Алгебра и анализ
Том. 16 (2004), вып. 1

St. Petersburg Math. J.
Vol. 16 (2005), No. 1, Pages 9–24
S 1061-0022(04)00842-8
Article electronically published on December 14, 2004

ISOMETRIC EMBEDDINGS OF FINITE-DIMENSIONAL ℓ_p-SPACES OVER THE QUATERNIONS

YU. I. LYUBICH AND O. A. SHATALOVA

Dedicated to M. Sh. Birman on the occasion of his 75th birthday

Abstract. The nonexistence of isometric embeddings $\ell_q^m \to \ell_p^n$ with $p \neq q$ is proved. The only exception is $q = 2$, $p \in 2\mathbb{N}$, in which case an isometric embedding exists if n is sufficiently large, $n \geq N(m,p)$. Some lower bounds for $N(m,p)$ are obtained by using the equivalence between the isometric embeddings in question and the cubature formulas for polynomial functions on projective spaces. Even though only the quaternion case is new, the exposition treats the real, complex, and quaternion cases simultaneously.

Contents

§1. Introduction

The quaternions $\xi = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ with real coefficients a, b, c, d form a field $\mathbb{H}$, the third of the classical triad $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, where the first two are commutative, in contrast to $\mathbb{H}$. This difference does not obstruct using $\mathbb{H}$ as a ground field in linear algebra and the theory of normed spaces, though, of course, we must distinguish between right and left linear spaces in the noncommutative setting. However, the quaternion situation becomes much more complicated as soon as some nonlinear objects (say, polynomials) become involved. In particular, this is true for the subject of this paper, but, fortunately, the difficulties can be overcome in our case. Since, actually, our approach is independent of the choice of the ground field, it is expedient to present a unified theory, where the ground field is denoted by an indifferent letter, say $\mathbb{K}$.

Regardless of $\mathbb{K}$, we denote by ℓ_p^n the (right, for definiteness) $\mathbb{K}$-linear space $\mathbb{K}^n$ equipped with the ℓ_p-norm

$$\|x\|_p = \left(\sum_{k=1}^{n} |\xi_k|^p \right)^{\frac{1}{p}}, \tag{1.1}$$

2000 *Mathematics Subject Classification.* Primary 46B04.
Key words and phrases. Isometric embeddings, cubature formulas, addition theorem.

where $x = (\xi_k)_1^n$, $\xi_k \in \mathbb{K}$ $(1 \le k \le n)$, and $1 \le p \le \infty$. (In fact, $\|x\|_\infty = \max|\xi_k|$, as usual.)

The space ℓ_2^n is a Euclidean space provided with the inner product

$$(x, y) = \sum_{k=1}^{n} \bar{\xi}_k \eta_k, \quad x = (\xi_k)_1^n, \ \ y = (\eta_k)_1^n, \tag{1.2}$$

where in the quaternion case the conjugate to ξ is $\bar{\xi} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$. (Note that $\overline{\xi\zeta} = \bar{\zeta}\bar{\xi}$ and $\bar{\xi}\xi = a^2 + b^2 + c^2 + d^2 = |\xi|^2$.) This inner product has usual properties; in particular, $(x\alpha, y) = \bar{\alpha}\,(x, y)$ while $(x, y\alpha) = (x, y)\,\alpha$. Of course, these properties determine an abstract Euclidean structure, but all Euclidean spaces of the same dimension are isometric, so ℓ_2 is a universal model for all of them. Basically, the quaternion Euclidean geometry is the same as the real or complex one. For instance, the general form of linear functionals is $x \mapsto (\cdot, x)$.

We consider the isometric embeddings $\ell_q^m \to \ell_p^n$, $1 \le p, q \le \infty$, assuming $p \ne q$ and $2 \le m \le n$ to avoid some trivial situations. It turns out that the existence of such embeddings is a special phenomenon.

Theorem 1. *If there exists an isometric embedding $\ell_q^m \to \ell_p^n$, then either $q = 2$ and $p \in 2\mathbb{N}$, or $\mathbb{K} = \mathbb{R}$ and $\{p, q\} = \{1, \infty\}$.*

In the real case the theorem was proved in [22] under the assumption $q = 2$ and later in [25] without this assumption. In §2 we present a proof over any $\mathbb{K}$.

After this result, only the isometric embeddings $\ell_2^m \to \ell_p^n$ with $2 \le m < n$, $p \in 2\mathbb{N}$, $p \ge 4$ remain in question. The simplest situation is $\mathbb{K} = \mathbb{R}$, $m = 2$, $p = 4$. It turns out that there exists an isometric embedding $f : \ell_2^2 \to \ell_4^3$ [22]. Indeed, if $x = (\xi_1, \xi_2) \in \ell_2^2$ and

$$fx = \frac{1}{\sqrt[4]{2}} \left(\xi_1 + \frac{1}{\sqrt{3}}\xi_2,\, \xi_1 - \frac{1}{\sqrt{3}}\xi_2,\, \sqrt{\frac{4}{3}}\xi_2 \right) \in \ell_4^3, \tag{1.3}$$

then $\|fx\| = \|x\|$ by the elementary identity

$$\left(\xi_1 + \frac{1}{\sqrt{3}}\xi_2\right)^4 + \left(\xi_1 - \frac{1}{\sqrt{3}}\xi_2\right)^4 + \left(\sqrt{\frac{4}{3}}\xi_2\right)^4 = 2\left(\xi_1^2 + \xi_2^2\right)^2. \tag{1.4}$$

Another identity (due to Lucas, 1876) yields an isometric embedding $\ell_2^4 \to \ell_4^{12}$ over $\mathbb{R}$ [26]. Many identities of similar kind were mentioned in [8] in their relation to Waring's problem in number theory. Each of them yields an isometric embedding $\ell_2^m \to \ell_p^n$.

Let $\{e_k\}_1^n$ be the canonical basis of ℓ_p^n, and let $f : \ell_2^m \to \ell_p^n$ be a linear mapping. Then

$$fx = \sum_{k=1}^{n} e_k \xi_k(x), \quad x \in \ell_2^m,$$

where the $\xi_k(x)$ are the coordinates of x. Since they are linear functionals of x, we have a uniquely determined system $\{u_k\}_1^n$ of vectors in ℓ_2^m such that $\xi_k(x) = (u_k, x)$, $1 \le k \le n$. Thus,

$$fx = \sum_{k=1}^{n} e_k\,(u_k, x)\,. \tag{1.5}$$

The system $\{u_k\}_1^n$ is called the *frame* of f [25]. Obviously, any system $\{u_k\}_1^n$ in ℓ_2^m is the frame of a linear mapping $f : \ell_2^m \to \ell_p^n$. The latter is an embedding (not isometric, in general) if and only if its frame is complete in the space ℓ_2^m.

