.

§3. Stratification of prime ideals

In this section, we stratify the prime spectrum of R and the prime $\mathcal{D}$-stable spectrum of R_ε (Theorem 3.2). We shall prove that any prime $\mathcal{D}$-stable ideal of R_ε is completely prime (Theorem 3.3).

Throughout this section, R is an NQS-algebra satisfying Conditions CN1 and CN2, and ε is subject to conditions 2) and 3) of Definition 2.18.

Consider the multiplicatively closed subset $\mathfrak{N} = \prod_\alpha \mathfrak{N}_{\alpha,l}$ (see Corollary 2.17). The polynomials of $\mathfrak{N}$ do not vanish at $q = \varepsilon$.

We fix an integer i_1 with $1 \le i_1 \le M$. If $i_1 \in \mathfrak{k}$, we put $R^{(1)} := R$. If $i_1 \notin \mathfrak{k}$, we consider the denominator subset S_1 generated by $y_1 := x_{i_1}$. In accordance with Proposition 2.16 and Corollary 2.17, $R^{(1)} := RS_1^{-1}\mathfrak{N}^{-1}$ is an NQS-algebra over $C\mathfrak{N}^{-1}$ with the same systems of exponents as for the algebra R. The algebra $R^{(1)}$ is generated by

$$x_1', \dots, x_{i_1-1}', x_{i_1}^{\pm 1}, x_{i_1+1}', \dots, x_M', \tag{3.1}$$

where $x_j' := \widehat{x}_j$. We recall that all generators q-commute with y_1 and are FA-elements in $R^{(1)}$. It follows that, for all i, the elements $(x_{i\varepsilon}')^l$ lie in the center of $R_\varepsilon^{(1)} = R_\varepsilon S_{1\varepsilon}^{-1}$ (see Proposition 2.20).

Let i_2 be any integer satisfying $i_1 < i_2 \le n$. There exists a positive integer t such that $y_2 := x_{i_2}' x_{i_1}^t \in R$. As in the first step of the stratification process, we consider the denominator subset S_2 generated by q-commuting elements y_1, y_2. As we have seen, the element $(x_{i_2\varepsilon}')^l$ belongs to the center of $R_\varepsilon^{(1)}$. By Corollary 2.17, the algebra $R^{(2)} := RS_2^{-1}\mathfrak{N}^{-1} = RS_1^{-1}S_{x_{i_2}'}^{-1}\mathfrak{N}^{-1}$ is an NQS-algebra with the generators

$$x_1'', \dots, x_{i_1-1}'', x_{i_1}^{\pm 1}, x_{i_1+1}'', \dots, x_{i_2-1}'', (x_{i_2}')^{\pm 1}, x_{i_2+1}'', \dots, x_M''. \tag{3.2}$$

After k steps we get the denominator subset $S := S_\mu$, where $\mu := \{i_1, \dots, i_k\}$ is generated by $\mathfrak{N}$ and a system of q-commuting elements $y_1, \dots, y_k \in R$. We call S the *standard denominator subset*. The algebra $\tilde{R} := R^{(k)} = RS^{-1}$ is an NQS-algebra over $C\mathfrak{N}^{-1}$ with the generators $\tilde{x}_j := x_j^{(k)}$ and $y_1^{\pm 1}, \dots, y_k^{\pm 1}$. All generators are FA-elements in $\tilde{R}$.

Let $Y := Y_\mu$ denote the subalgebra generated by $y_1^{\pm 1}, \dots, y_k^{\pm 1}$ (or $x_{i_1}^{\pm 1}$, $(x_{i_2}')^{\pm 1}$, $(x_{i_3}'')^{\pm 1}$, $\dots, (x_{i_k}^{(k-1)})^{\pm 1}$). The relations in Y look like this: $y_i y_j = q^{t_{ij}} y_j y_i$. The integral matrix $(t_{ij})_{i,j=1}^k$ is obtained by elementary transformations from the submatrix $\mathbb{S}_\mu = (s_{ij})_{ij \in \mu}$ of $\mathbb{S}$. The algebra Y is an algebra of twisted Laurent polynomials.

By Proposition 2.11, we may treat $\tilde{R}$ as an iterated q-skew extension

$$\tilde{R} := \tilde{R}_1 \supset \tilde{R}_2 \supset \cdots \supset \tilde{R}_{\tilde{M}} \supset \tilde{R}_{\tilde{M}+1} =: Y, \tag{3.3}$$

where $\tilde{M} := M - k$ and $\tilde{R}_i \cong \tilde{R}_{i+1}[x; \tilde{\tau}_i, \tilde{\delta}_i]$.

Definition 3.1. 1) We say that $S := S_\mu$ is C-*admissible* if the ideal $\mathcal{J} := \mathcal{J}_S$ of $\tilde{R}$ generated by $\tilde{x}_i$, $i \in [1, M] - \mu$, has zero intersection with C.

2) We say that $S := S_\mu$ is ε-*admissible* if the ideal $J := J_S$ of $\tilde{R}_\varepsilon$ generated by $\tilde{x}_{i\varepsilon}$, $i \in [1, M] - \mu$, is proper.

3) We say that $S := S_\mu$ is $(\varepsilon, \mathcal{D})$-*admissible* if the $\mathcal{D}$-stable ideal of $\tilde{R}_\varepsilon$ generated by J (we denote it by ${}_{\mathcal{D}}J$; see 2)) is proper.

Note that, in general, the ideal $\mathcal{J}$ (respectively, J) may have nonzero intersection with Y (respectively, Y_ε) and is not prime. For instance, this is so for the algebra R_f constructed starting from a polynomial f as follows. This algebra is generated by $x_1, x_2, y_1^{\pm 1}, \dots, y_k^{\pm 1}$ where the elements $\{y_i\}$ lie in the center and $x_1 x_2 - q x_2 x_1 = f(y_1, \dots, y_k, q)$. The ideal $\mathcal{J}$ generated by x_1 and x_2 has nonzero intersection with Y

and is not prime if the polynomial f is reducible. In the case where $f = f(q)$, the ideal $\mathcal{J}$ has nonzero intersection with C.

Theorem 3.2. *Let R be an* NQS*-algebra satisfying Conditions* CN1 *and* CN2, *and let ε be a specialization of C that satisfies conditions* 2) *and* 3) *of Definition* 2.18.

1) *For any $\mathcal{I} \in \operatorname{Spec}(R)$, $\mathcal{I} \cap C = 0$, there exists a unique C-admissible standard denominator subset $S := S_\mu$ such that $\mathcal{I} \cap S = \varnothing$ and $\mathcal{I}S^{-1} \supset \mathcal{J}_S$.*

2) *Let R and ε be as before. For any prime $\mathcal{D}$-stable ideal I of R_ε, there exists a unique $(\varepsilon, \mathcal{D})$-admissible standard denominator subset $S = S_\mu$ such that $I \cap S_\varepsilon = \varnothing$ and $IS_\varepsilon^{-1} \supset J_S$.*

Proof. Let $\mathcal{I} \in \operatorname{Spec}(R)$ and $\mathcal{I} \cap C = 0$. Suppose that $x_1, \dots, x_{i_1-1} \in \mathcal{I}$ and $y_1 := x_{i_1} \notin \mathcal{I}$. All prime ideals of R are completely prime (see [GL2, Theorem 2.3]; this is false for R_ε). Therefore, $\mathcal{I} \cap \{y_1^t\}_{t\in\mathbb{N}} = \varnothing$. The ideal $\mathcal{I}$ admits localization over $S_1 \cdot \mathfrak{N}$ (see the stratification process). By formula (2.3), $x'_1, \dots, x'_{i_1-1} \in \mathcal{I}S_1^{-1}\mathfrak{N}^{-1}$. Suppose that $x'_{i_1+1}, \dots, x'_{i_2-1} \in \mathcal{I}S_1^{-1}\mathfrak{N}^{-1}$ and $x'_{i_2} \notin \mathcal{I}S_1^{-1}\mathfrak{N}^{-1}$. Following the stratification process, finally we see that $\tilde{x}_i \in \mathcal{I}S^{-1}$ for $i \in [1, M] - \mu$. This proves 1).

For a prime $\mathcal{D}$-stable ideal I of R_ε, consider the greatest Θ'-stable ideal $I_{\Theta'}$ in I. The ideal $I_{\Theta'}$ is $\mathcal{D}$-stable (see [P3, Proposition 3.14]) and prime (see [MC-R, 14.2.3], [D, 3.3.2]). Consider the left filtration $R'_{1\varepsilon} \subset \dots \subset R'_{i\varepsilon} \subset \dots \subset R'_{M\varepsilon} = R_\varepsilon$. A prime $(\mathcal{D}, \Theta')$-stable ideal has prime intersections with all subalgebras $R'_{i\varepsilon}$ (see [P3, Theorem 2.12]). Suppose that I contains $x_{1\varepsilon}, \dots, x_{i_1-1,\varepsilon}$ but not $y_{1\varepsilon} = x_{i_1,\varepsilon}$. The ideal $I_{\Theta'} \cap R'_{i\varepsilon}$ is prime. Since

$$\frac{R'_{i\varepsilon}}{I_{\Theta'} \cap R'_{i\varepsilon}} \cong \frac{K[x_{i\varepsilon}]}{I_{\Theta'} \cap K[x_{i\varepsilon}]},$$

the ideal $I_{\Theta'} \cap R'_{i\varepsilon}$ is completely prime. It follows that $I_{\Theta'}$ has empty intersection with the subset $S_{1\varepsilon} := \{y_{1\varepsilon}^m\}_{m\in\mathbb{N}}$. Since $y_{1\varepsilon}$ is a Θ'-eigenvector, I does not intersect $S_{1\varepsilon}$.

