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CASSON INVARIANT OF KNOTS ASSOCIATED WITH DIVIDES

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Abstract. A formula for the Casson invariant of knots associated with divides is presented. The formula is written in terms of Arnold's invariants of pieces of the divide. Various corollaries are discussed.

Introduction

A *divide* P is the image of a generic relative immersion of a compact (not necessarily connected) 1-manifold into the standard unit disk

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

in the plane $\mathbb{R}^2$. The relativity of the immersion implies that the boundary of the manifold in mapped to the boundary of D. The condition that the immersion is generic means that a divide has only transversal double points as singularities and is transversal to ∂D at its boundary points. Divides are considered modulo ambient isotopy in D.

With every divide P, a link $L(P) \subset S^3$ can be associated by using the following construction. Consider a standard projection π of

$$S^3 = \{(x, y, u, v) \in \mathbb{R}^4 \mid x^2 + y^2 + u^2 + v^2 = 1\}$$

onto D, i.e., $\pi(x, y, u, v) = (x, y)$. The preimage of a point $p \in D$ under π is either a circle if p belongs to the interior of D, or a singleton.

Now $L(P)$ consists of all points $(x, y, u, v) \in \pi^{-1}(P) \subset S^3$ such that either $(x, y) \in \partial P \subset \partial D$, or (u, v) is tangent to P at (x, y). It is easily seen that for any $p \in P$ the preimage $\pi^{-1}(p) \cap L(P)$ in $L(P)$ consists of either 1, 4, or 2 points, depending on whether p is a boundary, double, or generic point of P. $L(P)$ is indeed a link with $2c + i$ components, where c and i are the numbers of closed and nonclosed components of P, respectively. Obviously, ambiently isotopic divides give rise to ambiently isotopic links.

Originally, divides and the associated links were considered by N. A'Campo [1]. They are closely related to the real morsifications of isolated complex plane curve singularities (see [2], [3]).

In this paper I present a formula for the Casson invariant of the knot associated with a given divide with no closed components and only one nonclosed component. Such divides are called the *I-divides* in this paper. The Casson invariant of a link L can be defined as $\frac{1}{2}\Delta''_L(1)$, where $\Delta_L(t)$ is the Alexander polynomial of L and $\Delta''_L(1)$ is the value of its second derivative at 1. It is also a unique Vassiliev invariant of degree 2 that takes the value 0 on the unknot and the value 1 on a trefoil. Since the Casson invariant of a link L is equal to the sum of its values on the components of L, only the case of knots is of interest.

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Amazingly enough, the formula is written in terms of Arnold's invariants $J^{\pm}$ and St of pieces of the divide (see Subsection 1.2 or [6] for the definitions).

In [7], S. Chmutov constructed a second order $J^{\pm}$-type invariant of long curves, called $J_2^{\pm}$, and proved that its value on an I-divide is equal to the Casson invariant of the associated knot. Since the invariant was defined by an actuality table only, the computation of its values on a curve (even with as few as 2 double points) was rather involved. Also, the original definition did not provide a mean to compute changes of $J_2^{\pm}$ under self-tangency *perestroĭkas*. My formula fills these gaps (see Lemmas 3.2.A and 3.2.C, and Subsection 3.3).

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§1. Main ingredients and definitions

1.1. Diagrams of divide links. Let $P \subset D$ be a divide. In this section, I present a way to draw a usual diagram with over- and under-crossings of the link $L(P)$. The following construction is due to M. Hirasawa [10] with some minor modifications (cf. [9]).

First, we perturb P slightly without changing its ambient isotopy type so that the following two conditions be met:

- at every double point the two branches of P are parallel to the main diagonals $y = \pm x$ of the plane;
- P has only finitely many points where the tangent vector is parallel to the y-axis, and at every such point the projection of P to the x-axis has a local extremum.

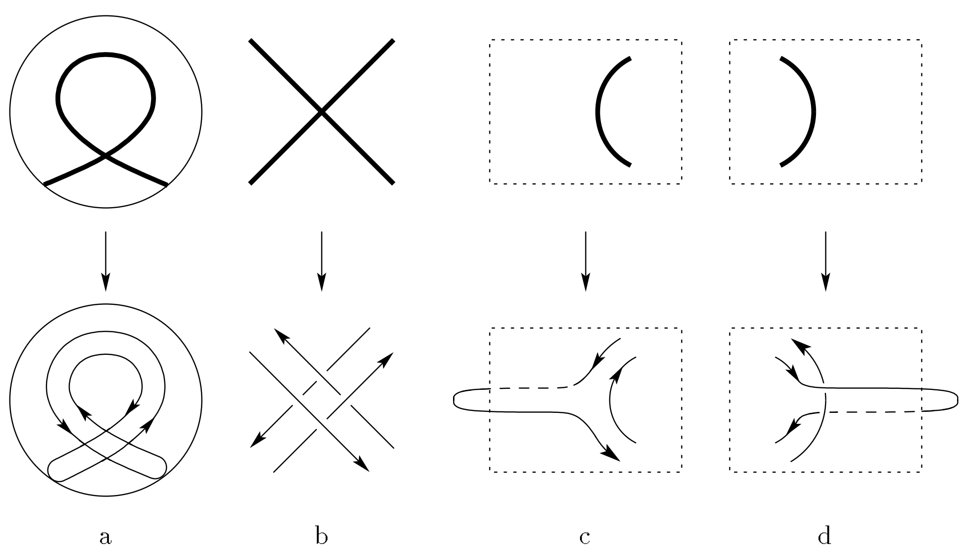


Figure 1. Drawing a diagram of the divide link.

