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MIKHAIL SHLEMOVICH BIRMAN
(on the occasion of his 75th birthday)

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Mikhail Shlemovich Birman reached the age of 75 on the 17th January, 2003. An eminent mathematician, he is the author of numerous fundamental results in the general spectral theory and in the spectral theory of differential operators, both ordinary and partial. He contributed much to several other fields of analysis: function space theory, approximation theory, integral operators, etc.

A detailed survey of M. B.'s achievements before 1998 can be found in [1*] (see also [3*]). For this reason, in the present note we restrict ourselves to the description of his results obtained after 1998, referring to earlier publications only when necessary.

Nowadays M. B. is as active as ever. Largely, his new results pertain to two fields: 1) the discrete spectrum of a perturbed operator in gaps of the unperturbed one, and 2) absolute continuity of the spectrum of periodic operators of mathematical physics. Before analyzing these results, we would like to emphasize a characteristic feature of M. B.'s entire work: he knows how to look at a specific problem from a bird's-eye view. Rising over individual peculiarities of a concrete question, he states a new and general problem that includes the initial one (and many others) as a particular case. Next he analyzes the general problem in abstract terms. This either automatically leads to a solution of the initial problem, or—in more complicated cases—reveals questions of analytic nature to be answered for that.

The bibliography completing this text consists of two parts. In the first part we give references to the papers [1*, 3*] mentioned above and to the list [2*] comprising the publications by M. B. over the same period of time. Five publications occurring in [2*] are also included in the first part, with the same numbers as in [2*]. For various reasons, the corresponding bibliographic data in [2*] were not quite complete, and now we give full references. The second part is the full list of subsequent publications by M. B. When making a reference to the first part, we mark the corresponding number with $*$. Also, in the text the reader may see references of the type [110*] that are absent in both parts of the list. Such a reference means the paper number 110 in the earlier list [2*].

§1. Spectrum in gaps. Threshold effects

Let A be a selfadjoint operator with a gap (λ_-, λ_+) in the spectrum. This means that the interval (λ_-, λ_+) is free of points of the spectrum, and the edges λ_- and λ_+ belong to the spectrum (or are infinite).

Let V be another operator (a "perturbation"); for definiteness, we assume that $V \geq 0$. Denote $A_\pm(\alpha) = A \mp \alpha V$. If V is subordinate to A in an appropriate sense, then for any $\alpha > 0$, the spectrum of $A_\pm(\alpha)$ in the gap of A either is empty, or consists of at most countably many eigenvalues (with regard to multiplicity), with the only possible accumulation point $\lambda_\pm$. Moreover, as α increases, the eigenvalues of the operator $A_+(\alpha)$ (of $A_-(\alpha)$) move leftwards (rightwards).

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Suppose a point λ inside the gap is chosen (the "observation point"). We denote by $N_\pm(\lambda, A, V, \alpha)$ the number of eigenvalues of $A_\pm(t)$ that cross the point λ as t grows from 0 to α. In a natural way, this definition extends to the case of $\lambda = \lambda_\pm$ (provided $\lambda_\pm \neq \pm\infty$). The problem is to study the behavior of the quantities $N_\pm(\lambda, A, V, \alpha)$ as $\alpha \to \infty$.

If the operator A is lower semibounded, and λ_+ is the leftmost point of its spectrum, then for $\lambda \leq \lambda_+$ the quantity $N_+(\lambda, A, V, \alpha)$ coincides with the number of eigenvalues of $A_+(\alpha)$ that lie to the left of λ. For instance, this is so in the well-studied case where $A = -\Delta$ (the Laplacian in $\mathbf{R}^d$), and V is the operator of multiplication by a function (potential). Then $\lambda_+ = 0$. At early stages of his work, M. B. himself contributed much to investigations devoted to this situation. First, the famous "Birman–Schwinger principle" should be mentioned, which was established by M. B. in the paper [15*] dating back to 1959. Numerous subsequent publications in this field, by many mathematicians and physicists, were based on the use of that principle.