Since y_1 is an FA-element in R, the element $y_{1\varepsilon}^l$ lies in the center of R_ε (see Proposition 2.20). By the proof of Corollary 2.9, we can reduce $x'_1, \dots, x'_{i_1-1}$ modulo $q - \varepsilon$, obtaining $x'_{1\varepsilon}, \dots, x'_{i_1-1,\varepsilon} \in IS_{1\varepsilon}^{-1}$.

In the second step, suppose that $x'_{i_1+1,\varepsilon}, \dots, x'_{i_2-1,\varepsilon} \in I_{\Theta'}S_{1\varepsilon}^{-1}$ and $x'_{i_2} \notin I_{\Theta'}S_{1\varepsilon}^{-1}$. The factor algebra

$$\frac{R'_{i_2}S_{1\varepsilon}^{-1}}{I_{\Theta'} \cap R'_{i_2}S_{1\varepsilon}^{-1}}$$

is a prime factor of the algebra generated by two q-commuting elements y_1 and x'_{i_2}. The image of x'_{i_2} is either zero or regular (see [P3, Lemma 3.11]). Since $x'_{i_2} \notin I_{\Theta'}$, the image is regular. The ideal $I_{\Theta'}$ (and $IS_{1\varepsilon}^{-1}$) has empty intersection with $S_{2\varepsilon}$, which is generated by $y_{1\varepsilon}$ and $y_{2\varepsilon}$ (see the stratification process). We consider localization over $S_{2\varepsilon}$. After k steps we arrive at 2). □

We say that an ideal of Y_ε is $\mathcal{D}_0$-*stable*, if it is stable with respect to all derivations $\mathcal{D}_{y_i^l}$ with $1 \le i \le k$. Direct calculations show that

$$\mathcal{D}_{y_i^l} y_{j\varepsilon} = t_{ij} l \varepsilon^{-1} y_{j\varepsilon} y_{i\varepsilon}^l,$$
$$\{y_{i\varepsilon}^l, y_{j\varepsilon}^l\} = t_{ij} l^2 \varepsilon^{-1} y_{i\varepsilon}^l y_{j\varepsilon}^l$$

(see [P3, Lemma 3.16]).

Theorem 3.3. *Let R, ε be as in Theorem* 3.2. *Any prime $\mathcal{D}$-stable ideal of R_ε is completely prime.*

Proof. Let I be a prime $\mathcal{D}$-stable ideal of R_ε. By the preceding theorem, we have

$$\frac{R_\varepsilon S_\varepsilon^{-1}}{IS_\varepsilon^{-1}} = \frac{Y_\varepsilon}{IS_\varepsilon^{-1} \cap Y_\varepsilon}, \tag{3.4}$$

where Y_ε is the algebra of twisted Laurent polynomials that is generated by $y_{1\varepsilon}, \dots, y_{k\varepsilon}$. It follows that the ideal $IS_\varepsilon^{-1} \cap Y_\varepsilon$ of Y_ε is prime. Since y_i is an FA-element, $y_{i\varepsilon}^l$ lies in the center Z_ε (see Proposition 2.20). Since the ideal I is $\mathcal{D}$-stable, it is stable with respect to $\mathcal{D}_{y_i^l} : R_\varepsilon \to R_\varepsilon$, $1 \le i \le k$. The same is true for $IS_\varepsilon^{-1} \cap Y_\varepsilon$. Any prime $\mathcal{D}_0$-stable ideal of an algebra of twisted Laurent polynomials is completely prime by [P3, Corollary 3.18]. The ideal $IS_\varepsilon^{-1} \cap Y_\varepsilon$ is completely prime, and therefore, I is completely prime. □

Till the end of this section we assume that ε is an admissible specialization of C (see Definition 2.18). A new system of generators (monomials) $h_1, \dots, h_k$ of Y can be chosen so as to satisfy the relations

$$h_1 h_2 = q^{m_1} h_2 h_1, \ \dots, \ h_{2r-1} h_{2r} = q^{m_r} h_{2r} h_{2r-1}, \tag{3.5}$$

where $m_1, \dots, m_r$ are relatively prime to l (see Definition 2.18) and $h_{2t+1}, \dots, h_k$ generate the center of Y. All the h_i are FA-elements in $\tilde{R}$. In what follows, we assume that the elements of C and the elements $z_1 := h_{2r+t+1}, \dots, z_p := h_k$, $p = k - 2r - t$, generate the intersection $\mathfrak{Z} := Y \cap \tilde{Z}$, where $\tilde{Z} := \mathrm{Center}(\tilde{R})$. Denote $u_1 := h_{2r+1}, \dots, u_t := h_{2r+t}$. We have $\mathfrak{Z} = K[z_1^{\pm 1}, \dots, z_p^{\pm 1}, q^{\pm 1}]$, $\mathfrak{Z}_\varepsilon = K[z_{1\varepsilon}^{\pm 1}, \dots, z_{p\varepsilon}^{\pm 1}]$, and

$$\begin{aligned} Z(Y) &= K[u_1^{\pm 1}, \dots, u_t^{\pm 1}, z_1^{\pm 1}, \dots, z_p^{\pm 1}, q^{\pm 1}], \\ Z(Y)_\varepsilon &:= Z(Y) \bmod (q - \varepsilon) = K[u_{1\varepsilon}^{\pm 1}, \dots, u_{t\varepsilon}^{\pm 1}, z_{1\varepsilon}^{\pm 1}, \dots, z_{p\varepsilon}^{\pm 1}], \\ Z(Y_\varepsilon) &:= \mathrm{Center}(Y_\varepsilon) = K[h_{1\varepsilon}^{\pm l}, \dots, h_{2r\varepsilon}^{\pm l}, u_{1\varepsilon}^{\pm 1}, \dots, u_{t\varepsilon}^{\pm 1}, z_{1\varepsilon}^{\pm 1}, \dots, z_{p\varepsilon}^{\pm 1}]. \end{aligned}$$

The algebra $Z(Y)_\varepsilon$ coincides with the subalgebra $Z(Y_\varepsilon)^{\mathcal{D}}$ that consists of all elements of $Z(Y_\varepsilon)$ annihilated by all $\mathcal{D}_{y_i^l}$. As before, $\tilde{Z}_\varepsilon := \mathrm{Center}(\tilde{R}_\varepsilon)$. The intersection $\tilde{Z}_\varepsilon \cap Y_\varepsilon$ is a polynomial algebra,

$$\tilde{Z}_\varepsilon \cap Y_\varepsilon = K[h_{1\varepsilon}^{\pm l}, \dots, h_{2r\varepsilon}^{\pm l}, u_{1\varepsilon}^{\pm l}, \dots, u_{t\varepsilon}^{\pm l}, z_{1\varepsilon}^{\pm 1}, \dots, z_{p\varepsilon}^{\pm 1}].$$

Notation 3.4. 1) G is the subgroup in $\tilde{R}$ generated by S (i.e., by $y_1, \dots, y_k$);
2) G^l is its subgroup generated by $y^l, \dots, y_k^l$;
3) $W := \{a \in \tilde{R} : ay = ya \text{ for all } y \in Y\}$;
4) $W_\varepsilon := W \bmod (q - \varepsilon)$.

The elements of G are FA-elements in $\tilde{R}$. It follows that, for any $y \in G$, the linear operator Ad_y is diagonalizable over $C\mathfrak{N}^{-1}$ (Proposition 2.20). Since the generators of G are q-commuting, the set $\{\mathrm{Ad}_y : y \in G\}$ is a commutative subgroup of $\mathrm{Aut}(\tilde{R})$. It follows that the $\{\mathrm{Ad}_y\}$ are simultaneously diagonalizable.

The map $\Delta_{y^l} := y_\varepsilon^{-l} \mathcal{D}_{y^l} : \tilde{R}_\varepsilon \to \tilde{R}_\varepsilon$ is a diagonalizable derivation. Moreover, if $\mathrm{Ad}_y v = q^\alpha v$, then $\Delta_{y^l}(v_\varepsilon) = \underline{\alpha} v_\varepsilon$, where $\underline{\alpha} := \alpha l \varepsilon^{-1}$. If $\mathcal{D}_{y^l}(v_\varepsilon) = 0$ for any $y \in G$, then $v_\varepsilon \in W_\varepsilon$.

The derivation Δ_{y^l} preserves the center $\tilde{Z}_\varepsilon$ and is diagonalizable in it.