The next step is to double the divide, i.e., to draw two parallel strands along the divide and to join them at the endpoints (see Figure 1.a). The resulting closed curves have a natural orientation in accordance with the "*keeping right*" rule. One puts the over- and under-crossings near the double points in accordance with the rule depicted in Figure 1.b. Finally, at every point of P with tangent vector parallel to the y-axis, the strand pointing downwards should make a "*jump through infinity*". This means to go far right (left) above (below) the rest of the diagram and to return back below (above) it, depending on whether the projection of P to the x-axis has a local maximum or minimum at that point (see Figure 1.c, d). We denote the resulting diagram by $D(P)$. For example, the divide depicted in Figure 1.a gives rise to a (right) trefoil (see Figure 2).

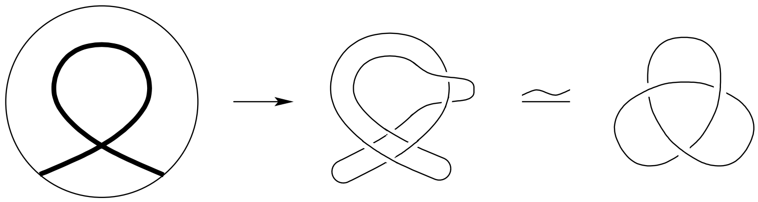


FIGURE 2. Example of a divide link.

1.1.A. Theorem (Hirasawa [10]). *The diagram $D(P)$ represents the link $L(P)$ in S^3.*

1.2. Arnold's invariants of plane curves. Let C be a generic plane curve, i.e., a (C^1-smooth) immersion of the circle into the plane that has only transversal double points as singularities. For any such curve, its *Whitney index* or simply *index* can be defined as the total rotation number of the vector tangent to the curve. Clearly, this number is the degree of the map that takes every point of the circle to the direction of the tangent vector. The index of C is denoted by $w(C)$. It is easily seen that the index does not change under a *regular homotopy* (i.e., a C^1-smooth homotopy in the class of C^1-immersions). Moreover, two plane curves are regularly homotopic if and only if their Whitney indices are the same.

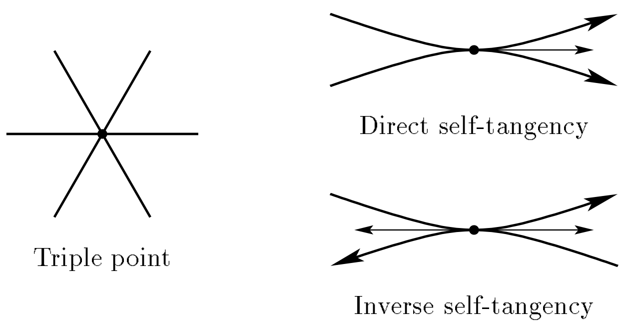


FIGURE 3. Three types of singularities that may appear in a generic regular homotopy.

A generic regular homotopy connecting two generic curves may contain only finitely many nongeneric curves. Each of the latter differs from a generic curve either at exactly one point of triple transversal self-intersection or at exactly one point of self-tangency. At a point of self-tangency, the velocity vectors of the tangent branches have either the same directions or opposite ones. In the first case the self-tangency is said to be *direct* and in the second *inverse* (see Figure 3). The type of self-tangency does not change under reversion of orientation.

Hence, there are three types of singular curves that may appear in a generic regular homotopy. Passages through these curves correspond to three *perestroĭkas* of generic curves.

Consider the triple point *perestroĭka* more carefully. Just before and just after the passage through a singular curve with a triple point, there is a small *vanishing* triangle close to the place of *perestroĭka*, which is formed by three branches of the curve. The orientation of the curve determines a cyclic order of the sides of the triangle. This is the order in which one meets the sides while traveling along the curve. This cyclic order determines an orientation of the triangle and, therefore, an orientation of its sides. We denote by q the number of sides of the vanishing triangle for which the orientation described above coincides with the orientation of the curve (obviously, $q \in \{0, 1, 2, 3\}$).

We define the *sign* of a vanishing triangle to be $(-1)^q$. This sign does not change under reversion of the orientation of the curve. It is easily seen that before and after the *perestroĭka* the signs of the vanishing triangles are different.

1.2.A. Definitions (Arnold [6]). 1. A triple point *perestroĭka* is said to be *positive* if the newborn vanishing triangle is positive.

2. A self-tangency *perestroĭka* is said to be *positive* if it increases (by 2) the number of self-intersection points of the curve.

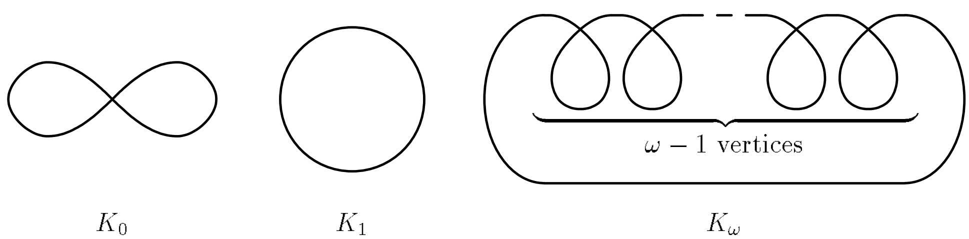


FIGURE 4. The standard curves with Whitney indices $0, \pm 1, \pm\omega$.

The following theorem provides a definition of invariants of generic (plane) curves.