In particular, it was discovered that two fundamentally different situations may occur in dimensions $d \geq 3$: if $V \in L^{d/2}$, then for any choice of the observation point $\lambda \leq 0$ the function $N_+(\lambda, -\Delta, V, \alpha)$ admits an asymptotic formula of Weyl type. Thus, this asymptotics is stable relative to the choice of λ, and remains valid at the edge of the gap, i.e., for $\lambda = 0$. On the other hand, if $V \notin L^{d/2}$ (but the negative spectrum of $A_+(\alpha)$ is still discrete for any $\alpha > 0$), then $N_+(\lambda, -\Delta, V, \alpha)$ may have different asymptotics for different $\lambda < 0$. For $\lambda = 0$ it may happen that $N_+ = \infty$ for all α. It is also possible that the quantity $N_+(0, -\Delta, V, \alpha)$ is finite, but grows faster than $N_+(\lambda, -\Delta, V, \alpha)$ for $\lambda < 0$, e.g., for $\lambda = -1$; namely, for some $q > d/2$ we may have

$$\begin{aligned} N_+(0, -\Delta, V, \alpha) &= O(\alpha^q), \\ N_+(-1, -\Delta, V, \alpha) &= o(\alpha^q). \end{aligned} \tag{1}$$

All this shows that such asymptotics are unstable.

M. B. suggested to call the asymptotics of the first type (and the corresponding perturbations) *regular* and to refer to the second type as *nonregular*. He argued that this classification is of general meaning and applies to other problems. This viewpoint turned out to be fruitful, which has been confirmed by M. B. in a series of papers since 1991.

In [110*], the regularity problem was stated and studied in abstract terms and in the general case where the unperturbed operator has gaps in its spectrum. General conditions that ensure the regular asymptotic behavior of $N_\pm$ were found. It is important that not only do these conditions allow one to shift the point λ inside a fixed gap, but also they make it possible to move λ from one gap to another. Moreover, conditions were established under which the asymptotics of $N_\pm$ does not change if we shift the observation point, and it also survives if we replace the unperturbed operator A by another operator close to it in the sense of quadratic forms.

The results of [110*] have found numerous applications, starting with those given in that paper itself (periodic second order operators perturbed by a decaying potential, perturbation of higher order operators by a second order differential operator) and in the papers [111*, 135*] (perturbation of the magnetic Schrödinger operator and of its pseudorelativistic analog).

The main results of [110*] pertain to the case where the observation point λ is located inside a gap. As a rule, the treatment of the gap edges requires a more detailed information about the unperturbed operator. For applications, particularly important is the case in which the unperturbed operator A is a periodic Schrödinger operator. Then, the necessary information about A is provided by the Floquet theory.

A detailed study of the behavior of the functions $N_\pm(\lambda, A, V, \alpha)$ for such operators A was accomplished by M. B. in two important papers [135*, 134*]. The investigation in [135*] (see also the survey [128*]) was devoted to the case where $d \geq 3$ and the potential perturbation is subject to the condition $V \in L^{d/2}$. It was shown that such perturbations remain regular also for λ lying at an edge of the gap.

The study of nonregular asymptotics is more difficult, in particular, because nonregularity can be caused by fairly different reasons. A possible reason (for $d \geq 3$) is in the violation of the condition $V \in L^{d/2}$. Another possible reason is more subtle. Namely, the regularity of perturbations $V \in L^{d/2}(\mathbf{R}^d)$ with $d \geq 3$ turns out to be closely related to the validity of the classical Hardy inequality $\int |x|^{-2}|u|^2\,dx \leq C(d)\int |\nabla u|^2\,dx$ for $u \in C_0^\infty(\mathbf{R}^d)$. If $d = 2$, this inequality fails on the "radial subspace" of functions $u(x) = \phi(|x|)$. This may also lead to the nonregularity of asymptotics. In order to distinguish these two types of nonregularity, we shall talk about nonregular perturbations of the first kind and the second kind, respectively.