Lemma 3.5. *Let $v \in \tilde{R}$ and $v_\varepsilon \in \tilde{Z}_\varepsilon$. Then*
1) *$\mathcal{D}_v(Y_\varepsilon)$ is contained in the ideal $\langle v_\varepsilon \rangle$ generated by v_ε;*
2) *if $v_\varepsilon \in W_\varepsilon \cap \tilde{Z}_\varepsilon$, then $\mathcal{D}_v(Y_\varepsilon) = 0$.*

Proof. We may assume that v_ε (respectively, v) is a Δ_{G^l}-eigenvector (respectively, an Ad_G-eigenvector). For $\mathrm{Ad}_y v = q^\alpha v$, we have $\Delta_{y^l} v_\varepsilon = \underline{\alpha} v_\varepsilon$ and $\underline{\alpha} y_\varepsilon^l v_\varepsilon = \mathcal{D}_{y_l} v_\varepsilon$. On the other hand,

$$\mathcal{D}_{y^l} v_\varepsilon = \{y_\varepsilon^l, v_\varepsilon\} = -\{v_\varepsilon, y_\varepsilon^l\} = -\mathcal{D}_v(y_\varepsilon^l) = -l y_\varepsilon^{l-1} \mathcal{D}_v(y_\varepsilon).$$

Consequently,

$$\mathcal{D}_v(y_\varepsilon) = -\underline{\alpha} l^{-1} y_\varepsilon v_\varepsilon. \tag{3.6}$$

Formula (3.6) implies 1).

To prove statement 2), we decompose v into a sum $v = v_0 + v_1 + \cdots + v_n$ of Ad_G-eigenvectors. Suppose that $v_0 \in W$ (i.e., $\mathrm{Ad}_y v = v$ for all $y \in G$) and $\mathrm{Ad}\, v_i = q^{\alpha_i} v_i$, $\alpha_i \neq 0$ for $1 \leq i \leq n$. Since $v_\varepsilon \in W_\varepsilon$, we have $v_{i\varepsilon} = 0$ for $1 \leq i \leq n$. Using (3.6), we obtain

$$\mathcal{D}_v(y_\varepsilon) = \mathcal{D}_{v_0}(y_\varepsilon) + \mathcal{D}_{v_1}(y_\varepsilon) + \cdots + \mathcal{D}_{v_n}(y_\varepsilon) = \mathcal{D}_{v_0}(y_\varepsilon) = 0.$$

This proves 2). □

Proposition 3.6. *Let $S := S_\mu$ be $(\varepsilon, \mathcal{D})$-admissible, and let ${}_{\mathcal{D}}J$ denote the lowest $\mathcal{D}$-stable ideal that contains $J := J_S$ (see Definition 3.1). Then*

$$\tilde{Z}_\varepsilon = {}_{\mathcal{D}}J \cap \tilde{Z}_\varepsilon + K[h_{1\varepsilon}^{\pm l}, \ldots, h_{2t,\varepsilon}^{\pm l}](W_\varepsilon \cap \tilde{Z}_\varepsilon). \tag{3.7}$$

Proof. Let $v_\varepsilon \in \tilde{Z}_\varepsilon$ be a common eigenvector for Δ_{G^l} . If $v_\varepsilon \notin {}_{\mathcal{D}}J \cap \tilde{Z}_\varepsilon$, then $v_\varepsilon = j_{0\varepsilon} + y_{0\varepsilon}$, where $j_{0\varepsilon} \in {}_{\mathcal{D}}J$, $y_{0\varepsilon}$ is a nonzero element of $Z(Y_\varepsilon)$, and $j_{0\varepsilon}$, $y_{0\varepsilon}$ are Δ_{G^l}-eigenvectors with a common eigenvalue. We can write $y_{0\varepsilon}$ in the form $y_{0\varepsilon} = h_\varepsilon y'_{0\varepsilon}$, where h_ε is some monomial:

$$h_\varepsilon := h_{1\varepsilon}^{m_1 l} \cdots h_{2t\varepsilon}^{m_{2t} l}$$

with $m_1, \ldots, m_{2t} \in \mathbb{Z}$, $y'_{0\varepsilon} \in Y_\varepsilon$, and $\Delta_{G^l} y'_{0\varepsilon} = 0$. Then for the element $v'_\varepsilon := h_\varepsilon^{-1} v_\varepsilon$ we have $\Delta_{G^l} v'_\varepsilon = 0$, whence $v'_\varepsilon \in W_\varepsilon$. □

Observe that ${}_{\mathcal{D}}J \cap Y_\varepsilon$ is a $\mathcal{D}_Y$-stable ideal in the algebra of twisted Laurent polynomials Y_ε. By [P3, Lemma 3.17], ${}_{\mathcal{D}}J \cap Y_\varepsilon$ is generated by its intersection with $Z(Y)_\varepsilon$.

Proposition 3.7. 1) *If $\mathfrak{m}$ is a maximal ideal of $Z(Y)_\varepsilon$ that lies over ${}_{\mathcal{D}}J \cap Z(Y)_\varepsilon$, then $L_\mathfrak{m} := {}_{\mathcal{D}}J + \mathfrak{m}Y_\varepsilon$ is a $\mathcal{D}$-stable ideal in $\tilde{R}_\varepsilon$.*

2) *If $\mathfrak{M}$ is a maximal ideal of $W_\varepsilon \cap \tilde{Z}_\varepsilon$ over ${}_{\mathcal{D}}J \cap W_\varepsilon \cap \tilde{Z}_\varepsilon$, then*

$$L_\mathfrak{M} := {}_{\mathcal{D}}J \cap \tilde{Z}_\varepsilon + K[h_{1\varepsilon}^{\pm l}, \ldots, h_{2t,\varepsilon}^{\pm l}]\mathfrak{M}$$

is a Poisson ideal of $\tilde{Z}_\varepsilon$.

Proof. By the formula $\tilde{R}_\varepsilon = {}_{\mathcal{D}}J + Y_\varepsilon$ (respectively, formula (3.7)), $L_\mathfrak{m}$ (respectively, $L_\mathfrak{M}$) is a two-sided ideal in $\tilde{R}_\varepsilon$ (respectively, $\tilde{Z}_\varepsilon$).

Let $v_\varepsilon \in \tilde{Z}_\varepsilon$ (respectively, $v \in \tilde{R}$) be a common Δ_{G^l}-eigenvector (respectively, Ad_G-eigenvector). We are going to prove that both ideals $L_\mathfrak{m}$ and $L_\mathfrak{M}$ are $\mathcal{D}_v$-stable.

If $v_\varepsilon \in {}_{\mathcal{D}}J \cap \tilde{Z}_\varepsilon$ or $v_\varepsilon \in W_\varepsilon \cap \tilde{Z}_\varepsilon$, the statement is a consequence of Lemma 3.5. For $v_\varepsilon \in K[h_{1\varepsilon}^{\pm l}, \ldots, h_{2t,\varepsilon}^{\pm l}]$, the derivation $\mathcal{D}_v$ is zero in $Z(Y)_\varepsilon$ and in W_ε. Both $\mathfrak{m}$ and $\mathfrak{M}$ are annihilated by $\mathcal{D}_v$. The ideals $L_\mathfrak{m}$ and $L_\mathfrak{M}$ are $\mathcal{D}_v$-stable. □

§4. Irreducible representations

Let R be an algebra and a free C-module. We consider a specialization R_ε of R. As before, Z_ε is the center of R_ε. This algebra has a Poisson structure via the quantum adjoint action (see §2).

Let χ be a central character, $\chi : Z_\varepsilon \to K$, and let $m(\chi)$ be the corresponding maximal ideal. We treat χ as a point of the variety $\mathcal{M} := \mathrm{Maxspec}(Z_\varepsilon)$. We stratify $\mathcal{M}$ (see [BG1]): $\mathcal{M} = \mathcal{M}_0 \supset \mathcal{M}_1 \supset \cdots \supset \mathcal{M}_m = \varnothing$, where $\mathcal{M}_{i+1} = (\mathcal{M}_i)_{\mathrm{sing}}$. All $\mathcal{M}_i$ are Poisson varieties. In the case where $K = \mathbb{C}$, each smooth locus $\mathcal{M}_i^0 := \mathcal{M}_i - \mathcal{M}_{i+1}$ is a complex analytic Poisson variety. Each locus is a disjoint union of symplectic leaves. The symplectic leaf corresponding to χ is denoted by Ω_χ.

Let $m(\chi, \mathcal{D})$ be the greatest Poisson (i.e., $\mathcal{D}$-stable) ideal in $m(\chi)$. The algebra $\mathcal{F} := Z_\varepsilon/m(\chi, \mathcal{D})$ can be viewed as the algebra of regular functions on the Zariski closure $\mathcal{M}_\chi$ of Ω_χ. Let $R_{\varepsilon,\chi}$ denote the finite-dimensional subalgebra $R_\varepsilon/m(\chi)R_\varepsilon$.

Lemma 4.1 ([P3, Lemma 5.1])**.** *Let $K = \mathbb{C}$. Let f be a nonzero element of $\mathcal{F}$. There exists $\chi' \in \Omega_\chi$ such that $f(\chi') \neq 0$ and the algebra $R_{\varepsilon,\chi'}$ is isomorphic to $R_{\varepsilon,\chi}$.*

Theorem 4.2. *Suppose $K = \mathbb{C}$, R is an* NQS-*algebra satisfying Conditions* CN1 *and* CN2, *and ε is an admissible specialization of C. Let π be an irreducible representation with central character χ. Then*:

1) $\dim(\pi) = l^{\frac{1}{2}\dim(\Omega_\chi)}$;
2) Ω_χ *is algebraic* (*i.e., Ω_χ is Zariski open in its Zariski closure*);
3) *the algebras $R_{\varepsilon,\chi'}$ and $R_{\varepsilon,\chi''}$ are isomorphic for any $\chi', \chi'' \in \Omega_\chi$.*

Proof. For an irreducible representation π with the central character χ, we consider its kernel $I(\pi)$ in R_ε. This ideal is prime, and the greatest $\mathcal{D}$-stable ideal $I(\pi, \mathcal{D})$ in $I(\pi)$ is completely prime (see Theorem 3.3). The ideal $m(\chi, \mathcal{D})$ coincides with $I(\pi, \mathcal{D}) \cap Z_\varepsilon$.

