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THETA HYPERGEOMETRIC INTEGRALS

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Dedicated to Mizan Rahman

Abstract. A general class of (multiple) hypergeometric type integrals associated with the Jacobi theta functions is defined. These integrals are related to theta hypergeometric series via the residue calculus. In the one variable case, theta function extensions of the Meijer function are obtained. A number of multiple generalizations of the elliptic beta integral associated with the root systems A_n and C_n is described. Some of the C_n-examples were proposed earlier by van Diejen and the author, but other integrals are new. An example of the biorthogonality relations associated with the elliptic beta integrals is considered in detail.

§1. Introduction

Exact integration formulas and integral representations of functions are important from various points of view. Such representations serve sometimes as definitions of functions, but more often they are needed for the better understanding of properties of functions defined beforehand. Due to numerous applications (see [AAR]), the Euler beta integral

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \operatorname{Re}\alpha, \operatorname{Re}\beta > 0, \tag{1.1}$$

where $\Gamma(z)$ is the standard gamma function, plays a fundamental role in classical analysis. Various q-generalizations of (1.1) involving q-gamma functions have been proposed within the theory of basic hypergeometric series [GR]. Recently, a "third floor" of the hierarchy of beta-type integrals, which is related to the elliptic gamma function, was discovered in [S2] (in the one variable case) and in [DS1, DS2] (multiple extensions). For a brief review of the results in this direction, see [DS4]. In this paper, we discuss a general class of hypergeometric type integrals associated with the Jacobi theta functions and propose several new multiple elliptic beta integrals admitting exact evaluation.

An infinite hierarchy of multiple gamma functions was proposed by Barnes long ago [Ba1]:

$$\Gamma_r^{-1}(u;\omega) = e^{\sum_{i=0}^r \gamma_{ri}\frac{u^i}{i!}} u \prod_{n_1,\dots,n_r=0}^{\infty}{}' \left(1+\frac{u}{\Omega}\right) e^{\sum_{i=1}^r (-1)^i \frac{u^i}{i\Omega^i}}, \tag{1.2}$$

where the γ_{ri} are some constants similar to the Euler constant, and $\Omega = n_1\omega_1 + \dots + n_r\omega_r$ (if some of the ratios ω_i/ω_k are real, then they must be positive). The prime in the

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product sign means that the point $n_1 = \cdots = n_r = 0$ is skipped. The function (1.2) satisfies a collection of first order difference equations

$$\frac{\Gamma_r(u+\omega_j;\omega)}{\Gamma_r(u;\omega)} = \frac{1}{\Gamma_{r-1}(u;\omega(j))}, \quad j = 1,\dots,r,$$

where $\omega(j) = (\omega_1,\dots,\omega_{j-1},\omega_{j+1},\dots,\omega_r)$ and $\Gamma_1(u;\omega) = \rho(\omega)\omega^{u/\omega}\Gamma(u/\omega)$ for some constant $\rho(\omega)$ (for a brief account of this function, see also Appendix A in [JM]).

Following Barnes' analysis, Jackson [J] considered the generalized gamma function in a slightly different way and proposed the q-gamma function and the elliptic gamma function. We recall the definition of the latter. Taking two complex variables q and p such that $|q|,|p| < 1$, we compose the following (convergent) Jackson double infinite product:

$$(z;q,p)_\infty = \prod_{j,k=0}^{\infty}(1 - zq^jp^k). \tag{1.3}$$

Two first order q- and p-difference equations for this product,

$$\frac{(z;q,p)_\infty}{(qz;q,p)_\infty} = (z;p)_\infty, \quad \frac{(z;q,p)_\infty}{(pz;q,p)_\infty} = (z;q)_\infty, \tag{1.4}$$

where $(z;p)_\infty = \prod_{k=0}^{\infty}(1-zp^k)$, are of major importance. Replacing z by pz^{-1} in the first equation and by qz^{-1} in the second, we get

$$\frac{(qp(qz)^{-1};q,p)_\infty}{(qpz^{-1};q,p)_\infty} = (pz^{-1};p)_\infty, \quad \frac{(qp(pz)^{-1};q,p)_\infty}{(qpz^{-1};q,p)_\infty} = (qz^{-1};q)_\infty. \tag{1.5}$$

We define a theta function as follows:

$$\theta(z;p) = (z;p)_\infty(pz^{-1};p)_\infty. \tag{1.6}$$

It is related to the standard Jacobi θ_1-function [WW] in a simple way:

$$\begin{aligned}\theta_1(u;\sigma,\tau) &= -i\sum_{n=-\infty}^{\infty}(-1)^n p^{(2n+1)^2/8}q^{(n+1/2)u}\\ &= p^{1/8}iq^{-u/2}(p;p)_\infty\theta(q^u;p), \quad u\in\mathbb{C},\end{aligned} \tag{1.7}$$

where we assume that $q = e^{2\pi i\sigma}, p = e^{2\pi i\tau}$. Sometimes, for brevity, it is convenient to drop q and p or the modular parameters σ and τ in the notation for theta functions and elliptic gamma functions, as well as for the elliptic analogs of shifted factorials to be defined below.

The function $\theta_1(u)$ is entire, odd ($\theta_1(-u) = -\theta_1(u)$), and doubly quasiperiodic:

$$\theta_1(u+\sigma^{-1}) = -\theta_1(u), \quad \theta_1(u+\tau\sigma^{-1}) = -e^{-\pi i\tau - 2\pi i\sigma u}\theta_1(u). \tag{1.8}$$

These transformation properties of the θ_1-function are used extensively in our formalism. For the $\theta(z;p)$ function, they take the form

$$\theta(pz;p) = \theta(z^{-1};p) = -z^{-1}\theta(z;p). \tag{1.9}$$

Now we multiply the left-hand sides and right-hand sides of the first identities in (1.4) and (1.5), respectively, and do the same with the second identities. This yields the difference equations

$$\Gamma(qz;q,p) = \theta(z;p)\Gamma(z;q,p), \quad \Gamma(pz;q,p) = \theta(z;q)\Gamma(z;q,p) \tag{1.10}$$

for the elliptic gamma function $\Gamma(z;q,p)$; in the explicit form, we have

$$\Gamma(z;q,p) = \prod_{j,k=0}^{\infty}\frac{1-z^{-1}q^{j+1}p^{k+1}}{1-zq^jp^k}. \tag{1.11}$$

Despite the fact that the general idea of associating a generalized gamma function with the elliptic theta function was formulated in the well-known paper [J], it did not get much attention. However, Jackson's double infinite product was used explicitly in the mathematical physics literature on integrable models of statistical mechanics starting with Baxter's work on the eight vertex model; see [Bax]. The name "elliptic gamma function" for the product (1.11) was proposed by Ruijsenaars in the recent paper [Ru1], where he reintroduced the function $\Gamma(z;q,p)$ anew and started a systematic investigation of its properties. A further detailed analysis of this function was performed by Felder and Varchenko in [FV].

In order to compare the elliptic gamma function with the Barnes multiple gamma function, in (1.11) we put

$$z = e^{2\pi i \frac{u}{\omega_2}}, \quad q = e^{2\pi i \frac{\omega_1}{\omega_2}}, \quad p = e^{2\pi i \frac{\omega_3}{\omega_2}},$$

where the ω_i are some constants satisfying the same constraints for the quasiperiods as in (1.2). Then it is not difficult to see that the set of zeros and poles of $\Gamma(z;q,p)$, viewed as a meromorphic function of the variable u, coincides with the set of zeros and poles of the following combination of Barnes Γ_3-functions:

$$\frac{\Gamma_3(u;\omega_1,\omega_2,\omega_3)\Gamma_3(u-\omega_2;\omega_1,-\omega_2,\omega_3)}{\Gamma_3(\omega_1+\omega_3-u;\omega_1,\omega_2,\omega_3)\Gamma_3(\omega_1+\omega_3-\omega_2-u;\omega_1,-\omega_2,\omega_3)}. \tag{1.12}$$

This means that the ratio of $\Gamma(z;q,p)$ and (1.12) is an entire function of u, and this function is seen to be given by an exponential of some polynomial of u of the third degree.

For arbitrary complex s, the elliptic shifted factorials are defined as ratios of elliptic gamma functions:

$$\theta(z;p;q)_s = \frac{\Gamma(zq^s;q,p)}{\Gamma(z;q,p)}.$$

We use also the following shorthand notation:

$$\Gamma(t_1,\dots,t_k;q,p) \equiv \prod_{j=1}^{k} \Gamma(t_j;q,p),$$

$$\theta(t_1,\dots,t_k;p;q)_n \equiv \prod_{j=1}^{k}\prod_{\ell=0}^{n-1} \theta(t_j q^\ell;p), \quad n\in\mathbb{N}.$$

The elliptic beta integral [S2] is the first exact integration formula involving the elliptic gamma function. We conclude this section by an explicit description of it.

Theorem 1. *Let five complex parameters t_m, $m=0,\dots,4$, satisfy the inequalities $|t_m|<1$, $|pq|<|A|$, where $A\equiv\prod_{r=0}^4 t_r$. Define the elliptic beta integral as the following contour integral*:

$$\mathcal{N}_E(\mathbf{t}) = \int_{\mathbb{T}} \Delta_E(z,\mathbf{t})\,\frac{dz}{z}, \tag{1.13}$$

where $\mathbb{T}$ is the positively oriented unit circle, and

$$\Delta_E(z,\mathbf{t}) = \frac{1}{2\pi i}\frac{\prod_{m=0}^4 \Gamma(zt_m, z^{-1}t_m;q,p)}{\Gamma(z^2,z^{-2},zA,z^{-1}A;q,p)}. \tag{1.14}$$

Then

$$\mathcal{N}_E(\mathbf{t}) = \frac{2\prod_{0\le m<s\le 4}\Gamma(t_m t_s;q,p)}{(q;q)_\infty (p;p)_\infty \prod_{m=0}^4 \Gamma(At_m^{-1};q,p)}. \tag{1.15}$$

Relation (1.13) determines a new Askey–Wilson type integral representing an elliptic extension of the Nassrallah–Rahman integral. Indeed, if we set $p = 0$, then the integral (1.13) is reduced to the Nassrallah–Rahman q-beta integral [NR, R1] which, in turn, is a one parameter extension of the celebrated Askey–Wilson q-beta integral [AW]. Theorem 1 was proved by the author with the help of an elliptic generalization of the method used by Askey in [As] for proving the Nassrallah–Rahman integral. A large list of known plain and q-hypergeometric beta integrals was given in [RS2].

As was shown in [DS1], a special finite-dimensional reduction of (1.13) that is associated with the residue calculus results in the elliptic generalization of the Jackson sum for a terminating ${}_8\Phi_7$ basic hypergeometric series, which was discovered by Frenkel and Turaev in [FT]. In [S4], the integral (1.13) was applied to the construction of a large family of continuous biorthogonal functions generalizing Rahman's ${}_{10}\Phi_9$ biorthogonal rational functions [R1, R2]. These functions are expressed in terms of products of two ${}_{12}E_{11}$ elliptic hypergeometric series with different modular parameters (for the definition of an appropriate system of notation for such series, see [S5]). It is believed that in the theory of biorthogonal functions they play a role similar to that played by the Askey–Wilson polynomials [AW] in the theory of orthogonal polynomials. We describe these biorthogonal functions in the last section and in Appendix A of the present paper and give complete proofs of some of the results announced in [S4]. An elliptic extension of Wilson's discrete (finite-dimensional) set of biorthogonal rational functions [Wi] was constructed earlier by Zhedanov and the author in [SZ1, SZ2]. The corresponding three term recurrence relation generates the most general example in the pool of known terminating continued fractions expressed in terms of series of hypergeometric type, namely, it is expressed via the ${}_{12}E_{11}$ series as well [SZ3, SZ4]. The integral (1.13) leads to integral representations for a terminating ${}_{12}E_{11}$ series and its particular bilinear form [S4] (the proofs are given in Appendix B). All these results open new ways of exploration of the world of elliptic functions and modular forms, which complement recent progress reached in the classical setting by Milne [Mi2].

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§2. A general definition of theta hypergeometric integrals

The right-hand side of (1.13) belongs to a general class of integrals related to the series of hypergeometric type built from Jacobi theta functions. In accordance with the theory of general theta hypergeometric series developed in [S5], we give the following definition.

Definition. Let C denote a smooth Jordan curve on the complex plane, and let $\Delta(y_1, \dots, y_n)$ be a meromorphic function of its arguments $y_1, \dots, y_n$. Consider the (multiple) integrals

$$I_n = \int_C dy_1 \cdots \int_C dy_n \, \Delta(y_1, \dots, y_n) \tag{2.1}$$

and the ratios

$$h_\ell(\mathbf{y}) = \frac{\Delta(y_1, \dots, y_\ell + 1, \dots, y_n)}{\Delta(y_1, \dots, y_\ell, \dots, y_n)}. \tag{2.2}$$

Then the integrals I_n are called:

1) the *plain hypergeometric integrals* if

$$h_\ell(\mathbf{y}) = R_\ell(\mathbf{y}) \tag{2.3}$$

are rational functions of $y_1, \dots, y_n$ for all $\ell = 1, \dots, n$;

2) the *q-hypergeometric integrals* if

$$h_\ell(\mathbf{y}) = R_\ell(q^{\mathbf{y}}) \tag{2.4}$$

are rational functions of $q^{y_1}, \dots, q^{y_n}$, $q \in \mathbb{C}$, for all $\ell = 1, \dots, n$;

3) the *elliptic hypergeometric integrals* if for all $\ell = 1, \dots, n$ the ratios $h_\ell(\mathbf{y})$ are elliptic functions of the variables $y_1, \dots, y_n$ with periods σ^{-1} and $\tau\sigma^{-1}$, $\mathrm{Im}(\tau) > 0$;

4) the general *theta hypergeometric integrals* if $h_\ell(\mathbf{y})$ and $1/h_\ell(\mathbf{y})$ are meromorphic functions obeying the double quasiperiodicity conditions

$$\begin{aligned} h_\ell(y_1, \dots, y_k + \sigma^{-1}, \dots, y_n) &= e^{\sum_{j=1}^n a_{\ell k}(j) y_j + b_{\ell k}} h_\ell(\mathbf{y}), \\ h_\ell(y_1, \dots, y_k + \tau\sigma^{-1}, \dots, y_n) &= e^{\sum_{j=1}^n c_{\ell k}(j) y_j + d_{\ell k}} h_\ell(\mathbf{y}) \end{aligned} \tag{2.5}$$

with the quasiperiodicity factors similar to those for the Weierstrass sigma function (which is related to $\theta_1(u)$ in a simple way; see [WW]).

If we assume that the variables $y_1, \dots, y_n$ are discrete, $\mathbf{y} \in \mathbb{N}^n$, and replace integrals by sums $\sum_{\mathbf{y} \in \mathbb{N}^n}$, then we get the definitions of the plain and q-hypergeometric series, which go back to Horn [GGR], and the definition of the elliptic hypergeometric series suggested in [S5], respectively. The theta hypergeometric series were defined in [S5] in a less general form because of the less general choice of quasiperiodicity factors. Evidently, if $a_{\ell k}(j) = b_{\ell k} = c_{\ell k}(j) = d_{\ell k} = 0$, then the theta hypergeometric functions are reduced to the elliptic ones. The integrals (or series) defined in this way do not form an algebra because, in general, sums of hypergeometric integrals do not fit the above definition.

The shifts $y_\ell \to y_\ell + 1$ in (2.2) may be replaced by translations by an arbitrary constant $y_\ell \to y_\ell + \omega_1$, $\omega_1 = \text{const}$. However, we can replace ω_1 by 1 after an appropriate rescaling of y_ℓ, which results in a simple deformation of the contour C in (2.1).

Consider the case of $n = 1$ in detail. The general rational function of y can be represented in the form

$$R(y) = \frac{\prod_{j=1}^n (1 - a_j + y) \prod_{j=n+1}^r (a_j - 1 - y)}{\prod_{j=1}^m (b_j - 1 - y) \prod_{j=m+1}^s (1 - b_j + y)} x,$$

where n, r, m, s are arbitrary integers, x is an arbitrary complex constant, and the a_j, b_j describe the positions of the zeros and poles of $R(y)$. The equation $\Delta(y+1) = R(y)\Delta(y)$ has the following general solution:

$$\Delta(y) = \frac{\prod_{j=1}^m \Gamma(b_j - y) \prod_{j=1}^n \Gamma(1 - a_j + y)}{\prod_{j=m+1}^s \Gamma(1 - b_j + y) \prod_{j=n+1}^r \Gamma(a_j - y)} x^y \varphi(y), \tag{2.6}$$

where $\Gamma(y)$ is the standard gamma function and $\varphi(y)$ is an arbitrary periodic function, $\varphi(y+1)=\varphi(y)$. If we set $\varphi(y)=1$, then for an appropriate choice of the contour C the integral I_1 (see (2.1)) is none other than the Meijer function [EMOT]. In this case we have no natural additional tools for fixing an infinite-dimensional (functional) freedom contained in the solution $\Delta(y)$.

In the q-case, in a similar way we can write

$$R(q^y)=\frac{\prod_{j=1}^n(1-t_jq^y)\prod_{j=n+1}^r(1-t_j^{-1}q^{-y})}{\prod_{j=1}^m(1-w_j^{-1}q^{-y})\prod_{j=m+1}^s(1-w_jq^y)}x.$$

For $0<|q|<1$, the general meromorphic solution of the equation $\Delta(y+1)=R(q^y)\Delta(y)$ is

$$\Delta(y)=\frac{\prod_{j=n+1}^r(t_j^{-1}q^{1-y};q)_\infty\prod_{j=m+1}^s(w_jq^y;q)_\infty}{\prod_{j=1}^n(t_jq^y;q)_\infty\prod_{j=1}^m(w_j^{-1}q^{1-y};q)_\infty}x^y\varphi(y), \tag{2.7}$$

where, again, $\varphi(y)$ is an arbitrary periodic function, $\varphi(y+1)=\varphi(y)$. In this case, for $\varphi(y)=1$ the integral I_1 describes a q-Meijer function, which was investigated by Slater in [Sl].

For $|q|>1$, the equation $\Delta(y+1)=R(q^y)\Delta(y)$ has the following general solution:

$$\Delta(y)=\frac{\prod_{j=1}^n(t_jq^{y-1};q^{-1})_\infty\prod_{j=1}^m(w_j^{-1}q^{-y};q^{-1})_\infty}{\prod_{j=n+1}^r(t_j^{-1}q^{-y};q^{-1})_\infty\prod_{j=m+1}^s(w_jq^{y-1};q^{-1})_\infty}x^y\varphi(y),$$

that is, we have an effective $q\to q^{-1}$ replacement and a reshuffling of parameters in (2.7).

We remind the reader that $q=e^{2\pi i\sigma}$. The parameter σ gives a second scale, which may be used for generating a natural additional restriction upon $\Delta(y)$. The function q^y is periodic under the shift $y\to y+\sigma^{-1}$, and (2.7) satisfies the equation $\Delta(y+\sigma^{-1})/\Delta(y)=x^{1/\sigma}\varphi(y+\sigma^{-1})/\varphi(y)$. We can fix $\varphi(y)$ by demanding that

$$\varphi(y+\sigma^{-1})=\tilde R(e^{2\pi iy})\varphi(y),$$

where $\tilde R$ is another rational function of its argument. In accordance with the periodicity condition $\varphi(y+1)=\varphi(y)$, we have

$$\tilde R(e^{2\pi iy})=\frac{\prod_{j=1}^{n'}(1-\tilde t_je^{-2\pi iy})\prod_{j=n'+1}^{r'}(1-\tilde t_j^{-1}e^{2\pi iy})}{\prod_{j=1}^{m'}(1-\tilde w_j^{-1}e^{2\pi iy})\prod_{j=m'+1}^{s'}(1-\tilde w_je^{-2\pi iy})},$$

where the $\tilde t_j$ and $\tilde w_j$ are arbitrary new parameters. Note that we cannot multiply the function $\tilde R$ by terms like $\rho e^{2\pi iky}, k\in\mathbb{Z},\rho\in\mathbb{C}$, if they are different from 1, because then the periodicity condition for $\varphi(y)$ will be broken. For $|q|<1$, the general meromorphic solution of the difference equation for $\varphi(y)$ is as follows:

$$\varphi(y)=\frac{\prod_{j=m'+1}^{s'}(\tilde w_je^{-2\pi iy};\tilde q)_\infty\prod_{j=n'+1}^{r'}(\tilde q\tilde t_j^{-1}e^{2\pi iy};\tilde q)_\infty}{\prod_{j=1}^{m'}(\tilde q\tilde w_j^{-1}e^{2\pi iy};\tilde q)_\infty\prod_{j=1}^{n'}(\tilde t_je^{-2\pi iy};\tilde q)_\infty}\tilde\varphi(y), \tag{2.8}$$

where $\tilde q=e^{-2\pi i/\sigma}$ is the modular partner of q. Indeed, for $\mathrm{Im}(\sigma)>0$ we have $\mathrm{Im}(\sigma^{-1})<0$, and (2.8) is well defined. The function $\tilde\varphi(y)$ in (2.8) is an arbitrary elliptic function with periods 1 and σ^{-1}. It is characterized uniquely by the position of its poles and zeros in the fundamental parallelogram of periods containing $2k-1$ free parameters, where k is the order of $\tilde\varphi(y)$. Thus, the space of solutions is not too large: it becomes finite-dimensional (in the sense of the number of free parameters).