In the paper [134*], M. B. analyzed the nonregular perturbations of the first kind. The role of the unperturbed operator is played, once again, by the periodic Schrödinger operator for $d \geq 3$. The requirements on the perturbation V are imposed in terms of the behavior of the function $N_+(\lambda, -\Delta, V, \alpha)$ for the "usual" Schrödinger operator. More precisely, it is assumed that condition (1) is fulfilled. In the case under consideration the point λ coincides with an edge of the gap. Under the previous assumption of regular behavior near λ of the band function $E_l(k)$ that determines the edge of the gap in question, a certain "model" differential operator is constructed. Roughly speaking, this is a vector-valued Schrödinger operator with matrix potential. The vector dimension of this operator, as well as the potential, are explicitly determined by the Floquet decomposition.

This model operator satisfies a condition of type (1), due to the initial requirement imposed on V. The basic result is that the asymptotic behavior of the functions $N_\pm(\lambda_\pm, A, V, \alpha)$ is the same as that of the analogous functions for the model operator constructed.

Qualitatively, this result means that the asymptotics of $N_\pm(\lambda, A, V, \alpha)$ is determined by interaction between the potential V and the Floquet data in an arbitrarily small neighborhood of λ. Therefore, it makes sense to talk about a "threshold effect" near the point λ; in the case under consideration, it lies at an edge of the gap.

The first fundamental result on nonregular asymptotics of the second kind was obtained in the paper [130*], which was devoted to the 2-dimensional Schrödinger operator ($A = -\Delta$, and V is the potential). The role of the gap of the unperturbed operator is played by the negative semiaxis. Under a certain assumption about V (for $d = 2$, the condition $V \in L^1 = L^{d/2}$ does not suffice) the function $N_+(\lambda, -\Delta, V, \alpha)$ has a Weyl type asymptotics for any $\lambda < 0$. However, this may fail for $\lambda = 0$. To describe the behavior of $N_+(0, -\Delta, V, \alpha)$, a certain auxiliary Schrödinger operator on the semiaxis should be considered. This operator (say, $A'(\alpha)$) is generated by the restriction of the quadratic form of the operator $A_+(\alpha)$ to the subspace of radial functions. Roughly speaking, the asymptotics of $N_+(\lambda, -\Delta, V, \alpha)$ for $\lambda = 0$ can be written as the sum of a Weyl type term and the asymptotics for $N_+(0, A'(\alpha))$. This phenomenon can be viewed as the emergence of a new "channel" that yields an independent contribution to the asymptotics as λ comes to the edge of the gap. Thus, here we have another realization of the threshold effect.

Sometimes, the function $N_+(0, A'(\alpha))$ grows more slowly than the Weyl term and does not influence the resulting asymptotics. In other cases, to the contrary, the additional term downbears the Weyl component. Also, a situation is possible where both terms are

of the same order, and then the function $N_+(0, -\Delta, V, \alpha)$ has a Weyl order $O(\alpha)$, but with a non-Weyl coefficient in the asymptotics.

In the paper [3], a similar result was obtained in the case where the unperturbed operator A is the 2-dimensional periodic Schrödinger operator, and the observation point λ coincides with the right edge of the semibounded gap. The full statement looks much more involved than in the previous "model" case because, like in the case of nonregular perturbations of the first kind, the construction of the auxiliary operator A' requires detailed information about the Floquet decomposition for A. In [5], waveguide type operators were considered (dimension was arbitrary, and periodicity was assumed along one of the coordinate axes). However, only the asymptotics near the left edge of the spectrum was treated in that paper.