Lemma 1. *A linear mapping $f : \ell_2^m \to \ell_p^n$ is isometric if and only if its frame satisfies the* basic identity

$$\sum_{k=1}^{n} |(u_k, x)|^p = (x, x)^{p/2}, \quad x \in \ell_2^m. \tag{1.6}$$

Proof. The identity (1.6) means precisely that $\|fx\|_p = \|x\|_2$, in view of (1.1) and (1.5). □

An important equivalent version of the basic identity follows from the formula

$$(x, x)^{p/2} = \frac{1}{\gamma_{m,p}} \int_S |(y, x)|^p \, \mathrm{d}\sigma(y), \quad x \in \ell_2^m, \tag{1.7}$$

where $S = S(m, \mathbb{K})$ is the unit sphere of ℓ_2^m, σ is the unitarily invariant normed measure on S, and $\gamma_{m,p}$ is a positive constant (depending on the ground field). In the real case, identity (1.7) goes back to Hilbert [14], who used it to solve Waring's problem. We shall call (1.7) the *Hilbert identity* regardless of the ground field. The constant $\gamma_{m,p}$ can be calculated explicitly but we do not need it in this paper. Modulo this detail, (1.7) can be proved easily. The point is that the right-hand side of (1.7) is a unitarily invariant homogeneous polynomial of x on the real $\ell_2^{\delta m}$-space with

$$\delta = \delta(\mathbb{K}) = 1, 2, 4 \tag{1.8}$$

for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively. The degree of this polynomial is p. Since the group of isometries of l_2^m acts on S transitively, the only such polynomial is $(x, x)^{p/2}$ up to a constant factor.

Corollary. *An isometric embedding $\ell_2^m \to \ell_p^n$ exists if and only if the identity*

$$\sum_{k=1}^{n} |(v_k, x)|^p \lambda_k = \int_S |(y, x)|^p \, \mathrm{d}\sigma(y), \quad x \in \ell_2^m, \tag{1.9}$$

is fulfilled for a system $\{v_k\}_1^n$ in ℓ_2^m and some positive coefficients λ_k, $1 \le k \le n$.

Proof. The substitution

$$v_k = u_k \left(\frac{\gamma_{m,p}}{\lambda_k} \right)^{1/p}, \quad 1 \le k \le n,$$

turns (1.9) into (1.6), and *vice versa.* □

In [29] and [25] (two independent works, both related to $\mathbb{K} = \mathbb{R}$) the following important interpretation of (1.9) was observed: this is a *cubature formula*

$$\int_S \phi(y) \, \mathrm{d}\sigma(y) = \sum_{k=1}^{n} \phi(w_k) \rho_k \tag{1.10}$$

with nodes $w_k = v_k / \|v_k\|$ on S and weights $\rho_k = \lambda_k \|v_k\|^p$ such that the formula is valid for all functions

$$\phi_x(y) = |(y, x)|^p, \quad x \in l_2^m. \tag{1.11}$$

In what follows, these functions will be called the *elementary polynomials.*

A fortiori, if a cubature formula (1.10) *with n nodes is valid for all homogeneous polynomials of degree p on $\mathbb{R}^{\delta m}$, then an isometric embedding $\ell_2^m \to \ell_p^n$ over $\mathbb{K}$ exists.* In such a way a rich collection of isometric embeddings over $\mathbb{R}$ was produced in [25, 29].

The simplest example is the *Gauss–Chebyshev formula* on the unit circle in the real Euclidean plane ℓ_2^2. With $p + 2$ nodes placed at the vertices of a regular $(p + 2)$-gon, this formula yields an isometric embedding $\ell_2^2 \to \ell_p^{p/2+1}$ (for $p = 4$ we return to (1.3)). Here

$n = p/2 + 1$ because the centrally symmetric nodes can be joined. Moreover, this value of n turns out to be minimal [25, 29].

Another minimal real isometric embedding is $\ell_2^3 \to \ell_4^6$ comes from the cubature formula constructed in [9] by taking the vertices of the regular icosahedron as nodes. This example was generalized by Sobolev [31] who suggested to place the nodes on an orbit of a finite subgroup of the group $O(2)$. Then the degree of polynomial exactness of the formula can be found as the maximal degree of the nonvanishing invariant harmonic polynomials. For a further development of Sobolev's method see, e.g., [1, 12, 32].

In all cubature formulas based on group orbits, the weights are equal, $\lambda_k = 1/n$, $1 \leq k \leq n$. In general, the set of nodes $\{w_k\}_1^n$ in (1.10) with equal weights is called a *spherical design*. This concept (over $\mathbb{R}$) was introduced by Delsart, Goethals, and Seidel in their seminal paper [7], and since then a large number of papers on this subject have been written. Note that the quadrature formulas with equal weights on the segment are classical, due to Chebyshev. Their multidimensional analogs (in general, not spherical) were studied recently in [21]. However, the spherical designs are especially important from the geometric point of view. On the other hand, they are continual counterparts of some purely combinatorial objects such as association schemes, etc. (see, e.g., [4, 11]). A unified concept of a *Delsarte space* was introduced by Neumaier in his fundamental (but, unfortunately, not available at present) Memorandum [28]. This includes an extension of the theory of spherical designs to all classical fields (and even the Cayley numbers), which actually was started in [6, 16] and further developed by Bannai and Hoggar [2, 3, 17, 18]. The information contained in these papers can be partially incorporated into the context of isometric embeddings. For instance, a minimal isometric embedding $\ell_2^2 \to \ell_4^{10}$ over $\mathbb{H}$ corresponds to a design described in [17] (see Example 3 on p. 243 therein).

A number of isometric embeddings over $\mathbb{C}$ were found by König [19], the minimal embeddings $\ell_2^2 \to \ell_4^4$ and $\ell_2^2 \to \ell_6^6$ among them. Later they were reconstructed in [24], where a complex version of Sobolev's method was elaborated and applied to finite subgroups of $SU(2)$. This also yields some new minimal isometric embeddings, namely, $\ell_2^2 \to \ell_8^{10}$ and $\ell_2^2 \to \ell_{10}^{12}$ over $\mathbb{C}$.

The existence of spherical designs with given m, p and large n follows from the general Seymour–Zaslavski theorem [30]. The proof of this theorem in [30] is rather complicated. However, in our setting the weights may fail to be equal. This allows for a simpler proof and (more importantly) leads to a reasonable upper bound for n. Namely, in §3 we prove the following statement.

Theorem 2. *For given m, p, there exists an isometric embedding $\ell_2^m \to \ell_p^n$ with $n \leq \Lambda(m,p)$, where*

$$\Lambda(m,p) = \begin{cases} \binom{m+p-1}{m-1} & \text{over } \mathbb{R}, \\ \binom{m+p/2-1}{m-1}^2 & \text{over } \mathbb{C}, \\ \frac{1}{2m-1}\binom{2m+p/2-2}{2m-2}\binom{2m+p/2-1}{2m-2} & \text{over } \mathbb{H}. \end{cases} \tag{1.12}$$

In the context of identities of the form (1.9), the bound

$$n \leq \binom{m+p-1}{m-1} + 1 \tag{1.13}$$

over $\mathbb{R}$ (and, formally, for $m = 5$) is a byproduct of Hilbert's key lemma from [14]. Hilbert's original proof was simplified by many authors. In particular, a geometric proof based on Carathéodory's theorem on convex combinations was given in [10]. Later this way was rediscovered in [26], and (1.13) for the isometric embeddings was obtained. The refined bound

$$n \le \binom{m+p-1}{m-1} \tag{1.14}$$

needs a small modification of that argument; see [25, 29]. (Note that (1.12) for $\mathbb{K} = \mathbb{R}$ and $\mathbb{C}$ can be improved by 1; see [5].) Actually, the binomial coefficient in (1.13) comes from Carathéodory's theorem as the dimension of the linear space of all homogeneous polynomials of degree p on $\mathbb{R}^m$.

The Carathéodory's theorem based approach to the complex and quaternion cases in Theorem B also requires to treat $\Lambda(m,p)$ as the dimension of a relevant polynomial space. In the complex case such a space was specified in [19] to get (1.12) over $\mathbb{C}$. The following definition (cf. [13]) describes the same spaces over $\mathbb{R}$ and $\mathbb{C}$ in equivalent terms, and yields what we need over $\mathbb{H}$.