Consider the regime $|q|=1$. Denoting $\sigma=\omega_1/\omega_2$ and assuming that $\mathrm{Re}(\sigma)>0$, we introduce the variable $u=y\omega_1$. Now it is possible to choose the parameters $t_j,\tilde t_j$, etc.

in a special way, so that the infinite products $(t_j q^y; q)_\infty$, $(\tilde{t}_j e^{-2\pi i y}; \tilde{q})_\infty$, etc. in (2.7) and (2.8) combine into the double sine functions $S(u + g_j; \omega_1, \omega_2)$ for some g_j, where

$$S(u; \omega_1, \omega_2) = \frac{(e^{2\pi i u/\omega_2}; q)_\infty}{(e^{2\pi i u/\omega_1} \tilde{q}; \tilde{q})_\infty} \tag{2.9}$$

is a well-defined function as $|q| \to 1$. Indeed, it can be checked that the zeros and poles of (2.9) coincide with the zeros and poles of the function $\Gamma_2(\omega_1 + \omega_2 - u; \omega)/\Gamma_2(u; \omega)$, which is a well-defined meromorphic function of u for $\omega_1/\omega_2 > 0$.

In this case σ is real, and if σ is incommensurable with 1, then $\tilde{\varphi}(y) = 1$ (i.e., the function $\Delta(y)$ is determined uniquely). For a description of the properties of the double sine function and some of its applications, see [JM, KLS, NU, Ru2]. In particular, the integrals introduced by Jimbo and Miwa in [JM] as solutions of some q-difference equations at $|q| = 1$ provided the first examples of q-hypergeometric integrals for q on the unit circle. Faddeev's concept of the modular double for quantum groups (see [F]) is also related to the function (2.9).

Thus, the world of q-Meijer functions appears to be richer than in the plain hypergeometric case. The introduction of the additional equation involving shifts by σ^{-1} has brought some new nontrivial structures in the integrals and reduced the functional freedom in the definition of the meromorphic function $\Delta(y)$ to an elliptic function $\tilde{\varphi}(y)$ containing a finite number of free parameters.

Now, we turn to the single variable elliptic hypergeometric integrals. The general elliptic function of order $r + 1$ can be factorized as follows [WW]:

$$\begin{gathered} h(y) = e^\gamma \prod_{j=0}^{r} \frac{\theta_1(u_j + y; \sigma, \tau)}{\theta_1(v_j + y; \sigma, \tau)} = e^\gamma \frac{\theta(t_0 q^y, \dots, t_r q^y; p)}{\theta(w_0 q^y, \dots, w_r q^y; p)}, \\ \theta(t_0, \dots, t_k; p) = \prod_{i=0}^{k} \theta(t_i; p), \end{gathered} \tag{2.10}$$

where $p = e^{2\pi i \tau}$, $\mathrm{Im}(\tau) > 0$, $q = e^{2\pi i \sigma}$. The parameter γ is an arbitrary complex number, but $t_i \equiv q^{u_i}$ and $w_i \equiv q^{v_i}$ satisfy the balancing constraint

$$\sum_{i=0}^{r} (u_i - v_i) = 0, \quad \text{or} \quad \prod_{i=0}^{r} t_i = \prod_{i=0}^{r} w_i, \tag{2.11}$$

which guarantees that the meromorphic function $h(y)$ is doubly periodic:

$$h(y + \sigma^{-1}) = h(y), \quad h(y + \tau\sigma^{-1}) = h(y).$$

For $\tau = \sigma$ (which requires that $\mathrm{Im}(\sigma) > 0$), the function $h(y)$ gives an explicit form of $\tilde{\varphi}(y)$ in (2.8).

In order to find the integrand $\Delta(y)$, it is necessary to solve the first order difference equation

$$\Delta(y + 1) = h(y)\Delta(y) \tag{2.12}$$

in the class of meromorphic functions. The theory of such equations was developed long ago (see, e.g., [Ba2]). Obviously, since $h(y)$ is factorized into the ratio of products of theta functions, it suffices to find a meromorphic solution of the equation

$$f(y + 1) = \theta(q^y; p) f(y), \tag{2.13}$$

which leads to various elliptic gamma functions [J]. The simplest function (1.11) of this sort is determined by equation (2.13) only up to a periodic function $\varphi(y+1) = \varphi(y)$ and, moreover, it requires that $\mathrm{Im}(\sigma) > 0$ (or $|q| < 1$), which was not assumed in (2.10).

We introduce the variable $z = q^y$, so that the shift $y \mapsto y+1$ becomes equivalent to the multiplication $z \mapsto qz$. Then the general solution of (2.12) looks like this:

$$\Delta(y) = \prod_{j=0}^{r} \frac{\Gamma(t_j z; q, p)}{\Gamma(w_j z; q, p)} e^{\gamma y + \delta} \varphi(y), \tag{2.14}$$

where the balancing condition (2.11) is assumed and $\varphi(y+1) = \varphi(y)$ is an arbitrary periodic function. Using the reflection formulas

$$\begin{aligned} \Gamma(pz, qz^{-1}; q, p) &= \Gamma(qz, pz^{-1}; q, p) \\ &= \Gamma(pqz, z^{-1}; q, p) = \Gamma(z, pqz^{-1}; q, p) = 1, \end{aligned} \tag{2.15}$$

in (2.14) we can replace several elliptic gamma functions containing z in the arguments by those with arguments containing z^{-1}. After that, $\Delta(y)$ would look closer to the integrands for the plain or q-Meijer functions, but in the elliptic case this does not increase generality because the right-hand side of (2.15) is trivial.

In the region $\mathrm{Im}(\sigma) < 0$, that is, for $|q| > 1$, the general solution of (2.12) can be written in the form

$$\Delta(y) = \prod_{j=0}^{r} \frac{\Gamma(w_j q^{y-1}; q^{-1}, p)}{\Gamma(t_j q^{y-1}; q^{-1}, p)} e^{\gamma y + \delta} \varphi(y). \tag{2.16}$$

Effectively, we have a permutation of parameters and a simple $q \mapsto q^{-1}$ substitution in the elliptic gamma functions in (2.14) (cf. the definition of this function for $|q| > 1$ given in [FV]).

Let us take $\varphi(y) = 1$. Then the function (2.14) satisfies two simple difference equations of the first order:

$$\Delta(y + \sigma^{-1}) = e^{\gamma/\sigma} \Delta(y), \tag{2.17}$$

$$\Delta(y + \tau\sigma^{-1}) = e^{\gamma\tau/\sigma} \prod_{j=0}^{r} \frac{\theta(t_j q^y; q)}{\theta(w_j q^y; q)} \Delta(y). \tag{2.18}$$

Suppose that $1, \sigma^{-1}, \tau\sigma^{-1}$ are pairwise incommensurable. Then the system of three equations (2.12), (2.17), and (2.18) determines $\Delta(y)$ uniquely up to a factor. As in the q-hypergeometric case, we can generalize equations (2.17) and (2.18), use them as natural tools for fixing the functional freedom in $\Delta(y)$, and get qualitatively different elliptic hypergeometric integrals in this way.

The ratio $\Delta(y + \tau\sigma^{-1})/\Delta(y)$ in (2.18) is an elliptic function with periods 1 and σ^{-1}. Therefore, it is natural to demand that $\Delta(y+\sigma^{-1})/\Delta(y)$ be also an elliptic function with periods that, by symmetry, are equal to 1 and $\tau\sigma^{-1}$.

Theorem 2. *Suppose that $\Delta(y)$ satisfies equation (2.12) and that 1, σ^{-1}, $\tau\sigma^{-1}$ are pairwise incommensurable. Denote $\tilde{q} = e^{-2\pi i/\sigma}$, $\tilde{p} = e^{2\pi i\tau/\sigma}$. For simplicity, assume that $\mathrm{Im}(\sigma) > 0$ (i.e., $|q| < 1$). If $\Delta(y + \tau\sigma^{-1})/\Delta(y)$ is an elliptic function with periods 1 and $\tau\sigma^{-1}$, then for $\mathrm{Im}(\tau/\sigma) > 0$ the most general form of the meromorphic function $\Delta(y)$ is as follows*:

$$\Delta(y) = \prod_{j=0}^{r} \frac{\Gamma(t_j q^y; q, p)}{\Gamma(w_j q^y; q, p)} \prod_{j=0}^{n} \frac{\Gamma(\tilde{t}_j e^{-2\pi i y}; \tilde{q}, \tilde{p})}{\Gamma(\tilde{w}_j e^{-2\pi i y}; \tilde{q}, \tilde{p})} e^{\gamma y + \delta}, \tag{2.19}$$

where $\prod_{j=0}^{r} t_j w_j^{-1} = \prod_{j=0}^{n} \tilde{t}_j \tilde{w}_j^{-1} = 1$. For $\mathrm{Im}(\tau/\sigma) < 0$, we have

$$\Delta(y) = \prod_{j=0}^{r} \frac{\Gamma(t_j q^y; q, p)}{\Gamma(w_j q^y; q, p)} \prod_{j=0}^{n} \frac{\Gamma(\tilde{w}_j e^{-2\pi i y} \tilde{p}^{-1}; \tilde{q}, \tilde{p}^{-1})}{\Gamma(\tilde{t}_j e^{-2\pi i y} \tilde{p}^{-1}; \tilde{q}, \tilde{p}^{-1})} e^{\gamma y + \delta}. \tag{2.20}$$

Proof. First, observe that for $\mathrm{Im}(\sigma) > 0$ we have $\mathrm{Im}(\sigma^{-1}) < 0$ automatically, that is, $|\tilde{q}| < 1$. Therefore, for $\mathrm{Im}(\tau/\sigma) > 0$ the function $\Gamma(z;\tilde{q},\tilde{p})$ is well defined.

The function $\Delta(y)$ in (2.14) gives the general solution of equation (2.12). Suppose that $\Delta(y+\sigma^{-1})/\Delta(y)$ is an elliptic function of order $n+1$ with periods 1 and $\tau\sigma^{-1}$. For $\mathrm{Im}(\tau/\sigma) > 0$, this demand is equivalent to the following equation for $\varphi(y)$:

$$\frac{\varphi(y+\sigma^{-1})}{\varphi(y)} = \prod_{j=0}^{n} \frac{\theta(\tilde{t}_j e^{-2\pi i y};\tilde{p})}{\theta(\tilde{w}_j e^{-2\pi i y};\tilde{p})}, \tag{2.21}$$

where $\prod_{j=0}^{n} \tilde{t}_j \tilde{w}_j^{-1} = 1$. Note that we cannot multiply the right-hand side of this equation by any constant different from 1, since this would violate the condition $\varphi(y+1) = \varphi(y)$.

The meromorphic solution of (2.21) is

$$\varphi(y) = \prod_{j=0}^{n} \frac{\Gamma(\tilde{t}_j e^{-2\pi i y};\tilde{q},\tilde{p})}{\Gamma(\tilde{w}_j e^{-2\pi i y};\tilde{q},\tilde{p})} \tilde{\varphi}(y), \tag{2.22}$$

where $\tilde{\varphi}(y)$ is an elliptic function with periods 1 and σ^{-1}. We can write

$$\tilde{\varphi}(y) = \prod_{j=1}^{m} \frac{\theta(a_j e^{-2\pi i y};\tilde{q})}{\theta(b_j e^{-2\pi i y};\tilde{q})} = \prod_{j=1}^{m} \frac{\Gamma(a_j e^{-2\pi i y}\tilde{p}, b_j e^{-2\pi i y};\tilde{q},\tilde{p})}{\Gamma(a_j e^{-2\pi i y}, b_j e^{-2\pi i y}\tilde{p};\tilde{q},\tilde{p})},$$

where $\prod_{j=1}^{m} a_j b_j^{-1} = 1$. Therefore, we can absorb the function $\tilde{\varphi}(y)$ into the ratio of elliptic gamma functions in (2.22) by changing $n \mapsto n+2m$ and identifying $\tilde{t}_k = \tilde{p} a_k, \tilde{w}_k = a_k$ for $k = n+1,\dots,n+m$ and $\tilde{t}_k = \tilde{b}_k, \tilde{w}_k = \tilde{p}\tilde{b}_k$ for $k = n+m+1,\dots,n+2m$. Since n, $\tilde{t}_j$, and $\tilde{w}_j$ are arbitrary, without loss of generality we can set $\tilde{\varphi}(y) = 1$, which yields the desired expression (2.19).

The function (2.19) satisfies the following equations:

$$\Delta(y+\sigma^{-1}) = e^{\gamma/\sigma} \prod_{j=0}^{n} \frac{\theta(\tilde{t}_j e^{-2\pi i y};\tilde{p})}{\theta(\tilde{w}_j e^{-2\pi i y};\tilde{p})} \Delta(y), \tag{2.23}$$

$$\Delta(y+\tau\sigma^{-1}) = e^{\gamma\tau/\sigma} \prod_{j=0}^{r} \frac{\theta(t_j q^y;q)}{\theta(w_j q^y;q)} \prod_{j=0}^{n} \frac{\theta(\tilde{w}_j e^{-2\pi i y}\tilde{p}^{-1};\tilde{q})}{\theta(\tilde{t}_j e^{-2\pi i y}\tilde{p}^{-1};\tilde{q})} \Delta(y). \tag{2.24}$$

The elliptic functions defined by the products $\prod_{j=0}^{r}$ and $\prod_{j=0}^{n}$ in (2.24) have different forms though both have periods 1 and σ^{-1}. They are related to each other by the modular transformation $\sigma \mapsto -1/\sigma$ for the corresponding theta functions.

Now, we consider the region $\mathrm{Im}(\tau/\sigma) < 0$. Equations (2.12) and (2.24) are well defined in this case. They can be used for the determining of $\Delta(y)$, and it can be checked that, indeed, the function (2.20) provides their general solution. Equation (2.23) is replaced now by the following one:

$$\frac{\Delta(y+\sigma^{-1})}{\Delta(y)} = e^{\gamma/\sigma} \prod_{j=0}^{n} \frac{\theta(\tilde{w}_j e^{-2\pi i y};\tilde{p}^{-1})}{\theta(\tilde{t}_j e^{-2\pi i y};\tilde{p}^{-1})}, \tag{2.25}$$

that is, $\tilde{p}$ in (2.21) is changed to $\tilde{p}^{-1}$, and the parameters $\tilde{t}_j, \tilde{w}_j$ are replaced by $\tilde{p}^{-1}\tilde{w}_j$ and $\tilde{p}^{-1}\tilde{t}_j$, respectively. Using (2.16), it is easy to construct $\Delta(y)$ satisfying (2.12), (2.23), and (2.24) in the region $|q| > 1$ as well. □

In order to be able to work with q on the unit circle $|q| = 1$, we need another elliptic gamma function: a kind of elliptic analog of the double sine function (2.9). Denote

$$\begin{aligned} q &= e^{2\pi i \frac{\omega_1}{\omega_2}}, \quad \tilde{q} = e^{-2\pi i \frac{\omega_2}{\omega_1}}, \\ p &= e^{2\pi i \frac{\omega_3}{\omega_2}}, \quad \tilde{p} = e^{2\pi i \frac{\omega_3}{\omega_1}}, \end{aligned} \tag{2.26}$$

where the ω_i are some complex numbers.

Suppose that $\omega_1/\omega_2 > 0$ and $\mathrm{Im}(\omega_3/\omega_2) > 0$ (i.e., $|p| < 1$). Then we have $\mathrm{Im}(\omega_3/\omega_1) = (\omega_2/\omega_1)\,\mathrm{Im}(\omega_3/\omega_2) > 0$, that is, $|\tilde{p}| < 1$ automatically. Therefore, in the analysis of equation (2.12) for $|q| = 1$ it is necessary to assume that $|p|, |\tilde{p}| < 1$.

Definition. Let $|q|, |p|, |\tilde{p}| < 1$. Then we define a new elliptic gamma function by the formula

$$G(u;\omega) = \prod_{j,k=0}^{\infty} \frac{(1 - e^{-2\pi i \frac{u}{\omega_2}} q^{j+1} p^{k+1})(1 - e^{2\pi i \frac{u}{\omega_1}} \tilde{q}^{j+1} \tilde{p}^{k})}{(1 - e^{2\pi i \frac{u}{\omega_2}} q^{j} p^{k})(1 - e^{-2\pi i \frac{u}{\omega_1}} \tilde{q}^{j} \tilde{p}^{k+1})}. \tag{2.27}$$

In the limit $p \to 0$ taken in such a way that simultaneously $\tilde{p} \to 0$, we get

$$G(u;\omega_1,\omega_2,\omega_3) \to S^{-1}(u;\omega_1,\omega_2),$$

where the double sine function $S(u;\omega)$ is fixed in (2.9).

The function $G(u;\omega)$ satisfies the following three difference equations:

$$G(u+\omega_1;\omega) = \theta(e^{2\pi i \frac{u}{\omega_2}};p)G(u;\omega), \tag{2.28}$$

$$G(u+\omega_2;\omega) = \theta(e^{2\pi i \frac{u}{\omega_1}};\tilde{p})G(u;\omega), \tag{2.29}$$

$$\begin{aligned} G(u+\omega_3;\omega) &= \frac{\theta(e^{2\pi i \frac{u}{\omega_2}};q)}{\theta(e^{2\pi i \frac{u}{\omega_1}}\tilde{q};\tilde{q})} G(u;\omega) \\ &= S(u;\omega_1,\omega_2)S(\omega_1+\omega_2-u;\omega_1,\omega_2)G(u;\omega). \end{aligned} \tag{2.30}$$

For pairwise incommensurable ω_1, ω_2, ω_3, these equations determine the meromorphic function $G(u;\omega)$ uniquely up to multiplication by a constant, which follows from the nonexistence of triply periodic functions.

The first equation requires that $|p| < 1$, the second requires $|\tilde{p}| < 1$, and both of them do not impose any constraint upon q. The third equation (2.30) involves only the function $S(u;\omega_1,\omega_2)$, which is well defined for $\omega_1/\omega_2 > 0$, that is, $|q| = |\tilde{q}| = 1$. This means that the function $G(u;\omega)$ may be well defined in this unit circle region as well.

In essence, the original elliptic gamma function (1.11) has the same properties as the function (1.12). In a similar way, the function (2.27) can be expressed as the following combination of the Barnes Γ_3-functions up to an exponential of some polynomial in u of the third degree:

$$\begin{aligned} &\frac{\Gamma_3(u;\omega_1,\omega_2,\omega_3)\Gamma_3(\omega_3-\omega_1-u;-\omega_1,-\omega_2,\omega_3)}{\Gamma_3(\omega_1+\omega_3-u;\omega_1,\omega_2,\omega_3)\Gamma_3(u-\omega_1-\omega_2;-\omega_1,-\omega_2,\omega_3)} \\ &\qquad \times \Gamma_2(\omega_3-\omega_2-u;-\omega_2,\omega_3)\Gamma_2(\omega_3-u;\omega_1,\omega_3). \end{aligned} \tag{2.31}$$

From this representation it follows that, indeed, (2.27) is well defined for real ω_1, ω_2 with $\omega_1/\omega_2 > 0$ (and any complex ω_3), like in the double sine function case. A more detailed analysis of this correspondence and an investigation of other properties of the function $G(u;\omega)$ will be given elsewhere. In particular, it is expected that $G(u;\omega)$ is the key function for an elliptic extension of the modular doubling principle for q-deformed algebras [F, KLS].