Theorem (Arnold [6]). *There exist three integers* $\mathrm{St}(C)$, $J^+(C)$, *and* $J^-(C)$ *assigned to an arbitrary generic plane curve* C *that are uniquely determined by the following properties*:

(i) St, J^+, *and* J^- *are invariant under a regular homotopy in the class of generic curves.*

(ii) St *does not change under self-tangency perestroĭkas and increases by* 1 *under a positive triple point perestroĭka.*

(iii) J^+ *does not change under triple point and inverse self-tangency perestroĭkas and increases by* 2 *under a positive direct self-tangency perestroĭka.*

(iv) J^- *does not change under triple point and direct self-tangency perestroĭkas and decreases by* 2 *under a positive inverse self-tangency perestroĭka.*

(v) St, J^+, *and* J^- *take the following values on the standard curves* K_ω *shown in Figure 4*:

$$\begin{aligned} \mathrm{St}(K_0) &= 0, & \mathrm{St}(K_{\omega+1}) &= \omega & (\omega &= 0, 1, 2, \dots); \\ J^+(K_0) &= 0, & J^+(K_{\omega+1}) &= -2\omega & (\omega &= 0, 1, 2, \dots); \\ J^-(K_0) &= -1, & J^-(K_{\omega+1}) &= -3\omega & (\omega &= 0, 1, 2, \dots). \end{aligned}$$

Remark. The normalization of St and $J^\pm$, which is fixed by the last property, makes them additive with respect to the connected sum of curves. It is easily seen that the invariants are independent of the orientation of a curve.

1.3. Formulas for Arnold's invariants. It is quite complicated to use the definition given above to compute Arnold's invariants for a given curve with many double points. Two theorems stated below provide explicit formulas, which make such computations easy and straightforward.

Consider a generic plane curve C. For every region r of C, the *index* of r with respect to C is the total rotation number of the radius vector that connects an arbitrary interior point of r with a point traveling along C. Clearly, this number is independent of the choice of the point in r; it is denoted by $\operatorname{ind}_C(r)$. The *index* $\operatorname{ind}_C(e)$ of an edge e of C is the half-sum of the indices of the two regions adjacent to e. The *index* $\operatorname{ind}_C(v)$ of a double point v of C is the quarter-sum of the indices of the four regions adjacent to v.

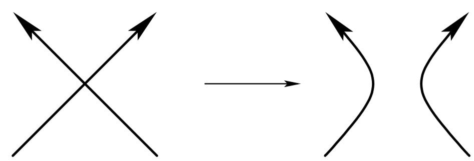


FIGURE 5. The curve smoothing at a self-intersection point.

1.3.A. Theorem (Viro [13]). *Let C be a generic plane curve and $\widetilde{C}$ a family of embedded circles obtained as a result of smoothing C at each double point with respect to the orientation (see Figure 5). Then*

$$J^+(C) = 1 - \sum_{r\in\mathcal{R}_{\widetilde{C}}} \operatorname{ind}^2_{\widetilde{C}}(r)\chi(r) + n, \tag{1.1}$$

$$J^-(C) = 1 - \sum_{r\in\mathcal{R}_{\widetilde{C}}} \operatorname{ind}^2_{\widetilde{C}}(r)\chi(r), \tag{1.2}$$

where $\mathcal{R}_{\widetilde{C}}$ is the set of all regions of $\widetilde{C}$, χ is the Euler characteristic, and n is the number of double points of C.

Now, on a generic plane curve C we fix a base point f that is not a double point. One can enumerate all edges by numbers from 1 to $2n$ (where n is again the number of double points of C) following the orientation and assigning 1 to the edge containing f.

Consider an arbitrary double point v of C. There are two edges pointing to v. Let them have numbers i and j such that the tangent vector of the edge i and the tangent vector of the edge j give a positive orientation of the plane. We define the *sign* $s(v)$ of v to be $\operatorname{sgn}(i-j)$.

1.3.B. Theorem ([12]). *Let C be a generic plane curve. Then*

$$\operatorname{St}(C) = \sum_{v\in\mathcal{V}_C} \operatorname{ind}_C(v)s(v) + \operatorname{ind}^2_C(f) - \frac{1}{4}, \tag{1.3}$$

where $\mathcal{V}_C$ is the set of all double points of C.

Remark. If the point f is chosen on an exterior edge (i.e., an edge that bounds the exterior region), then $\operatorname{ind}_C(f) = \pm\frac{1}{2}$, and

$$\operatorname{St}(C) = \sum_{v\in\mathcal{V}_C} \operatorname{ind}_C(v)s(v). \tag{1.4}$$

Remark. Other formulas for Arnold's invariants can be found in [8], [11], [12].

§2. Formula for the Casson invariant

2.1. The main results. Let $P \subset D$ be an I-divide. Choose an arbitrary orientation on P. For any double point v of P we denote by O_v and I_v the closed and the nonclosed curves[1] obtained by smoothing P at v with regard for the orientation (see Figure 5). The endpoints of P split the boundary ∂D into two arcs. We complete P with one of those arcs so that the orientation of P induce the counterclockwise orientation on the arc. Let $\overline{P}$ be the resulting closed curve. Finally, for a given plane curve C, we denote $1 - J^-(C)$ by $\widetilde{J}(C)$.

Remark. Clearly, the closure $\overline{P}$ of P depends on the orientation of P (see Figure 6).

