ON SPACES OF POLYNOMIAL GROWTH
WITH NO CONJUGATE POINTS

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Abstract. The following generalization of the Hopf conjecture is proved: if the fundamental group of an n-dimensional compact polyhedral space M without boundary and with no conjugate points has polynomial growth, then there exists a finite covering of M by a flat torus.

§1. Introduction

By an n-dimensional polyhedral space we mean a metric space M (with an inner metric) covered by n-simplexes; each simplex is endowed with a smooth Riemannian metric, and these metrics coincide on the common (n−1)-faces of the n-simplexes. The precise definition is given at the end of this section. In the definitions below, it is assumed that we deal with a fixed triangulation.

A polyhedral pseudomanifold is an n-dimensional polyhedral space in which the (n−1)-simplexes of the triangulation are adjacent to at most two n-simplexes. The boundary of a polyhedral space is the union of the (n−1)-simplexes of the triangulation that are adjacent to only one n-simplex. We say that M has no conjugate points if any two points in the universal covering space of M are connected by a unique geodesic. All polyhedral spaces considered in this paper are assumed to be connected.

Let M be a compact polyhedral space without boundary and with no conjugate points. It is well known that M is isometric to the quotient space \( \tilde{M}/\Gamma \), where \( \tilde{M} \) is the universal covering space of M, and \( \Gamma \) is a subgroup of the group of isometries of \( \tilde{M} \); recall that \( \Gamma \) is isomorphic to \( \pi_1(M) \).

Our aim in this paper is to prove the following two theorems.

Theorem 1. Let M be an n-dimensional compact polyhedral space without boundary and with no conjugate points. If the fundamental group \( \pi_1(M) \) of M is nilpotent, then M is a flat torus.

Theorem 2. Let M be an n-dimensional compact polyhedral space without boundary and with no conjugate points. If the fundamental group \( \pi_1(M) \) of M is of polynomial growth, then there exists a finite covering of M by a flat torus.

Theorem 2 can be derived from Theorem 1. Indeed, let M satisfy the assumptions of Theorem 1. Then \( \pi_1(M) \) is of polynomial growth. The well-known result by Gromov (see [G2]) says that \( \pi_1(M) \) is virtually nilpotent, i.e., \( \pi_1(M) \) contains a nilpotent subgroup \( G \) of finite index. Consequently, there exists a finite covering \( \tilde{M} \to M \) such that \( \pi_1(\tilde{M}) = G \).

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Since $\overline{M}$ is a compact polyhedral space without boundary and with no conjugate points, $\overline{M}$ is flat by Theorem 1. In the remaining part of the paper we prove Theorem 1. The proof is organized as follows.

In §2 we prove that $M^n$ is a pseudomanifold and that it is homotopy equivalent to an $n$-dimensional torus.

In §3 we construct a map $f : M \to T^n$, where $T^n$ is a flat torus. We show that $f$ is a local isometry on the complement of the $(n-2)$-skeleton of $M$. This step of the proof is similar to a version of the proof of the Hopf conjecture (see [1]). For the first time, the Hopf conjecture was proved by D. Burago and S. Ivanov in [BI].

In §4 we prove that the map $f : M \to T^n$ is an isometry. In contrast to the case of Riemannian manifolds considered in [I], this step is not trivial for Riemannian polyhedra.

Now we explain more precisely what we mean by polyhedral spaces.

An $n$-dimensional Riemanian simplex is an $n$-simplex in $\mathbb{R}^n$ equipped with a smooth Riemannian metric (as usual, we assume that the metric is defined in a neighborhood of this simplex), as well as any metric space isometric to such a simplex.

An $n$-dimensional polyhedral space is a connected metric space that can be obtained by gluing together $n$-dimensional Riemanian simplexes along some isometries between their faces.

§2. Homotopy type of $M$

In the proof of Theorem 1 we use the following results obtained earlier (see [L1, L2]).