Thus, if A is the periodic Schrödinger operator, the study of the behavior of the quantity $N_\pm(0, A, V, \alpha)$ for nonregular perturbations of both kinds requires a detailed analysis of the spectral expansion of the unperturbed operator near the edges of the gap. The same situation occurs if the unperturbed operator is the periodic Pauli operator or another periodic operator of mathematical physics. In all these cases we encounter threshold effects near the edges of the gap. Investigation of such phenomena is of considerable interest for applications.

The technique of investigating the threshold effects that was elaborated within the study of the behavior of the functions $N_\pm(0, A, V, \alpha)$ has turned out to be useful in a completely different field of applied analysis, namely, in homogenization theory. A typical problem there is the study of the limit behavior as $\varepsilon \to 0$ of the operator family $A_\varepsilon = -\operatorname{div} g(x/\varepsilon) \operatorname{grad}$ in $L^2(\mathbf{R}^d)$. Here $g(x)$ is a positive definite matrix-valued function periodic with respect to the standard lattice. In this case, the spectrum of the periodic operator A_1 lies on the positive semiaxis and contains the point $\lambda = 0$. It is known that, as $\varepsilon \to 0$, the resolvents $(A_\varepsilon + \varkappa^2 I)^{-1}$ converge weakly to the resolvent of some operator A_0 independent of ε; the homogenization procedure consists in the proof of the existence of such an operator and in the construction of it.

In [2] it was shown that the possibility of homogenization is a manifestation of the threshold effect near the left edge of the spectrum of the operator A_1. This was done not only for the problem described above, but also for a wide class of problems of mathematical physics. The recent paper [11] was devoted to the possibility of homogenization near the edges of inner gaps. It turned out that in this case the picture is more involved: the limit behavior of resolvents cannot be described in terms of a "constant" limit operator. This is caused by the fact that threshold effects near edges of inner gaps are much more complicated than near the left edge of the spectrum.

When the preparation of this paper was at its final stage, the large survey [12] appeared, devoted precisely to the relationship between the idea of a threshold effect and that of homogenization. This survey can be viewed as an important development of [2]. We can only briefly describe this publication, in which many new ideas were introduced in the vast field of homogenization theory and which, apparently, will find a broad response among the experts in homogenization. Much attention is paid (both conceptually and in volume) to developing a general pattern in order to absorb a major part of the constructions related to applications to specific operators of mathematical physics. However, as a rule, the analysis of every particular model of mathematical physics reveals specific facts that follow from the general pattern, but are not dictated by it straightforwardly. In this connection, we mention the results obtained for the Maxwell system, which are most unconventional.

As a whole, the method is oriented towards general periodic differential operators of the form $A = f(x)^* b(D)^* g(x) b(D) f(x)$ (with matrix-valued components) acting on

vector-valued functions in $\mathbf{R}^n$. On one hand, such operators admit a natural factorization $A = X^*X$; on the other hand, as usual, the periodicity of the coefficients allows one to write such operators as direct integrals over the quasimomentum k of their restrictions (supplemented by quasiperiodic boundary conditions) to the periodicity cell. These restrictions admit factorizations of the same form as above, but, unlike the initial operator, they have discrete spectrum. For them, a completely abstract perturbation method relative to the parameter $t = |k|$ can be developed near the point 0. With further applications in mind, it does not suffice to look only at the leading order of perturbation theory. Due to factorization mentioned above, this version of perturbation theory possesses many specific properties, which allow for a detailed study of the spectral characteristics for the operator A in L^2 near the lowest point 0 of its spectrum. In particular, this makes it possible to explicitly describe the singularities of the resolvent of A near 0. The role of the model in this description is played by the resolvent of an operator that is similar to A but is determined in terms of a certain effective constant matrix g^0 replacing the initial g; the explicit form of g^0 is dictated by the new version of perturbation theory described above. Ultimately, this part of [12] is aimed at applications to homogenization theory and is absolutely new in this context. Having proved such a result, the authors of [12] come to justification of the homogenization procedure rather quickly. It should be mentioned separately that the model indicated above can be used to distinguish between the leading singularities of the resolvent of A in the standard operator norm.