Definition. A function $\phi : \mathbb{K}^m \to \mathbb{C}$ belongs to the class $\Phi_{\mathbb{K}}(m,p)$ if

a) ϕ is a polynomial on the real space $\mathbb{R}^{\delta m}$;

b) ϕ is *absolutely homogeneous* of degree p in the sense that

$$\phi(x\alpha) = \phi(x)|\alpha|^p \quad (x \in \mathbb{K}^m, \alpha \in \mathbb{K}). \tag{1.15}$$

In particular, $\phi(x\alpha) = \phi(x)$ for $|\alpha| = 1$, i.e., ϕ is invariant under the natural action of the multiplicative group

$$U(\mathbb{K}) = \{\alpha \in \mathbb{K} : |\alpha| = 1\}.$$

For this reason, the restrictions $\phi|S$, $\phi \in \Phi_{\mathbb{K}}(m,p)$, are well defined on the projective space $\mathbb{KP}^{m-1}$, which can be identified with $S/U(\mathbb{K})$ in a natural way.

Obviously, $\Phi_{\mathbb{K}}(m,p)$ is a finite-dimensional complex linear space. In particular, the space $\Phi_{\mathbb{R}}(m,p)$ consists of all (complex-valued) homogeneous polynomials of degree p on $\mathbb{R}^m$. The monomials

$$\xi_1^{i_1} \cdots \xi_m^{i_m}, \quad (\xi_k)_1^m \subset \mathbb{R}^m, \tag{1.16}$$

with $i_1 + \cdots + i_m = p$ constitute a basis of $\Phi_{\mathbb{R}}(m,p)$.

In the space $\Phi_{\mathbb{C}}(m,p)$, a natural basis consists of all "monomials"

$$\xi_1^{i_1} \cdots \xi_m^{i_m} \bar{\xi}_1^{j_1} \cdots \bar{\xi}_m^{j_m}, \quad (\xi_k)_1^m \subset \mathbb{C}^m, \tag{1.17}$$

where $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_m)$ run independently over all nonnegative m-tuples such that $i_1 + \cdots + i_m = j_1 + \cdots + j_m = p/2$. Thus, the space $\Phi_{\mathbb{C}}(m,p)$ coincides with that of [19].

The structure of $\Phi_{\mathbb{H}}(m,p)$ is much more complicated, since the quaternion analogs of (1.17) are not absolutely homogeneous because of the noncommutativity of $\mathbb{H}$. However, all elementary polynomials are absolutely homogeneous.

Fact. *The set of elementary polynomials is complete in* $\Phi_{\mathbb{K}}(m,p)$.

In other words, *any function* $\phi \in \Phi_{\mathbb{K}}(m,p)$ *is a linear combination of elementary polynomials.*

In the real case this fact is well known (see [29] for a proof and references). Another proof given in [25] can be extended to the complex case; see [23]. This is impossible to do in the quaternion case for the reason mentioned above. Nevertheless, there is a general proof based on the addition theorem for the harmonic functions in $\Phi_{\mathbb{K}}(m,p)$. The latter is formulated in §4.

The addition theorem for $\Phi_{\mathbb{R}}(m,p)$ is classical (see, e.g., [27]), but its generalization to $\Phi_{\mathbb{C}}(m,p)$ is rather recent [20]. In [15], an $\mathbb{H}$-version of the addition theorem was established for the symplectically (i.e., l_2-isometrically over $\mathbb{H}$) invariant irreducible subspaces in $L_2(\mathbb{KP}^{m-1})$. The equivalence of this result to our version is not obvious; see [13].

In §4 we formulate our addition theorem and use it to prove the dimension formula

$$\dim \Phi_{\mathbb{K}}(m,p) = \Lambda(m,p). \tag{1.18}$$

Over $\mathbb{R}$ and $\mathbb{C}$ this formula is elementary in view of the existence of the monomial bases (1.16) and (1.17). Note that the real-valued functions in $\Phi_{\mathbb{K}}(m,p)$ constitute a real subspace of the same dimension.

So far we have focused on how the cubature formulas lead to isometric embeddings. Remarkably, this way is invertible; see [25, 29] for $\mathbb{K}=\mathbb{R}$ and [19] for $\mathbb{K}=\mathbb{C}$.

We shall say that a cubature formula (1.10) is of *index p over $\mathbb{K}$* if it is valid for all functions $\phi \in \Phi_{\mathbb{K}}(m,p)$. In this case an isometric embedding $\ell_2^m \to \ell_p^n$ over $\mathbb{K}$ does exist because all ϕ_x belong to $\Phi_{\mathbb{K}}(m,p)$. Conversely, the Fact formulated above implies the following statement.

Lemma 2. *Any isometric embedding $\ell_2^m \to \ell_p^n$ is generated by a cubature formula* (1.10) *of index p.*

In its turn, Lemma 2 leads to the following result (again, see [25, 29] for $\mathbb{K}=\mathbb{R}$ and [19] for $\mathbb{K}=\mathbb{C}$).

Theorem 3. *For any isometric embedding $\ell_2^m \to \ell_p^n$, we have the inequality*

$$n \geq \Lambda(m,p/2). \tag{1.19}$$

Over $\mathbb{C}$ and $\mathbb{H}$ the quantity $\Lambda(m,p/2)$ is defined by (1.12) only if $p \equiv 0 \pmod 4$. In the case where $p \equiv 2 \pmod 4$, we must replace $p/2$ by the integral part $[p/4]$ in the corresponding expressions.

In terms of the minimal $n = N(m,p)$ such that an isometric embedding $\ell_2^m \to \ell_p^n$ exists, Theorems 2 and 3 can be joined in the inequality

$$\Lambda(m,p/2) \leq N(m,p) \leq \Lambda(m,p). \tag{1.20}$$

Obviously, for any $n \geq N(m,p)$ an isometric embedding $\ell_2^m \to \ell_p^n$ exists.

The proof of Theorem 3 is presented in §5. The method we use goes back to [7] but with a reference to our version of the addition theorem. Furthermore, we must allow for cubature formulas with nonequal weights (i.e., not only designs) when dealing with an arbitrary isometric embedding. In this context, we prove that *the weights must be equal for* tight *embeddings*, i.e., in the case where the lower bound (1.19) is attained. Thus, we see that, actually, the frames of tight embeddings are designs, so the design theory is applicable automatically to this case.

Note that, formally, the same lower bound for designs exists over any $\mathbb{K}$ (see [3]), but with a different construction of the function spaces involved, as has already been said. Therefore, that bound is not applicable immediately to the isometric embeddings.

In conclusion, we emphasize that the image of any isometric embedding of a Euclidean space in a normed space X is a Euclidean subspace of X. Thus, all results mentioned above admit an immediate reformulation in terms of Euclidean subspaces of ℓ_p^n.

§2. Proof of Theorem 1

We assume that an isometric embedding $\ell_q^m \to \ell_p^n$ with $1 \leq p,q \leq \infty$, $p \neq q$, $2 \leq m \leq n$ exists. In view of the canonical isometric embedding $\ell_q^2 \to \ell_p^n$, we have an isometric embedding $f : \ell_q^2 \to \ell_p^n$. Let

$$f \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}, \quad f \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix}.$$

Then

$$f\begin{bmatrix}1\\ \zeta\end{bmatrix}=\begin{bmatrix}\xi_1+\eta_1\zeta\\ \vdots\\ \xi_n+\eta_n\zeta\end{bmatrix},\quad \zeta\in\mathbb{K}.$$

Since f is isometric, we have the identity

$$\sum_{k=1}^{n}|\xi_k+\eta_k\zeta|^p=(1+|\zeta|^q)^{\frac{p}{q}},\quad \zeta\in\mathbb{K}, \tag{2.1}$$

in the case where $p,q<\infty$.