Claim 1 ([L1]). Let $M$ be a compact locally simply connected space without conjugate points. Then every nilpotent subgroup of the fundamental group of $M$ is Abelian and torsion free.

Claim 2 ([L2]). Let $M$ be an $n$-dimensional compact polyhedral space without boundary and with no conjugate points. If the triangulation of $M$ contains three $n$-simplexes with a common $(n-1)$-face, then the fundamental group $\pi_1(M)$ is of exponential growth.

Our aim in this section is to prove the following auxiliary statement.

Lemma 1. Let $M$ be as in Theorem 1. Then $M$ is a pseudomanifold that is homotopy equivalent to an $n$-dimensional torus.

Proof. Since the fundamental group of a compact metric space with intrinsic metric is finitely generated, from Claim 1 it follows that $\pi_1(M) = \mathbb{Z}^m$ for some $m$. Applying Claim 2, we see that at most two $n$-simplexes of $M$ may have a common $(n-1)$-face, i.e., $M$ is a pseudomanifold. Since the universal covering space of $M$ is contractible, the fundamental group of $M$ determines the homotopy type of $M$. Hence, $M$ is homotopy equivalent to an $m$-torus $T^m$. It follows that $H_k(M, \cdot) = H_k(T^m, \cdot)$ for every $k$.

We prove that $m = n$, where $n$ is the dimension of $M$.

Suppose that $n > m$. Since $M$ is a pseudomanifold, we have $H_n(M, \mathbb{Z}_2) = \mathbb{Z}_2$. This contradicts the relation $H_n(T^m, \mathbb{Z}_2) = 0$.

Suppose $n < m$; then $H_m(M, \mathbb{Z}) = 0$. This contradicts the relation $H_m(T^m, \mathbb{Z}) = \mathbb{Z}$. Thus, $\pi_1(M) = \mathbb{Z}^n$. □

§3. Constructing a local isometry

We denote by $M'$ the complement of the $(n-2)$-skeleton of $M$; then $M'$ is an open dense subset of $M$. In this section we shall prove the following statement.

Proposition 1. Under the assumptions of Theorem 1, there exists a map $f : M \to T^n$, where $T^n$ is a flat $n$-torus, with the following properties:
(1) \( f|_{M'} \) is a local isometry on \( M' \), i.e., \( f|_{M'} \) is an open map preserving distances;
(2) \( f \) is Lipschitz;
(3) \( f \) induces an isomorphism between the corresponding fundamental groups.

We start with several lemmas.

Let \( SM \) denote the space of all unit tangent vectors of \( M \). A canonical measure \( \mu_L \) on the space \( SM \) is defined in a standard way as the product of two measures: the normalized Riemannian volume on \( M \) and the normalized Riemannian volume on the unit \((n-1)\)-sphere. This measure is called the \textit{Liouville measure}.

Since for almost every unit vector \( e \in SM \) there exists a unique generic geodesic \( \gamma \) with \( \gamma'(0) = e \) (see \cite{L1}), the geodesic flow transformation is well defined almost everywhere on \( SM \), and it is known that the Liouville measure is invariant with respect to this transformation (see \cite{L1}).

We recall that \( M \) is isometric to the quotient space \( \tilde{M}/\Gamma \), where \( \tilde{M} \) is the universal covering space of \( M \) and \( \Gamma \) is a deck transformation group isomorphic to \( \pi_1(M) = \mathbb{Z}^n \) and acting by isometries on \( \tilde{M} \).