§2. Absolute continuity of the spectrum of periodic operators

We now turn to the description of the new results on the absolute continuity of the spectrum of periodic operators of mathematical physics obtained by M. B. All these results were established by M. B. in collaboration with T. A. Suslina.

This field of investigation originates in the following. In accordance with long-known classical facts, the Hill operator (one-dimensional Schrödinger operator on the axis with periodic potential) has absolutely continuous spectrum in $L^2(\mathbf{R})$ of multiplicity 2, which consists, in general, of an infinite system of intervals. The first multidimensional result (for the Schrödinger operator with periodic potential in $L^2(\mathbf{R}^3)$) was obtained in 1973 by L. Thomas, who proved that, under certain assumptions on the potential, the spectrum of this operator is also absolutely continuous. Afterwards, the Thomas approach and the result on absolute continuity were extended to the Schrödinger operator in arbitrary dimension $d \geq 2$. We dwell on the Thomas approach, because it has a component preserved in almost all subsequent studies of the absolute continuity property of the spectrum of differential operators with periodic coefficients, including M. B.'s work.

This component can be described as follows. The operator A under consideration expands into a direct integral over the quasimomentum. As mentioned before, the operator $A(k)$ arising in this expansion and depending on k (the initial differential operator restricted to the fundamental domain and accompanied by the corresponding quasiperiodicity conditions) has discrete spectrum. The constructions described above imply that the spectrum of A is always of band nature. For the verification of absolute continuity, it must be shown that there are no bands degenerating to a point. This a key moment of the method, which immediately implies the absolute continuity of the spectrum of the initial operator. In motivations, an important role is played by the analytic dependence on k of the eigenvalues indicated above. Due to this property, it suffices to check that the eigenvalues are nonconstant for large complex k. In its turn, this can be done with the help of perturbation theory, which shows that, in the case of the Schrödinger operator, the contribution made by the potential for such k is small.

Attempts to carry this method over to the "magnetic" periodic Schrödinger operator showed, however, that the approach does not work at the very last point: the contribution of a periodic magnetic potential cannot be controlled in the framework of perturbation theory. In [131*] this difficulty was overcome in the dimension $d = 2$: it was shown that, in a sense, the influence of the magnetic potential can be handled explicitly, and after that, again, the impact of the electric potential can be controlled with the help of perturbation theory. To realize this idea, the auxiliary operator $M_P = (\frac{1}{i}\nabla - A(x))^2 + B(x)$, $B(x) = \partial_1 A_2 - \partial_2 A_1$, was considered. It can be viewed as one of the blocks of the two-dimensional Pauli operator. The operator M_P and the corresponding operator $M_P(k)$ on the fundamental domain admit explicit factorization in the form of a product of first order periodic operators that can be studied and estimated explicitly. In particular, this made it possible to establish the fact that the norm of the operator $(M_P((\mu+iy)e_1 + ke_2))^{-1}$ decays as $o(|y|^{-1})$ as $|y| \to \infty$. Here e_1 and e_2 constitute the basis of the dual cell of periods. This is a key estimate, which allows one to apply perturbation theory to estimation of the effect produced by the initial potential V (now the latter should be replaced with $V - B$).

In the paper [138*], the results of [131*] were substantially refined. In [131*] it was assumed that the potential A is continuous and $V \in L^2$. In [138*] the same result was extended to the case where $A \in L^r, r > 2$, and $V \in L^p, p > 1$. Of course, passage to these free conditions required overcoming substantial and new analytic difficulties. However, the general pattern survived.