Let ν denote the number of $\xi_k\neq 0$; we assume that $\xi_{\nu+1}=\cdots=\xi_n=0$ for definiteness. Then (2.1) can be rewritten as

$$\sum_{k=1}^{\nu}\alpha_k|1+\gamma_k\zeta|^p+\alpha_0|\zeta|^p=(1+|\zeta|^q)^{\frac{p}{q}},\quad \zeta\in\mathbb{K}, \tag{2.2}$$

with

$$\alpha_0=\sum_{k=\nu+1}^{n}|\eta_k|^p,\quad \alpha_k=|\xi_k|^p,\ \ \gamma_k=\xi_k^{-1}\eta_k\ \ (1\le k\le\nu). \tag{2.3}$$

For a while we restrict (2.2) to $\zeta\in\mathbb{R}$. Passing to the limit as $\zeta\to\infty$, we get

$$\alpha_0+\sum_{k=1}^{\nu}\alpha_k|\gamma_k|^p=1. \tag{2.4}$$

Since all quantities in (2.4) are nonnegative, we have $0\le\alpha_0\le 1$, and $\alpha_0=1$ if and only if all γ_k are equal to 0.

Yet another relation, namely,

$$\sum_{k=1}^{\nu}\alpha_k=1, \tag{2.5}$$

follows at $\zeta=0$.

Since

$$|1+\gamma_k\zeta|^p=(1+2\lambda_k\zeta+\mu_k\zeta^2)^{\frac{p}{2}}, \tag{2.6}$$

where

$$\lambda_k=\mathrm{Re}(\gamma_k),\ \ \mu_k=|\gamma_k|^2,\quad 1\le k\le\nu, \tag{2.7}$$

we obtain

$$\begin{aligned}&\sum_{k=1}^{\nu}\alpha_k|1+\gamma_k\zeta|^p\\ &\quad=\sum_{k=1}^{\nu}\alpha_k+p\Bigg(\sum_{k=1}^{\nu}\alpha_k\lambda_k\Bigg)\zeta+\frac{p}{2}\Bigg(\sum_{k=1}^{\nu}\alpha_k\left(\mu_k+(p-2)\lambda_k^2\right)\Bigg)\zeta^2+O\left(|\zeta|^3\right)\end{aligned} \tag{2.8}$$

by Taylor's expansion at $\zeta=0$.

On the other hand,

$$(1+|\zeta|^q)^{\frac{p}{q}}-\alpha_0|\zeta|^p=1-\alpha_0|\zeta|^p+\frac{p}{q}|\zeta|^q+O\left(|\zeta|^{2q}\right). \tag{2.9}$$

By (2.2), the expansions (2.8) and (2.9) coincide in a real neighborhood of the origin.

Now we observe that $\gamma_k \neq 0$ for some k. Indeed, if all γ_k are equal to zero, then $\alpha_0 = 1$ and (2.2) takes the form

$$1 = (1 + |\zeta|^q)^{\frac{p}{q}} - |\zeta|^p,$$

which is obviously incorrect.

As a result, the coefficient of ζ^2 in (2.8) is not zero.

Let $p > q$ and $\alpha_0 \neq 0$. Since the left-hand sides of (2.8) and (2.9) coincide locally, we conclude that $q = 2$. Moreover, p is an even integer since the function (2.8) is real analytic near $\zeta = 0$.

Now, let $p < q$ and $\alpha_0 \neq 0$. Then comparing (2.9) and (2.8) shows that $p = 2$ and q is an even integer. Our identity turns into

$$(1 + \zeta^q)^2 = [\omega(\zeta)]^q, \tag{2.10}$$

where $\omega(\zeta)$ is a quadratic polynomial. The left-hand side of (2.10) has q distinct complex roots against at most 2 distinct roots on the right. Therefore, $q = 2$, whence $q = p$, the case excluded from the very beginning.

Finally, let $\alpha_0 = 0$. Then $q = 2$ again, and

$$\sum_{k=1}^{\nu} \alpha_k \left(1 + 2\lambda_k \zeta + \mu_k \zeta^2\right)^{\frac{p}{2}} = \left(1 + \zeta^2\right)^{\frac{p}{2}}. \tag{2.11}$$

Since the local analyticity argument fails, we resort to analytic continuation.

Suppose, to the contrary, that p is not an even integer. Then the singularities in (2.11) are $\zeta = \pm\mathbf{i}$ and the roots of the polynomials $1 + 2\lambda_k\zeta + \mu_k\zeta^2$ (except for the double roots in the case of an odd integer p). Hence, at least one of the latter polynomials must be $1 + \zeta^2$. In fact, all of them are $1 + \zeta^2$. Indeed, let $1 + 2\lambda_k\zeta + \mu_k\zeta^2 \neq 1 + \zeta^2$ for some k, and let Q be the set of all such k, $1 \leq k \leq \nu$. Then

$$\sum_{k\in Q} \alpha_k \left(1 + 2\lambda_k \zeta + \mu_k \zeta^2\right)^{\frac{p}{2}} = \Big(1 - \sum_{k\notin Q} \alpha_k\Big) \left(1 + \zeta^2\right)^{\frac{p}{2}}, \tag{2.12}$$

whence,

$$\sum_{k\notin Q} \alpha_k = 1 \tag{2.13}$$

(otherwise, the roots $\zeta = \pm\mathbf{i}$ would appear on the left-hand side of (2.12)). Comparing (2.13) and (2.5), we get

$$\sum_{k\in Q} \alpha_k = 0,$$

which contradicts the conditions $\alpha_k > 0$, $Q \neq \varnothing$. Thus, all μ_k are equal to 1 and all λ_k are equal to 0, i.e., all $\mathrm{Re}(\gamma_k)$ are equal to 0. This means that all γ_k are equal to 0 if $\mathbb{K} = \mathbb{R}$, a contradiction. If $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$, we need an additional step. Namely, we return to the initial identity (2.2) and replace ζ by $\zeta\mathbf{i}$. Then $\gamma_k \mapsto \gamma_k\mathbf{i}$, $1 \leq k \leq \nu$. Hence, if $\mathbb{K} = \mathbb{C}$, then $\mathrm{Im}(\gamma_k) = 0$, $1 \leq k \leq \nu$. Ultimately, we see that all γ_k are equal to 0 if $\mathbb{K} = \mathbb{C}$. The same result for $\mathbb{K} = \mathbb{H}$ follows by substitutions $\zeta \mapsto \zeta\mathbf{i}, \zeta\mathbf{j}, \zeta\mathbf{k}$.