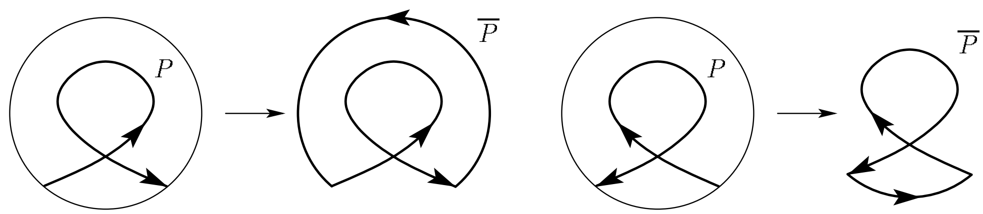


FIGURE 6. Closure of a divide depending on the orientation.

Remark. $1 - J^-(C)$ can be expressed via the following very simple formula: $1 - J^-(C) = \sum_{r \in \mathcal{R}_{\widetilde{C}}} \operatorname{ind}^2_{\widetilde{C}}(r)\chi(r)$. Therefore, it deserves a special notation.

2.1.A. Theorem. *Let P be an I-divide (i.e., a divide with only one nonclosed component). Then the Casson invariant v_2 of the knot $L(P)$ is given by*

$$v_2(L(P)) = \sum_{v \in \mathcal{V}_P} \left(\widetilde{J}(O_v) + \frac{1}{4}\#(O_v \cap I_v)\right) + \frac{J^+(\overline{P}) + 2\operatorname{St}(\overline{P})}{4}, \tag{2.1}$$

where $\mathcal{V}_P$ is the set of all double points of P and $\#(O_v \cap I_v)$ is the number of intersection points of O_v and I_v.

Remark. Obviously, $J^+(\overline{P})$ and $\operatorname{St}(\overline{P})$ depend on the orientation of P, but the sum $J^+(\overline{P}) + 2\operatorname{St}(\overline{P})$ does not, since all the other ingredients of (2.1) are independent of the orientation. This fact can also be verified directly.

2.2. Special cases. It is well known [5] that for a tree-like generic plane curve C the sum $J^+(C) + 2\operatorname{St}(C)$ is always 0. Here a curve C is said to be *tree-like* if smoothing at every double point of C produces two disjoint curves. A *tree-like divide* can be defined in a similar way. From the definition it follows that $\#(O_v \cap I_v) = 0$ for every double point v of a tree-like divide P. Hence, formula (2.1) can be simplified.

2.2.A. *Let P be a tree-like divide. Then*

$$v_2(L(P)) = \sum_{v \in \mathcal{V}_P} \widetilde{J}(O_v). \tag{2.2}$$

A slalom divide (see [4] for the definitions) is a special case of tree-like divides. In this case the numbers $\widetilde{J}(O_v)$ can be calculated directly.

[1] Notation rationale: the letters O and I were chosen because they conveniently represent the topological type of the corresponding curves.

2.2.B. *Let P be a slalom divide. Then*

$$v_2(L(P)) = \sum_{v\in\mathcal{V}_P}(1+n(O_v)), \tag{2.3}$$

where $n(O_v)$ is the number of the double points of O_v. Since this number is always nonnegative, the Casson invariant of a slalom knot is always strictly positive.

Remark. Formula (2.1) can easily be rewritten in terms of the Gauss diagrams of divides (see [11] for the definitions and examples of the technique). In particular, the term $\sum_{v\in\mathcal{V}_P}\frac{1}{4}\#(O_v\cap I_v)$ is half the number of the generic intersections of chords in the Gauss diagram of a divide P.

§3. Proof of Theorem 2.1.A.

3.1. Normalization. We denote the right-hand side of (2.1) by $\mathcal{X}(P)$. It is easy to check that the required identity $v_2(L(P)) = \mathcal{X}(P)$ is true for the standard divides D_0, D_1, etc. Here a standard divide D_k looks like a standard curve K_{k+1} cut at the external edge and immersed appropriately into the unit disk. Indeed, $v_2(L(D_0)) = 0 = \mathcal{X}(D_0)$ and $v_2(L(D_1)) = 1 = \mathcal{X}(D_1)$, because $L(D_1)$ is a trefoil knot (see Figure 2). Furthermore, $L(D_n)$ is the connected sum of n copies of $L(D_1)$ for $n > 1$, whence $v_2(L(D_n)) = v_2(\#nL(D_1)) = nv_2(L(D_1)) = n = n\widetilde{J}(K_1) = \sum_{v\in\mathcal{V}_{D_n}}\widetilde{J}(O_v) = \mathcal{X}(D_n)$.

3.2. Invariance under perestroĭkas. Since any divide can be transformed into a standard one with a finite sequence of Arnold's *perestroĭkas*, it suffices to check that $v_2(L(P))$ and $\mathcal{X}(P)$ change in the same way under the *perestroĭkas*. This is proved in the lemmas that follow.

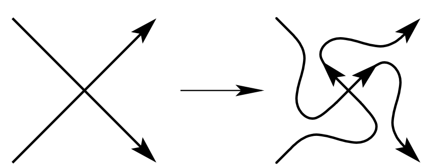


Figure 7. Small perturbation of a divide near a double point.

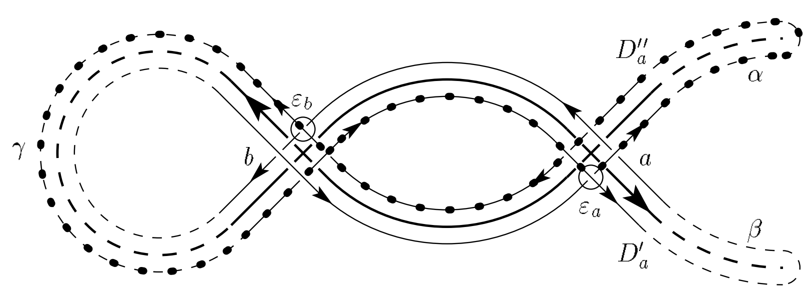
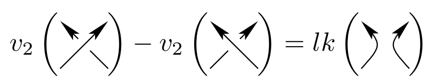


Figure 8. Positive inverse self-tangency *perestroĭka* of a divide.