As a result, for $|q| = 1$ we get a solution $\Delta(y)$ of equation (2.12) by the mere replacement of $\Gamma(q^y;q,p)$ in (2.14) by $G(y\omega_1;\omega)$. In the rest of this paper we limit ourselves to the case where $|q| < 1$. Note that the region $|p| = 1$ is not well defined in the elliptic functions setting. In a sense, the region of real ω_3/ω_2 is reachable only at the level of the original Barnes multiple gamma functions.

§3. A theta analog of the Meijer function

The integral corresponding to (2.14) may be regarded as a kind of elliptic extension of a particular Meijer function. The general Jacobi theta function analog of the Meijer

function arises in the case where $h(y)$ is a quasiperiodic function corresponding to the fourth case of the definition given at the beginning of §2 (see (2.5)).

Let $P_3(y) = \sum_{i=1}^{3} \alpha_i y^i$ be an arbitrary polynomial of the third degree with the property $P_3(0) = 0$. The function defined by the integral

$$G_r^s\left(\begin{smallmatrix}\mathbf{t}\\ \mathbf{w}\end{smallmatrix}; \alpha; q, p\right) = \int_C \frac{\prod_{j=0}^{s}\Gamma(t_j q^y; q, p)}{\prod_{k=0}^{r}\Gamma(w_k q^y; q, p)} e^{P_3(y)}\, dy, \tag{3.1}$$

where C is some contour on the complex plane, may be called a theta analog of the Meijer function whenever the integral is well defined. Note that no constraints are imposed in (3.1) upon the integers r, s and the complex parameters t_j, w_k.

We have the following equation for the integrand $\Delta(y)$ of (3.1):

$$\frac{\Delta(y+1)}{\Delta(y)} = h(y) = e^{P_2(y)} \frac{\theta(t_0 q^y, \dots, t_s q^y; p)}{\theta(w_0 q^y, \dots, w_r q^y; p)}, \tag{3.2}$$

where $P_2(y) = P_3(y+1) - P_3(y)$ is a polynomial in y of the second degree. From the considerations of [S5] it follows that this $h(y)$ is the most general function such that h is meromorphic in y (together with its inverse $1/h(y)$) and satisfies the quasiperiodicity conditions

$$h(y + \sigma^{-1}) = e^{ay+b} h(y), \quad h(y + \tau\sigma^{-1}) = e^{cy+d} h(y) \tag{3.3}$$

for some constants a, b, c, d. The function $h(y)$ may also be interpreted as a general meromorphic modular Jacobi form in the sense of Eichler and Zagier [EZ].

However, the integral (3.1) is not the most general integral leading to (3.2). Using appropriate modifications of the integrands (2.19) and (2.20) and replacing y by y/ω_1, we arrive at the general theta analog of the Meijer function.

Definition. In the definitions (2.26) of the bases, assume that $|q|, |p| < 1$. Then, for $|\tilde{p}| < 1$, the integral

$$G_{rm}^{sn}\left(\begin{smallmatrix}\mathbf{t}, \tilde{\mathbf{t}}\\ \mathbf{w}, \tilde{\mathbf{w}}\end{smallmatrix}; \alpha; \omega\right) = \int_C \frac{\prod_{k=0}^{s}\Gamma(t_k e^{\frac{2\pi i y}{\omega_2}}; q, p)\prod_{j=0}^{n}\Gamma(\tilde{t}_j e^{-\frac{2\pi i y}{\omega_1}}; \tilde{q}, \tilde{p})}{\prod_{k=0}^{r}\Gamma(w_k e^{\frac{2\pi i y}{\omega_2}}; q, p)\prod_{j=0}^{m}\Gamma(\tilde{w}_j e^{-\frac{2\pi i y}{\omega_1}}; \tilde{q}, \tilde{p})} e^{P_3(y)}\, dy \tag{3.4}$$

is called the general theta hypergeometric integral of one variable whenever it is well defined. For $|\tilde{p}| > 1$, we set

$$G_{rm}^{sn}\left(\begin{smallmatrix}\mathbf{t}, \tilde{\mathbf{t}}\\ \mathbf{w}, \tilde{\mathbf{w}}\end{smallmatrix}; \alpha; \omega\right) = \int_C \frac{\prod_{k=0}^{s}\Gamma(t_k e^{\frac{2\pi i y}{\omega_2}}; q, p)\prod_{j=0}^{m}\Gamma(\tilde{w}_j e^{-\frac{2\pi i y}{\omega_1}}\tilde{p}^{-1}; \tilde{q}, \tilde{p}^{-1})}{\prod_{k=0}^{r}\Gamma(w_k e^{\frac{2\pi i y}{\omega_2}}; q, p)\prod_{j=0}^{n}\Gamma(\tilde{t}_j e^{-\frac{2\pi i y}{\omega_1}}\tilde{p}^{-1}; \tilde{q}, \tilde{p}^{-1})} e^{P_3(y)}\, dy. \tag{3.5}$$

There are no constraints upon the integers $r, s, n, m \in \mathbb{N}$ and the complex parameters $t_j, \tilde{t}_j, w_k, \tilde{w}_k$.

Both integrands in (3.4) and (3.5) satisfy the equations $\Delta(y + \omega_i)/\Delta(y) = h_i(y)$, $i = 1, 2, 3$, where the h_i are some quasiperiodic functions: $h_i(y + \omega_k) = e^{a_{ik}y + b_{ik}} h(y)$, $i \neq k$, with a_{ik}, b_{ik} being some constants related to the parameters $\mathbf{t}, \tilde{\mathbf{t}}, \mathbf{w}, \tilde{\mathbf{w}}, \alpha$, and ω. The integral (3.5) was determined by the condition that it has the same functions $h_1(y), h_3(y)$ as (3.4). For a special choice of the parameters $\mathbf{t}, \tilde{\mathbf{t}}, \mathbf{w}, \tilde{\mathbf{w}}, \alpha$, in the limits $|p|, |\tilde{p}| \to 0$ or $|p|, |\tilde{p}|^{-1} \to 0$ the function $G_{rm}^{sn}\left(\begin{smallmatrix}\mathbf{t}, \tilde{\mathbf{t}}\\ \mathbf{w}, \tilde{\mathbf{w}}\end{smallmatrix}; \alpha; \omega\right)$ is reduced to the general q-hypergeometric integral considered in §2 (see (2.7) and (2.8)).

The general single variable theta hypergeometric series is defined by the following formula [S5]:

$${}_{s+1}E_r\left(\begin{smallmatrix}t_0, \dots, t_s\\ w_1, \dots, w_r\end{smallmatrix}; \alpha; q, p\right) = \sum_{n=0}^{\infty} \frac{\theta(t_0, t_1, \dots, t_s; p; q)_n}{\theta(q, w_1, \dots, w_r; p; q)_n} e^{P_3(n)}. \tag{3.6}$$

Actually, these series are slightly more general than those introduced in [S5], because in that paper we considered only the case where $\alpha_3 = 0$, but the generalization to (3.6) is straightforward. We note that the presence of cubics of the independent variable y in (3.1) or n in (3.6) is natural since we are working at the level of the Barnes multiple gamma function (1.2) of the third order.

Writing (3.6) in the form of the sum $\sum_{n=0}^{\infty} c_n$ with $c_0 = 1$, we easily see that $c_{n+1}/c_n = h(n)$, where $h(n)$ is given by (3.2) with $w_0 = q$ and $y = n$. This coincidence is not artificial. Consider the sequence of poles of the integrand in (3.1) located at $y = y_0 + n,\ n \in \mathbb{N}$, for some y_0. We denote by $\kappa c_n,\ c_0 = 1$, the residues of these poles. As $y \to y_0 + n$, we have $\Delta(y) \to \kappa c_n/(y - y_0 - n) + O(1)$. Now it is not difficult to see that

$$\lim_{y\to y_0+n} \frac{\Delta(y+1)}{\Delta(y)} = \frac{c_{n+1}}{c_n} = \lim_{y\to y_0+n} h(y) = h(y_0+n).$$

In particular, this means that the sums of the residues in the integral (3.1) that appear from appropriate deformations of the contour C, form the theta hypergeometric series (3.6) for some choices of the parameters.

In accordance with the classification introduced in [S5], the elliptic hypergeometric series correspond, by definition, to $h(n)$ equal to an elliptic function of n. Such series are called also the balanced theta hypergeometric series. They are defined by the following constraints imposed upon (3.6):

$$s = r, \quad \alpha_3 = \alpha_2 = 0, \quad \prod_{j=0}^{r} t_j = \prod_{j=0}^{r} w_j. \tag{3.7}$$

Similarly, the integral (3.1) will be called the *elliptic* (or *balanced theta*) hypergeometric integral if conditions (3.7) are satisfied. Evidently, in this case $h(y)$ in (3.2) becomes an elliptic function of y.

When $h(y)$ is an elliptic function of y and of all parameters u_j, v_j (we remind the reader that $t_j = q^{u_j}, w_j = q^{v_j}$), we call (3.6) and (3.1) the *totally elliptic hypergeometric series* and *integrals*, respectively. As was shown in [S5], in addition to the balancing requirement, such a property imposes the following constraints on the parameters:

$$t_j w_j = \rho = \text{const}, \quad j = 0, \dots, r, \tag{3.8}$$

which are known as the *well-posedness* conditions in the theory of basic hypergeometric series [GR]. The explicit form of the integrand $\Delta(y)$ for well-poised balanced theta hypergeometric integrals is

$$\Delta(y) = \prod_{j=0}^{m-1} \frac{\Gamma(t_j z; q, p)}{\Gamma(\rho t_j^{-1} z; q, p)} \frac{\Gamma(\rho^{\frac{m+1}{2}} \prod_{j=0}^{m-1} t_j^{-1} z; q, p)}{\Gamma(\rho^{\frac{1-m}{2}} \prod_{j=0}^{m-1} t_j z; q, p)} e^{\gamma y}, \tag{3.9}$$

where we have denoted $z = q^y$ and $\gamma = \alpha_1$. The parameter ρ is redundant; it can be eliminated by the rescalings $t_i \mapsto \rho^{1/2} t_i, z \to \rho^{-1/2} z$, but we keep it for further needs. Observe that without loss of generality one of the parameters in (3.1) can be set equal to one by a shift of y.

Without the balancing condition, a theta hypergeometric series ${}_{r+1}E_r$ is said to be *well poised* if the constraints (3.8) are valid with $w_0 = q$, and *very well poised* if, in addition to (3.8), we have

$$\begin{aligned} t_{r-3} &= t_0^{1/2} q, & t_{r-2} &= -t_0^{1/2} q, \\ t_{r-1} &= t_0^{1/2} q p^{-1/2}, & t_r &= -t_0^{1/2} q p^{1/2}. \end{aligned} \tag{3.10}$$

Such series take a simpler form

$$
\begin{aligned}
&{}_{r+1}E_r\left(\begin{matrix} t_0,t_1,\dots,t_{r-4},qt_0^{1/2},-qt_0^{1/2},qp^{-1/2}t_0^{1/2},-qp^{1/2}t_0^{1/2} \\ qt_0/t_1,\dots,qt_0/t_{r-4},t_0^{1/2},-t_0^{1/2},p^{1/2}t_0^{1/2},-p^{-1/2}t_0^{1/2}\end{matrix};\alpha;q,p\right) \\
&\quad=\sum_{n=0}^{\infty}\frac{\theta(t_0q^{2n};p)}{\theta(t_0;p)}\prod_{m=0}^{r-4}\frac{\theta(t_m;p;q)_n}{\theta(qt_0/t_m;p;q)_n}(-q)^n e^{P_3(n)}.
\end{aligned}
\tag{3.11}
$$

The essence of (3.10) consists in the replacement of the product of four $\theta(t_iz;p)$ by one theta function $\propto\theta(t_0q^2z^2;p)$ (this corresponds to doubling the argument of the θ_1-function). Very-well-poised series play a distinguished role in applications; in particular, they admit an appropriate generalization of the Bailey chain technique of generating infinite sequences of summation or transformation formulas [S6].

In the case of integrals, we call (3.1) the *very-well-poised* theta hypergeometric integral if, in addition to conditions (3.8), eight parameters t_i are fixed in the following way:

$$
(t_{m-8},\dots,t_{m-1})=(\pm(pq)^{1/2},\pm q^{1/2}p,\pm p^{1/2}q,\pm pq).
\tag{3.12}
$$

These constraints lead to squaring the argument of the elliptic gamma function

$$
\prod_{j=m-8}^{m-1}\Gamma(t_jz;q,p)=\Gamma(pqz^2;q,p)=\frac{1}{\Gamma(z^{-2};q,p)}
\tag{3.13}
$$

(such a relation was used already in [S2] in the derivation of the elliptic beta integral (1.13)). As a result, the integrand of the very-well-poised balanced theta hypergeometric integral takes the form

$$
\Delta(y)=\prod_{j=0}^{m-9}\frac{\Gamma(t_jz;q,p)}{\Gamma(\rho t_j^{-1}z;q,p)}\frac{\Gamma(\rho^{\frac{m+1}{2}}p^{-6}q^{-6}\prod_{j=0}^{m-9}t_j^{-1}z;q,p)e^{\gamma y}}{\Gamma(z^{-2},(\rho/pq)^2z^2,\rho^{\frac{1-m}{2}}p^6q^6\prod_{j=0}^{m-9}t_jz;q,p)}.
\tag{3.14}
$$

After imposing conditions (3.12), the parameter ρ is no longer redundant, and its choice plays an important role. If we fix it as $\rho=pq$, then $\Delta(y)$ takes a particularly symmetric form

$$
\Delta(y)=\frac{\prod_{j=0}^{m-9}\Gamma(t_jz,t_jz^{-1};q,p)}{\Gamma(z^2,z^{-2},Az,Az^{-1};q,p)}e^{\gamma y},
\tag{3.15}
$$

where $A=(pq)^{\frac{13-m}{2}}\prod_{j=0}^{m-9}t_j$. Clearly, the cases where m is odd or even differ from each other in a qualitative way. The choice $m=13$ gives the simplest expression for $\Delta(y)$ and plays a distinguished role. Other simple choices, $m=9$ or 11, correspond to particular subcases of the situation with $m=13$. For $m=13$ and $\gamma=0$ we get the integrand Δ_E of the elliptic beta integral (1.14), that is, the simplest very-well-poised elliptic hypergeometric integral turns out to be exactly calculable when C is taken to be a special cycle corresponding to the unit circle on the z-plane.

The sums of the residues of the function (3.15) for $m=13$ are expected to form a ${}_{14}E_{13}$ theta hypergeometric series. However, the very-well-arithmetic condition (3.12) results in the cancellation of theta functions in the corresponding ratios of the series coefficients $h(n)$. As a result, we get only a ${}_{10}E_9$ very-well-poised elliptic hypergeometric series which, for $\gamma=0$, corresponds to the left-hand side of (A.11) or (6.8) with $n=1$ (for more details, see [DS1, S5]). This shift of indices $m\mapsto m-4$ brings in another intriguing point related to the origins of the very-well-arithmetic condition. It is necessary to find some deeper algebraic geometry explanations of the fact that in the single variable case "the nice things" (summation or integration formulas) are related to the number 14, the order of the initial elliptic function $h(y)$. As was shown in [S5, S7], in the multiple case this order raises to higher numbers, but in quite an intriguing way as well.

Since $\Gamma((pq)^{1/2}z,(pq)^{1/2}z^{-1};q,p)=1$, we may drop the two parameters $\pm\sqrt{pq}$ in the condition (3.12). For $\rho=pq,\gamma=0$, this yields

$$\Delta(y)=\frac{\prod_{j=0}^{m-7}\Gamma(t_jz,t_jz^{-1};q,p)}{\Gamma(z^2,z^{-2},-Az,-Az^{-1};q,p)}, \tag{3.16}$$

where $A=(pq)^{\frac{11-m}{2}}\prod_{j=0}^{m-7}t_j$. For $m=11$ this expression looks similar to (1.14), but the different sign in front of A changes the things drastically, and it is not known whether the corresponding integral gets any closed form expression.

In a more general setting, we can impose balancing and very-well-arithmetic conditions upon the general theta hypergeometric integrals (3.4) and (3.5). However, at the moment it is not known whether the simplest integrals appearing in this way admit exact evaluation.

As far as the multivariable integrals of hypergeometric type are concerned, the general form of $\Delta(\mathbf{y})$ in the plain and q-hypergeometric cases can be deduced from the Ore–Sato theorem for Horn's series (see, e.g., [GGR] for a detailed discussion). The general form of the multiple elliptic hypergeometric series or integrals is not established yet. We formulate it as an open problem—to find an elliptic or general theta function analog of the Ore–Sato characterization theorem. In the following sections we give a series of examples of multivariable extensions of the very-well-poised balanced theta hypergeometric integrals associated with the root systems A_n and C_n.

As to the further possible generalizations, it is natural to consider integrals of hypergeometric type for arbitrary algebraic curves or general Abelian varieties. Both would involve Riemann theta functions of many variables, appropriate generalizations of gamma functions and theta hypergeometric series. Some preliminary discussion of ideas in this direction can be found in [S7]; in particular, a special subcase of the ${}_8\Phi_7$ Jackson summation formula was generalized there to Riemann surfaces of an arbitrary genus.

§4. Known C_n elliptic beta integrals

The following multivariable generalization of the Euler beta integral (1.1) was introduced by Selberg [AAR]:

$$\begin{aligned}&\int_0^1\cdots\int_0^1\prod_{1\le j\le n}x_j^{\alpha-1}(1-x_j)^{\beta-1}\prod_{1\le j<k\le n}|x_j-x_k|^{2\gamma}\,dx_1\cdots dx_n\\&\quad=\prod_{1\le j\le n}\frac{\Gamma(\alpha+(j-1)\gamma)\Gamma(\beta+(j-1)\gamma)\Gamma(1+j\gamma)}{\Gamma(\alpha+\beta+(n+j-2)\gamma)\Gamma(1+\gamma)},\end{aligned} \tag{4.1}$$

where

$$\mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0,$$

and

$$\mathrm{Re}(\gamma)>-\min(1/n,\mathrm{Re}(\alpha)/(n-1),\mathrm{Re}(\beta)/(n-1)).$$

This integral has found many important applications in mathematical physics.

At first glance, the integrand of (4.1) does not fit the definition of the hypergeometric integrals introduced in the preceding section. However, we can rescale $y_i\mapsto y_i\epsilon$ in (2.1), and choose rational functions $R_i(\mathbf{y})$ appropriately so that the limit $\epsilon\to 0$ becomes well defined and yields a system of linear differential equations of the first order. As a result, we get $\partial\Delta(\mathbf{y})/\partial y_i=R_i(\mathbf{y})\Delta(\mathbf{y})$ and, apparently, (4.1) satisfies these conditions.

Two types of multidimensional generalizations of the elliptic beta integral (1.13) to the root system C_n were proposed by van Diejen and the author in [DS1, DS2]. One of them reduces in a special limit to the Selberg integral (4.1). Here we describe these

elliptic Selberg integrals explicitly. For brevity, we drop the bases p, q in the notation for elliptic gamma functions from now on. We introduce the Type I integrand as

(4.2)
$$\Delta^I(\mathbf{z}; C_n) = \frac{1}{(2\pi i)^n} \prod_{1\le j<k\le n} \Gamma^{-1}(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}) \prod_{j=1}^{n} \frac{\prod_{r=0}^{2n+2} \Gamma(t_r z_j, t_r z_j^{-1})}{\Gamma(z_j^2, z_j^{-2}, A z_j, A z_j^{-1})},$$

where $t_r \in \mathbb{C}$, $r = 0, \dots, 2n+2$, are free parameters and $A \equiv \prod_{r=0}^{2n+2} t_r$. The Type II integrand has the form

(4.3)
$$\Delta^{II}(\mathbf{z}; C_n) = \frac{1}{(2\pi i)^n} \prod_{1\le j<k\le n} \frac{\Gamma(t z_j z_k, t z_j z_k^{-1}, t z_j^{-1} z_k, t z_j^{-1} z_k^{-1})}{\Gamma(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1})} \prod_{j=1}^{n} \frac{\prod_{r=0}^{4} \Gamma(t_r z_j, t_r z_j^{-1})}{\Gamma(z_j^2, z_j^{-2}, B z_j, B z_j^{-1})},$$

where $t, t_r \in \mathbb{C}$, $r = 0, \dots, 4$, are free parameters and $B \equiv t^{2n-2} \prod_{s=0}^{4} t_s$. By $\mathbb{T}$ we denote the unit circle with positive orientation.