Further generalization was considered in [141*], namely, the magnetic Schrödinger operator with variable positive periodic metric was treated: $M = (-i\nabla - A(x))^* g(x)(-i\nabla - A(x)) + V(x)$. That paper contains also a complete survey of the results (obtained by that time and belonging not only to the authors) on the absolute continuity of the spectrum of the Schrödinger operator (not only magnetic). In almost all cases, the authors refine the statements under discussion by weakening the requirements imposed on coefficients. We cite the final results. In particular, for the periodic Schrödinger operator itself, the spectrum is absolutely continuous if $V \in L^2$, $r > 0$ $(d = 2)$; if $V \in L^0_{d/2,\infty}$ $(d = 3, 4)$; and if $V \in L^0_{d-2,\infty}$ $(d \geq 5)$. Here the so-called weak L^p-classes are used, which can be characterized briefly as the weak $L_{p,\infty}$-classes in the Lorentz scale $L_{p,q}$. In the case of the magnetic Schrödinger operator, the spectrum is shown to be absolutely continuous under the same assumptions about the electric potential and the following additional requirements on the magnetic potential A: $A \in L^{2r}, r > 1$ $(d = 2)$; $A \in C^{2d+3}$ $(d \geq 3)$. For $d = 2$ this repeats the result of the preceding paper, and for $d \geq 3$ the authors lean upon Sobolev's result in which the resolvent of the pure magnetic Schrödinger operator was estimated with the help of a different technique not involving factorization ideas. As to variable metrics, the metric studied in [141*] was of the special form $g = \omega^2 a$, where ω is a measurable periodic function bounded and bounded away from 0, and a is a constant positive matrix. The absolute continuity of the spectrum was established under the same conditions on V and A as above, and under the following additional conditions on ω: $V_\omega \in L^r, r > 1$ $(d = 2)$; $V_\omega \in L^0_{\delta,\infty}$, $\delta = \max\{d/2, d-2\}$ $(d \geq 3)$, where $V_\omega = \omega^{-1}(\operatorname{div} a\nabla > \omega)$.

In the next paper [4], again the two-dimensional periodic magnetic Schrödinger operator with variable metric was treated, but this time the electric potential was the sum of two terms: a term satisfying the usual conditions, and a delta-function type term concentrated on a system of smooth curves. Not dwelling on details, we note that for this operator the absolute continuity of its spectrum was also established. As motivations for considering such a model, possible applications should be mentioned, and also attempts

to "experimentally" find the border beyond which the spectrum fails to be absolutely continuous.

Besides the magnetic Schrödinger operator, some other periodic operators of mathematical physics were also treated in M. B.'s publications, in particular, the Dirac operator [140*] and the elasticity theory operator [7]. Each time, the basic problem was in estimating the resolvent for large values of the quasimomentum; such an estimate must show that the operator on the fundamental domain has no spectral bands reducing to a point. Here, relaxation of the conditions imposed on the coefficients is also of significant interest, and M. B.'s achievements in this direction are rather involved technically and mobilize all the wealth of his analytic experience. Our comments will not be full if we do not mention a recent result of N. D. Filonov, a student of M. B. The operator studied by Filonov was the Schrödinger operator with neither magnetic field, nor exterior potential, but with a nonsmooth variable periodic metric; he showed that under an appropriate choice of the metric the spectrum may fail to be absolutely continuous, i.e., some bands in the spectrum can degenerate to a point.

As a publication standing somewhat apart from those pertaining to the two main topics described above, we mention the survey on double operator integrals. The theory of double operators was one of the issues on which M. B. had been working in 1960–70s. These investigations continue to attract attention of experts in operator theory. In particular, in recent years new papers appeared in which various aspects of the theory of operator integrals were carried over to the context of Banach spaces. Some new applications to the Hilbert space case were also found. All this prompted M. B. and M. Z. Solomyak to publish the paper [13] where a general survey of that theory and its applications (in the Hilbert space framework) was given.

In conclusion, we would like to note that, as before, in the past years M. B. has been an active and productive member of two Dissertation Councils and the Editorial Boards of the journals "Algebra i Analiz" and "Funktsional′nyĭ Analiz i ego Prilozheniya".

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