Lemma. *Let P' be the result of a positive inverse self-tangency perestroĭka of P. We define $\Delta\mathcal{X} = \mathcal{X}(P') - \mathcal{X}(P)$ and $\Delta v_2 = v_2(L(P')) - v_2(L(P))$. Then $\Delta\mathcal{X} = \Delta v_2$.*

Proof. We recall that the divides under consideration are equipped additionally with an orientation. Without loss of generality, we assume that at every double point the two branches of P are oriented upwards. It is always possible to perturb P slightly without changing its isotopy type in order to satisfy this condition, as is shown in Figure 7. We also assume that the inverse self-tangency *perestroĭka* is performed "horizontally", i.e., the two new double points are created side by side as depicted in Figure 8. Denote these two points by a and b. The common part of P and P' consists of 3 nonclosed curves. Two of them contain the initial and final points of P (or P'), and we denote these curves by α and β, respectively. The third curve is denoted by γ. Without loss of generality, we may assume that α, β, and γ are placed as in Figure 8. Of course, they may intersect each other and may have multiple self-intersections.

Now we observe that there are two crossing points of $L(P')$ close to a and b and such that changing the over- and under-crossings at them allows one to pull 4 strands of $L(P')$ placed between a and b away from each other and to obtain $L(P)$. Those points are marked with small circles in Figure 8 and are denoted by ε_a and ε_b, respectively. More precisely, let K_a be the result of the over- and under-crossing change of $L(P')$ at ε_a, and let K_b be the result of such a change of K_a at ε_b. Then K_b and $L(P)$ are ambiently isotopic, whence $v_2(L(P)) = v_2(K_b)$. It is well known how the value of v_2 behaves under these crossing changes. Namely, the following skein relation is true:

(3.1) 

where lk is the linking number of a two-component link.

Now, let L_a be the result of smoothing of $L(P')$ at ε_a with regard for the orientation, and let L_b be the result of smoothing of K_a at ε_b. Both L_a and L_b are links with two components L'_a, L''_a and L'_b, L''_b, respectively. Then (3.1) implies that $v_2(L(P')) - v_2(K_a) = lk(L'_a, L''_a)$ and $v_2(K_a) - v_2(K_b) = lk(L'_b, L''_b)$.

Thus, $\Delta v_2 = lk(L'_a, L''_a) + lk(L'_b, L''_b)$. We compute the first linking number. Denote the diagrams of L'_a and L''_a by D'_a and D''_a, respectively. Then the linking number in question is half the sum of signs of crossings in $D'_a \cap D''_a$. Here the *sign of a crossing* is $+1$ or -1 if it looks like the first or the second summand in (3.1), respectively. In Figure 8, D''_a is depicted with a dotted line.

D'_a and D''_a may intersect each other either near a double point of the divide, or where a branch of $D(P')$ jumps through infinity. The self-intersections of α and β do not contribute to twice the linking number, because the corresponding pieces of $D(P)$ belong entirely to D''_a and D'_a, respectively. The contribution of other double points is shown in Figure 9. It is 2 for every intersection point of β with α and γ and every self-intersection of γ. The contribution is 0 for every intersection of α with γ. The points a and b both contribute 2. Hence, the part of $lk(L'_a, L''_a)$ coming from the crossings located near self-intersections of P' is $\#(\alpha \cap \beta) + \#(\beta \cap \gamma) + n(\gamma) + 2$.

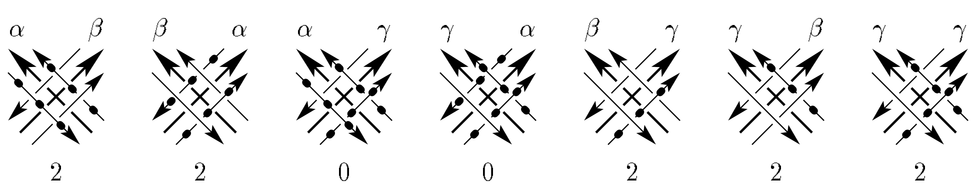


FIGURE 9. Contribution of crossing points to $2lk(L'_a, L''_a)$.

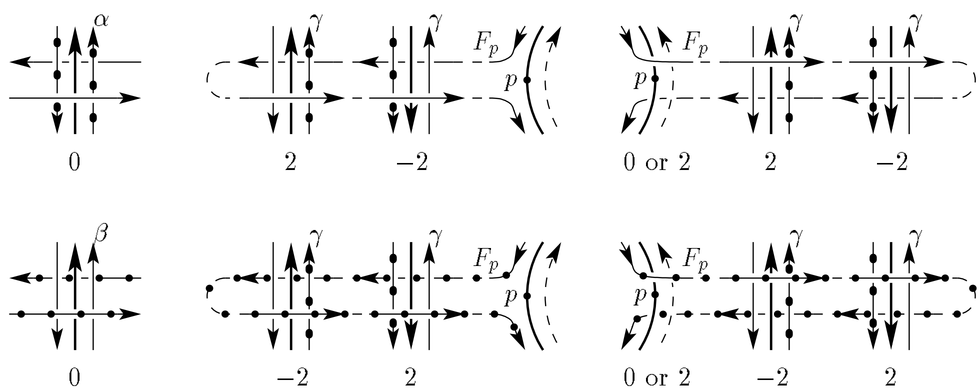


FIGURE 10. Contribution of "jumps through infinity" to $2lk(L'_a, L''_a)$.