It remains to consider the limit cases where $q = \infty$ and $p < \infty$, or $p = \infty$ and $q < \infty$. Accordingly, we have two limit forms of (2.2): either

$$\sum_{k=1}^{\nu} \alpha_k |1 + \gamma_k\zeta|^p + \alpha_0|\zeta|^p = \left(\max\left(1, |\xi|\right)\right)^p, \quad 1 \leq p < \infty, \tag{2.14}$$

or

$$\max\left(\alpha_0|\zeta|, \max_{1\leq k\leq\nu} \alpha_k|1 + \gamma_k\zeta|\right) = \left(1 + |\zeta|^q\right)^{\frac{1}{q}}, \quad 1 \leq q < \infty, \tag{2.15}$$

both identities with $\zeta \in \mathbb{K}$ and $\alpha_0 \geq 0$, $\alpha_k > 0$, $\gamma_k \in \mathbb{K}$ $(1 \leq k \leq \nu)$ as before. First, we prove that *identity* (2.14) *is impossible over* $\mathbb{C}$ *or* $\mathbb{H}$*, and the only case over* $\mathbb{R}$ *is where* $p = 1$. From (2.14) it follows that

$$\sum_{k=1}^{\nu} \alpha_k |1 + \gamma_k \zeta|^p \leq 1, \quad |\zeta| \leq 1. \tag{2.16}$$

We see that the function on the left attains its maximum on $[-1, 1]$ at $\zeta = 0$. Consequently, its second derivative at this point is nonpositive, i.e.,

$$\sum_{k=1}^{\nu} \alpha_k \left(|\mathrm{Im}(\gamma_k)|^2 + (p-1)\mathrm{Re}(\gamma_k)^2\right) \leq 0.$$

Since $\alpha_k > 0$ for all k, and $p \geq 1$, we obtain

$$\mathrm{Im}(\gamma_k) = 0, \;\; (p-1)\,\mathrm{Re}(\gamma_k) = 0 \quad (1 \leq k \leq \nu).$$

This implies $\gamma_k = 0$, $1 \leq k \leq \nu$, if $p > 1$, but with $p = 1$ we have only $\mathrm{Im}(\gamma_k) = 0$, $1 \leq k \leq \nu$, i.e., all γ_k are real in this case. This says nothing for $\mathbb{K} = \mathbb{R}$, but for $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$ one can replace γ_k by $\gamma_k \mathbf{i}$ (by the substitution $\zeta \mapsto \zeta \mathbf{i}$ in (2.16)). Therefore, all γ_k are equal to 0 again. With this property, (2.14) becomes trivially false.

Now we prove that *identity* (2.15) *is impossible over* $\mathbb{C}$ *or* $\mathbb{H}$*, and the only case over* $\mathbb{R}$ *is where* $q = 1$. Indeed, if $\mathbb{K} = \mathbb{R}$, then the function on the left-hand side of (2.15) is piecewise linear. In particular, we have

$$(1 + \zeta^q)^{\frac{1}{q}} = \alpha + \beta\zeta \tag{2.17}$$

on a small interval $(0, \varepsilon)$. If $q > 1$, then (2.17) is incompatible with Taylor's expansion.

Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$. Then

$$(1 + |\zeta|^q)^{\frac{1}{q}} = \max_{1 \leq k \leq \nu} \alpha_k |1 + \gamma_k \zeta| \tag{2.18}$$

on a small "disk" $D_\varepsilon = \{\zeta \in \mathbb{K} : |\zeta| < \varepsilon\}$. Geometrically, this means that

$$D_\varepsilon = \bigcup_{k=1}^{\nu} D_\varepsilon^{(k)}, \tag{2.19}$$

where

$$D_\varepsilon^{(k)} = \left\{\zeta \in D_\varepsilon : \alpha_k |1 + \gamma_k \zeta| = (1 + |\zeta|^q)^{\frac{1}{q}}\right\}.$$

If for some k either $0 \notin D_\varepsilon^{(k)}$ or 0 is an isolated point in $D_\varepsilon^{(k)}$, then we can reduce ε so that $D_\varepsilon^{(k)} = \varnothing$. We may assume that such k's are missing in (2.19).

We fix k and let $\zeta \in D_\varepsilon^{(k)}$. Then

$$\alpha_k^2 \left(1 + 2\,\mathrm{Re}\,(\gamma_k \zeta) + |\gamma_k|^2 |\zeta|^2\right) = 1 + \frac{2}{q}|\zeta|^q + O\left(|\zeta|^{2q}\right).$$

Letting $\zeta \to 0$, we see that $\alpha_k = 1$, $\gamma_k \neq 0$, and

$$\mathrm{Re}\left(\gamma_k \frac{\zeta}{|\zeta|}\right) = \frac{1}{q}|\zeta|^{q-1} + o\,(1)\,,$$

so that

$$\lim_{\zeta \to 0, \zeta \in D_\varepsilon^{(k)}} \mathrm{Re}\left(\gamma_k \frac{\zeta}{|\zeta|}\right) = \frac{1 - \mathrm{sgn}(q-1)}{q}.$$

Taking any ω with $|\omega| = 1$, we consider the sequence

$$\left\{\frac{\varepsilon\omega}{n+1}\right\}_{n=1}^{\infty} \subset D_\varepsilon.$$

By (2.19), there is a number k, $1 \le k \le \nu$, such that $D_\varepsilon^{(k)}$ contains an infinite subsequence of this sequence. Then

$$\operatorname{Re}(\gamma_k \omega) = \frac{1 - \operatorname{sgn}(q-1)}{q}, \quad 1 \le k \le \nu. \tag{2.20}$$

We conclude that the disk D_ε is covered by the real "hyperplanes" (2.20), a contradiction. □

§3. Proof of Theorem 2

The Hilbert identity shows that the function

$$y \mapsto \int_S |(y,x)|^p \, \mathrm{d}\sigma(x), \quad y \in S, \tag{3.1}$$

belongs to the closed convex hull of the family of real-valued functions $\phi_x(y) = |(y,x)|^p$, $x \in S$. By Carathéodory's theorem, there exists a number ν,

$$1 \le \nu \le \dim \Phi_{\mathbb{K}}(m,p), \tag{3.2}$$

and a subset $\{x_k\}_1^{\nu+1} \in S$ such that the function (3.1) is a convex combination of functions ϕ_x, i.e.,

$$\int_S |(y,x)|^p \, \mathrm{d}\sigma(x) = \sum_{k=1}^{\nu+1} |(y,x_k)|^p \varrho_k,$$

where

$$\varrho_k \ge 0, \quad \sum_{k=1}^{\nu+1} \varrho_k = 1.$$

Without loss of generality, we can assume that

$$\int_S |(y,x)|^p \, \mathrm{d}\sigma(x) = \sum_{k=1}^{n} |(y,x_k)|^p \varrho_k, \tag{3.3}$$

where $n \le \nu + 1$, and

$$\varrho_k > 0, \quad \sum_{k=1}^{n} \varrho_k = 1. \tag{3.4}$$

Suppose that $n = \nu + 1$, where $\nu = \dim \Phi_{\mathbb{K}}(m,p)$. Then the functions $\phi_{x_k}(y)$, $1 \le k \le n$, are linearly dependent, i.e.,

$$\sum_{k=1}^{n} |(y,x_k)|^p \alpha_k = 0 \tag{3.5}$$

with some real α_k, $1 \le k \le n$. From (3.3) and (3.5) it follows that

$$\int_S |(y,x)|^p \, \mathrm{d}\sigma(x) = \sum_{k=1}^{n} |(y,x_k)|^p (\varrho_k - \xi \alpha_k) \tag{3.6}$$

for any real ξ. With

$$\xi^{-1} = \max\left\{ \frac{\alpha_k}{\varrho_k} : 1 \le k \le n \right\} > 0,$$

the number of nodes in (3.6) reduces to some $\widetilde{n} \le n - 1$, i.e., $\widetilde{n} \le \nu$. It remains to refer to the dimension formula (1.18). We prove the latter in the next section, and then the proof of Theorem B will be complete.