By Theorem 3.2, there exists an $(\varepsilon, \mathcal{D})$-admissible denominator set $S := S_\mu$ not intersecting $I(\pi, \mathcal{D})$. The ideal $I(\pi, \mathcal{D})$ admits a localization $\tilde{I}(\pi, \mathcal{D}) := I(\pi, \mathcal{D})S_\varepsilon^{-1}$, and $\tilde{I}(\pi, \mathcal{D}) \supset {}_{\mathcal{D}}J$ (see §3). The subset $S_l := \{h^l : h \in S\}$ is a denominator subset consisting of q-commuting FA-elements, and $\tilde{R} := RS^{-1} = RS_l^{-l}$. The subset $S_{l\varepsilon}$ lies in the center Z_ε, it is a denominator subset in R_ε, and $\tilde{R}_\varepsilon := R_\varepsilon S_\varepsilon^{-1} = R_\varepsilon S_{l\varepsilon}^{-1}$. The ideal $m(\chi, \mathcal{D})$ has empty intersection with S_{l_ε}. We denote $\tilde{Z}_\varepsilon = Z_\varepsilon S_{l\varepsilon}^{-1}$ and $\tilde{m}(\pi, \mathcal{D}) := m(\chi, \mathcal{D})S_{l\varepsilon}^{-1} = \tilde{I}(\pi, \mathcal{D}) \cap \tilde{Z}_\varepsilon \supset {}_{\mathcal{D}}J \cap \tilde{Z}_\varepsilon$. By Lemma 4.1, we may require that $\chi(y_\varepsilon^l) \neq 0$ for any $y \in S_l$. Since π is an irreducible representation, we have $\pi(y_\varepsilon^l) = \chi(y_\varepsilon^l) \cdot \mathrm{id}$. The ideal $I(\pi)$ admits localization over S_ε, and π is an irreducible representation of $\tilde{R}_\varepsilon$.

We recall that $\tilde{R}_\varepsilon := {}_{\mathcal{D}}J + Y_\varepsilon$ (see §3). We can treat $\tilde{R}$ as a free left (and right) Y-module. Consider a free basis that consists of monomials (in the lexicographic order) over $\{\tilde{x}_i\}$. Consider the natural projection $\rho : \tilde{R} \to Y$. The projection ρ is a morphism of left (and right) Y-modules. Similarly, $\rho_\varepsilon : \tilde{R}_\varepsilon \to Y_\varepsilon$ is a morphism of Y_ε-modules, and $\rho_\varepsilon \mathcal{D}_{y^l}(a) = \mathcal{D}_{y^l}\rho_\varepsilon(a)$ for any $y \in G$ and any $a \in \tilde{R}_\varepsilon$. It follows that $\rho_\varepsilon(W_\varepsilon) = Z(Y)_\varepsilon$.

The representation π passes through ρ_ε and is determined by

$$\nu_\alpha,\ 1 \leq \alpha \leq 2r; \quad \lambda_\beta,\ 1 \leq \beta \leq t; \quad \xi_\gamma,\ 1 \leq \gamma \leq p,$$

where $\pi(h_{\alpha\varepsilon}^l) = \nu_\alpha \cdot \mathrm{id}$, $\pi(u_{\beta\varepsilon}) = \lambda_\beta \cdot \mathrm{id}$, and $\pi(z_{\gamma\varepsilon}) = \xi_\gamma \cdot \mathrm{id}$.

The ideal $\tilde{I}(\pi) := I(\pi)S_\varepsilon^{-l}$ is the maximal ideal of $Z(Y_\varepsilon)$ generated by all $h_{\alpha\varepsilon}^l - \nu_\alpha$, $u_{\beta\varepsilon} - \lambda_\beta$, $z_\gamma - \xi_\gamma$. We obtain

$$\dim(\pi) = l^{2r}. \tag{4.1}$$

Let λ be the character of $Z(Y)_\varepsilon$ determined by π. The ideal

$$\mathfrak{m}_\pi := \mathrm{Ker}(\lambda) = \sum_{1 \leq \beta \leq t} Y_\varepsilon(u_{\beta\varepsilon} - \lambda_\beta) + \sum_{1 \leq \gamma \leq p} Y_\varepsilon(z_\gamma - \xi_\gamma) \tag{4.2}$$

is a maximal $\mathcal{D}_Y$-stable ideal in Y_ε.

The character λ satisfies the condition $\lambda|_{{}_{\mathcal{D}}J \cap Z(Y)_\varepsilon} = 0$. We have

$$\tilde{I}(\pi, \mathcal{D}) \subset {}_{\mathcal{D}}J + Y_\varepsilon \mathfrak{m}_\pi \subset \tilde{I}(\pi).$$

By Proposition 3.17, the mid-term ideal is $\mathcal{D}$-stable. This implies that

$$\tilde{I}(\pi, \mathcal{D}) = {}_{\mathcal{D}}J + Y_\varepsilon \mathfrak{m}_\pi. \tag{4.3}$$

Similarly, by (3.7), the central character χ also passes through ρ_ε and is determined by $\nu_\alpha = \chi(h^l_{\alpha\varepsilon})$ and $\chi|_{W_\varepsilon \cap \tilde{Z}_\varepsilon}$.

Denoting

$$\mathfrak{M}_\chi := \operatorname{Ker} \chi|_{W_\varepsilon \cap \tilde{Z}_\varepsilon},$$

we obtain

$$\tilde{m}(\pi, \mathcal{D}) = {}_{\mathcal{D}}J \cap \tilde{Z}_\varepsilon + K[h^{\pm l}_{1\varepsilon}, \ldots, h^{\pm l}_{2t,\varepsilon}]\mathfrak{M}_\chi. \tag{4.4}$$

Comparison of (3.7) and (4.4) shows that, as a Poisson algebra, the algebra $\tilde{Z}_\varepsilon/\tilde{m}(\pi, \mathcal{D})$ is isomorphic to $\mathbb{C}[h^{\pm l}_{1\varepsilon}, \ldots, h^{\pm l}_{2t\varepsilon}]$ with the Poisson bracket

$$\{h^l_{1\varepsilon}, h^l_{2\varepsilon}\} = m_1 l^2 \varepsilon^{-1} h^l_{1\varepsilon} h^l_{2\varepsilon}, \ldots, \{h^l_{2r-1,\varepsilon}, h^l_{2r\varepsilon}\} = m_r l^2 \varepsilon^{-1} h^l_{2r-1,\varepsilon} h^l_{2r\varepsilon}.$$

The maximal spectrum of the above Poisson algebra has a single symplectic leaf. It follows that the symplectic leaf Ω_χ contains a subset $\mathcal{O}$ that is Zariski-open in the Zariski closure $\mathcal{M}_\chi := \overline{\Omega}_\chi$. Consequently, $\dim \Omega_\chi = \dim \mathcal{O} = 2r$, and $\dim(\pi) = l^{\frac{1}{2}\dim(\Omega_\chi)}$ by (4.3). This proves 1).

2) For any $\chi' \in \mathcal{M}_\chi - \Omega_\chi$, we have

$$\dim \Omega_{\chi'} \leq \dim(\mathcal{M}_\chi - \mathcal{O}) < \dim \mathcal{O} = \dim \Omega_\chi.$$

Then $\mathcal{M}_\chi - \Omega_\chi = \{\chi' \in \mathcal{M}_\chi : \dim \Omega_{\chi'} < 2r\}$. On the other hand, the subset $\mathcal{M}_{<2r} := \{\chi' \in \mathcal{M} : \dim \Omega_{\chi'} < 2r\}$ is Zariski closed in $\mathcal{M}$. The subset $\mathcal{M}_\chi - \Omega_\chi$ coincides with $\mathcal{M}_{<2r} \cap \mathcal{M}_\chi$ and is Zariski-closed. Thus, Ω_χ is Zariski-open in $\mathcal{M}_\chi$.

3) Consider the following equivalence relation on Ω_χ: $\chi \cong \chi'$ if and only if $R_\varepsilon/m(\chi)R_\varepsilon \cong R_\varepsilon/m(\chi')R_\varepsilon$. Any equivalence class $[\chi] = \{\chi' : \chi' \cong \chi\}$ is an open subset of Ω_χ in the topology of a complex analytic manifold (see [P3, Lemma 5.1]). The manifold Ω_χ is connected. This proves 3). □

Theorem 4.3. *Let K be an algebraically closed field of zero characteristic. Let R and ε be as in Theorem 4.2. Any two vertices e_i, e_j, $i \neq j$, of the quiver of the algebra $R_{\varepsilon,\chi}$ are linked by the edges (e_i, e_j) and (e_j, e_i). In particular, the quiver is connected.*

Proof. Step 1. We shall prove that all irreducible representations over a common central character χ can pass through a suitable localization $\tilde{R}_\varepsilon$ such that $\pi({}_{\mathcal{D}}J) = 0$ for any π over χ.