Let p be a point of local extremum on P' with respect to the projection to the x-axis. Denote by F_p the corresponding piece of $D(P')$ that makes a jump through the infinity. We may assume that all these pieces are pairwise disjoint and intersect P' transversely at finitely many points. The intersections of F_p with the branches of $D(P')$ corresponding to α and β do not contribute to twice the linking number (see the two leftmost pictures in Figure 10). The contribution from the intersections with γ is 2 if F_p belongs to D'_a (D''_a) and the intersected branch of γ is oriented upwards (downwards), and -2 otherwise (see Figure 10). Moreover, we must add 2 if $p \in \gamma$ and p is a local maximum. The total contribution of F_p to twice the linking number is summarized in the following table.

Location of p	Contribution of F_p	Location of p	Contribution of F_p
$p \in {}^{\alpha}\langle$	$2\,\mathrm{ind}_{\bar\gamma}(p)$	$p \in {}^{\gamma}\langle_{\uparrow}$	$-2\,\mathrm{ind}_{\bar\gamma}(p) - 1$
$p \in \rangle^{\alpha}$	$-2\,\mathrm{ind}_{\bar\gamma}(p)$	$p \in {}^{\gamma}\langle_{\downarrow}$	$2\,\mathrm{ind}_{\bar\gamma}(p) - 1$
$p \in {}^{\beta}\langle$	$-2\,\mathrm{ind}_{\bar\gamma}(p)$	$p \in \rangle^{\gamma}_{\uparrow}$	$2\,\mathrm{ind}_{\bar\gamma}(p) + 1$
$p \in \rangle^{\beta}$	$2\,\mathrm{ind}_{\bar\gamma}(p)$	$p \in \rangle^{\gamma}_{\downarrow}$	$-2\,\mathrm{ind}_{\bar\gamma}(p) + 1$

Here $\bar\gamma$ is the natural closure of γ, and ${}^{C}\langle$ and $\rangle^{C}$ denote the sets of all local minima and maxima on the curve C, respectively (with an appropriate orientation if necessary). We recall that $\mathrm{ind}_{\gamma}(p)$ is a half-integer for $p \in \gamma$. Consequently, the contribution of F_p is always even, as it should be.

It is obvious that $\#(\rangle^{\bar\gamma}) - \#({}^{\bar\gamma}\langle) = 0$, because $\bar\gamma$ is closed. The difference between γ and $\bar\gamma$ is a local maximum near the point b. Therefore, $\#(\rangle^{\gamma}) - \#({}^{\gamma}\langle) = -1$. Finally,

$$
\begin{aligned}
lk(L'_a, L''_a) = & \sum_{p \in {}^{\alpha}\langle \cup \rangle^{\beta}} \mathrm{ind}_{\bar\gamma}(p) - \sum_{p \in {}^{\beta}\langle \cup \rangle^{\alpha}} \mathrm{ind}_{\bar\gamma}(p) + \sum_{p \in {}^{\gamma}\langle_{\downarrow} \cup \rangle^{\gamma}_{\uparrow}} \mathrm{ind}_{\bar\gamma}(p) - \sum_{p \in {}^{\gamma}\langle_{\uparrow} \cup \rangle^{\gamma}_{\downarrow}} \mathrm{ind}_{\bar\gamma}(p) \\
& + \#(\alpha \cap \beta) + \#(\beta \cap \gamma) + n(\gamma) + 3/2.
\end{aligned}
$$

Computation of $lk(L'_b, L''_b)$ is almost the same. We only interchange the roles of α and β throughout and take into account that a does not contribute to the linking number anymore. The contribution of b remains 2. Therefore,

$$\begin{aligned}\Delta v_2 = \sum_{p\in {}^{\gamma}\!\swarrow\cup\searrow^{\gamma}} \operatorname{ind}_{\bar\gamma}(p) - 2\sum_{p\in {}^{\gamma}\!\nearrow\cup\nwarrow^{\gamma}} \operatorname{ind}_{\bar\gamma}(p) \\ + 2\#(\alpha\cap\beta) + \#(\alpha\cap\gamma) + \#(\beta\cap\gamma) + 2n(\gamma) + 2.\end{aligned} \tag{3.2}$$

It is simpler to compute $\Delta\mathcal{X}$. First, $J^+(\overline{P}) = J^+(\overline{P'})$ and $\mathrm{St}(\overline{P}) = \mathrm{St}(\overline{P'})$, because neither J^+ nor St changes under an inverse self-tangency *perestroĭka*. Let v be a double point of P. If v is a self-intersection point of either α, β, or γ, then neither $\widetilde{J}(O_v)$ nor $\#(O_v\cap I_v)$ changes. If $v\in\alpha\cap\gamma$ or $v\in\beta\cap\gamma$, then $\widetilde{J}(O_v)$ does not change, but $\#(O_v\cap I_v)$ increases by 2 under the *perestroĭka*. Finally, if $v\in\alpha\cap\beta$, then $\#(O_v\cap I_v)$ does not change, but O_v experiences a positive inverse self-tangency *perestroĭka* and, consequently, $\widetilde{J}(O_v)$ increases by 2. Moreover,

$$\#(O_a\cap I_a) = \#(O_b\cap I_b) = \#(\alpha\cap\gamma) + \#(\beta\cap\gamma).$$