§4. The addition theorem and the dimension formula for $\Phi_{\mathbb{K}}(m,p)$

For any even integer l we consider the space $\mathrm{Harm}_{\mathbb{K}}(l)$ of harmonic functions in $\Phi_{\mathbb{K}}(m,l)$. Note that $\Phi_{\mathbb{K}}(m,l) \subset \Phi_{\mathbb{K}}(m,p)$, $p \geq l$, in view of the natural embedding $\phi(x) \mapsto \phi(x)\,(x,x)^{(p-l)/2}$.

In this space, let $\{s_{j,l}\}_1^{h_{m,l}}$ be a basis orthonormal with respect to the inner product

$$\langle \phi, \psi \rangle = \int_S \bar{\phi}\psi \,\mathrm{d}\sigma. \tag{4.1}$$

Addition Theorem. *The formula*

$$\sum_{j=1}^{h_{m,l}} \overline{s_{j,l}(x)}s_{j,l}(y) = b_{m,l/2}^{(\delta)} P_{l/2}^{(\alpha,\beta)}(2|(x,y)|^2 - 1), \quad x, y \in S, \tag{4.2}$$

is valid with

$$\alpha = \frac{\delta m - \delta - 2}{2}, \quad \beta = \frac{\delta - 2}{2}, \quad b_{m,l/2}^{(\delta)} = \frac{h_{m,l}}{P_{l/2}^{(\alpha,\beta)}(1)}, \tag{4.3}$$

where $P_{l/2}^{(\alpha,\beta)}(u)$ is the Jacobi polynomial of degree $l/2$ normalized so that

$$P_{l/2}^{(\alpha,\beta)}(1) = \binom{a + l/2}{l/2}. \tag{4.4}$$

We employ this theorem to prove the dimension formula (1.18) by using the orthogonal decomposition

$$\Phi_{\mathbb{K}}(m,p) = \mathrm{Harm}_{\mathbb{K}}(0) \oplus \mathrm{Harm}_{\mathbb{K}}(2) \oplus \cdots \oplus \mathrm{Harm}_{\mathbb{K}}(p), \tag{4.5}$$

which is classical in the real case and can be extended to the other two fields.

Since in our notation $\dim \mathrm{Harm}_{\mathbb{K}}(2k) = h_{m,2k}$, $0 \leq k \leq p/2$, we have

$$\dim \Phi_{\mathbb{K}}(m,p) = \sum_{k=0}^{p/2} h_{m,2k}. \tag{4.6}$$

The dimension $\Phi_{\mathbb{K}}(m,p)$ can be expressed in terms of Jacobi polynomials as follows.

We replace l in the addition theorem by $2k$ and then set $y = x$. This yields

$$\sum_{j=1}^{h_{m,2k}} |s_{j,2k}(x)|^2 = h_{m,2k}. \tag{4.7}$$

On the other hand, (4.2) is an orthogonal decomposition with fixed x. By Parseval's equation,

$$\sum_{j=1}^{h_{m,2k}} |s_{j,l}(x)|^2 = \left(b_{m,k}^{(\delta)}\right)^2 \int_S \left[P_k^{(\alpha,\beta)}\left(2|(x,y)|^2 - 1\right)\right]^2 \mathrm{d}\sigma(y). \tag{4.8}$$

Combining this formula with (4.7) and (4.3), we obtain

$$h_{m,2k} = \frac{\left[P_k^{\alpha,\beta}(1)\right]^2}{\int_S \left[P_k^{\alpha,\beta}(2|(x,y)|^2 - 1)\right]^2 \mathrm{d}\sigma(y)}.$$

The integral in the denominator can be rewritten as

$$\int_{-1}^{1} \left[P_k^{(\alpha,\beta)}(u)\right]^2 \Omega_{\alpha,\beta}(u)\,\mathrm{d}u = \|P_k^{(\alpha,\beta)}\|_{\Omega_{\alpha,\beta}}^2, \tag{4.9}$$

where $\Omega_{\alpha,\beta}$ is the normalized Jacobi weight, so that

$$\Omega_{\alpha,\beta}(u) = \frac{\omega_{\alpha,\beta}(u)}{\tau_{\alpha,\beta}}, \tag{4.10}$$

where

$$\omega_{\alpha,\beta}(u) = (1-u)^{\alpha}(1+u)^{\beta}, \quad \tau_{\alpha,\beta} = \int_{-1}^{1} \omega_{\alpha,\beta}(v)\,\mathrm{d}v. \tag{4.11}$$

Thus,

$$h_{m,2k} = \left(\frac{P_k^{(\alpha,\beta)}(1)}{\|P_k^{(\alpha,\beta)}\|_{\Omega_{\alpha,\beta}}} \right)^2$$

and

$$\dim \Phi_{\mathbb{K}}(m,p) = \sum_{k=0}^{p/2} \left(\frac{P_k^{(\alpha,\beta)}(1)}{\|P_k^{(\alpha,\beta)}\|_{\Omega_{\alpha,\beta}}} \right)^2 \tag{4.12}$$

by (4.6).

Now the required result follows from (4.12) by the explicit formula for the Christoffel–Darboux kernel

$$K_t^{(\alpha,\beta)}(u,v) = \sum_{k=0}^{t} \frac{P_k^{(\alpha,\beta)}(u)P_k^{(\alpha,\beta)}(v)}{\|P_k^{(\alpha,\beta)}\|^2_{\omega_{\alpha,\beta}}} \tag{4.13}$$

(see [33]). We omit the further details.

§5. Proof of Theorem 3

First, we note that the addition theorem immediately yields the formula

$$\sum_{x,y\in X} F\left(2|(x,y)|^2-1\right)\bar{\lambda}(x)\lambda(y) = \sum_{i\geq 0} c_{m,i}^{(\delta)}(F) \sum_{j=1}^{h_{m,2k}} \left| \sum_{x\in X} s_{j,2i}(x)\lambda(x) \right|^2, \tag{5.1}$$

where X is a finite subset of the sphere S, λ is a complex-valued function on X, and

$$F(u) = \sum_{i\geq 0} c_{m,i}^{(\delta)}(F) b_{m,i}^{(\delta)} P_i^{(\alpha,\beta)}(u) \tag{5.2}$$

is an arbitrary polynomial, $-1 \leq u \leq 1$.