Consider the first type of the C_n multivariable elliptic beta integral. We take $|p|, |q| < 1$ and $|t_r| < 1$, $r = 0, \dots, 2n+2$, and assume that $|pq| < |A|$. Then

$$\int_{\mathbb{T}^n} \Delta^I(\mathbf{z}; C_n) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} = \frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n} \frac{\prod_{0\le r<s\le 2n+2} \Gamma(t_r t_s)}{\prod_{r=0}^{2n+2} \Gamma(t_r^{-1} A)}. \tag{4.4}$$

The second type of the multiple elliptic beta integral has the following form. We take $|p|, |q|, |t| < 1$ and $|t_r| < 1$, $r = 0, \dots, 4$, and assume that $|pq| < |B|$. Then

$$\int_{\mathbb{T}^n} \Delta^{II}(\mathbf{z}; C_n) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} = \frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n} \prod_{j=1}^{n} \frac{\Gamma(t^j)}{\Gamma(t)} \frac{\prod_{0\le r<s\le 4} \Gamma(t^{j-1} t_r t_s)}{\prod_{r=0}^{4} \Gamma(t^{1-j} t_r^{-1} B)}. \tag{4.5}$$

The elliptic Selberg integral of Type II (4.5) can be deduced from the Type I integral with the help of an interesting trick [DS2]. The type I integral (4.2) was proved in [DS2] under a vanishing hypothesis, namely, that its left-hand side vanishes on the hypersurface of the parameters $A = t_{2n+2}$ after an appropriate deformation of $\mathbb{T}$ to an integration contour C that separates the sequences of poles in $\mathbf{z}$ converging to zero from those diverging to infinity. As $p \to 0$, both Type I and II integrals are reduced to Gustafson's well-known q-Selberg integrals [G1, G2], which are related (for particular choices of the parameters) to the Macdonald–Morris constant term identities and the Macdonald polynomials for various root systems [M], including the Koornwinder polynomials [K].

As was shown in [DS1, DS3], the sums of residues of the functions (4.2) and (4.3) form some multiple elliptic hypergeometric series. In particular, the multivariable ${}_{10}E_9$ sum conjectured by Warnaar in [Wa] can be deduced from (4.5). The residue calculus for (4.4) yields an elliptic extension of the basic hypergeometric series summation formula proved in [DG] and [ML]. Recursive proofs of these multivariable Frenkel–Turaev sums were given by Rosengren in [Ro1, Ro2]. The well-arithmetic property (or total ellipticity, in the case of elliptic hypergeometric series) plays an important role in such summation formulas. First examples of (plain) multiple hypergeometric series well-poised on classical groups were introduced by Biedenharn, Holman, and Louck in [HBL].

§5. A new C_n integration formula

We define

$$\Delta^{III}(\mathbf{z};C_n) = \frac{1}{(2\pi i)^n}\prod_{1\le i<j\le n} z_j\theta\Big(\frac{z_i}{z_j},\frac{1}{z_iz_j};p\Big)\prod_{i=1}^n\prod_{\nu=\pm1}\frac{\Gamma(z_i^\nu x_i, z_i^\nu t_1, z_i^\nu t_2, z_i^\nu t_3, z_i^\nu t/x_i)}{\Gamma(z_i^{2\nu}, z_i^\nu A)}, \tag{5.1}$$

where $A = tt_1t_2t_3q^{n-1}$.

Theorem 3. *Impose the following restrictions upon the parameters*: $|x_i|$, $|t_k|<1$, $|t|<|x_i|$, *where* $i=1,\dots,n$, $k=1,2,3$, *and* $|pq|<|A|$. *Then*

$$\begin{aligned}\int_{\mathbb{T}^n}\Delta^{III}(\mathbf{z};C_n)\,\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n} &= \frac{2^n}{(p;p)_\infty^n(q;q)_\infty^n}\prod_{1\le i<j\le n} x_j\theta\left(x_i/x_j, t/x_ix_j;p\right)\\ &\quad\times\Gamma^n(t)\prod_{i=1}^n\left(\frac{\prod_{1\le r<s\le 3}\Gamma(t_rt_sq^{i-1})}{\Gamma(A/x_i, Ax_i/t)}\prod_{k=1}^3\frac{\Gamma(x_it_k, tt_k/x_i)}{\Gamma(Aq^{1-i}/t_k)}\right).\end{aligned} \tag{5.2}$$

Proof. Consider the determinant

$$\begin{aligned}&\det_{1\le i,j\le n}\left(\int_{\mathbb{T}}\Delta_E(z,x_i,t_1q^{n-j},t_2q^{j-1},t_3,tx_i^{-1})\,\frac{dz}{z}\right)\\ &\quad= \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n}\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}\prod_{i=1}^n G_i(z_i)G_i(z_i^{-1})D(\mathbf{z}),\end{aligned} \tag{5.3}$$

where Δ_E is the integrand of the elliptic beta integral (1.14) with an appropriate choice of the parameters. The expression standing on the right-hand side of (5.3) appears after taking the integral signs outside the determinant symbol, so that we get a multiple integral with

$$\begin{aligned}G_i(z_i) &= \frac{\Gamma(z_ix_i, z_it_1, z_it_2, z_it_3, z_it/x_i)}{\Gamma(z_i^2, z_iA)},\\ D(\mathbf{z}) &= \det_{1\le i,j\le n}\left(\theta(z_it_1, z_i^{-1}t_1;p;q)_{n-j}\theta(z_it_2, z_i^{-1}t_2;p;q)_{j-1}\right).\end{aligned}$$

The determinant $D(\mathbf{z})$ can be rewritten as follows:

$$D(\mathbf{z}) = \frac{\prod_{i=1}^n\theta(z_it_2, z_i^{-1}t_2;p;q)_{n-1}}{t_2^{2\binom{n}{2}}q^{4\binom{n}{3}}}\det_{1\le i,j\le n}\left(\frac{\theta(z_it_1, z_i^{-1}t_1;p;q)_{n-j}}{\theta(q^{2-n}/z_it_2, q^{2-n}z_i/t_2;p;q)_{n-j}}\right).$$

In [Wa], Warnaar computed the following elliptic generalization of a Krattenthaler determinant [Kr]:

$$\begin{aligned}&\det_{1\le i,j\le n}\left(\frac{\theta(aX_i, ac/X_i;p;q)_{n-j}}{\theta(bX_i, bc/X_i;p;q)_{n-j}}\right)\\ &\quad= a^{\binom{n}{2}}q^{\binom{n}{3}}\prod_{1\le i<j\le n}X_j\theta(X_iX_j^{-1}, cX_i^{-1}X_j^{-1};p)\prod_{i=1}^n\frac{\theta(b/a, abcq^{2n-2i};p;q)_{i-1}}{\theta(bX_i, bc/X_i;p;q)_{n-1}}.\end{aligned} \tag{5.4}$$

Using this identity for $X_i = z_i$, $a = t_1, b = q^{2-n}/t_2$, and $c = 1$, we find

$$D(\mathbf{z}) = (t_1 t_2^2)^{\binom{n}{2}} q^{3\binom{n}{3}} \prod_{1\le i<j\le n} z_j \theta(z_i z_j^{-1}, z_i^{-1} z_j^{-1}; p) \times \prod_{i=1}^{n} \theta(q^{2-n}/t_1 t_2, t_1 q^{n+2-2i}/t_2; p; q)_{i-1}. \tag{5.5}$$

As a result, the determinant (5.3) yields an expression proportional to the left-hand side of the C_n multiple integral (5.2) in question.

Now we substitute the result of computation of the elliptic beta integral (1.13) in the determinant (5.3). This yields

(5.6)

$$\begin{aligned} &\text{l.h.s. of (5.3)} \\ &= \frac{2^n}{(q;q)_\infty^n (p;p)_\infty^n} \\ &\quad \times \prod_{i=1}^{n} \frac{\Gamma(x_i t_1, t_1 t/x_i, x_i t_2, t_2 t/x_i, x_i t_3, t, t_1 t_2 q^{n-1}, t_1 t_3 q^{n-i}, t_2 t_3 q^{i-1}, t_3 t/x_i)}{\Gamma(A/x_i, A/t_3, x_i A/t, A q^{1-i}/t_2, A q^{i-n}/t_1)} \\ &\quad \times \det_{1\le i,j\le n} \left(\theta(x_i t_1, t_1 t/x_i; p; q)_{n-j} \theta(x_i t_2, t_2 t/x_i; p; q)_{j-1}\right). \end{aligned}$$

By (5.4), the determinant in the last line takes the form

$$(t_1 t_2^2 t)^{\binom{n}{2}} q^{3\binom{n}{3}} \prod_{1\le i<j\le n} x_j \theta\left(x_i/x_j, t/x_i x_j; p\right) \prod_{i=1}^{n} \theta(q^{2-n}/t_1 t_2 t, t_1 q^{n+2-2i}/t_2; p; q)_{i-1}.$$

Equating the resulting expression in (5.6) with the right-hand side of (5.3), we get the desired integral evaluation (5.2). □

This is the first nontrivial multiple elliptic beta integral with a complete proof. It is not symmetric in p and q, unlike all other cases considered in the present paper. This fact suggests that there should exist yet another integral of similar nature that would be symmetric in p, q.

The above method of computation of the C_n integral taken represents a next step in the logical development of applications of determinant formulas to multiple basic hypergeometric series; see, e.g., the paper [GK] of Gustafson and Krattenthaler, which was followed by Schlosser [Sc1, Sc2] and Warnaar [Wa]. Similar considerations for computing some multiple q-hypergeometric integrals were given by Tarasov and Varchenko in [TV].

§6. An elliptic beta integral for the A_n root system

In this section we conjecture a multiple elliptic beta integral for the A_n root system, which will be used in the next section for derivation of other nontrivial A_n integrals.

Conjecture. *Let z_i, $i = 1, \dots, n$, t_k, $k = 1, \dots, n+1$, and f_j, $j = 1, \dots, n+2$, be independent complex variables (n is an arbitrary positive integer). We denote $A \equiv \prod_{k=1}^{n+1} t_k$, $B \equiv \prod_{j=1}^{n+2} f_j$, and*

$$\Delta^I(\mathbf{z}; A_n) = \frac{1}{(2\pi i)^n} \frac{\prod_{k=1}^{n+1} \left(\prod_{i=1}^{n+1} \Gamma(t_i z_k^{-1}) \prod_{j=1}^{n+2} \Gamma(f_j z_k)\right)}{\prod_{i,j=1; i\ne j}^{n+1} \Gamma(z_i z_j^{-1}) \prod_{k=1}^{n+1} \Gamma(A B z_k)}, \tag{6.1}$$

where $z_1 z_2 \cdots z_{n+1} = 1$.

Suppose that the parameters t_k, f_j satisfy the constraints $|t_k|$, $|f_j| < 1$, $|pq| < |AB|$. Then the following integration formula is conjectured to hold true:

$$\begin{aligned}\int_{\mathbb{T}^n} \Delta^I(\mathbf{z}; A_n)\, \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} \\ = \frac{(n+1)!}{(q;q)_\infty^n (p;p)_\infty^n} \frac{\Gamma(A) \prod_{j=1}^{n+2} \Gamma(f_j^{-1}B) \prod_{k=1}^{n+1} \prod_{j=1}^{n+2} \Gamma(t_k f_j)}{\prod_{k=1}^{n+1} \Gamma(t_k B) \prod_{j=1}^{n+2} \Gamma(f_j^{-1}AB)}.\end{aligned} \tag{6.2}$$

For $n = 1$ this conjecture is reduced to the elliptic beta integral (1.13). For arbitrary n and $p = 0$, we get a Gustafson integral proved in [G1]. Let us show that the two sides of (6.2) satisfy one and the same difference equation.

Theorem 4. *Let $I_n(\mathbf{t}, \mathbf{f})$ denote either side of* (6.2)*. Then this function satisfies the q-difference equation*

$$\sum_{r=1}^{n+1} \frac{\theta(Bt_r; p)}{\theta(A; p)} \prod_{\substack{j=1 \\ j \neq r}}^{n+1} \frac{\theta(ABt_j; p)}{\theta(t_r t_j^{-1}; p)} I_n(t_1, \ldots, qt_r, \ldots, t_{n+1}, \mathbf{f}) = I_n(\mathbf{t}, \mathbf{f}) \tag{6.3}$$

and its counterpart obtained by the permutation of q and p.

Proof. Denote the function (6.1) by $\Delta^I(\mathbf{z}; t_1, \ldots, t_{n+1}; A_n)$. It is not difficult to see that

$$\frac{\Delta^I(\mathbf{z}; t_1, \ldots, qt_r, \ldots, t_{n+1}; A_n)}{\Delta^I(\mathbf{z}; t_1, \ldots, t_{n+1}; A_n)} = \prod_{k=1}^{n+1} \frac{\theta(t_r z_k^{-1}; p)}{\theta(ABz_k; p)},$$

so that equation (6.3) for the left-hand side of (6.2) is satisfied if the following theta function identity is fulfilled:

$$\sum_{r=1}^{n+1} \frac{\theta(Bt_r; p)}{\theta(A; p)} \prod_{\substack{j=1 \\ j \neq r}}^{n+1} \frac{\theta(ABt_j; p)}{\theta(t_r t_j^{-1}; p)} \prod_{k=1}^{n+1} \frac{\theta(t_r z_k^{-1}; p)}{\theta(ABz_k; p)} = 1. \tag{6.4}$$

For $n = 1$ this identity is equivalent to the well-known relation for products of four theta functions

$$\begin{aligned}\theta(xw, x/w, yz, y/z; p) - \theta(xz, x/z, yw, y/w; p) \\ = yw^{-1}\theta(xy, x/y, wz, w/z; p),\end{aligned} \tag{6.5}$$

and equation (6.3) coincides with the one used in [S2] for proving the integral (1.13).

For $n > 1$, identity (6.4) can be established with the help of the Liouville theorem, much as in the arguments presented in [DS2]. A simpler proof follows from the general theta function identity given in [WW]. As was shown by Rosengren in [Ro2], that identity can be rewritten as the following generalized partial fraction expansion of a ratio of theta functions:

$$\prod_{k=1}^{n} \frac{\theta(t/b_k; p)}{\theta(t/a_k; p)} = \sum_{r=1}^{n} \frac{\theta(ta_1 \cdots a_n / a_r b_1 \cdots b_n; p)}{\theta(t/a_r, a_1 \cdots a_n / b_1 \cdots b_n; p)} \frac{\prod_{j=1}^{n} \theta(a_r/b_j; p)}{\prod_{\substack{j=1 \\ j \neq r}}^{n} \theta(a_r/a_j; p)}, \tag{6.6}$$

where $a_1 \cdots a_n \neq b_1 \cdots b_n$. Here we replace n by $n+1$ and substitute $a_k = t_k^{-1}$, $b_k = z_k^{-1}$, and $t = AB$. As a result, we get an identity which is seen to coincide with (6.4) due to the relation $z_1 \cdots z_{n+1} = 1$.

In a similar way, for the right-hand side of (6.2) we get

$$\frac{I_n(t_1, \ldots, qt_r, \ldots, t_{n+1}, \mathbf{f})}{I_n(\mathbf{t}, \mathbf{f})} = \frac{\theta(A; p)}{\theta(t_r B; p)} \prod_{j=1}^{n+2} \frac{\theta(t_r f_j; p)}{\theta(ABf_j^{-1}; p)},$$

and in this case equation (6.3) is equivalent to the identity

$$\frac{\prod_{j=1}^{n+2}\theta(ABf_j^{-1};p)}{\prod_{j=1}^{n+1}\theta(ABt_j;p)}=\sum_{r=1}^{n+1}\frac{\prod_{j=1}^{n+2}\theta(t_rf_j;p)}{\prod_{\substack{j=1\\ j\neq r}}^{n+1}\theta(t_rt_j^{-1};p)}\frac{1}{\theta(ABt_r;p)}. \tag{6.7}$$

If we substitute here $f_{n+2}=B/f_1\cdots f_{n+1}$ and divide both sides by $\theta(\frac{AB}{f_{n+2}};p)$, then we get (6.6) with n replaced by $n+1$ and with $a_j=t_j^{-1}, b_j=f_j, t=AB$. □

The equation derived above works in the space of parameters t_k, whereas in [G1] Gustafson used an equation in the variables f_j for proving the $p=0$ case of the integral (6.2). It is of interest that the latter equation does not admit a straightforward elliptic generalization, namely, the corresponding partial fraction expansion cannot be lifted to the theta functions level.

Another argument in favor of the validity of formula (6.2) consists in the fact that, via the residue calculus, it generates a multivariable ${}_{10}E_9$ elliptic hypergeometric series sum for the A_n root system, which was proved by Rosengren in [Ro2] and which was considered independently by the author in [S7].

Theorem 5. *The residue calculus for the integral* (6.2) *yields the following summation formula*:

$$\begin{aligned}\sum_{\substack{0\le\lambda_j\le N_j\\ j=1,\dots,n}} q^{\sum_{j=1}^n j\lambda_j}\prod_{j=1}^n\frac{\theta(t_jq^{\lambda_j+|\lambda|};p)}{\theta(t_j;p)}\prod_{1\le i<j\le n}\frac{\theta(t_it_j^{-1}q^{\lambda_i-\lambda_j};p)}{\theta(t_it_j^{-1};p)}\\ \times\prod_{i,j=1}^n\frac{\theta(t_it_j^{-1}q^{-N_j};p;q)_{\lambda_i}}{\theta(qt_it_j^{-1};p;q)_{\lambda_i}}\prod_{j=1}^n\frac{\theta(t_j;p;q)_{|\lambda|}}{\theta(t_jq^{1+N_j};p;q)_{|\lambda|}}\\ \times\frac{\theta(b,c;p;q)_{|\lambda|}}{\theta(q/d,q/e;p;q)_{|\lambda|}}\prod_{j=1}^n\frac{\theta(dt_j,et_j;p;q)_{\lambda_j}}{\theta(t_jq/b,t_jq/c;p;q)_{\lambda_j}}\\ =\frac{\theta(q/bd,q/cd;p;q)_{|N|}}{\theta(q/d,q/bcd;p;q)_{|N|}}\prod_{j=1}^n\frac{\theta(t_jq,t_jq/bc;p;q)_{N_j}}{\theta(t_jq/b,t_jq/c;p;q)_{N_j}},\end{aligned} \tag{6.8}$$

where $|\lambda|=\lambda_1+\cdots+\lambda_n$, $|N|=N_1+\cdots+N_n$, *and* $bcde=q^{1+|N|}$. *For* $n=1$ *this is the Frenkel–Turaev sum* [FT], *and for* $p=0$ *it is reduced to Milne's multiple* ${}_8\Phi_7$ *sum for the* A_n *root system* [Mi1].

Proof. We scale t_i for $i=1,\dots,n$ from the region $|t_i|<1$ to $|t_i|>1$, and keep $|t_{n+1}|,|f_j|<1$ together with the condition $|pq|<|AB|$. During this procedure, some poles of the integrand $\Delta^I(\mathbf{z};A_n)$ in (6.2) go out of the unit disk and, vice versa, some of them cross over $\mathbb{T}$ entering inside. The outgoing poles are located at the following points: $z_k=\{t_iq^{\lambda_i}, i=1,\dots,n\}$ for each $k=1,\dots,n$, and the number of such poles is determined by the conditions $|t_iq^{\lambda_i}|>1$. The ingoing poles correspond to the points $z_1\cdots z_n=\{t_i^{-1}q^{-\lambda_i}, i=1,\dots,n\}$.

We denote by C a deformed contour of integration such that none of the poles mentioned above crosses over C during the change of parameters. By analyticity, the value of the integral (6.2) is not changing when C replaces $\mathbb{T}$, that is, the right-hand side of (6.2) remains the same. If we deform the contour C back to $\mathbb{T}$, we begin to pick up residues from the poles by the Cauchy theorem. As a result, the following formula arises:

$$\int_{C^n}\Delta^I(\mathbf{z};A_n)\prod_{k=1}^n\frac{dz_k}{z_k}=\sum_{j=0}^n\int_{\mathbb{T}^j}R_j(z_1,\dots,z_j)\prod_{k=1}^j\frac{dz_k}{z_k}, \tag{6.9}$$

where $R_n = \Delta^I(\mathbf{z}; A_n)$, and $R_j(z_1, \dots, z_j)$ for $j < n$ are sums of the residues of $\Delta^I(\mathbf{z}; A_n)$ corresponding to the poles crossing C.