For any irreducible representation π there exists an $(\varepsilon, \mathcal{D})$-admissible standard denominator subset $S := S_\mu$ such that $I(\pi, \mathcal{D}) \cap S_{l\varepsilon} = \varnothing$ and $\tilde{I}(\pi, \mathcal{D}) \supset {}_{\mathcal{D}}J$. We may assume that $I(\pi) \cap S_{l\varepsilon} = \varnothing$ (see the proof of Theorem 4.2). As before, χ is the central character of π. The ideal $\tilde{m}(\pi, \mathcal{D})$ admits localization over $S_{l\varepsilon}$, and $\tilde{I}(\pi) \supset R_\varepsilon \tilde{m}(\chi) \supset R_\varepsilon \tilde{m}(\chi, \mathcal{D})$.

For another irreducible representation π' over χ, we also have $I(\pi') \supset R_\varepsilon m(\chi) \supset R_\varepsilon m(\pi, \mathcal{D})$. If $I(\pi', \mathcal{D})$ is the greatest $\mathcal{D}$-stable ideal in $I(\pi')$, then $I(\pi') \supset I(\pi', \mathcal{D}) \supset R_\varepsilon m(\chi, \mathcal{D})$ and $I(\pi', \mathcal{D}) \cap Z_\varepsilon = m(\chi, \mathcal{D})$. This implies that $I(\pi', \mathcal{D}) \cap S_{l\varepsilon} = \varnothing$ and $I(\pi', \mathcal{D}) \supset \{\tilde{x}^l_{i\varepsilon}, i \in [1, \tilde{M} - \mu]\}$ (see §3). That is, $I(\pi', \mathcal{D})$ contains ${}_{\mathcal{D}}J$ and admits localization over $S_{l\varepsilon}$. The ideal $I(\pi')$ also admits localization over $S_{l\varepsilon}$. This proves the claim of Step 1.

Step 2. In accordance with Step 1, any irreducible representation π over χ is a representation of $\tilde{R}_\varepsilon$, and its kernel contains ${}_{\mathcal{D}}J$. Then ${}_{\mathcal{D}}J \bmod m(\chi)$ is contained in the radical of $R_\varepsilon/m(\chi)R_\varepsilon$.

By (3.7), π lies over χ if and only if $\pi(h^l_{i\varepsilon}) = \chi(h^l_{i\varepsilon}) \cdot \mathrm{id}$, and

$$\pi|_{W_\varepsilon \cap \tilde{Z}_\varepsilon} = \chi|_{W_\varepsilon \cap \tilde{Z}_\varepsilon} \cdot \mathrm{id}\,. \tag{4.5}$$

We denote $Z(Y_\varepsilon)' := \rho_\varepsilon(W_\varepsilon \cap \tilde{Z}_\varepsilon)$. Since $\rho_\varepsilon(W_\varepsilon) = Z(Y)_\varepsilon$, the set $Z(Y_\varepsilon)'$ is a subalgebra of $Z(Y)_\varepsilon$.

The character χ determines a character χ' on $Z(Y_\varepsilon)'$ such that $\chi'\rho_S(w_\varepsilon) = \chi(w_\varepsilon)$ for any $w_\varepsilon \in W_\varepsilon \cap \tilde{Z}_\varepsilon$. In particular, $\chi'(z_{i\varepsilon}) = \chi(z_{i\varepsilon})$ and $\chi'(u_{i\varepsilon}^l) = \chi(u_{i\varepsilon}^l)$.

The proof of Theorem 4.2 shows that there exists a one-to-one correspondence between the irreducible representations over χ and the characters λ of $Z(Y)_\varepsilon$ such that

$$\lambda|_{Z(Y_\varepsilon)'} = \chi'|_{Z(Y_\varepsilon)'}. \tag{4.6}$$

We shall say that such a λ is comparable with χ. In particular, $\lambda(z_{i\varepsilon}) = \chi'(z_{i\varepsilon})$, and $\lambda_i^l = \lambda(u_{i\varepsilon}^l) = \chi'(u_{i\varepsilon}^l) =: \chi_i$.

For two characters λ and λ' over χ we have $\lambda_i' = \varepsilon_i\lambda_i$, $1 \le i \le r$, where $\varepsilon_1, \dots, \varepsilon_r$ are lth roots of unity. We denote

$$e_\lambda = l^{-r}\prod_{i=1}^{r}((\lambda_i^{-1}u_{i\varepsilon})^{l-1} + (\lambda_i^{-1}u_{i\varepsilon})^{l-2} + \cdots + 1).$$

The elements $\{e_\lambda\}$ satisfy $e_\lambda^2 = e_\lambda$. If λ is comparable (respectively, noncomparable) with χ, then e_λ is a primitive idempotent corresponding to π (respectively, is the zero element of $\tilde{R}_\varepsilon/\tilde{m}(\chi)\tilde{R}_\varepsilon$).

By the choice of $u_1, \dots, u_t$ (see (3.5) and below), there exist $v_1, \dots, v_t$ such that $v_iu_j = q^{n_{ij}}u_jv_i$, where $d := \det(n_{ij})_{ij=1}^t \ne 0$ and d is relatively prime to l (see Definition 3.1). For any system $(\varepsilon_1, \dots, \varepsilon_r)$ of lth roots of unity, there exists $v \in \tilde{R}_\varepsilon$ such that

$$vu_i = \varepsilon_iu_iv. \tag{4.7}$$

We prove that v can be chosen in such a way that $v \notin \tilde{m}(\chi)\tilde{R}_\varepsilon$. Since the Ad-action of the subgroup U_ε generated by $u_{i\varepsilon}$, $1 \le i \le r$, is diagonalizable, we can decompose $\tilde{R}_\varepsilon = \tilde{m}(\chi)\tilde{R}_\varepsilon \oplus V$, where V is some finite-dimensional $\mathrm{Ad}_{U_\varepsilon}$-stable subspace. Consider the completion $\widehat{R_\varepsilon}$ (respectively, $\widehat{Z_\varepsilon}$) of $\tilde{R}_\varepsilon$ (respectively, $\tilde{Z}_\varepsilon$) in the $\tilde{m}(\chi)$-adic topology. We can write $\widehat{R_\varepsilon} = \widehat{Z_\varepsilon} \otimes \tilde{R}_\varepsilon \cong \widehat{Z_\varepsilon} \oplus V$. The $\mathrm{Ad}_{U_\varepsilon}$-action is the identity in $\widehat{Z_\varepsilon}$. We can choose $v \in V$.

Put $\hat{v} := v \bmod \tilde{m}(\chi)\tilde{R}_\varepsilon$. We have proved that $\hat{v} \ne 0$. For $(\varepsilon_1, \dots, \varepsilon_r) \ne (1, \dots, 1)$, the element v is in ${}_{\mathcal{D}}J$. Formula (4.7) implies that for different primitive idempotents e_λ, $e_{\lambda'}$ of $\tilde{R}_\varepsilon/\tilde{m}(\chi)\tilde{R}_\varepsilon$ in the radical there exists a nonzero element $\hat{v}$ such that

$$\hat{v}e_\lambda = e_{\lambda'}\hat{v}.$$

The idempotents λ and λ' are linked by an edge (as vertices of the quiver of the algebra $R_{\varepsilon,\chi}$; see [Pie, 6.4]). □

§5. On the number of irreducible representations

Our goal in this section is to prove some statements about the number of irreducible representations over the common central character.

We begin with the proof of formula (5.1) for some ideal in the iterated skew polynomial extension. Property (5.1) is well known for commutative rings (see [AM, Corollary 10.18]). Note that, in general, (5.1) is false for noncommutative iterated extensions (for instance, take $R = U(\mathfrak{g})$ for the two-dimensional Lie algebra with $[x, y] = y$ and $I = \langle x, y\rangle$).

Lemma 5.1. *Suppose we have an iterated q-skew extension $\mathfrak{R} = \mathfrak{R}_1 \supset \cdots \supset \mathfrak{R}_n = \mathfrak{Y}$ of an algebra $\mathfrak{Y}$ over the field $\mathfrak{F}$. Let $\mathfrak{R}_i = \mathfrak{R}_{i+1}[x_i; \tau_i, \delta_i]$, where τ_i is a diagonal automorphism of $\mathfrak{R}_{i+1}$ and $\tau_i\delta_i = q_i\delta_i\tau_i$ with $q_i \in \mathfrak{F}^*$. We impose the following requirements*:

1) *$\mathfrak{Y}$ is a free module over its center and a Noetherian domain*;
2) *any ideal of $\mathfrak{Y}$ is generated by its intersection with the center*;
3) $\delta_i(\mathfrak{Y}) = 0$;
4) *any δ_i is locally nilpotent in $\mathfrak{R}_{i+1}$.*

Let $\mathfrak{I}$ be an ideal of $\mathfrak{R}$ that contains $x_1, \dots, x_n$. Then

$$\bigcap_{m=1}^{\infty} \mathfrak{I}^m = 0. \tag{5.1}$$

Proof. We proceed by induction on n. If $n = 1$, then $\mathfrak{R} = \mathfrak{Y}$. Let $\{f_\alpha\}$ be a free basis of $\mathfrak{Y}$ over its center $Z(\mathfrak{Y})$. The ideal $\mathfrak{I}$ is generated by its intersection with $Z(\mathfrak{Y})$. Then $\mathfrak{I} = \{\sum c_\alpha f_\alpha : c_\alpha \in \mathfrak{I} \cap Z(\mathfrak{Y})\}$. Consequently, $\mathfrak{I}^m \subset \{\sum c_\alpha f_\alpha : c_\alpha \in (\mathfrak{I} \cap Z(\mathfrak{Y}))^m\}$. Property (5.1) is true for $\mathfrak{I} \cap Z(\mathfrak{Y})$, and it is true for $\mathfrak{I}$.