Collecting these facts, we obtain

$$\Delta\mathcal{X} = 2\#(\alpha\cap\beta) + \#(\alpha\cap\gamma) + \#(\beta\cap\gamma) + \widetilde{J}(O_a) + \widetilde{J}(O_b). \tag{3.3}$$

Subtracting (3.3) from (3.2), we get

$$\Delta v_2 - \Delta\mathcal{X} = 2\sum_{p\in {}^{\gamma}\!\swarrow\cup\searrow^{\gamma}} \operatorname{ind}_{\bar\gamma}(p) - 2\sum_{p\in {}^{\gamma}\!\nearrow\cup\nwarrow^{\gamma}} \operatorname{ind}_{\bar\gamma}(p) + 2n(\gamma) + 2 - \widetilde{J}(O_a) - \widetilde{J}(O_b).$$

Lemma 3.2.B below readily implies that this expression is always zero. Indeed,

$$\widetilde{J}(O_a) = \widetilde{J}(\gamma) + 1 - (\operatorname{ind}_{\bar\gamma}(a) - 1/2) \quad\text{and}\quad \widetilde{J}(O_b) = \widetilde{J}(\gamma) + \operatorname{ind}_{\bar\gamma}(b) + 1/2$$

with $\operatorname{ind}_{\bar\gamma}(a) = \operatorname{ind}_{\bar\gamma}(b)$, where

$$\widetilde{J}(\gamma) := \sum_{p\in {}^{\gamma}\!\swarrow\cup\searrow^{\gamma}} \operatorname{ind}_{\bar\gamma}(p) - \sum_{p\in {}^{\gamma}\!\nearrow\cup\nwarrow^{\gamma}} \operatorname{ind}_{\bar\gamma}(p) + n(\gamma). \qquad \square$$

3.2.B. Lemma. *Let C be an oriented closed plane curve such that*

- *at every double point, the two branches of C are parallel to the main diagonals $y=\pm x$ of the plane*;
- *C has only finitely many points where the tangent vector is parallel to the y-axis, and at every such point the projection of C to the x-axis has either a local minimum or a local maximum.*

Then

$$\widetilde{J}(C) = \#(\underset{C}{\text{⤨}}) + \#(\overset{C}{\text{⤧}}) + \sum_{p\in {}^{C}\!\swarrow\cup\searrow^{C}} \operatorname{ind}_C(p) - \sum_{p\in {}^{C}\!\nearrow\cup\nwarrow^{C}} \operatorname{ind}_C(p).$$

The proof is elementary and is left to the reader. Hint: smoothing at a double point of C with regard for the orientation does not change the above expression.

3.2.C. Lemma. *Let P' be the result of a positive direct self-tangency perestroĭka of P. We put $\Delta\mathcal{X} = \mathcal{X}(P') - \mathcal{X}(P)$ and $\Delta v_2 = v_2(L(P')) - v_2(L(P))$. Then $\Delta\mathcal{X} = \Delta v_2$.*

The proof is similar to that of Lemma 3.2.A. The corresponding picture of a divide and its link close to the place of *perestroĭka* is shown in Figure 11. In this case,

$$\begin{aligned}\Delta v_2 &= 2\sum_{p\in {}^{\gamma}\!\swarrow\cup\searrow^{\gamma}} \operatorname{ind}_{\bar\gamma}(p) - 2\sum_{p\in {}^{\gamma}\!\nearrow\cup\nwarrow^{\gamma}} \operatorname{ind}_{\bar\gamma}(p) + \#(\alpha\cap\gamma) + \#(\beta\cap\gamma) + 2n(\gamma) + 1 \\ &= \#(\alpha\cap\gamma) + \#(\beta\cap\gamma) + \widetilde{J}(O_a) + \widetilde{J}(O_b) + 1 = \Delta\mathcal{X}.\end{aligned} \tag{3.4}$$

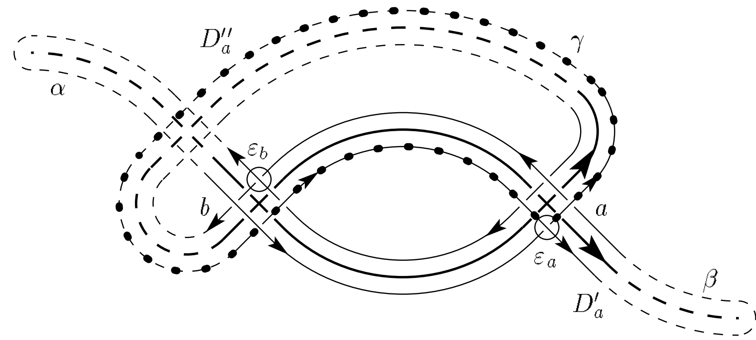


FIGURE 11. Positive direct self-tangency *perestroĭka* of a divide.

Now we turn to triple point *perestroĭkas*. The definition implies that if a divide P experiences a triple point *perestroĭka*, then the ambient isotopy type of the corresponding link does not change.

3.2.D. Lemma. *Let P' be the result of a positive triple point perestroĭka of P. We put $\Delta\mathcal{X} = \mathcal{X}(P') - \mathcal{X}(P)$. Then $\Delta\mathcal{X} = 0$.*

Proof. We distinguish two kinds of triple points, depending on whether the three branches of a curve (or a divide) at the point are directed into the same half-plane or not. A triple point is said to be of *type B* in the former case and of *type A* in the latter (see Figure 12). Accordingly, there are two kinds of triple point *perestroĭkas*. It is easy to check that a triple point *perestroĭka* of type B is equivalent to a sequence of triple point *perestroĭkas* of type A and several self-tangency *perestroĭkas*. Therefore, it suffices to consider only type A *perestroĭkas* in the proof.