We shall not derive explicit expressions for all coefficients R_j as it was done for the C_n integrals in [DS1, DS3]. For our purposes, it suffices to pick up only the residues that diverge in the limits $f_j \to q^{-N_j} t_j^{-1}$, $N_j \in \mathbb{N}$, for all $j = 1, \dots, n$ simultaneously. First, consider the residues appearing from the poles $z_j = t_j q^{\lambda_j}$, where the λ_j are some integers such that $|t_j q^{\lambda_j}| > 1$. Straightforward computations yield

$$
\begin{aligned}
R_0^{\mathrm{div}}(\lambda) &\equiv \prod_{j=1}^{n} \lim_{z_j \to t_j q^{\lambda_j}} (1 - t_j q^{\lambda_j} z_j^{-1}) \Delta^I(\mathbf{z}; A_n) \\
&= \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \Gamma^{-1}(t_i t_j^{-1} q^{\lambda_i - \lambda_j}) \\
&\quad \times \frac{\prod_{i=1}^{n+1} \Gamma(t_i D q^{|\lambda|}) \prod_{j=1}^{n+2} \Gamma(f_j D^{-1} q^{-|\lambda|})}{\prod_{k=1}^{n} \Gamma(t_k q^{\lambda_k} D q^{|\lambda|}, D^{-1} q^{-|\lambda|} t_k^{-1} q^{-\lambda_k})} \\
&\quad \times \frac{\prod_{k=1}^{n} \left(\prod_{\substack{i=1 \\ i \neq k}}^{n+1} \Gamma(t_i t_k^{-1} q^{-\lambda_k}) \prod_{j=1}^{n+2} \Gamma(f_j t_k q^{\lambda_k}) \right)}{\prod_{k=1}^{n} \Gamma(ABt_k q^{\lambda_k}) \Gamma(ABD^{-1} q^{-|\lambda|})} \\
&\quad \times \prod_{k=1}^{n} \frac{(-1)^{\lambda_k} q^{\lambda_k(\lambda_k+1)/2}}{(q;q)_\infty (p;p)_\infty \theta(q;p;q)_{\lambda_k}},
\end{aligned}
\tag{6.10}
$$

where $D = A/t_{n+1}$. The factors $\Gamma(f_j t_j q^{\lambda_j})$ provide the required divergence in the limits $f_j \to q^{-N_j} t_j^{-1}$. We write $R_0^{div}(\lambda) = \kappa_n \Delta(\lambda; A_n)$, where

$$
\begin{aligned}
\kappa_n = &\prod_{1 \le i < j \le n} \Gamma^{-1}(t_i t_j^{-1}, t_i^{-1} t_j) \frac{\prod_{i=1}^{n+1} \Gamma(t_i D) \prod_{j=1}^{n+2} \Gamma(f_j D^{-1})}{\prod_{k=1}^{n} \Gamma(t_k D, t_k^{-1} D^{-1})} \\
&\times \frac{\prod_{k=1}^{n} \left(\prod_{\substack{i=1 \\ i \neq k}}^{n+1} \Gamma(t_i t_k^{-1}) \prod_{j=1}^{n+2} \Gamma(f_j t_k) \right)}{(q;q)_\infty^n (p;p)_\infty^n \prod_{k=1}^{n} \Gamma(ABt_k) \Gamma(ABD^{-1})},
\end{aligned}
$$

and after a chain of simplifying calculations, $\Delta(\lambda; A_n)$ takes the form

$$
\begin{aligned}
\Delta(\lambda; A_n) = q^{\sum_{j=1}^{n} j\lambda_j} &\prod_{1 \le i < j \le n} \frac{\theta(t_i t_j^{-1} q^{\lambda_i - \lambda_j}; p)}{\theta(t_i t_j^{-1}; p)} \\
&\times \prod_{j=1}^{n} \left(\frac{\theta(t_j D; p; q)_{|\lambda|}}{\theta(q f_j^{-1} D; p; q)_{|\lambda|}} \frac{\theta(t_j D q^{|\lambda| + \lambda_j}; p)}{\theta(t_j D; p)} \prod_{k=1}^{n} \frac{\theta(f_j t_k; p; q)_{\lambda_k}}{\theta(q t_j^{-1} t_k; p; q)_{\lambda_k}} \right) \\
&\times \frac{\theta(cD, qD/AB; p; q)_{|\lambda|}}{\theta(qD/d, qD/e; p; q)_{|\lambda|}} \prod_{k=1}^{n} \frac{\theta(dt_k, et_k; p; q)_{\lambda_k}}{\theta(qt_k/c, ABt_k; p; q)_{\lambda_k}},
\end{aligned}
$$

where we have denoted $c = t_{n+1}, d = f_{n+1}, e = f_{n+2}$. Now we substitute $f_j = q^{-N_j} t_j^{-1}$ (which assumes that $AB = q^{-|N|} cde$) in this expression and introduce the parameter $b \equiv q^{1+|N|}/cde$. As a result, the function $\Delta(\lambda; A_n)$ becomes equal to the summand on the left-hand side of (6.8) after the transformations $t_j \mapsto t_j/D$, $b \mapsto b/D$, $c \mapsto c/D$, $d \mapsto Dd$, $e \mapsto De$.

Now, we find the total number of residues of such type. There is permutational symmetry between the variables $z_1, \dots, z_n$. Therefore, there are $n!$ ways to satisfy the equations $z_k = t_j q^{\lambda_j}$ using each t_j only once. The residues of the other outgoing poles

located at $z_k = \{t_i q^{\lambda_i}, i = 1, \dots, n\}$, where at least one t_i enters twice, do not diverge as $f_j \to q^{-N_j} t_j^{-1}$ for some j.

We pass to the ingoing poles. It is not difficult to verify that the residues of $\Delta^I(\mathbf{z}; A_n)$ for the poles located at $z_n = t_j^{-1} q^{-\lambda_j}/z_1 \cdots z_{n-1}$ for some fixed j (or, equivalently, for $z_{n+1} = t_j q^{\lambda_j}$) are equal to the residues for the poles at $z_n = t_j q^{\lambda_j}$. Among the remaining poles in the variables $z_1, \dots, z_{n-1}$, we must consider only the outgoing ones because only they may diverge as $f_k \to q^{-N_k} t_k^{-1}$ with $k = 1, \dots, n, k \neq j$. There are n ways to fix the variable z_k for which we shall consider ingoing poles, there are n ways to fix the parameter t_j in the equation $z_{n+1} = t_j q^{\lambda_j}$, and there are $(n-1)!$ appropriate outgoing poles with the required residue divergence. As a result, the contribution of these combined ingoing and outgoing poles is equal to $n^2(n-1)!$, and the total number of the diverging residues (6.10) is equal to $(n+1)!$. Roughly speaking, the incoming poles imitate the $(n+1)$st independent contour of integration over z_{n+1}, which enters symmetrically with $z_1, \dots, z_n$, and there are $(n+1)!$ ways to order these variables in the residue calculus.

As has already been mentioned, (6.9) is equal to the right-hand side of (6.2). Now we divide both these expressions by $(n+1)!\kappa_n$ and take the limits as $f_j \to q^{-N_j} t_j^{-1}, j = 1, \dots, n$. Since $\kappa_n \to \infty$ in this limit, only the residues considered above survive in (6.9), and their sum is given by the elliptic Milne series (6.8). As to the right-hand side, we get

$$\lim_{f_j \to q^{-N_j} t_j^{-1}} \frac{\text{r.h.s. of (6.2)}}{(n+1)!\kappa_n} = \frac{\theta(q/bd, q/cd; p; q)_{|N|}}{\theta(qD/d, q/Dbcd; p; q)_{|N|}} \prod_{j=1}^{n} \frac{\theta(qt_j D, qt_j/Dbc; p; q)_{N_j}}{\theta(qt_j/c, qt_j/b; p; q)_{N_j}},$$

which coincides with the right-hand side of (6.8) after the appropriate changes of parameters indicated above. The theorem is proved. □

Formula (6.2) generates the following symmetry transformation for integrals:

$$\begin{aligned} &\prod_{j=1}^{n+2} \frac{\Gamma(Bf_j^{-1})}{\Gamma(t^{n+1}Bf_j^{-1})} \int_{\mathbb{T}^n} \frac{\prod_{k=1}^{n+1}\prod_{j=1}^{n+2}\Gamma(tf_j z_k^{-1}, s_j z_k)\prod_{j=1}^{n} dz_j/z_j}{\prod_{\substack{i,j=1\\ i\neq j}}^{n+1}\Gamma(z_i z_j^{-1})\prod_{k=1}^{n+1}\Gamma(t^{n+1}Sz_k, tBz_k^{-1})} \\ &\quad = \prod_{j=1}^{n+2} \frac{\Gamma(Ss_j^{-1})}{\Gamma(t^{n+1}Ss_j^{-1})} \int_{\mathbb{T}^n} \frac{\prod_{k=1}^{n+1}\prod_{j=1}^{n+2}\Gamma(ts_j z_k^{-1}, f_j z_k)\prod_{j=1}^{n} dz_j/z_j}{\prod_{\substack{i,j=1\\ i\neq j}}^{n+1}\Gamma(z_i z_j^{-1})\prod_{k=1}^{n+1}\Gamma(t^{n+1}Bz_k, tSz_k^{-1})}. \end{aligned} \tag{6.11}$$

Here $t, f_j, s_j, j = 1, \dots, n+2$, are free independent variables, $B = \prod_{j=1}^{n+2} f_j, S = \prod_{j=1}^{n+2} s_j$, and it is assumed that $|t|, |f_j|, |s_j| < 1$, $|pq| < |t^{n+1}B|, |t^{n+1}S|$. In order to derive this identity, it is necessary to consider the $2n$-tuple integral

$$\begin{aligned} &\frac{1}{(2\pi i)^n} \int_{\mathbb{T}^{2n}} \frac{\prod_{k=1}^{n+1}\prod_{j=1}^{n+2}\Gamma(f_j z_k, s_j w_k^{-1})\prod_{i,k=1}^{n+1}\Gamma(tz_k^{-1}w_i)}{\prod_{\substack{i,j=1\\ i\neq j}}^{n+1}\Gamma(z_i z_j^{-1}, w_i w_j^{-1})\prod_{k=1}^{n+1}\Gamma(t^{n+1}Bz_k, t^{n+1}Sw_k^{-1})} \\ &\qquad \times \frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}\frac{dw_1}{w_1}\cdots\frac{dw_n}{w_n}, \end{aligned}$$

where $z_1 \cdots z_{n+1} = w_1 \cdots w_{n+1} = 1$. Integration with respect to the variables z_k with the help of (6.2) makes this expression proportional to the left-hand side of (6.11) (after the replacements $w_k \mapsto z_k^{-1}$). Changing the order of integration, which is allowed because the integrand is bounded on $\mathbb{T}$, we arrive at the integral standing on the right-hand side of (6.11), up to some coefficient. After cancelling common factors, we get the required identity. For $p = 0$ this reduces to the Denis–Gustafson transformation formula

[DG], which describes a Bailey transformation for a terminating multivariable ${}_{10}\Phi_9$ series. It is natural to expect that in our case formula (6.11) yields a Bailey transformation for some terminating A_n multiple ${}_{12}E_{11}$ series appearing from sums of residues of the corresponding integrals.

§7. Some other A_n integrals

Now from conjecture (6.2) we derive a number of different multiple A_n-integrals. Denote

$$\Delta^{II}(\mathbf{z};A_n) = \frac{1}{(2\pi i)^n}\prod_{1\le i<j\le n+1}\frac{\Gamma(tz_iz_j, sz_i^{-1}z_j^{-1})}{\Gamma(z_iz_j^{-1}, z_i^{-1}z_j)}\prod_{j=1}^{n+1}\frac{\Gamma(t_1z_j,t_2z_j,t_3z_j,t_4z_j^{-1},t_5z_j^{-1})}{\Gamma(z_j(ts)^{n-1}\prod_{j=1}^5 t_j)}. \tag{7.1}$$

Theorem 6. *Suppose the validity of the conjectured A_n and C_n multiple elliptic beta integrals* (6.2) *and* (4.4), *respectively. Then the following two integration formulas are true. For odd $n=2m-1$, we have*

$$\begin{aligned}\int_{\mathbb{T}^n}\Delta^{II}(\mathbf{z};A_n)\,\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n} &= \frac{(n+1)!}{(q;q)_\infty^n(p;p)_\infty^n}\frac{\Gamma(t^m,s^m,s^{m-1}t_4t_5)\prod_{1\le i<j\le 3}\Gamma(t^{m-1}t_it_j)}{\prod_{k=4}^5\Gamma(t^{2m-2}s^{m-1}t_1t_2t_3t_k)}\\ &\times\prod_{j=1}^m\frac{\prod_{i=1}^3\prod_{k=4}^5\Gamma((ts)^{j-1}t_it_k)}{\prod_{1\le i<\ell\le 3}\Gamma((ts)^{m+j-2}t_it_\ell t_4t_5)}\\ &\times\prod_{j=1}^{m-1}\frac{\Gamma((ts)^j,t^js^{j-1}t_4t_5)\prod_{1\le i<\ell\le 3}\Gamma(t^{j-1}s^jt_it_\ell)}{\prod_{k=4}^5\Gamma(t^{m+j-2}s^{m+j-1}t_1t_2t_3t_k)}.\end{aligned} \tag{7.2}$$

For even $n=2m$, we have

$$\begin{aligned}\int_{\mathbb{T}^n}\Delta^{II}(\mathbf{z};A_n)\,\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n} &= \frac{(n+1)!}{(q;q)_\infty^n(p;p)_\infty^n}\\ &\times\frac{\prod_{i=1}^3\Gamma(t^mt_i)\prod_{k=4}^5\Gamma(s^mt_k)\Gamma(t^{m-1}t_1t_2t_3)}{\Gamma(t^{2m-1}s^{m-1}\prod_{i=1}^5 t_i, t^{2m-1}s^mt_1t_2t_3)}\\ &\times\prod_{j=1}^m\frac{\Gamma((ts)^j,t^js^{j-1}t_4t_5)\prod_{i=1}^3\prod_{k=4}^5\Gamma((ts)^{j-1}t_it_k)}{\prod_{k=4}^5\Gamma(t^{m+j-2}s^{m+j-1}t_k^{-1}\prod_{i=1}^5 t_i)}\\ &\times\prod_{j=1}^m\prod_{1\le i<\ell\le 3}\frac{\Gamma(t^{j-1}s^jt_it_\ell)}{\Gamma((ts)^{m+j-1}t_it_\ell t_4t_5)}.\end{aligned} \tag{7.3}$$

Proof. The proofs follow the procedure used by Gustafson [G2] for proving the $p=0$ cases of the integrals (7.2) and (7.3). We start with the case of odd $n=2m-1$. Consider

the following $(4m-1)$-tuple integral:

$$\begin{aligned}
&\int_{\mathbb{T}^{4m-1}} \frac{\prod_{i=1}^{2m}\prod_{j=1}^{m}\Gamma(t^{1/2}z_iw_j, t^{1/2}z_iw_j^{-1}, s^{1/2}z_i^{-1}x_j, s^{1/2}z_i^{-1}x_j^{-1})}{\prod_{i,j=1;i\neq j}^{2m}\Gamma(z_iz_j^{-1})\prod_{\nu=\pm1}\prod_{1\le i<j\le m}\Gamma(w_i^\nu w_j^\nu, w_i^\nu w_j^{-\nu}, x_i^\nu x_j^\nu, x_i^\nu x_j^{-\nu})} \\
&\quad\times\prod_{i=1}^{2m}\frac{\Gamma(z_i(ts)^{m-2}\prod_{k=1}^5 t_k)}{\Gamma(z_i(ts)^{2m-2}\prod_{k=1}^5 t_k)}\prod_{k=1}^{2m-1}\frac{dz_k}{z_k} \\
&\quad\times\prod_{j=1}^{m}\left(\frac{dw_j}{w_j}\frac{dx_j}{x_j}\prod_{\nu=\pm1}\left(\frac{\prod_{k=1}^3\Gamma\left(\frac{t_kw_j^\nu}{t^{1/2}}\right)\Gamma(x_j^\nu t^{m-2}s^{-\frac12}t_1t_2t_3)\prod_{k=4}^5\Gamma\left(\frac{t_kx_j^\nu}{s^{1/2}}\right)}{\Gamma\left(w_j^\nu t^{m-\frac32}t_1t_2t_3, x_j^\nu t^{m-2}s^{m-\frac32}\prod_{k=1}^5 t_k, w_j^{2\nu}, x_j^{2\nu}\right)}\right)\right),
\end{aligned} \tag{7.4}$$

where $\prod_{i=1}^{2m} z_j = 1$. Using the exact C_n integration formula of type I (see (4.4)), we first take integrals in (7.4) with respect to the variables $w_j, j=1,\dots,m$, and after that with respect to $x_j, j=1,\dots,m$. The resulting integral is equal to the left-hand side of (7.2) up to the factor

$$\begin{aligned}
&(2\pi i)^{4m-1}\frac{2^{2m}(m!)^2}{(p;p)_\infty^{2m}(q;q)_\infty^{2m}}\frac{\Gamma(s^{-1}t_4t_5)}{\Gamma(s^{m-1}t_4t_5)} \\
&\quad\times\prod_{1\le i<k\le 3}\frac{\Gamma(t^{-1}t_it_k)}{\Gamma(t^{m-1}t_it_k)}\prod_{k=4,5}\frac{\Gamma(t^{m-2}s^{-1}t_k^{-1}\prod_{i=1}^5 t_k)}{\Gamma(t^{m-2}s^{m-1}t_k^{-1}\prod_{i=1}^5 t_k)}.
\end{aligned}$$

In this two step procedure, we need the following restrictions upon the parameters:

$$|t|<1,\quad |t_{1,2,3}|<|t|^{1/2},\quad |pq|<|t^{m-3/2}t_1t_2t_3|$$

and

$$|s|<1,\quad |t_{4,5}|<|s|^{1/2},\quad |pq|<\left|t^{m-2}s^{m-3/2}\prod_{k=1}^5 t_k\right|,$$

respectively. However, the resulting expression can be extended analytically to the region $|t_k|<1$, $k=1,\dots,5$, $|pq|<|(ts)^{2m-2}\prod_{k=1}^5 t_k|$ without changing the integral value.

Since the integrand in (7.4) is bounded on the unit circle, we can change the order of integrations. First, we take the integrals over z_i, $i=1,\dots,2m-1$, using the A_n-formula (6.2). Then we apply formula (4.4) in order to take the integrals over $x_j, j=1,\dots,m$. Finally, we use the intrinsic elliptic Selberg integral (4.5) for taking the integrals over w_j, $j=1,\dots,m$,; this leads to the following expression:

$$\begin{aligned}
&\frac{(2\pi i)^{4m-1}(2m)!(2^m m!)^2}{((p;p)_\infty(q;q)_\infty)^{4m-1}}\frac{\Gamma(t^m, s^m, t_4t_5s^{-1})}{\Gamma((ts)^m, t^ms^{m-1}t_4t_5)}\prod_{k=4}^5\frac{\Gamma(t^{m-2}s^{-1}t_k^{-1}\prod_{i=1}^5 t_i)}{\Gamma(t^{2m-2}s^{m-1}t_k^{-1}\prod_{i=1}^5 t_i)} \\
&\times\prod_{j=1}^{m}\frac{\Gamma((ts)^j, t^js^{j-1}t_4t_5)\prod_{1\le i<k\le 3}\Gamma(t^{j-2}s^{j-1}t_it_k)\prod_{k=4}^5\prod_{i=1}^3\Gamma((ts)^{j-1}t_kt_i)}{\prod_{k=1}^3\Gamma((ts)^{m+j-2}t_k^{-1}\prod_{i=1}^5 t_i)\prod_{k=4}^5\Gamma(t^{m+j-3}s^{m+j-2}t_k^{-1}\prod_{i=1}^5 t_i)}.
\end{aligned}$$

As a result, we get the needed integration formula for odd $n=2m-1$.