Suppose that (5.1) is true for the extensions of length at most n. We prove (5.1) for the extensions of length $n+1$. Let $\mathfrak{R}$ be an iterated extension of $\mathfrak{Y}$ that satisfies the requirements of the lemma,

$$\mathfrak{R} = \mathfrak{R}_*[x; \tau, \delta] \supset \mathfrak{R}_* = \mathfrak{R}_1 \supset \cdots \supset \mathfrak{R}_n = \mathfrak{Y}.$$

By the induction hypothesis, (5.1) is true for the ideal $\mathfrak{I}_* = \mathfrak{I} \cap \mathfrak{R}_*$ of $\mathfrak{R}_*$.

Since $x \in \mathfrak{I}$, we have $\delta(\mathfrak{R}_*) \subset \mathfrak{I}_*$. Any element of $\mathfrak{I}$ is of the form $r_0 + xr_1 + x^2 r_2 + \cdots$, where $r_0 \in \mathfrak{I}_*$ and $r_i \in \mathfrak{R}_*$, $i \geq 1$. Therefore, $\mathfrak{I}^m$ is the span of $x^k b_k$,

$$b_k = \delta^{\alpha_1}(r_1) \cdots \delta^{\alpha_n}(r_n) j_1^{\beta_1} \cdots j_t^{\beta_t}, \tag{5.2}$$

where k, α_i, β_i, n, and t are nonnegative integers, $r_i \in \mathfrak{R}_*$, $j_i \in \mathfrak{I}_*$, and

$$k + \alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_t \geq m. \tag{5.3}$$

Suppose that $a \in \bigcap_{m=1}^{\infty} \mathfrak{I}^m$ and $a \neq 0$. Then $a = x^k b_k + x^{k+1} b_{k+1} + \cdots$, $b_k \neq 0$. For any m we can present b_k in the form (5.2), where the α_i, β_i, t, n depend on the choice of m, and (5.3) is fulfilled.

On the other hand, since $\bigcap \mathfrak{I}_*^m = 0$, there exists m_0 such that

$$b_k \in \mathfrak{I}_*^{m_0} \quad \text{and} \quad b_k \notin \mathfrak{I}_*^{m_0+1}. \tag{5.4}$$

Condition (5.4) yields $\beta_1 + \cdots + \beta_t \leq m_0$, and the number of the nonzero α_i is also at most m_0.

We recall that δ is a locally nilpotent τ-derivation; there exists N such that $\delta^N(x_i) = 0$ for all i. This implies that $\delta^{nN}(\mathfrak{R}_*) \subset \mathfrak{I}_*^n$ for all n. In particular, $\delta^{(m_0+1)N}(\mathfrak{R}_*) \subset \mathfrak{I}_*^{m_0+1}$. Since $b_k \notin \mathfrak{I}_*^{m_0+1}$, we have $\alpha_i < (m_0+1)N$ for any i.

We conclude that the left-hand side of inequality (5.3) is bounded as $m \to \infty$. This leads to a contradiction. The ideals $\mathfrak{I}^m$ have zero intersection. □

Let $S := S_\mu$ be the standard denominator subset (see §3). We recall that, after localization of an NQS-algebra, we obtain an iterated q-skew extension $\tilde{R} := \tilde{R}_1 \supset \cdots \supset \tilde{R}_{\tilde{M}} \supset Y$, where $\tilde{M} := M - k$ and $\tilde{R}_i \cong \tilde{R}_{i+1}[x; \tilde{\tau}_i, \tilde{\delta}_i]$ (see (3.3)). As before, we denote by $\mathcal{J} = \mathcal{J}_S$ the smallest ideal of $\tilde{R}$ that contains $\tilde{x}_i$ for all i. Let $\{\mathcal{Q}_1, \dots, \mathcal{Q}_m\}$ be the set of all minimal prime ideals over $\mathcal{J}$. We denote

$$X_1 := X_{1S} := \{\mathcal{Q}_i : \mathcal{Q}_i \cap C = 0\},$$
$$X_2 := X_{2S} := \{\mathcal{Q}_i : \mathcal{Q}_i \cap C \neq 0\}.$$

Proposition 5.2. *Suppose that $X_1 \neq \varnothing$ and $\mathcal{Q} \in X_1$.*

1) *The ideal $\mathcal{Q} \cap Y$ is generated by $\mathcal{Q} \cap \mathfrak{Z}$.*
2) *$\mathcal{Q} = \mathcal{J} + \tilde{R}(\mathcal{Q} \cap Y)$.*

Proof. The second statement is an easy consequence of the first. We prove statement 1).

First, we note that any prime ideal of X_1 is completely prime (see [GL2, Theorem 3.2]). We proceed by induction on $\tilde{M}$. Obviously, the statement is true if $\tilde{M} = 0$.

Assume that statement 1) is true for any algebra of length $\tilde{M}$. Our aim is to prove the statement for $\tilde{R}$ of length $\tilde{M}+1$. Let $\tilde{R}_*$ be the subalgebra generated by Y and all $\tilde{x}_i$

apart from the first, let $\tilde{R} = \tilde{R}_*[\tilde{x}; \tilde{\tau}, \tilde{\delta}]$, $\tilde{\tau}\tilde{\delta} = q^s\tilde{\delta}\tilde{\tau}$ with $s \neq 0$, and let $\mathcal{J}_*$ be the minimal ideal of $\tilde{R}_*$ that contains $\{\tilde{x}_i\}$.

Since the completely prime ideal $\mathcal{Q} \cap \tilde{R}_*$ of $\tilde{R}_*$ contains $\mathcal{J}_*$, there exists a minimal prime ideal $\mathcal{Q}_*$ of $\tilde{R}_*$ such that $\mathcal{Q} \cap \tilde{R}_* \supset \mathcal{Q}_* \supset \mathcal{J}_*$.

Since $0 = \mathcal{Q} \cap C \supset \mathcal{Q}_* \cap C$, we have $\mathcal{Q}_* \cap C = 0$. Therefore, $\mathcal{Q}_*$ satisfies the requirements of the proposition. In particular, $\mathcal{Q}_*$ is a completely prime ideal of $\tilde{R}_*$. The ideal $\mathfrak{q} := \tilde{R}(\mathcal{Q}_* \cap \mathfrak{Z})$ of $\mathfrak{Z}$ is completely prime. We retain the former notation $\tilde{R}$, Y, $\mathcal{Q}$ for $\frac{\tilde{R}}{\tilde{R}\mathfrak{q}}$, $\frac{Y}{Y\mathfrak{q}}$, $\frac{\mathcal{Q}}{\tilde{R}\mathfrak{q}}$. Then $\mathcal{Q}_* \cap Y = 0$. We obtain a natural projection $\pi_S : \tilde{R}_* \to Y$ with the kernel $\mathcal{Q}_*$. Let B denote the denominator subset $Y - \{0\}$. The algebra $\mathfrak{R}_* := \tilde{R}_* B^{-1}$ is an iterated q-skew polynomial extension of $\mathfrak{Y} := YB^{-1}$. The ideal $\mathfrak{I}_* := \mathcal{Q}_* B^{-1}$ of $\mathfrak{R}_*$ satisfies the requirements of Lemma 5.1. We can choose a subset $\Psi \subset \{\tilde{x}_i \in \tilde{R}_*\}$ that is a $\mathfrak{Y}$-basis of $\mathfrak{I}_*$ over $\mathfrak{I}_*^2$. By (5.1), the set Ψ^m of products of arbitrary m elements of Ψ generates $\mathfrak{I}_*$ over $\mathfrak{I}_*^{m+1}$. This implies that if an element of $Z(Y)$ commutes with all elements of Ψ, then it belongs to the center of $\tilde{R}$.

We put $\tilde{\tau}(u_i) = q^{\alpha_i}u_i$ and denote by $\mathfrak{Z}_*$ the intersection of the center of $\tilde{R}_*$ with Y.

The subalgebra $\mathfrak{Z}_*$ is contained in $Z(Y)$. The following situations are possible.

Case 1. $\mathfrak{Z}_* = \mathfrak{Z}$. There exist elements $\Phi := \{v_1, \dots, v_t\} \subset \Psi$ such that $u_i v_j = q^{n_{ij}} v_j u_i$ with $d := \det(n_{ij})_{ij=1}^t \neq 0$. We put $\Phi_* = \Phi$ if there is no element $v_{j_0} \in \Phi$ such that $u_i v_{j_0} = q^{\alpha_i} v_{j_0} u_i$ for all i. If such an element exists, then it is unique, and we put $\Phi_* := \Phi - \{v_{j_0}\}$.