Triple points of type A can further be classified into types A_+ or A_-, depending on whether the cyclic order of the branches determines the positive or the negative orientation of the plane (see Figure 12). Without loss of generality, we shall restrict our consideration to the *perestroĭkas* of type A_+ only. The corresponding picture of divides before and after a *perestroĭka* is shown in Figure 13.

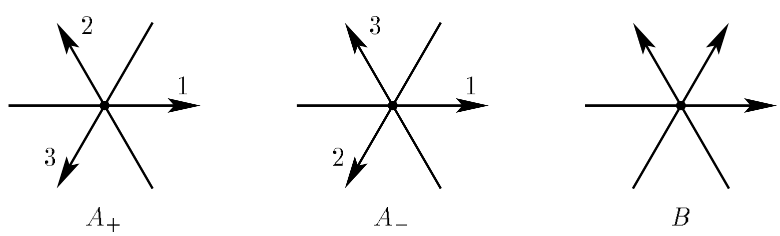


FIGURE 12. Different types of triple points.

Now, it is straightforward to check that

$$\mathrm{St}(\overline{P'}) = \mathrm{St}(\overline{P}) + 1, \quad \widetilde{J}(O_{a'}) = \widetilde{J}(O_a) - 2, \quad \widetilde{J}(O_{b'}) = \widetilde{J}(O_b), \quad \widetilde{J}(O_{c'}) = \widetilde{J}(O_c)$$

and

$$\#(O_{v'} \cap I_{v'}) = \#(O_v \cap I_v) + 2 \quad \text{for } v \in \{a, b, c\}.$$

Hence, $\Delta\mathcal{X} = 1/2 - 2 + 6/4 = 0$. □

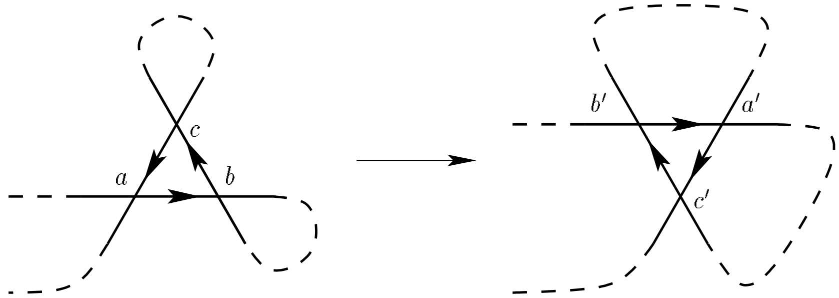


FIGURE 13. Positive triple point *perestroĭka* of type A_+.

3.3. Applications to Chmutov's $J_2^\pm$ invariant. $J_2^\pm$ is a second order $J^\pm$-type invariant of long curves (or I-divides). Chmutov [7] defined it by explicitly specifying its actuality table (i.e., the values of the invariant on all chord diagrams with two chords) and its values on the standard divides with at most one self-tangency point. He also proved that $v_2(L(P)) = J_2^\pm(P)$ for any I-divide P. The definition of $J_2^\pm$ did not allow one to integrate this invariant, i.e., to compute its values on chord diagrams with one chord. Since these values are none other than the changes of $J_2^\pm$ under self-tangency *perestroĭkas*, formulas (3.3) and (3.4) provide an answer to this question.

3.3.A. Corollary. *Suppose that P is an* I-*divide and P' is the result of a positive inverse self-tangency perestroĭka of P. Let α, β, a, and b be as in the proof of Lemma 3.2.A* (*see Figure 8*). *Then*

$$\begin{aligned} J_2^\pm(P') - J_2^\pm(P) &= 2\widetilde{J}(O_a) + 2\,\mathrm{ind}_{O_a}(b) + \#(O_a \cap I_a)) + 2\#(\alpha\cap\beta) - 1 \\ &= 2\widetilde{J}(O_b) - 2\,\mathrm{ind}_{O_b}(a) + \#(O_b \cap I_b)) + 2\#(\alpha\cap\beta) + 1. \end{aligned} \tag{3.5}$$

The proof follows from (3.3) and the fact that

$$\#(O_a\cap I_a) = \#(O_b\cap I_b) = \#(\alpha\cap\gamma) + \#(\beta\cap\gamma), \quad \widetilde{J}(O_a) = \widetilde{J}(O_b) - 2\,\mathrm{ind}_{O_a}(b) + 1$$

and $\mathrm{ind}_{O_a}(b) = \mathrm{ind}_{O_b}(a)$.

3.3.B. Corollary. *Suppose that P is an* I-*divide and P' is the result of a positive direct self-tangency perestroĭka of P. Let a and b be as in the proof of Lemma 3.2.C* (*see Figure 11*). *Then*

$$J_2^\pm(P') - J_2^\pm(P) = 2\widetilde{J}(O_a) + \#(O_a\cap I_a) = 2\widetilde{J}(O_b) + \#(O_b\cap I_b).$$

The proof follows from (3.4) and the fact that

$$\#(O_a\cap I_a) = \#(O_b\cap I_b) = \#(\alpha\cap\gamma) + \#(\beta\cap\gamma) + 1$$

and $\widetilde{J}(O_a) = \widetilde{J}(O_b)$.

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