In order to prove (7.3), we consider the $4m$-tuple integral

$$\begin{aligned}
&\int_{\mathbb{T}^{4m}} \frac{\prod_{i=1}^{2m+1}\prod_{j=1}^{m}\Gamma(t^{1/2}z_iw_j, t^{1/2}z_iw_j^{-1}, s^{1/2}z_i^{-1}x_j, s^{1/2}z_i^{-1}x_j^{-1})}{\prod_{i,j=1;i\neq j}^{2m+1}\Gamma(z_iz_j^{-1})\prod_{\nu=\pm1}\prod_{1\le i<j\le m}\Gamma(w_i^\nu w_j^\nu, w_i^\nu w_j^{-\nu}, x_i^\nu x_j^\nu, x_i^\nu x_j^{-\nu})} \\
&\quad\times\prod_{i=1}^{2m+1}\frac{\Gamma(t_3z_i, s^{m-1}t_4t_5z_i, t^{m-1}t_1t_2z_i^{-1})}{\Gamma(z_i(ts)^{2m-1}\prod_{k=1}^5 t_k)} \\
&\quad\times\prod_{j=1}^{m}\left(\frac{dw_j}{w_j}\frac{dx_j}{x_j}\prod_{\nu=\pm1}\left(\frac{\prod_{k=1}^2\Gamma(t^{-1/2}t_kw_j^\nu, s^{-1/2}t_{k+3}x_j^\nu)}{\Gamma\left(t^{m-\frac12}t_1t_2w_j^\nu, s^{m-\frac12}t_4t_5x_j^\nu, w_j^{2\nu}, x_j^{2\nu}\right)}\right)\right)\prod_{k=1}^{2m}\frac{dz_k}{z_k},
\end{aligned} \tag{7.5}$$

where $\prod_{i=1}^{2m+1} z_j = 1$. Repeating the same trick as in the case of odd n (that is, integrating successively with respect to the variables w_j and x_j and then changing the order of integrations in this expression), we get (7.3). $\square$

In a similar way, we can establish elliptic analogs of the A_n basic hypergeometric integrals of Gustafson and Rakha [GuR].

Theorem 7. *Suppose the validity of the A_n and C_n multiple elliptic beta integrals* (6.2) *and* (4.4), *respectively. Denote*

(7.6)
$$\Delta^{III}(\mathbf{z}; A_n) = \frac{1}{(2\pi i)^n} \prod_{1\le i<j\le n+1} \frac{\Gamma(tz_iz_j)}{\Gamma(z_iz_j^{-1}, z_i^{-1}z_j)} \frac{\prod_{i=1}^{n+1}\Big(\prod_{k=1}^{n+1}\Gamma(t_kz_i^{-1})\prod_{k=n+2}^{n+4}\Gamma(tt_kz_i)\Big)}{\prod_{j=1}^{n+1}\Gamma(Az_j^{-1})},$$

where $A = t^{n+2}\prod_{i=1}^{n+4} t_i$ and $\prod_{j=1}^{n+1} z_j = 1$. Then the following two integration formulas are true. For odd $n = 2l-1$, we have

$$\begin{aligned}\int_{\mathbb{T}^n} \Delta^{III}(\mathbf{z}; A_n)\,\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n} &= \frac{(n+1)!}{(q;q)_\infty^n (p;p)_\infty^n}\,\frac{\Gamma(t^l, \prod_{k=1}^{2l} t_k)}{\Gamma(t^l\prod_{k=1}^{2l} t_k)}\\ &\times \frac{\prod_{i=1}^{2l}\prod_{j=2l+1}^{2l+3}\Gamma(tt_it_j)\prod_{1\le i<j\le 2l}\Gamma(tt_it_j)\prod_{2l+1\le i<j\le 2l+3}\Gamma(t^{l+1}t_it_j)}{\prod_{i=1}^{2l}\Gamma(t^{2l+1}t_i^{-1}\prod_{k=1}^{2l+3} t_k)\prod_{i=2l+1}^{2l+3}\Gamma(t^{l+1}t_i^{-1}\prod_{k=1}^{2l+3} t_k)}.\end{aligned} \tag{7.7}$$

For even $n = 2l$, we have

$$\begin{aligned}\int_{\mathbb{T}^n} \Delta^{III}(\mathbf{z}; A_n)\,\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n} &= \frac{(n+1)!}{(q;q)_\infty^n (p;p)_\infty^n}\,\frac{\Gamma(\prod_{k=1}^{2l+1} t_k, t^{l+2}\prod_{k=2l+2}^{2l+4} t_k)}{\Gamma(t^{l+2}\prod_{k=1}^{2l+4} t_k)}\\ &\times \frac{\prod_{i=1}^{2l+1}\prod_{j=2l+2}^{2l+4}\Gamma(tt_it_j)\prod_{1\le i<j\le 2l+1}\Gamma(tt_it_j)\prod_{i=2l+2}^{2l+4}\Gamma(t^{l+1}t_i)}{\prod_{i=1}^{2l+1}\Gamma(t^{2l+2}t_i^{-1}\prod_{k=1}^{2l+4} t_k)\prod_{i=2l+2}^{2l+4}\Gamma(t^{l+1}t_i\prod_{k=1}^{2l+1} t_k)}.\end{aligned} \tag{7.8}$$

Proof. In accordance with the procedure used in [GuR], we consider the $(3l-1)$-tuple integral

$$\begin{aligned}&\int_{\mathbb{T}^{3l-1}} \frac{\prod_{i=1}^{2l}\prod_{j=1}^{l}\Gamma(t^{1/2}z_iw_j, t^{1/2}z_iw_j^{-1})\prod_{i=0}^{2l}\prod_{j=1}^{2l}\Gamma(t_iz_j^{-1})}{\prod_{i,j=1;i\ne j}^{2l}\Gamma(z_iz_j^{-1})\prod_{j=1}^{2l}\Gamma(t^l\prod_{i=0}^{2l} t_iz_j^{-1})}\\ &\quad\times \prod_{\nu=\pm1}\prod_{1\le i<j\le l}\Gamma^{-1}(w_i^\nu w_j^\nu, w_i^\nu w_j^{-\nu})\prod_{j=1}^{l}\frac{\prod_{k=2l+1}^{2l+3}\Gamma(t^{1/2}t_kw_j^\nu)}{\Gamma(w_j^{2\nu}, t^{l+3/2}\prod_{k=2l+1}^{2l+3} t_kw_j^\nu)}\\ &\quad\times \frac{dw_1}{w_1}\cdots\frac{dw_l}{w_l}\frac{dz_1}{z_1}\cdots\frac{dz_{2l-1}}{z_{2l-1}},\end{aligned}$$

where $\prod_{i=1}^{2l} z_i = 1$ and $t_0 = t^{l+1}\prod_{k=2l+1}^{2l+3} t_k$. Integrating with respect to the variables w_j with the help of formula (4.4), we get the left-hand side of (7.7) up to some factor. Changing the order of integration, we can integrate over z_i using (6.2) (where it is necessary to change z_k to z_k^{-1}) and then over w_j using (4.4). Equating two expressions, we arrive at formula (7.7).

In a similar way, in the case of even $n=2l$ we consider the $(3l+1)$-tuple integral

$$\int_{\mathbb{T}^{3l+1}} \frac{\prod_{i=1}^{2l+1}\Big(\prod_{j=1}^{l+1}\Gamma(t^{1/2}z_iw_j,t^{1/2}z_iw_j^{-1})\prod_{j=1}^{2l+1}\Gamma(t_iz_j^{-1})\Big)}{\prod_{i,j=1;i\neq j}^{2l+1}\Gamma(z_iz_j^{-1})\prod_{j=1}^{2l+1}\Gamma(t^{l+1}\prod_{i=1}^{2l+1}t_iz_j)}$$
$$\times\prod_{\nu=\pm1}\prod_{1\le i<j\le l+1}\Gamma^{-1}(w_i^\nu w_j^\nu,w_i^\nu w_j^{-\nu})\prod_{j=1}^{l+1}\frac{\prod_{k=2l+1}^{2l+5}\Gamma(t^{1/2}t_kw_j^\nu)}{\Gamma(w_j^{2\nu},t^{l+5/2}\prod_{k=2l+1}^{2l+5}t_kw_j^\nu)}$$
$$\times\frac{dw_1}{w_1}\cdots\frac{dw_{l+1}}{w_{l+1}}\frac{dz_1}{z_1}\cdots\frac{dz_{2l}}{z_{2l}},$$

where $\prod_{i=1}^{2l+1}z_i=1$ and $t_{2l+5}=t^l\prod_{k=1}^{2l+1}t_k$. Repeating the same trick as in the preceding case, we get the desired integration formula (7.8). □

Sums of residues for the derived integrals (7.2)–(7.8) form elliptic hypergeometric series on the A_n root system that differ from the series (6.8) introduced in [Ro2, S7]. We skip their consideration and formulate only a conjecture concerning the elliptic extension of Theorem 1.2 in [GuR].

Conjecture. *Suppose that N is a positive integer and $\prod_{k=1}^n t_k=q^{-N}$. Then*

$$\sum_{\substack{\lambda_k=0,\dots,N\\ \lambda_1+\cdots+\lambda_n=N}}\frac{\prod_{1\le i<j\le n}\theta(tt_it_j)_{\lambda_i+\lambda_j}\prod_{i=1}^n\prod_{j=n+1}^{n+3}\theta(tt_it_j)_{\lambda_i}\prod_{i,j=1}^n\theta(t_it_j^{-1})_{-\lambda_j}}{\prod_{i,j=1;i\neq j}^n\theta(t_it_j^{-1})_{\lambda_i-\lambda_j}\prod_{j=1}^n\theta(t^{n+1}t_j^{-1}\prod_{k=1}^{n+3}t_k)_{-\lambda_j}}$$
$$=\begin{cases}\dfrac{\theta(1)_{-N}}{\theta(t^{n/2})_{-N}\prod_{n+1\le i<j\le n+3}\theta(t^{(n+2)/2}t_it_j)_{-N}}, & n \text{ is even},\\[2ex] \dfrac{\theta(1)_{-N}}{\prod_{i=n+1}^{n+3}\theta(t^{(n+1)/2}t_i)_{-N}\theta(t^{(n+3)/2}\prod_{i=n+1}^{n+3}t_i)_{-N}}, & n \text{ is odd},\end{cases}\tag{7.9}$$

where $\theta(a)_\lambda\equiv\theta(a;p;q)_\lambda$.

These summation formulas are expected to follow from the residue calculus for the integrals (7.7) and (7.8). Some evidence in favor of (7.9) is provided by the following theorem.

Theorem 8. *Denote $t=q^g,t_i=q^{g_i},i=1,\dots,n+3$ (so that $\sum_{j=1}^n g_j+N=0$). The series $\sum_\lambda c(\lambda)$ standing on the left-hand side of (7.9) is a totally elliptic hypergeometric series, that is, the ratios of successive series coefficients*

$$h_k(\lambda)=\frac{c(\lambda_1,\dots,\lambda_k+1,\dots,\lambda_n)}{c(\lambda_1,\dots,\lambda_n)}$$

are elliptic functions of all unconstrained variables in the set $(\lambda_1,\dots,\lambda_n,g,g_1,\dots,g_{n+3})$. Moreover, the functions $h_k(\lambda)$ are $SL(2,\mathbb{Z})$ modular invariant. The ratios of the expressions standing on the two sides of (7.9) are elliptic functions of g and $n+2$ free parameters in the set $(g_1,\dots,g_{n+3})$, and these ratios are modular invariant as well.

We skip the proof of this theorem, because it consists of quite long but straightforward computations, whose structure was described in detail in [S5, S7] in the process of similar considerations for different elliptic hypergeometric series summation formulas. Using the fact that there are no cusp forms of weight less than 12, we deduce from this theorem, as in [DS1, S7], that relations (7.9) are valid in the small σ expansion up to the terms of the order of σ^{12}. This also gives yet another example in favor of the general conjecture of [S5] that all totally elliptic hypergeometric series are automatically modular invariant.

The integrals (7.2) and (7.3) are expected to generate A_n summation formulas similar to (7.9). It is also natural to expect that all multiple elliptic beta integrals described above lead to integral representations for various multiple ${}_{12}E_{11}$ elliptic hypergeometric series generalizing the single variable formula announced in [S4] (see Appendix B for the proof of it). We suppose that, as in the A_n-case, there exist several types of elliptic beta integrals and elliptic hypergeometric series sums associated with the D_n root system (see, e.g., [Ro2, S7]), but their consideration lies beyond the scope of the present paper.

§8. Relationships with the generalized eigenvalue problems

Consider the very-well-poised theta hypergeometric series (3.11) with the additional constraint $(-q)^n e^{P_3(n)} = (qx)^n$ for some $x \in \mathbb{C}$. Special notation for such series was introduced in [S6]:

$$ {}_{r+1}V_r(t_0; t_1, \dots, t_{r-4}; q, p; x) \equiv \sum_{n=0}^{\infty} \frac{\theta(t_0 q^{2n}; p)}{\theta(t_0; p)} \prod_{m=0}^{r-4} \frac{\theta(t_m; p; q)_n}{\theta(q t_0 t_m^{-1}; p; q)_n} (qx)^n. \tag{8.1} $$

For (8.1), the balancing condition (3.7) reduces to

$$ \prod_{m=1}^{r-4} t_m = t_0^{(r-5)/2} q^{(r-7)/2}. $$

As $p \to 0$, the ${}_{r+1}V_r$ series are reduced to ${}_{r-1}W_{r-2}$ very-well-poised q-hypergeometric series of the argument qx (in the notation of [GR]).

The balanced ${}_{12}V_{11}$ series with $x = 1$ plays an important role in applications. For instance, the elliptic solutions of the Yang–Baxter equation derived by Date et al. in [D-O1, D-O2] are expressed in terms of such series for a particular choice of the parameters [FT]. In all cases of the ${}_{12}V_{11}$ function to be considered below, we have $x = 1$; therefore, we omit the dependence on this unit argument from now on.

We denote $\mathcal{E}(\mathbf{t}) \equiv {}_{12}V_{11}(t_0; t_1, \dots, t_7; q, p)$, where $\prod_{m=1}^{7} t_m = t_0^3 q^2$, and assume that this series terminates due to the condition $t_m = q^{-n}$, $n \in \mathbb{N}$, for some m. In [SZ1, SZ2], the following two contiguous relations for $\mathcal{E}(\mathbf{t})$ were derived:

$$ \begin{aligned} &\mathcal{E}(\mathbf{t}) - \mathcal{E}(t_0; t_1, \dots, t_5, q^{-1} t_6, q t_7) \\ &\quad = \frac{\theta(q t_0, q^2 t_0, q t_7/t_6, t_6 t_7/q t_0; p)}{\theta(q t_0/t_6, q^2 t_0/t_6, t_0/t_7, t_7/q t_0; p)} \prod_{r=1}^{5} \frac{\theta(t_r; p)}{\theta(q t_0/t_r; p)} \mathcal{E}(q^2 t_0; q t_1, \dots, q t_5, t_6, q t_7), \end{aligned} \tag{8.2} $$

$$ \begin{aligned} &\frac{\theta(t_7; p)}{\theta(t_6/q t_0, t_6/q^2 t_0, t_6/t_7; p)} \prod_{r=1}^{5} \theta(t_r t_6/q t_0; p) \mathcal{E}(q^2 t_0; q t_1, \dots, q t_5, t_6, q t_7) \\ &\qquad + \frac{\theta(t_6; p)}{\theta(t_7/q t_0, t_7/q^2 t_0, t_7/t_6; p)} \prod_{r=1}^{5} \theta(t_r t_7/q t_0; p) \mathcal{E}(q^2 t_0; q t_1, \dots, q t_6, t_7) \\ &\quad = \frac{1}{\theta(q t_0, q^2 t_0; p)} \prod_{r=1}^{5} \theta(q t_0/t_r; p) \mathcal{E}(\mathbf{t}). \end{aligned} \tag{8.3} $$

Their combination yields

$$\begin{aligned}
&\frac{\theta(t_7,t_0/t_7,qt_0/t_7;p)}{\theta(qt_7/t_6,t_7/t_6;p)}\prod_{r=1}^{5}\theta(qt_0/t_6t_r;p)\left(\mathcal{E}(t_0;t_1,\dots,t_5,q^{-1}t_6,qt_7)-\mathcal{E}(\mathbf{t})\right)\\
&\quad+\frac{\theta(t_6,t_0/t_6,qt_0/t_6;p)}{\theta(qt_6/t_7,t_6/t_7;p)}\prod_{r=1}^{5}\theta(qt_0/t_7t_r;p)\left(\mathcal{E}(t_0;t_1,\dots,t_5,qt_6,q^{-1}t_7)-\mathcal{E}(\mathbf{t})\right)\\
&\quad+\theta(qt_0/t_6t_7;p)\prod_{r=1}^{5}\theta(t_r;p)\mathcal{E}(\mathbf{t})=0.
\end{aligned}\tag{8.4}$$

As $p\to 0$, these three identities are reduced to the contiguous relations for the terminating very-well-poised balanced ${}_{10}\Phi_9$ series of Gupta and Masson [GM]. Similar contiguous relations at the level of ${}_8\Phi_7$ functions were constructed earlier by Ismail and Rahman [IR].

We change the parametrization of the $\mathcal{E}$-function and consider relation (8.4) for the following function:

$$R_n(z;q,p)\equiv {}_{12}V_{11}\left(\frac{t_3}{t_4};\frac{q}{t_0t_4},\frac{q}{t_1t_4},\frac{q}{t_2t_4},t_3z,\frac{t_3}{z},q^{-n},\frac{Aq^{n-1}}{t_4};q,p\right),\tag{8.5}$$

where $A=\prod_{r=0}^{4}t_r$. We replace the parameter t_0 in (8.4) by t_3/t_4; the variables t_1,t_2, and t_3 are replaced by $q/t_0t_4, q/t_1t_4$, and q/t_2t_4; the variables t_4 and t_5 by q^{-n} and Aq^{n-1}/t_4; and the variables t_6 and t_7 by t_3z and t_3/z, respectively. As a result, we see that $R_n(z;q,p)$ provides a particular solution of the following finite-difference equation:

$$\mathcal{D}_\mu f(z)=0,\quad \mathcal{D}_\mu=V_\mu(z)(T-1)+V_\mu(z^{-1})(T^{-1}-1)+\kappa_\mu,\tag{8.6}$$

where T is the q-shift operator, $Tf(z)=f(qz)$, and

$$V_\mu(z)=\theta\left(\frac{t_4}{q\mu z},\frac{A\mu}{q^2z},\frac{t_4z}{q};p\right)\frac{\prod_{r=0}^{4}\theta(t_rz;p)}{\theta(z^2,qz^2;p)},\tag{8.7}$$

$$\kappa_\mu=\theta\left(\frac{A\mu}{qt_4},\mu^{-1};p\right)\prod_{r=0}^{3}\theta\left(\frac{t_rt_4}{q};p\right).\tag{8.8}$$

The functions $f(z)=R_n(z;q,p)$ solve (8.6) for $\mu=q^n, n\in\mathbb{N}$.

Equation (8.6) looks like a nonstandard eigenvalue problem with the "spectral parameter" μ; indeed, it can be rewritten as the generalized eigenvalue problem

$$\mathcal{D}_\eta f(z)=\lambda\mathcal{D}_\xi f(z),\tag{8.9}$$

where the spectral parameter is

$$\lambda=\frac{\theta\left(\frac{\mu A\eta}{qt_4},\frac{\mu}{\eta};p\right)}{\theta\left(\frac{\mu A\xi}{qt_4},\frac{\mu}{\xi};p\right)},\tag{8.10}$$

and the operators $\mathcal{D}_\xi$, $\mathcal{D}_\eta$ are obtained from $\mathcal{D}_\mu$ after the replacement of μ by arbitrary gauge parameters $\xi,\eta\in\mathbb{C}$, $\xi\neq\eta p^k, qt_4p^k/A\eta$, $k\in\mathbb{Z}$. Application of the theta function identity (6.5) to equation (8.9) yields

$$\mathcal{D}_\eta-\lambda\mathcal{D}_\xi=\frac{\theta(A\eta\xi/qt_4,\xi/\eta;p)}{\theta(A\mu\xi/qt_4,\xi/\mu;p)}\mathcal{D}_\mu,$$

which shows that the gauge parameters ξ,η drop out completely from the equation determining $f(z)$, $\mathcal{D}_\mu f(z)=0$.