Case 2. $\mathfrak{Z}_* \neq \mathfrak{Z}$. We can assume that $\mathfrak{Z}_*$ is generated by $\mathfrak{Z}$ and u_t. Observe that $\alpha_t \neq 0$ (otherwise $u_t \in \mathfrak{Z}$). There exist elements $\Phi_* = \{v_1, \dots, v_{t-1}\} \subset \Psi$ such that all v_i commute with u_t, and $u_i v_j = q^{n_{ij}} v_j u_i$, $d' := \det(n_{ij})_{ij=1}^{t-1} \neq 0$. In this case we put $\Phi := \Phi_* \cup \{\tilde{x}\}$.

Step 1. We are going to prove that $\tilde{\delta}(v_j) \in \mathcal{Q}_*$ for any $v_j \in \Phi_*$. This means that $b_j := \pi_S(\tilde{\delta}(v_j)) = 0$ (for $v_j \in \Phi_*$).

Since $u_i v_j = q^{n_{ij}} v_j u_i$, from the relation $\tilde{\delta}(u_i) = 0$ we deduce that $q^{\alpha_i} u_i \tilde{\delta}(v_j) - q^{n_{ij}} \tilde{\delta}(v_j) u_i = 0$. The element u_i lies in the center $Z(Y)$ and is invertible. We have $(q^{\alpha_i} - q^{n_{ij}}) b_j = 0$. Recall that $v_j \in \Phi_*$; there exists i_0 such that $q^{\alpha_{i_0}} \neq q^{n_{i_0 j}}$. It follows that $b_j = 0$. This concludes Step 1.

Step 2. We recall that $\tilde{\tau}$ (but not an arbitrary $\tilde{\tau}_i$) is an automorphism of $\tilde{R}$. Then the ideal $\mathcal{Q}$ (and $\mathcal{Q} \cap Y$) is $\tilde{\tau}$-stable. As to $\tilde{\tau}_i$, $1 \leq i$, this map is an automorphism of $\tilde{R}_{i+1}$ (but not of $\tilde{R}$). We are going to prove that the ideal $\mathcal{Q} \cap Y$ is $\tilde{\tau}_i$-stable for $v_i \in \Phi_*$. It suffices to verify that, for any generator $a \in \{\tilde{x}_i\}$ of $\tilde{R}_*$, the element $b := \pi_S(\tilde{\delta}(a))$ is a $\tilde{\tau}_i$-eigenvector. Each a is an Ad_G-eigenvector; then b is also an Ad_G-eigenvector with the same eigenvalue. Multiplying a (and b) by a suitable monomial $h_1^{m_1} \cdots h_{2r}^{m_{2r}}$, we may assume that $\mathrm{Ad}_g\, b = b$, i.e., $b \in Z(Y)$.

Each generator a is an FA-element (actually, an FA_q-element) of $\tilde{R}$. Let $v_i \in \Phi_*$. Since all generators are $\tilde{\tau}$-eigenvectors, we have $\tilde{\tau}(v_i) = q^{\beta_i} v_i$. There exists a polynomial $f(t) = c_0 t^N + c_1 t^{N-1} + \cdots + c_N$ with $c_0 \neq 0$, $c_N \neq 0$, $c_i \in C$ (which is decomposable, $f(t) := c_0 \prod_{m=1}^N (t - q^{\gamma_m})$) such that

$$c_0 a^N v_i + c_1 a^{N-1} v_i a + \cdots + c_N v_i a^N = 0. \tag{5.5}$$

We apply $\tilde{\delta}$ to (5.5) N times to obtain

$$c_0' \tilde{\delta}(a)^N v_i + c_1' \tilde{\delta}(a)^{N-1} v_i \tilde{\delta}(a) + \cdots + c_N' v_i \tilde{\delta}(a)^N = 0 \bmod (\mathcal{Q}_*^2),$$

where $c'_j := q^{\beta_i} c_j$. Then

$$c'_0 b^N v_i + c'_1 b^{N-1} v_i b + \cdots + c'_N v_i b^N = 0 \bmod (\mathcal{Q}_*^2),$$
$$(c'_0 b^N + c'_1 b^{N-1} \tilde{\tau}_i(b) + \cdots + c'_N \tilde{\tau}_i(b)^N) v_i = 0 \bmod (\mathcal{Q}_*^2).$$

Each v_i is an element of a $\mathfrak{Y}$-basis of $\mathfrak{I}$ over $\mathfrak{I}^2$. Therefore,

$$c'_0 b^N + c'_1 b^{N-1} \tilde{\tau}_i(b) + \cdots + c'_N \tilde{\tau}_i(b)^N = 0,$$
$$\prod_{m=1}^{N} (b - q^{\gamma_m + \beta_i} \tilde{\tau}_i(b)) = 0.$$

Since Y is a domain, b is a $\tilde{\tau}_i$-eigenvector. The ideal $\mathcal{Q} \cap \mathcal{Y}$ is $\tilde{\tau}_i$-stable.

Step 3. Any ideal of Y is generated by its intersection with the center; $\mathcal{Q} \cap Y$ is generated by $\mathcal{Q} \cap Z(Y)$. We have proved that $\mathcal{Q} \cap Y$ is $\tilde{\tau}_i$-stable for $v_i \in \Phi$, i.e., $\mathcal{Q} \cap Y$ is generated by $\tilde{\tau}_i$-eigenvectors (for $v_i \in \Phi$). All these eigenvectors have the form $u_1^{m_1} \cdot u_t^{m_t} z$ with $z \in \mathfrak{Z}$. The elements u_i are invertible; the ideal $\mathcal{Q} \cap Y$ is generated by $\mathcal{Q} \cap \mathfrak{Z}$. □

Notation 5.3. For any $S := S_\mu$ and any $\mathcal{Q} \in X_{2S}$, we consider the finite subset $\mathbb{E}_{S,\mathcal{Q}} \subset K$ that consists of all elements $\varepsilon \in K$ such that $\mathcal{Q} \cap C|_{q=\varepsilon} = 0$. We denote

$$\mathbb{E}_S := \bigcup_{\mathcal{Q} \in X_{2S}} \mathbb{E}_{S,\mathcal{Q}},$$
$$\mathbb{E} := \bigcup_S \mathbb{E}_S.$$

Observe that the sets $\mathbb{E}_S$ and $\mathbb{E}$ are finite.

We consider the specialization of the NQS-algebra R_ε at an admissible root of unity. Let $J := J_S$ and ${}_{\mathcal{D}}J$ be the ideals of R_ε that were given in Definition 3.1. Let P be some minimal prime ideal of $\tilde{R}_\varepsilon$ over J.

Proposition 5.4. *Let R, ε, and P be as before. Suppose that $\varepsilon \notin \mathbb{E}_S$. Then*:
1) *$P \cap Y_\varepsilon$ is generated by $P \cap \mathfrak{Z}_\varepsilon$, $P = {}_{\mathcal{D}}J + \tilde{R}_\varepsilon(P \cap \mathfrak{Z}_\varepsilon)$;*
2) *P is a $\mathcal{D}$-stable ideal.*

Proof. The ideal $\mathcal{P} := \pi_\varepsilon^{-1}(P)$ is prime and $\mathcal{P} \cap C = (q-\varepsilon)C$. By the definition of P, the ideal $\mathcal{P}$ contains $\mathcal{J}$. Then $\mathcal{P}$ contains some minimal prime ideal $\mathcal{Q}$ over $\mathcal{J}$. If $\mathcal{Q} \in X_{2S}$, then $(q-\varepsilon)C = \mathcal{P} \cap C \supset \mathcal{Q} \cap C$. Therefore, $\mathcal{Q} \cap C$ is zero at $q = \varepsilon$, which leads to a contradiction.

Hence, $\mathcal{Q} \in X_{1S}$. By Proposition 5.2, $\mathcal{Q} \cap Y$ is generated by $\mathcal{Q} \cap \mathfrak{Z}$. Then $\mathcal{Q} = \mathcal{J} + \tilde{R}(\mathcal{Q} \cap \mathfrak{Z})$. Specializing modulo $q - \varepsilon$, we obtain $\mathcal{Q}_\varepsilon = J + \tilde{R}_\varepsilon \mathfrak{q}$, where $\mathfrak{q} := \mathcal{Q}_\varepsilon \cap \mathfrak{Z}_\varepsilon$. We have $P \supset \mathcal{Q}_\varepsilon \supset J$. The ideal $\mathcal{Q}_\varepsilon$ is $\mathcal{D}$-stable (see [P3, Lemma 3.12]). This implies that $\mathcal{Q}_\varepsilon \supset {}_{\mathcal{D}}J$, $\mathcal{Q}_\varepsilon = {}_{\mathcal{D}}J + \tilde{R}_\varepsilon \mathfrak{q}$, and ${}_{\mathcal{D}}J \cap Y_\varepsilon \subset Y_\varepsilon \mathfrak{q}$.