We apply (5.1) to $X = \{w_k\}_1^n$, $\lambda(w_k) = \varrho_k$ $(1 \leq k \leq n)$, taking into account the fact that

$$\sum_{k=1}^{n} s_{j,2i}(w_k)\varrho_k = \int_S s_{j,2i}(y)\,\mathrm{d}\sigma(y), \quad 1 \leq j \leq h_{m,2k},\ \ 0 \leq i \leq p/2,$$

by (1.10). For $i > 0$ these integrals are equal to zero, because the integrands are harmonic and vanish at the origin. Now (5.1) takes the form

$$\sum_{i,k=1}^{n} F\left(2|(w_i,w_k)|^2-1\right)\varrho_i\varrho_k = c_{m,0}^{(\delta)}(F)\left(\sum_{k=1}^{n}\varrho_k\right)^2, \quad \deg F \leq p/2,$$

since $h_{m,0} = 1$, $s_{1,0}(x) \equiv 1$. In fact,

$$\sum_{k=1}^{n} \varrho_k = 1 \tag{5.3}$$

because of (1.10) for $\phi(y) \equiv 1$. (Recall that the measure σ is normalized.) By the orthogonality of the Jacobi polynomials with respect to the weight $\Omega_{\alpha,\beta}$, we have

$$c^{(\delta)}_{m,0}(F) = \int_{-1}^{1} F(u)\Omega_{\alpha,\beta}(u)\,\mathrm{d}u,$$

since

$$b^{(\delta)}_{m,0}\int_{-1}^{1} P_0^{(\alpha,\beta)}(u)\Omega_{\alpha,\beta}(u)\,\mathrm{d}u = b^{(\delta)}_{m,0}P_0^{(\alpha,\beta)}(1)\int_{-1}^{1}\Omega_{\alpha,\beta}(u)\,\mathrm{d}u = h_{m,0} = 1.$$

(Recall that the weight $\Omega_{\alpha,\beta}$ is normalized.) Thus, we have the identity

$$\sum_{i,k=1}^{n} F\left(2|(w_i,w_k)|^2 - 1\right)\varrho_i\varrho_k = \int_{-1}^{1} F(u)\Omega_{\alpha,\beta}(u)\,\mathrm{d}u$$

for all polynomials of degree at most $p/2$. Hence,

$$F(1)\sum_{k=1}^{n}\varrho_k^2 + \sum_{i\neq k} F\left(2|(w_i,w_k)|^2 - 1\right)\varrho_i\varrho_k = \int_{-1}^{1} F(u)\Omega_{\alpha,\beta}(u)\,\mathrm{d}u.$$

Assume that $F(u) \geq 0$ for $-1 \leq u \leq 1$. Then

$$F(1)\sum_{k=1}^{n}\varrho_k^2 \leq \int_{-1}^{1} F(u)\Omega_{\alpha,\beta}(u)\,\mathrm{d}u,$$

and finally

$$n \geq \frac{F(1)}{\int_{-1}^{1} F(u)\Omega_{\alpha,\beta}(u)\,\mathrm{d}u}, \quad \deg F \leq p/2, \tag{5.4}$$

because

$$\sum_{k=1}^{n}\varrho_k^2 \geq \frac{1}{n}$$

under (5.3). Obviously, the weights ϱ_k must be equal if the bound (5.4) is attained with some F.

In order to optimize the lower bound (5.4), we must choose F as a solution of the following linear programming problem in the space $\Pi_{p/2}$ of real-valued polynomials of degree $\leq p/2$:

$$\begin{cases} F(u) \geq 0, \quad -1 \leq u \leq 1; \\ \displaystyle\int_{-1}^{1} F(u)\Omega_{\alpha,\beta}(u)\,du = 1; \\ F(1) \to \max. \end{cases} \tag{5.5}$$

The solution of this problem is known (see [33, Theorem 7.71.3]); so, formally, we could finish at this point. However, it is of interest to note that (5.5) can be reduced to a standard problem of Euclidean geometry, which, in turn, can easily be solved. Let us present this way.

By the Markov–Lukács theorem (see [33, Theorem 1.21.1]), a polynomial $F(u) \in \Pi_{p/2}$ is nonnegative on $[-1,1]$ if and only if

$$F(u) = (1+u)^{\varepsilon}P^2(u) + (1-u^{2-\varepsilon})Q^2(u), \tag{5.6}$$

where P and Q are real polynomials of degrees $\leq [p/4]$ and $\leq [p/4]+\varepsilon-1$, respectively, and ε is the remainder of $p/2$ modulo 2. Therefore, $F(1) = 2^{\varepsilon}P^2(1)$, so that the value

$F(1)$ is independent of Q. Thus, for any fixed P the maximum of the fractional linear functional

$$\frac{F(1)}{\int_{-1}^{1} F(u)\Omega_{\alpha,\beta}(u)\,du} \tag{5.7}$$

is attained at $Q = 0$. Consequently, $Q = 0$ for the maximizer $F_{\max}$. For this reason, (5.6) can be replaced by

$$F(u) = (1+u)^{\varepsilon} P^2(u). \tag{5.8}$$

Then

$$\int_{-1}^{1} F(u)\Omega_{\alpha,\beta}(u)\,du = \int_{-1}^{1} P^2(u)\Omega^{(\varepsilon)}_{\alpha,\beta}(u)\,du, \tag{5.9}$$

where

$$\Omega^{(\varepsilon)}_{\alpha,\beta}(u) = (1+u)^{\varepsilon}\Omega_{\alpha,\beta}(u) = \frac{1}{\tau_{\alpha,\beta}}\omega_{\alpha,\beta+\varepsilon}(u) \tag{5.10}$$

(see (4.10) and (4.11)).

Since $P(1)$ is a linear functional on the space $\Pi_{[p/4]}$, there exists a unique real polynomial $R \in \Pi_{[p/4]}$ such that

$$P(1) = (P, R),$$

where the inner product corresponds to the weight $\Omega^{(\varepsilon)}_{\alpha,\beta}$. By (5.9),

$$\int_{-1}^{1} F(u)\Omega_{\alpha,\beta}(u)\,du = (P, P),$$

while

$$F(1) = 2^{\varepsilon} P^2(1) = 2^{\varepsilon}(P, R)^2.$$

The problem (5.5) takes a simple geometric form: find $P \in \Pi_{[p/4]}$ such that

$$\begin{cases} (P, P) = 1, \\ |(P, R)| \to \max. \end{cases} \tag{5.11}$$

The only solution of (5.11) is

$$P_{\max} = \widehat{R} = \pm\frac{R}{\|R\|}. \tag{5.12}$$

The corresponding maximal value is

$$F_{\max}(1) = 2^{\varepsilon} P^2_{\max}(1) = 2^{\varepsilon}\widehat{R}^2(1) = \frac{2^{\varepsilon} R^2(1)}{(R, R)} = 2^{\varepsilon} R(1). \tag{5.13}$$

Let $(T_k)_0^{[p/4]}$ be an arbitrary orthonormal real basis in $\Pi_{[p/4]}$. Then

$$R(u) = \sum_{k=0}^{[p/4]} (T_k, R) T_k(u) = \sum_{k=0}^{[p/4]} T_k(1) T_k(u).$$

In particular,

$$R(1) = \sum_{k=0}^{[p/4]} \left(T_k(1)\right)^2,$$

and (5.13) turns into

$$F_{\max}(1) = 2^{\varepsilon} \sum_{k=0}^{[p/4]} (T_k(1))^2.$$

The most relevant basis is

$$T_k = \lambda_k P_k^{(\alpha,\beta+\varepsilon)}, \quad 0 \le k \le [p/4],$$

where

$$\lambda_k = \frac{1}{\|P_k^{(\alpha,\beta+\varepsilon)}\|}$$

and the norm is related to the weight $\Omega_{\alpha,\beta}^{(\varepsilon)}$. Passing to the weight $\omega_{\alpha,\beta+\varepsilon}$ by (5.10), we obtain

$$\lambda_k = \sqrt{\tau_{\alpha,\beta}} \cdot \frac{1}{\|P_k^{(\alpha,\beta+\varepsilon)}\|_{\omega_{\alpha,\beta+\varepsilon}}}.$$

Hence,

$$F_{\max}(1) = 2^{\varepsilon}\tau_{\alpha,\beta} \sum_{k=0}^{[p/4]} \left(\frac{(P_k^{(\alpha,\beta+\varepsilon)}(1))}{\|P_k^{(\alpha,\beta+\varepsilon)}\|_{\omega_{\alpha,\beta+\varepsilon}}} \right)^2. \tag{5.14}$$