A three term recurrence relation for the functions $R_n(z;q,p)$ was derived in [SZ1, SZ2]. It is obtained from formula (8.4) if we replace there t_6 by q^{-n} and t_7 by Aq^{n-1}/t_4, and

substitute $t_1 \mapsto q/t_0t_4, t_2 \mapsto q/t_1t_4, t_3 \mapsto q/t_2t_4,\ t_4 \mapsto t_3z,\ t_5 \mapsto t_3/z$. After some work, this formula can be represented as

$$\begin{aligned}(\gamma(z)-\alpha_{n+1})B(Aq^{n-1}/t_4)\,(R_{n+1}(z;q,p)-R_n(z;q,p))\\ +(\gamma(z)-\beta_{n-1})B(q^{-n})\,(R_{n-1}(z;q,p)-R_n(z;q,p))\\ +\delta\,(\gamma(z)-\gamma(t_3))\,R_n(z;q,p)=0,\end{aligned} \tag{8.11}$$

where

$$B(x)=\frac{\theta\left(x,\frac{t_3}{t_4x},\frac{qt_3}{t_4x},\frac{qx}{t_0t_1},\frac{qx}{t_0t_2},\frac{qx}{t_1t_2},\frac{q^2\eta x}{A},\frac{q^2x}{A\eta};p\right)}{\theta\left(\frac{qt_4x^2}{A},\frac{q^2t_4x^2}{A};p\right)}, \tag{8.12}$$

$$\delta=\theta\left(\frac{q^2t_3}{A},\frac{q}{t_0t_4},\frac{q}{t_1t_4},\frac{q}{t_2t_4},t_3\eta,\frac{t_3}{\eta};p\right), \tag{8.13}$$

$$\gamma(z)=\frac{\theta(z\xi,z/\xi;p)}{\theta(z\eta,z/\eta;p)}, \tag{8.14}$$

$$\alpha_n=\gamma(q^n/t_4),\quad \beta_n=\gamma(q^{n-1}A). \tag{8.15}$$

Here ξ and $\eta\neq\xi p^k,\xi^{-1}p^k$, $k\in\mathbb{Z}$, are arbitrary gauge parameters (they are not related to ξ,η in the difference equation, but we use the same notation). Substituting (8.12)–(8.14) in (8.11) and applying identity (6.5), we see that the auxiliary gauge parameters ξ,η drop out completely from the resulting recurrence relation.

Since $B(q^{-n})=0$ for $n=0$, the indeterminate R_{-1} is not involved in (8.11) for $n=0$. We can say that $R_n(z;q,p)$ are generated by the three term recurrence relation (8.11) for the initial conditions $R_{-1}=0, R_0=1$. All recurrence coefficients in (8.11) depend linearly on the variable $\gamma(z)$, which absorbs z-dependence. Therefore, the $R_n(z;q,p)$ are rational functions of $\gamma(z)$ with n being the degree of the polynomials in $\gamma(z)$ in the numerator and denominator of R_n. Moreover, the poles of these functions are located at $\gamma(z)=\alpha_1,\dots,\alpha_n$.

For a particular choice of one of the parameters and discretization of the values of z, the functions $R_n(z;q,p)$ yield elliptic generalizations of Wilson's finite-dimensional ${}_9F_8$ and ${}_{10}\Phi_9$ rational functions [Wi]. They were derived in [SZ1] from the theory of self-similar solutions of nonlinear integrable discrete time chains (for a brief review of the corresponding approach to special functions, see [S1, S3]). Discrete analogs of equations (8.6), (8.9) valid for the latter finite-dimensional system of functions were derived in [SZ3] with the help of self-duality. An equation satisfied by ${}_{10}\Phi_9$ functions, appearing from $R_n(z;q,p)$ in the limit as $p\to 0$, was investigated by Rahman and Suslov in [RS1]. The general three term recurrence relations (8.11) were considered in [Zh] and, in a different form related to R_{II} continued fractions, in [IM].

The solutions of the generalized eigenvalue problems are known to be biorthogonal to each other; see, e.g., [SZ1, SZ4, Zh] and the references therein. Here we would like to demonstrate that the elliptic beta integral (1.13) serves as the biorthogonality measure for the solutions of equation (8.9). Consider the scalar product

$$\int_C \Delta_E(z;\mathbf{t})\Psi(z)\,(\mathcal{D}_\eta-\lambda\mathcal{D}_\xi)\,\Phi(z)\,\frac{dz}{z}, \tag{8.16}$$

where $\Delta_E(z;\mathbf{t})$ is the integrand of (1.13) and $\Phi(z), \Psi(z)$ are some complex functions. The expression (8.16) can be rewritten as

$$\begin{aligned}\int_C \Delta_E(z;\mathbf{t})(\kappa_\eta - V_\eta(z) - V_\eta(z^{-1}))\Psi(z)\Phi(z)\,\frac{dz}{z}\\ + \int_{C_-} \Delta_E(q^{-1}z;\mathbf{t})V_\eta(q^{-1}z)\Psi(q^{-1}z)\Phi(z)\,\frac{dz}{z}\\ + \int_{C_+} \Delta_E(qz;\mathbf{t})V_\eta(q^{-1}z^{-1})\Psi(qz)\Phi(z)\,\frac{dz}{z} - \lambda\{\eta\to\xi\},\end{aligned} \tag{8.17}$$

where $\{\eta\to\xi\}$ means the preceding expression with η replaced by ξ. The integration contours $C_\pm$ are obtained from C after the scaling transformations $z\to q^{\pm1}z$. Suppose that the poles of $\Delta_E(z;\mathbf{t})$ and the singularities of the functions $\Phi(z), \Psi(q^{\pm1}z)$ do not lie in the region swept by the contours $C_\pm$ during their deformations to C. Then (8.17) takes the form

$$\int_C \Delta_E(z;\mathbf{t})\Phi(z)\left(\mathcal{D}_\eta^T - \lambda\mathcal{D}_\xi^T\right)\Psi(z)\,\frac{dz}{z}, \tag{8.18}$$

where the adjoint (or transposed) operator $\mathcal{D}_\xi^T$ has the form

$$\begin{aligned}\mathcal{D}_\xi^T = \frac{\Delta_E(qz;\mathbf{t})}{\Delta_E(z;\mathbf{t})}V_\xi(q^{-1}z^{-1})T + \frac{\Delta_E(q^{-1}z;\mathbf{t})}{\Delta_E(z;\mathbf{t})}V_\xi(q^{-1}z)T^{-1}\\ - V_\xi(z) - V_\xi(z^{-1}) + \kappa_\xi.\end{aligned} \tag{8.19}$$

Suppose that $\Phi_\lambda(z)$ is a solution of the equation $(\mathcal{D}_\eta - \lambda\mathcal{D}_\xi)\Phi(z) = 0$ and that $\Psi_{\lambda'}(z)$ solves the conjugate equation $(\mathcal{D}_\eta^T - \lambda'\mathcal{D}_\xi^T)\Psi(z) = 0$ for some λ'. Both these functions can be multiplied by arbitrary functions $f(z)$ satisfying the condition of periodicity in the logarithmic scale, $f(qz) = f(z)$. After the replacement of $\Phi(z)$ and $\Psi(z)$ in (8.16) and (8.18) by $\Phi_\lambda(z)$ and $\Psi_{\lambda'}(z)$, these expressions become equal to zero. In particular, (8.18) yields the relation

$$\begin{aligned}\int_C \Delta_E(z;\mathbf{t})\Phi_\lambda(z)\left(\mathcal{D}_\eta^T - \lambda\mathcal{D}_\xi^T\right)\Psi_{\lambda'}(z)\,\frac{dz}{z}\\ = (\lambda'-\lambda)\int_C \Delta_E(z;\mathbf{t})\Phi_\lambda(z)\mathcal{D}_\xi^T\Psi_{\lambda'}(z)\,\frac{dz}{z} = 0,\end{aligned} \tag{8.20}$$

which shows that for $\lambda'\neq\lambda$ the function $\Phi_\lambda(z)$ is orthogonal to $\mathcal{D}_\xi^T\Psi_{\lambda'}(z)$.

We find a function $g(z)$ such that

$$g^{-1}(z)\left(\mathcal{D}_\eta^T - \lambda\mathcal{D}_\xi^T\right)g(z) = \mathcal{D}_\eta - \lambda\mathcal{D}_\xi.$$

After substitution of the known expressions for $\Delta_E(qz;\mathbf{t})/\Delta_E(z;\mathbf{t})$ and the spectral parameter (see λ in (8.10)), we get the following equation for $g(z)$:

$$g(qz) = \frac{\theta\left(\frac{q}{t_4z}, \frac{q\mu z}{t_4}, Az, \frac{A\mu}{q^2z}; p\right)}{\theta\left(\frac{q^2z}{t_4}, \frac{\mu}{t_4z}, \frac{A}{qz}, \frac{A\mu z}{q}; p\right)}g(z),$$

which is solved easily:

$$g(z) = \frac{\Gamma(\frac{q\mu z}{t_4}, \frac{\mu q}{t_4z}, Az, \frac{A}{z}; q, p)}{\Gamma(\frac{q^2z}{t_4}, \frac{q^2}{t_4z}, \frac{A\mu z}{q}, \frac{A\mu}{qz}; q, p)}. \tag{8.21}$$

Here we have neglected the arbitrary factor $f(qz) = f(z)$, which has already been mentioned. As a result, we get a direct relation between $\Phi_\lambda(z)$ and $\Psi_\lambda(z)$: $\Psi_\lambda(z) = g(z)\Phi_\lambda(z)$, where $g(z)$ depends on λ as well.

We denote $\lambda_n \equiv \lambda|_{\mu=q^n}$ and

$$g_n(z) \equiv g(z)|_{\mu=q^n} = \frac{\theta\left(\frac{q^2 z}{t_4}, \frac{q^2}{t_4 z}; p; q\right)_{n-1}}{\theta\left(Az, \frac{A}{z}; p; q\right)_{n-1}}.$$

Substituting $\Phi_{\lambda_n}(z) = R_n(z;q,p)$ in the derived biorthogonality relations, we see that the functions $R_n(z;q,p)$ are formally orthogonal to $\mathcal{D}_\xi^T g_m(z) R_m(z;q,p)$ for $n \neq m$.

The general considerations of [SZ4, Zh] show that the $R_n(z;q,p)$, which are rational functions of $\gamma(z)$ with the poles at $\gamma(z) = \alpha_1, \dots, \alpha_n$, are orthogonal to other rational functions of $\gamma(z)$, which we denote by $T_m(z;q,p)$, with the poles at $\gamma(z) = \beta_1, \dots, \beta_n$. The choice of α_n, β_n and the other recurrence coefficients in (8.11) determine R_n and T_n uniquely, so that permutation of all α_n with β_n permutes R_n and T_n. In our case, we see that the parameters β_n are obtained from $\alpha_n = \gamma(q^n/t_4)$ after the replacement of t_4 by pq/A. Equivalently, this replacement converts β_n to α_n. An important point is that the weight function $\Delta_E(z,\mathbf{t})$ is invariant under such a transformation. Therefore, we can get T_n out of R_n simply by the $t_4 \to pq/A$ involution, which yields

$$T_n(z;q,p) = {}_{12}V_{11}\left(\frac{At_3}{q}; \frac{A}{t_0}, \frac{A}{t_1}, \frac{A}{t_2}, t_3 z, \frac{t_3}{z}, q^{-n}, \frac{Aq^{n-1}}{t_4}; q, p\right), \tag{8.22}$$

where the dependence on p in the parameters drops out due to the total ellipticity of this ${}_{12}V_{11}$ series. Comparing with the previous consideration, we see that

$$\mathcal{D}_\xi^T g_n(z) R_n(z;q,p) = \rho_n T_n(z;q,p)$$

for some proportionality constants ρ_n, which are of no importance for us here.

Thus, the operator formalism developed above leads to the following formal biorthogonality relation:

$$\int_C T_n(z;q,p) R_m(z;q,p) \Delta_E(z,\mathbf{t}) \frac{dz}{z} = \tilde{h}_n \delta_{nm} \tag{8.23}$$

for some constants $\tilde{h}_n$. Suppose that $C = \mathbb{T}$ and $|t_r| < 1, |qp| < |A|$. Some poles of the functions R_n, T_m cancel with zeros of $\Delta_E(z;\mathbf{t})$. The remaining ones approach $\mathbb{T}$ as the indices n and m increase. Starting with sufficiently high values of n and m, the contour $\mathbb{T}$ stops to satisfy the conditions used in the derivation of (8.18) and must be deformed. As is shown in Appendix A, (8.23) is true if C separates the points $z = t_{0,1,2,3} p^a q^b$, $t_4 p^a q^{b-m}$, and $A^{-1} p^{a+1} q^{b+1-n}$, $a, b \in \mathbb{N}$, from their partners with the inverse $z \mapsto z^{-1}$ coordinates.

Relation (8.23) seems to remain true even if we multiply $R_n(z;q,p)$ or $T_n(z;q,p)$ by an arbitrary function $f(z)$ with the property $f(qz) = f(z)$. However, such nontrivial $f(z)$ must have singularities that are crossed over when the contours $C_\pm$ are deformed to C (otherwise $f(z) = \text{const}$). Therefore, the influence of such additional factors should be considered more carefully. Moreover, only for very special $f(z)$ the normalization constants $\tilde{h}_n$ may admit exact evaluation.

The weight function $\Delta_E(z,\mathbf{t})$ is symmetric in q and p, whereas neither $R_n(z;q,p)$ nor $T_n(z;q,p)$ possess such a property. We can try to restore this symmetry using the freedom in the factor $f(z) = f(qz)$. Take $f(z)$ equal to $R_k(z;p,q)$, $k \in \mathbb{N}$, that is, to the functions $R_n(z;q,p)$ themselves with the permuted bases q and p. Then the product

$$R_{nk}(z) \equiv R_n(z;q,p) R_k(z;p,q)$$

satisfies two generalized eigenvalue problems: (8.6) and the p-difference equation obtained from (8.6) by the permutation of q and p. For (8.6) we should have $\mu = q^n$, and for its partner $\mu = p^k$. The function (8.10) does not change under the substitution $\mu \mapsto p\mu$. Therefore, the choice $\mu = q^n p^k$, $n, k \in \mathbb{N}$, gives "the spectrum" for both generalized

eigenvalue problems. The first factor of $R_{nk}(z)$ is a rational function of $\gamma(z;p)$ (we indicate the dependence on the base p explicitly), but the second is rational in $\gamma(z;q)$. Therefore, for generic q,p it is necessary to view the functions $R_{nk}(z)$ not as rational functions of some variable but as meromorphic functions of z.

In the same way, the series termination condition $t_6 = q^{-n}$ in (8.5) may be replaced by $t_6 = q^{-n}p^{-k}$, which terminates simultaneously the ${}_{12}V_{11}$ series for $R_k(z;p,q)$. The property of the total ellipticity of the balanced ${}_{r+1}V_r(t_0;\, t_1,\dots,t_{r-4};q,p)$ series plays a crucial role at this place: any parameter $t_1,\dots,t_{r-5}$ may be multiplied by an arbitrary integral power of p without changes (note that the parameter t_0 plays a distinguished role and the series are invariant under the transformation $t_0 \mapsto p^2 t_0$).

As a result of the doubling of eigenvalue problems, the functions $R_{nk}(z)$ turn out to satisfy a quite unusual biorthogonality relation (A.13) characteristic of functions of *two* independent variables, which was announced in [S4]. A rigorous consideration of all these biorthogonalities associated with the function $R_n(z;q,p)$ with complete proofs is given in Appendix A.

In fact, there is a deeper relationship between the structure of the elliptic beta integral (1.13) and the biorthogonal functions $R_n(z;q,p), T_n(z;q,p)$ and the solutions of equation (8.6) than it is indicated in this section. We hope to address this later on. As far as the multivariable generalizations are concerned, there are some multidimensional analogs of equation (8.6), for the solutions of which the multiple elliptic beta integrals on root systems described in this paper may serve as biorthogonality measures. However, their consideration lies beyond the scope of the present work.

Appendix A. Proof of a biorthogonality relation

We define a pair of functions of $z \in \mathbb{C}$ as products of two terminating ${}_{12}V_{11}$ very-well-poised balanced theta hypergeometric series with the argument $x = 1$ and *different modular parameters*:

$$\begin{aligned} R_{nm}(z) &= {}_{12}V_{11}\left(\frac{t_3}{t_4};\frac{q}{t_0t_4},\frac{q}{t_1t_4},\frac{q}{t_2t_4},t_3z,\frac{t_3}{z},q^{-n},\frac{Aq^{n-1}}{t_4};q,p\right) \\ &\quad\times {}_{12}V_{11}\left(\frac{t_3}{t_4};\frac{p}{t_0t_4},\frac{p}{t_1t_4},\frac{p}{t_2t_4},t_3z,\frac{t_3}{z},p^{-m},\frac{Ap^{m-1}}{t_4};p,q\right), \end{aligned} \tag{A.1}$$

$$\begin{aligned} T_{nm}(z) &= {}_{12}V_{11}\left(\frac{At_3}{q};\frac{A}{t_0},\frac{A}{t_1},\frac{A}{t_2},t_3z,\frac{t_3}{z},q^{-n},\frac{Aq^{n-1}}{t_4};q,p\right) \\ &\quad\times {}_{12}V_{11}\left(\frac{At_3}{p};\frac{A}{t_0},\frac{A}{t_1},\frac{A}{t_2},t_3z,\frac{t_3}{z},p^{-m},\frac{Ap^{m-1}}{t_4};p,q\right), \end{aligned} \tag{A.2}$$

where $n,m \in \mathbb{N}$. Obviously, these functions are symmetric with respect to the permutation of p and q. The balancing conditions are used already in these series in order to express one of the parameters in terms of the others.

For $m \neq 0$ and the fixed parameters $q, t_r, r = 0,\dots,4$, the limit as $p \to 0$ is not well defined for $R_{nm}(z)$ and $T_{nm}(z)$. The reason for this comes from the quasiperiodicity of $\theta(z;p)$ because the limits $z \to 0$ or $z \to \infty$ are not defined for it.

Now, we consider the following integral:

$$J_{mn,kl} \equiv \int_{C_{mn,kl}} T_{nl}(z)R_{mk}(z)\Delta_E(z,\mathbf{t})\,\frac{dz}{z}, \tag{A.3}$$

where $\Delta_E(z,\mathbf{t})$ denotes the weight function (1.14) and $C_{mn,kl}$ is some contour of integration. We want to show that a particular choice of $C_{mn,kl}$ (to be specified below) leads

to the formula

$$J_{mn,kl} = h_{nl}\delta_{mn}\delta_{kl}, \tag{A.4}$$

where the h_{nl} are some normalization constants.