We recall that the ideal P is prime; $P \cap \mathfrak{Z}_\varepsilon$ is a prime ideal of $\mathfrak{Z}_\varepsilon$. There exists a minimal prime ideal $\mathfrak{p}$ of $\mathfrak{Z}_\varepsilon$ such that $P \cap \mathfrak{Z}_\varepsilon \supset \mathfrak{p} \supset \mathfrak{q}$. We have $P \supset \tilde{R}_\varepsilon \mathfrak{p}$. Since $P \supset \mathcal{Q}_\varepsilon \supset {}_{\mathcal{D}}J$, we see that $P \supset {}_{\mathcal{D}}J + \tilde{R}_\varepsilon \mathfrak{p} \supset J$. Since the middle-term ideal is prime, $P = {}_{\mathcal{D}}J + \tilde{R}_\varepsilon \mathfrak{p}$. As in Proposition 3.7, P is a $\mathcal{D}$-stable ideal. □

Theorem 5.5. *Let R be an* NQS-*algebra satisfying Conditions* CN1 *and* CN2, *and let ε be an admissible root of* 1. *Moreover, suppose that $\varepsilon \notin \mathbb{E}$. Then the number of irreducible representations over a central character χ is equal to l^t for some nonnegative integer t.* (*For the geometrical meaning of t, see Theorem* 5.7.)

Proof. As in §4, we may assume that $\chi(S_{l\varepsilon}) \neq 0$. The ideals $I(\pi)$ and $I(\pi, \mathcal{D})$ admit localization by $S_{l\varepsilon}$. After localization, we obtain ideals $\tilde{I}(\pi)$ and $\tilde{I}(\pi, \mathcal{D})$ of $\tilde{R}_\varepsilon$ (see the proof of Theorem 4.2), and $\tilde{I}(\pi) \supset \tilde{I}(\pi, \mathcal{D}) \supset {}_{\mathcal{D}}J \supset J$. For some minimal prime ideal P (see Proposition 5.4) we have

$$\tilde{I}(\pi) \supset \tilde{I}(\pi, \mathcal{D}) \supset P = {}_{\mathcal{D}}J + \tilde{R}_\varepsilon \mathfrak{p} \supset J.$$

As above, we put $\chi(z_{j\varepsilon}) = \xi_j$. For the maximal ideal $\Xi := \sum_{i=1}^p \mathfrak{Z}_\varepsilon(z_{j\varepsilon} - \xi_j)$ of $\mathfrak{Z}_\varepsilon$ we have $\Xi \supset \mathfrak{p} \supset {}_{\mathcal{D}}J \cap \mathfrak{Z}_\varepsilon \supset J \cap \mathfrak{Z}_\varepsilon$.

We denote $\tilde{R}_\xi := \frac{\tilde{R}_\varepsilon}{\tilde{R}_\varepsilon \Xi}$, $Y_\xi := \frac{Y_\varepsilon}{Y_\varepsilon \Xi} \cong K[h_{1\varepsilon}^{\pm 1}, \dots, h_{2r\varepsilon}^{\pm 1}, u_{1\varepsilon}^{\pm 1}, \dots, u_{t\varepsilon}^{\pm 1}]$, and $J_\xi := ({}_{\mathcal{D}}J + \tilde{R}_\varepsilon \Xi) \bmod \Xi = (J + \tilde{R}_\varepsilon \Xi) \bmod \Xi$. Observe that

$$\tilde{R}_\xi = J_\xi \oplus Y_\xi. \tag{5.6}$$

The subset $B := Y_\xi - \{0\}$ is a denominator subset in $\tilde{R}_\xi$. After localization, we get algebras $\mathfrak{R}_\xi := \tilde{R}_\xi B^{-1}$, $\mathfrak{Y}_\xi := Y_\xi B^{-1}$, and an ideal $\mathfrak{J}_\xi := J_\xi B^{-1}$ that satisfies the requirements of Lemma 5.1. Therefore, $\bigcap_m \mathfrak{J}_\xi^m = 0$. The notation for the generators $\tilde{x}_{i\varepsilon}$, $h_{i\varepsilon}$, $u_{i\varepsilon}$ of $\tilde{R}_\varepsilon$ will be retained for their images in $\mathfrak{R}_\xi$. The basis $\Psi := \{\tilde{x}_{i\varepsilon}\}$ of the $\mathfrak{Y}_\xi$-linear space $\mathfrak{J}_\xi$ over $\mathfrak{J}_\xi^2$ generates $\mathfrak{J}_\xi$ in the following sense: for any element a of $\mathfrak{J}$ and any $m \in \mathbb{N}$ there exists an expression b of the elements Ψ with coefficients in $\mathfrak{Y}_\xi$ such that $a = b \bmod \mathfrak{J}^m$. A monomial in $\mathfrak{Y}_\xi$ lies in the center of $\mathfrak{R}_\xi$ if it commutes with all elements of Ψ.

The central character χ determines a character χ' of the subalgebra $Z(Y_\varepsilon)'$ (see §4), which is contained in $Z(Y)_\varepsilon$. We recall (see the proof of Proposition 4.4) that there exists a one-to-one correspondence between the irreducible representations that lie over χ and the characters λ of $Z(Y)_\varepsilon$ such that $\lambda|_{Z(Y_\varepsilon)'} = \chi'|_{Z(Y_\varepsilon)'}$. After factorization over Ξ, we obtain a similar statement for the subalgebra $Z(Y_\xi)' := Z(Y_\varepsilon)'\Xi$.

We show that

$$Z(Y_\xi)' = K[u_{1\varepsilon}^{\pm l}, \dots, u_{t\varepsilon}^{\pm l}]. \tag{5.7}$$

Suppose that $y_0 \in Z(Y_\xi)'$. Then there exists an element $w \in (W_\varepsilon \cap \tilde{Z}_\varepsilon) \bmod \Xi$ such that $w = j_0 + y_0$, where $j_0 \in J_\xi$ and $y_0 \in Y_\xi$.

Let $a = \tilde{x}_{i_\varepsilon} \in \Psi$. Since $aw = wa$ and $aj_0 - j_0 a \bmod \Xi \in \mathfrak{J}_\xi^2$, the element $aj_0 - j_0 a = y_0 a - a y_0 = (y_0 - \tilde{\tau}_i(y_0))a$ lies in $\mathfrak{J}_\xi^2$ modulo $\tilde{R}_\varepsilon \Xi$. By the definition of Ψ, $y_0 = \tilde{\tau}_i(y_0)$. Thus, Ad_{y_0} is identical on Ψ, whence $y_0 \in Z(R_\xi)$. This proves (5.7). By (3.7),

$$Z(R_\xi) = (J_\xi \cap Z(R_\xi)) \oplus K[h_{1\varepsilon}^{\pm l}, \dots, h_{2r\varepsilon}^{\pm l}, u_{1\varepsilon}^{\pm l}, \dots, u_{t\varepsilon}^{\pm l}]. \tag{5.8}$$

The number of irreducible representations over χ is equal to l^t. □

Definition 5.6. For a point $\chi \in \mathcal{M}$, we denote by $G(\chi)$ the Poisson subalgebra $\{a \in m(\chi) : \{a, m(\chi)\} \subset m(\chi)\}$ in Z_ε. It can be seen that $G(\chi) \supset m(\chi)^2$. The finite-dimensional Lie algebra $\mathfrak{g}(\chi) := G(\chi)/m(\chi)^2$ is called the stabilizer of χ (see [KM, 1.1]).

Theorem 5.7. *Let R, ε, and t be as in Theorem 5.5. For any central character χ, the stabilizer $\mathfrak{g}(\chi)$ is a semidirect sum $\mathfrak{g}(\chi) := \mathfrak{j} + \mathfrak{t}$, where $\mathfrak{j}$ is an ideal and $\mathfrak{t}$ is a toric subalgebra of dimension t.*

Proof. We may assume that the central character χ and all π over χ admit localization over $S_{l\varepsilon}$ for some standard denominator subset $S = S_\mu$. After specialization by $S_{l\varepsilon}$, there is no loss of generality in factoring the algebra $\tilde{R}_\varepsilon$ modulo the ideal Ξ (see Theorem 5.5).

The cotangent subspace $T_\chi^*(\mathcal{M})$ is the span of the images under $D : \tilde{Z}_\varepsilon \to m(\chi)/m(\chi)^2$ of the elements $h_{i\varepsilon}^l$, $u_{i\varepsilon}^l$, and $j_\varepsilon \in J_\xi \cap \tilde{Z}_\xi$. We put $\mathfrak{j} := D(J_\xi \cap Z(R_\xi))$ and $\mathfrak{t} := D(K[u_{1\varepsilon}^{\pm l}, \dots, u_{t\varepsilon}^{\pm l}])$. The subalgebra $\mathfrak{g}(\chi)$ decomposes into the direct sum of linear subspaces, $\mathfrak{g}(\chi) := \mathfrak{j} + \mathfrak{t}$.

Since the ideal J_ξ is $\mathcal{D}$-stable (see Proposition 5.4), the subspace $\mathfrak{j}$ is an ideal of $\mathfrak{g}(\chi)$. Since Ad_{u_i} is diagonalizable, the subalgebra $\mathfrak{t}$ is toric. □

Conjecture 5.8. *The number of irreducible representations over χ is equal to $l^{\operatorname{rank}\mathfrak{g}(\chi)}$.*

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Received 10/FEB/2003

Translated by THE AUTHOR