For $\varepsilon = 0$ this expression is precisely (4.12) with $p/2$ in place of p. For $\varepsilon = 1$ this yields $\Lambda(m, p/2)$ in accordance with the modified definition of the latter quantity. $\square$

References

[1] E. Bannai, *Spherical t-designs which are orbits of finite groups*, J. Math. Soc. Japan **36** (1984), 341–354. MR0740321 (86b:05016)

[2] ———, *On extremal finite sets in the sphere and other metric spaces*, Algebraic, Extremal and Metric Combinatorics (Montreal, PQ, 1986), London Math. Soc. Lecture Note Ser., vol. 131, Cambridge Univ. Press, Cambridge, 1988, pp. 13–38. MR1052659 (91i:05035)

[3] E. Bannai and S. G. Hoggar, *On tight t-designs in compact symmetric spaces of rank one*, Proc. Japan Acad. Ser. A Math. Sci. **61** (1985), 78–82. MR0796472 (87b:05040)

[4] E. Bannai and T. Ito, *Algebraic combinatorics.* I. *Associative schemes*, Benjamin/Cummings, Mehlo Park, CA, 1984. MR0882540 (87m:05001)

[5] F. Delbaen, H. Jarchow, and A. Pełczyński, *Subspaces of L_p isometric to subspaces of ℓ_p*, Positivity **2** (1998), 339–367. MR1656109 (99m:46071)

[6] P. Delsarte, J.-M. Goethals, and J. J. Seidel, *Bounds for systems of lines and Jacobi polynomials*, Philips Res. Rep. (Issue in honor of C. J. Bouwkamp) **30** (1975), 91*–105*.

[7] ———, *Spherical codes and designs*, Geom. Dedicata **6** (1977), 363–388. MR0485471 (58:5302)

[8] L. E. Dickson, *History of the theory of numbers.* Vol. 2, Carnegie Inst. Washington Publ., No. 256, New York, 1952. MR0245500 (39:6807b)

[9] V. A. Ditkin, *On some approximate formulas for computation of triple integrals*, Dokl. Akad. Nauk SSSR **62** (1948), no. 4, 445–447. (Russian) MR0027603 (10:331a)

[10] W. J. Ellison, *Waring's problem*, Amer. Math. Monthly **78** (1971), 10–36. MR0414510 (54:2611)

[11] C. D. Godsil, *Algebraic combinatorics*, Chapman and Hall, New York, 1993. MR1220704 (94e:05002)

[12] J. M. Goethals and J. J. Seidel, *Cubature formulas, polytopes, and spherical designs*, The Geometric Vein, Springer, New York–Berlin, 1981, pp. 203–218. MR0661779 (83k:05033)

[13] E. L. Grinberg, *Spherical harmonics and integral geometry on projective spaces*, Trans. Amer. Math. Soc. **279** (1983), 187–203. MR0704609 (84m:53071)

[14] D. Hilbert, *Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n-ter Potenzen (Waringsches Problem)*, Math. Ann. **67** (1909), 281–300.

[15] S. G. Hoggar, *Zonal functions and the symplectic group*, Preprint, Mat. Inst. Aarhus Univ., 1977, 1–22.

[16] ———, *Bounds for quaternionic line systems and reflection groups*, Math. Scand. **43** (1978), 241–249. MR0531304 (80k:94032)

[17] ———, *t-designs in projective spaces*, European J. Combin. **3** (1982), 233–254. MR0679208 (85b:05052)

[18] ———, *t-designs in Delsarte spaces*, Coding Theory and Design Theory, Part II, IMA Vol. Math. Appl., vol. 21, Springer-Verlag, New York, 1990, pp. 144–165. MR1056533 (91c:05051)

[19] H. König, *Isometric imbeddings of Euclidean spaces into finite-dimensional ℓ_p-spaces*, Panoramas of Mathematics (Warsaw, 1992/1994), Banach Center Publ., vol. 34, Polish Acad. Sci., Warsaw, 1995, pp. 79–87. MR1374341 (97f:46018)

[20] T. H. Koornwinder, *The addition formula for Jacobi polynomials and spherical harmonics*, SIAM J. Appl. Math. **25** (1973), 236–246. MR0346212 (49:10938)

[21] J. Korevaar and J. L. H. Meyers, *Chebyshev-type quadrature on multidimensional domains*, J. Approx. Theory **79** (1994), 144–164. MR1294326 (95i:41055)

[22] Yu. I. Lyubich, *The boundary spectrum of contractions in Minkowski spaces*, Sibirsk. Mat. Zh. **11** (1970), no. 2, 358–369; English transl., Siberian Math. J. **11** (1970), no. 2, 271–279. MR0270191 (42:5083)

[23] ———, *Almost Euclidean subspaces of real* ℓ_p^n *with* p *even integer*, Geometric Aspects of Functional Analysis, Lecture Notes in Math., Springer-Verlag, Berlin (to appear).

[24] Y. I. Lyubich and O. A. Shatalova, *Euclidean subspaces of the complex spaces* ℓ_p^n *constructed by orbits of finite subgroups of* $SU(m)$, Geom. Dedicata **86** (2001), 169–178. MR1856423 (2002h:46014)

[25] Y. I. Lyubich and L. N. Vaserstein, *Isometric embeddings between classical Banach spaces, cubature formulas, and spherical designs*, Geom. Dedicata **47** (1993), 327–362. MR1235223 (94j:46017)

[26] V. D. Milman, *A few observations on the connections between local theory and some other fields*, Geometric Aspects of Functional Analysis (1986/87), Lecture Notes in Math., vol. 1317, Springer-Verlag, Berlin–New York, 1988, pp. 283–289. MR0950988 (89h:52004)

[27] C. Müller, *Spherical harmonics*, Lecture Notes in Math., vol. 17, Springer-Verlag, Berlin–New York, 1966. MR0199449 (33:7593)

[28] A. Neumaier, *Combinatorial configuration in terms of distances*, Memorandum 81–09, Dept. of Math. Eindhoven Univ., 1981.

[29] B. Reznick, *Sums of even powers of real linear forms*, Mem. Amer. Math. Soc. **96** (1992), no. 463. MR1096187 (93h:11043)

[30] P. D. Seymour and T. Zaslavsky, *Averaging sets: a generalization of mean values and spherical designs*, Adv. in Math. **52** (1984), 213–240. MR0744857 (85m:05031)

[31] S. L. Sobolev, *Cubature formulas of the sphere which are invariant under transformations of finite rotation groups*, Dokl. Akad. Nauk SSSR **146** (1962), no. 2, 310–313; English transl., Soviet Math. Dokl. **3** (1962), 1307–1310. MR0141225 (25:4635)

[32] S. L. Sobolev and V. L. Vaskevich, *Cubature formulas*, Ross. Akad. Nauk, Sibirsk. Otdel., Inst. Mat., Novosibirsk, 1996; English transl., *The theory of cubature formulas*, Math. Appl., vol. 415, Kluwer, Dordrecht, 1997. MR1463155 (99g:65029a); MR1462617 (99g:65029b)

[33] G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, RI, 1959. MR0106295 (21:5029)

Department of Mathematics, Technion, Haifa 32000, Israel
E-mail address: `lyubich@tx.technion.ac.il`

Department of Mathematics, Technion, Haifa 32000, Israel
E-mail address: `oksana@tx.technion.ac.il`

Received 31/OCT/2003
Originally published in English