An elliptic extension of the Bailey transformation for the ${}_{10}\Phi_9$ series, which was derived by Frenkel and Turaev in [FT], has the following form (an alternative proof of it was given in [S6]):

$$\begin{aligned} &{}_{12}V_{11}(t_0;t_1,\dots,t_7;q,p) \\ &\quad = \frac{\theta(qt_0,qs_0/t_4,qs_0/t_5,qt_0/t_4t_5;p;q)_N}{\theta(qs_0,qt_0/t_4,qt_0/t_5,qs_0/t_4t_5;p;q)_N}\,{}_{12}V_{11}(s_0;s_1,\dots,s_7;q,p), \end{aligned} \tag{A.5}$$

where

$$\prod_{m=1}^{7} t_m = t_0^3q^2, \quad t_6 = q^{-N}, \quad N\in\mathbb{N},$$

$$\begin{aligned} s_0 &= \frac{qt_0^2}{t_1t_2t_3}, \qquad & s_1 &= \frac{s_0t_1}{t_0}, \\ s_2 &= \frac{s_0t_2}{t_0}, \qquad & s_3 &= \frac{s_0t_3}{t_0}, \end{aligned} \tag{A.6}$$

and s_4,s_5,s_6,s_7 form an arbitrary permutation of t_4,t_5,t_6,t_7. Using this identity, we can rewrite the functions $R_{mk}(z)$ as follows:

$$\begin{aligned} R_{mk}(z) &= \kappa_m(p;q)\kappa_k(q;p) \\ &\quad\times {}_{12}V_{11}\left(\frac{t_0}{t_4};\frac{q}{t_1t_4},\frac{q}{t_2t_4},\frac{q}{t_3t_4},t_0z,\frac{t_0}{z},q^{-m},\frac{Aq^{m-1}}{t_4};q,p\right) \\ &\quad\times {}_{12}V_{11}\left(\frac{t_0}{t_4};\frac{p}{t_1t_4},\frac{p}{t_2t_4},\frac{p}{t_3t_4},t_0z,\frac{t_0}{z},p^{-k},\frac{Ap^{k-1}}{t_4};p,q\right), \end{aligned} \tag{A.7}$$

where

$$\kappa_m(p;q) = \frac{\theta(qt_3/t_4,t_0t_1,t_0t_2,A/qt_0;p;q)_m}{\theta(qt_0/t_4,t_1t_3,t_2t_3,A/qt_3;p;q)_m}.$$

Substituting the explicit series expressions (A.2) and (A.7) in the definition of $J_{mn,kl}$ (A.3) yields

$$\begin{aligned} J_{mn,kl} &= \kappa_m(p;q)\kappa_k(q;p)\sum_{r=0}^{n}\sum_{r'=0}^{l}\sum_{s=0}^{m}\sum_{s'=0}^{k} q^{r+s}p^{r'+s'} \\ &\quad\times\frac{\theta(At_3q^{2r-1},t_0q^{2s}/t_4;p)}{\theta(At_3/q,t_0/t_4;p)}\frac{\theta(At_3p^{2r'-1},t_0p^{2s'}/t_4;q)}{\theta(At_3/p,t_0/t_4;q)} \\ &\quad\times\frac{\theta(At_3/q,A/t_0,A/t_1,A/t_2,q^{-n},Aq^{n-1}/t_4;p;q)_r}{\theta(q,t_0t_3,t_1t_3,t_2t_3,At_3q^n,t_3t_4q^{1-n};p;q)_r} \\ &\quad\times\frac{\theta(At_3/p,A/t_0,A/t_1,A/t_2,p^{-l},Ap^{l-1}/t_4;q;p)_{r'}}{\theta(p,t_0t_3,t_1t_3,t_2t_3,At_3p^l,t_3t_4p^{1-l};q;p)_{r'}} \\ &\quad\times\frac{\theta(t_0/t_4,q/t_1t_4,q/t_2t_4,q/t_3t_4,q^{-m},Aq^{m-1}/t_4;p;q)_s}{\theta(q,t_0t_1,t_0t_2,t_0t_3,t_0q^{m+1}/t_4,t_0q^{2-m}/A;p;q)_s} \\ &\quad\times\frac{\theta(t_0/t_4,p/t_1t_4,p/t_2t_4,p/t_3t_4,p^{-k},Ap^{k-1}/t_4;q;p)_{s'}}{\theta(p,t_0t_1,t_0t_2,t_0t_3,t_0p^{k+1}/t_4,t_0p^{2-k}/A;q;p)_{s'}}I_{rs,r's'}, \end{aligned}$$

where

$$I_{rs,r's'} = \int_{C_{mn,kl}} \Delta_E(z,\mathbf{t}) \frac{\theta(zt_3, z^{-1}t_3; p; q)_r}{\theta(zA, z^{-1}A; p; q)_r} \frac{\theta(zt_0, z^{-1}t_0; p; q)_s}{\theta(zqt_4^{-1}, z^{-1}qt_4^{-1}; p; q)_s} \times \frac{\theta(zt_3, z^{-1}t_3; q; p)_{r'}}{\theta(zA, z^{-1}A; q; p)_{r'}} \frac{\theta(zt_0, z^{-1}t_0; q; p)_{s'}}{\theta(zpt_4^{-1}, z^{-1}pt_4^{-1}; q; p)_{s'}} \frac{dz}{z}. \tag{A.8}$$

We introduce the notation

$$\tilde{t}_0 = t_0 q^s p^{s'}, \quad \tilde{t}_1 = t_1, \quad \tilde{t}_2 = t_2, \quad \tilde{t}_3 = t_3 q^r p^{r'}, \quad \tilde{t}_4 = t_4 q^{-s} p^{-s'},$$

so that

$$\tilde{A} = \prod_{m=0}^{4} \tilde{t}_m = A q^r p^{r'}.$$

Then, using the transformation property

$$\theta(z; p; q)_l = (-z)^l q^{l(l-1)/2} \theta(z^{-1}q^{-l+1}; p; q)_l = \frac{(-z)^l q^{l(l-1)/2}}{\theta(qz^{-1}; p; q)_{-l}},$$

we can rewrite the integral (A.8) in the form

$$I_{rs,r's'} = \left(\frac{t_0 t_4}{pq}\right)^{2ss'} \left(\frac{t_3}{A}\right)^{2rr'} \frac{t_4^{2(s+s')}}{q^{s(s+1)} p^{s'(s'+1)}} \int_{C_{mn,kl}} \Delta_E(z, \tilde{\mathbf{t}}) \frac{dz}{z}. \tag{A.9}$$

But the integral on the right-hand side of (A.9) coincides with the elliptic beta integral (1.13), provided we identify $C_{mn,kl} = \mathbb{T}$ and impose the constraints $|\tilde{t}_m| < 1$, $|pq| < |\tilde{A}|$. However, the values of the integers $r, s, r', s' \in \mathbb{N}$ are not limited; starting with their sufficiently large values, we shall have either $|\tilde{A}| = |q^r p^{r'} A| < |pq|$ or $|\tilde{t}_4| = |q^{-s} p^{-s'} t_4| > 1$. Now is the moment to specify the contour $C_{mn,kl}$. We choose it in such a way that formula (1.13) remains applicable. More precisely, let $C_{mn,kl}$ be a deformation of $\mathbb{T}$ such that it separates the poles at $z = t_{0,1,2,3} p^a q^b$, $t_4 p^{a-k} q^{b-m}$, and $A^{-1} p^{a+1-l} q^{b+1-n}$, $a, b \in \mathbb{N}$, that lie inside $C_{mn,kl}$ and converge to zero, from the poles diverging to infinity, which are obtained from the poles inside $C_{mn,kl}$ by the inversion transformation $z \to z^{-1}$. The subscripts m, n, k, l in the notation for such a contour indicate that, evidently, the shape of $C_{nm,kl}$ depends on the indices of the functions $T_{nl}(z), R_{mk}(z)$.

For such a contour $C_{mn,kl}$, we have

$$\begin{aligned} I_{rs,r's'} &= \left(\frac{t_0 t_4}{pq}\right)^{2ss'} \left(\frac{t_3}{A}\right)^{2rr'} \frac{t_4^{2(s+s')}}{q^{s(s+1)} p^{s'(s'+1)}} \mathcal{N}_E(\tilde{\mathbf{t}}) \\ &= \mathcal{N}_E(\mathbf{t}) \frac{\theta(t_1t_3, t_2t_3, t_3t_4; p; q)_r}{\theta(A/t_0, A/t_1, A/t_2; p; q)_r} \\ &\quad \times \frac{\theta(t_0t_3; p; q)_{r+s}}{\theta(A/t_4; p; q)_{r+s}} \frac{\theta(t_0t_1, t_0t_2, q^{1-r}t_0/A; p; q)_s}{\theta(q/t_1t_4, q/t_2t_4, q^{1-r}/t_3t_4; p; q)_s} \\ &\quad \times \frac{\theta(t_1t_3, t_2t_3, t_3t_4; q; p)_{r'}}{\theta(A/t_0, A/t_1, A/t_2; q; p)_{r'}} \\ &\quad \times \frac{\theta(t_0t_3; q; p)_{r'+s'}}{\theta(A/t_4; q; p)_{r'+s'}} \frac{\theta(t_0t_1, t_0t_2, p^{1-r'}t_0/A; q; p)_{s'}}{\theta(p/t_1t_4, p/t_2t_4, p^{1-r'}/t_3t_4; q; p)_{s'}}, \end{aligned}$$

where the function $\mathcal{N}_E(\mathbf{t})$ is fixed in (1.15).

As a result of these manipulations, the quantity $J_{mn,kl}$ splits into a product of two double series, each depending only on the indices m, n and k, l separately. After an application of the relation $(a; p; q)_{r+s} = (aq^r; p; q)_s (a; p; q)_r$ and various simplifications, we can write

$$J_{mn,kl} = \mathcal{N}_E(\mathbf{t}) J_{mn}(p; q) J_{kl}(q; p),$$

where

$$J_{mn}(p;q)=\kappa_m(p;q)\sum_{r=0}^{n}q^r\frac{\theta(At_3q^{2r-1};p)}{\theta(At_3/q;p)}\frac{\theta(At_3/q,q^{-n},Aq^{n-1}/t_4,t_3t_4;p;q)_r}{\theta(q,At_3q^n,t_3t_4q^{1-n},A/t_4;p;q)_r}$$
$$\times\,{}_{10}V_9\left(\frac{t_0}{t_4};\frac{q}{t_3t_4},t_0t_3q^r,\frac{q^{1-r}t_0}{A},\frac{Aq^{m-1}}{t_4},q^{-m};q,p\right). \tag{A.10}$$

The constraint $t_2t_3=qt_0$ imposed on relation (A.5) converts the ${}_{12}V_{11}$-series on its left-hand side into a terminating ${}_{10}V_9$ series, whereas on the right-hand side only the first term of the corresponding ${}_{12}V_{11}$-series survives. As a result, we get the Frenkel–Turaev sum, or an elliptic generalization of the Jackson sum for terminating very-well-poised balanced ${}_8\Phi_7$-series:

$${}_{10}V_9(t_0;t_1,t_4,t_5,t_6,t_7;q,p)$$
$$=\frac{\theta(qt_0,qt_0/t_1t_4,qt_0/t_1t_5,qt_0/t_4t_5;p;q)_N}{\theta(qt_0/t_1t_4t_5,qt_0/t_1,qt_0/t_4,qt_0/t_5;p;q)_N}, \tag{A.11}$$

where $t_1t_4t_5t_6t_7=qt_0^2$ and $t_6=q^{-N}$, $N\in\mathbb{N}$. Application of this sum to the ${}_{10}V_9$ series in (A.10) yields

$${}_{10}V_9(\cdots)=\frac{\theta(qt_0/t_4,t_1t_2,At_3q^{r-1},q^{-r};p;q)_m}{\theta(t_0t_3,A/qt_0,Aq^r/t_4,q^{1-r}/t_3t_4;p;q)_m}.$$

Clearly, this expression vanishes for $m>r$. This means that $J_{mn}=0$ for $m>n$. For $m\le n$, we get

$$J_{mn}(p;q)=\kappa_m(p;q)\frac{\theta(At_3;p;q)_{2m}}{\theta(A/t_4;p;q)_{2m}}\frac{\theta(Aq^{n-1}/t_4,qt_0/t_4,t_1t_2,q^{-n};p;q)_m}{\theta(t_3t_4q^{1-n},t_0t_3,A/qt_0,At_3q^n;p;q)_m}$$
$$\times(t_3t_4)^m{}_8V_7(At_3q^{2m-1};t_3t_4,Aq^{n+m-1}/t_4,q^{m-n};q,p).$$

Applying the summation formula (A.11) to the latter ${}_8V_7$ series, we get

$${}_8V_7(\cdots)=\frac{\theta(At_3q^{2m},q^{m-n+1},Aq^{m+n}/t_4,qt_3t_4;p;q)_{n-m}}{\theta(q,Aq^{2m}/t_4,t_3t_4q^{m-n+1},At_3q^{m+n};p;q)_{n-m}},$$

which is equal to zero for $n>m$ due to the factor $\theta(q^{m-n+1};p;q)_{n-m}$. As a result, $J_{mn}(p;q)=h_n(p;q)\delta_{mn}$, where the normalization constants have the form

$$h_n(p;q)=\frac{\theta(A/qt_4;p)\theta(q,qt_3/t_4,t_0t_1,t_0t_2,t_1t_2,At_3;p;q)_nq^{-n}}{\theta(Aq^{2n-1}/t_4;p)\theta(1/t_3t_4,t_0t_3,t_1t_3,t_2t_3,A/qt_3,A/qt_4;p;q)_n}. \tag{A.12}$$

The fact that $J_{mn}=0$ for $n\ne m$ provides the desired biorthogonality relation (A.4). We summarize the result obtained in the form of the theorem that was announced in [S4] (it is necessary to apply the series notation introduced in [S5, S6] and permute the parameters t_3 and t_4 in [S4] in order to match with the current presentation).

Theorem 9. *Let t_m, $\Delta_E(z,\mathbf{t})$, $\mathcal{N}_E(\mathbf{t})$ be the same as in Theorem 1. Let $C_{mn,kl}$ denote a positively oriented contour separating the points $z=\{t_{0,1,2,3}p^aq^b,\ t_4p^{a-k}q^{b-m},\ A^{-1}p^{a+1-l}q^{b+1-n}\}_{a,b\in\mathbb{N}}$ from the points with the inverse $(z\mapsto z^{-1})$ coordinates. Then $R_{mk}(z)$ and $T_{nl}(z)$ satisfy the biorthogonality relation*

$$\int_{C_{mn,kl}}T_{nl}(z)R_{mk}(z)\Delta_E(z,\mathbf{t})\,\frac{dz}{z}=h_{nl}\mathcal{N}_E(\mathbf{t})\delta_{mn}\delta_{kl}, \tag{A.13}$$

where the h_{nl} are normalization constants,

$$h_{nl} = \frac{\theta(A/qt_4;p)\theta(q,qt_3/t_4,t_0t_1,t_0t_2,t_1t_2,At_3;p;q)_n q^{-n}}{\theta(Aq^{2n}/qt_4;p)\theta(1/t_3t_4,t_0t_3,t_1t_3,t_2t_3,A/qt_3,A/qt_4;p;q)_n} \times \frac{\theta(A/pt_4;q)\theta(p,pt_3/t_4,t_0t_1,t_0t_2,t_1t_2,At_3;q;p)_l p^{-l}}{\theta(Ap^{2l}/pt_4;q)\theta(1/t_3t_4,t_0t_3,t_1t_3,t_2t_3,A/pt_3,A/pt_4;q;p)_l}. \tag{A.14}$$

As is clear from (8.5) and (8.22), we have $R_m(z;q,p) = R_{m0}(z)$ and $T_n(z;q,p) = T_{n0}(z)$. These functions R_m and T_n are equal to ${}_{12}V_{11}$ elliptic hypergeometric series with particular choices of the parameters.

Corollary 10. *The functions $R_m(z;q,p)$ and $T_n(z;q,p)$ satisfy the biorthogonality condition*

$$\int_{C_{mn}} T_n(z;q,p)R_m(z;q,p)\Delta_E(z,\mathbf{t})\,\frac{dz}{z} = h_n\mathcal{N}_E(\mathbf{t})\delta_{mn}, \tag{A.15}$$

where the constants h_n are fixed in (A.12), *and the contour C_{mn} encircles the poles of the integrand located at*

$$z = \{t_{0,1,2,3}q^a p^b, t_4 p^a q^{b-m}, A^{-1}p^{a+1}q^{b+1-n}\}_{a,b\in\mathbb{N}}$$

and separates them from the poles with the inverse $(z \mapsto z^{-1})$ coordinates.

The biorthogonal rational functions $R_m(z;q,p)$ and $T_n(z;q,p)$ describe elliptic generalizations of the Rahman set of continuous ${}_{10}\Phi_9$ functions [R1] to which they are reduced in the limit as $p \to 0$. Accordingly, in this limit, formula (A.15) is reduced to the Rahman biorthogonality condition.

Appendix B. Integral representations for ${}_{12}E_{11}$ series

Here we derive an integral representation for the product of two terminating ${}_{12}E_{11}$ (more precisely, ${}_{12}V_{11}$) series with some particular choice of the parameters. For this, we apply an elliptic generalization of the technique used in [R1] for the derivation of the contour integral representation for a terminating ${}_{10}\Phi_9$ series.

Theorem 11. *Suppose that five parameters $t_k, k = 0,\dots,4$, satisfy the conditions of Theorem* 1. *Denote by m,n two positive integers and by C_{mn} a positively oriented contour such that for all $a,b \in \mathbb{N}$ it separates the points $z = \{t_k p^a q^b,\ A^{-1}q^{b+1-m}p^{a+1-n}\}$ from their partners with the inverse coordinates $(z \mapsto z^{-1})$. Under these conditions, the following integral representation for the product of two ${}_{12}V_{11}$ terminating very-well-poised balanced theta hypergeometric series at $x = 1$ holds true*:

$$\begin{aligned} &{}_{12}V_{11}\left(\frac{At_0}{q};\alpha,t_0t_1,t_0t_2,t_0t_3,t_0t_4,q^{-m},\frac{A^2q^{m-1}}{\alpha};q,p\right) \\ &\quad\times {}_{12}V_{11}\left(\frac{At_0}{p};\beta,t_0t_1,t_0t_2,t_0t_3,t_0t_4,p^{-n},\frac{A^2p^{n-1}}{\beta};p,q\right) \\ &= \frac{1}{\mathcal{N}_E(\mathbf{t})}\frac{\theta(At_0,\frac{A}{t_0};p;q)_m\theta(At_0,\frac{A}{t_0};q;p)_n}{\theta(\frac{A}{\alpha t_0},\frac{At_0}{\alpha};p;q)_m\theta(\frac{A}{\beta t_0},\frac{At_0}{\beta};q;p)_n} \\ &\quad\times\int_{C_{mn}}\Delta_E(z,\mathbf{t})\frac{\theta(\frac{Az}{\alpha},\frac{A}{\alpha z};p;q)_m\theta(\frac{Az}{\beta},\frac{A}{\beta z};q;p)_n}{\theta(Az,\frac{A}{z};p;q)_m\theta(Az,\frac{A}{z};q;p)_n}\,\frac{dz}{z}, \end{aligned} \tag{B.1}$$

where α and β are arbitrary complex parameters.

Proof. Under the conditions imposed upon the parameters in the formulation of this theorem, the following relations are true:

$$\begin{aligned}\int_{C_{ij}} \Delta_E(z,\mathbf{t})\frac{\theta(zt_0,z^{-1}t_0;p;q)_i\theta(zt_0,z^{-1}t_0;q;p)_j}{\theta(zA,z^{-1}A;p;q)_i\theta(zA,z^{-1}A;q;p)_j}\,\frac{dz}{z}\\ =\left(\frac{t_0}{A}\right)^{2ij}\mathcal{N}_E(t_0q^ip^j,t_1,\dots,t_4)\\ =\frac{\theta(t_0t_1,\dots,t_0t_4;p;q)_i\theta(t_0t_1,\dots,t_0t_4;q;p)_j}{\theta(A/t_1,\dots,A/t_4;p;q)_i\theta(A/t_1,\dots,A/t_4;q;p)_j}\mathcal{N}_E(\mathbf{t}).\end{aligned} \tag{B.2}$$

We multiply (B.2) by the factor

$$\begin{aligned}q^i\frac{\theta(At_0q^{2i-1};p)}{\theta(At_0/q;p)}\frac{\theta(At_0/q,\alpha,q^{-m},A^2q^{m-1}/\alpha;p;q)_i}{\theta(q,At_0/\alpha,At_0q^m,\alpha q^{1-m}t_0/A;p;q)_i}\\ \times p^j\frac{\theta(At_0p^{2j-1};q)}{\theta(At_0/p;q)}\frac{\theta(At_0/p,\beta,p^{-n},A^2p^{n-1}/\beta;q;p)_j}{\theta(p,At_0/\beta,At_0p^n,\beta p^{1-n}t_0/A;q;p)_j},\end{aligned}$$

where α and β are arbitrary complex parameters, and sum over i from 0 to m and over j from 0 to n. As a result, we get the relation

$$\begin{aligned}\int_{C_{mn}} \Delta_E(z,\mathbf{t})\,{}_{10}V_9(At_0/q;t_0z,t_0z^{-1},\alpha,q^{-m},A^2q^{m-1}/\alpha;q,p)\\ \times {}_{10}V_9(At_0/p;t_0z,t_0z^{-1},\beta,p^{-n},A^2p^{n-1}/\alpha;p,q)\,\frac{dz}{z}\\ = {}_{12}V_{11}(At_0/q;\alpha,q^{-m},A^2q^{m-1}/\alpha,t_0t_1,\dots,t_0t_4;q,p)\\ \times {}_{12}V_{11}(At_0/p;\beta,p^{-n},A^2p^{n-1}/\beta,t_0t_1,\dots,t_0t_4;p,q)\mathcal{N}_E(\mathbf{t}).\end{aligned}$$

Application of the Frenkel–Turaev sum to the ${}_{10}V_9$ series in the integrand leads to (B.1). □

For $n=0$ we get an integral representation of a single terminating ${}_{12}V_{11}$ series, which can be reduced further to the ${}_{10}\Phi_9$ q-series level [R1] by letting $p\to 0$. However, for $n\neq 0$ the limit as $p\to 0$ is not well defined, and formula (B.1) exists only on the elliptic level.

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