Consider the vector space \( V = \Gamma \otimes \mathbb{R} \); it is isomorphic to \( \mathbb{R}^n \). There exists a canonical immersion of \( \Gamma = \mathbb{Z}^n \hookrightarrow V \), and its image is an integral lattice in \( V = \mathbb{R}^n \). Below we shall denote elements of \( \Gamma \) and the corresponding points of the lattice by the same symbol. Fix a point \( x_0 \in \tilde{M} \). The orbit of \( \Gamma \) is a lattice in \( \tilde{M} \); there is a one-to-one correspondence between the points of the lattice and the elements of \( \Gamma \). For \( k \in \Gamma \) and \( x \in \tilde{M} \), we denote by \( x + k \) the image of \( x \) under the isometry \( k \). When studying distances between remote points, it is convenient to approximate points of \( \tilde{M} \) by elements of the lattice. We define a map \( \tilde{\kappa} : \tilde{M} \to \Gamma \) commuting with \( \Gamma \). For this, we fix a bounded fundamental domain \( F \) containing the point \( x_0 \). For an arbitrary \( x \in \tilde{M} \), we put \( \tilde{\kappa}(x) = k \), where \( k \) is a unique element of \( \Gamma \) such that \( x \in F + k \).

Consider the function \( \| \cdot \| : \Gamma \to [0, \infty) \) given by the formula
\[
\|k\| = \lim_{n \to \infty} \frac{\tilde{\rho}(x_0, x_0 + nk)}{n},
\]
where \( \tilde{\rho} \) is the lift of the metric \( \rho \). The function \( \| \cdot \| \) is well known to be a norm on \( \Gamma \); therefore, it extends to a norm on \( V \), called the stable norm. For a linear function \( L : V \to \mathbb{R} \) we set \( \|L\| = \max\{L(x)\\|\\|x\\| = 1\} \).

**Lemma 2.** Let \( L : V \to \mathbb{R} \) be a linear function with \( \|L\| = 1 \). There exists a function \( \tilde{B}_L : \tilde{M} \to \mathbb{R} \) such that
1) \( \tilde{B}_L \) is Lipschitz with Lipschitz constant 1;
2) \( \tilde{B}_L(x + k) = \tilde{B}_L(x) + L(k) \) for every \( x \in \tilde{M} \), \( k \in \Gamma \).

**Proof.** Indeed, let
\[
\tilde{B}_L(x) = \inf_{k \in \Gamma} (L(k) + \rho(x, x_0 + k)).
\]

We prove that the function \( \tilde{B}_L \) is well defined. Since \( \|L\| = 1 \), from the definition of the stable norm it follows that
\[
-\rho(x_0 + k, x_0) \leq -\|k\| \leq L(k),
\]
whence
\[
L(k) + \rho(x, x_0 + k) \geq -\rho(x_0 + k, x_0) + \rho(x, x_0 + k) \geq -\rho(x, x_0).
\]
The required properties of \( \tilde{B}_L \) immediately follow from the definition. \( \square \)
For a linear function \( L : V \to \mathbb{R} \), let \( \tilde{B}_L \) denote the function constructed in Lemma 2. Since \( \tilde{B}_L \) is Lipschitz, it has a gradient almost everywhere; this gradient will be denoted by \( \tilde{v}_L \).

For \( \tilde{v} \in \tilde{S}M \), let \( \tilde{\gamma} : \mathbb{R} \to \tilde{M} \) be a geodesic with \( \gamma'(0) = \tilde{v} \). We define the direction at infinity \( \tilde{R}(\tilde{v}) = \tilde{R}(\tilde{\gamma}) \) in \( V \) by

\[
\tilde{R}(\tilde{v}) = \lim_{T \to -\infty} \frac{\mathcal{E}(\tilde{\gamma}(T)) - \mathcal{E}(\tilde{\gamma}(0))}{T}.
\]

By definition, for \( v \in SM \) we put \( R(v) = \tilde{R}(\tilde{v}) \), where \( \tilde{v} \) is a lifting of \( v \).

Since \( M \) has no conjugate points, it is clear that \( \| R(v) \| = 1 \).

**Lemma 3.** The functions \( R \) and \( \tilde{R} \) are defined almost everywhere on \( SM \) and \( \tilde{S}M \), respectively.

**Proof.** Let \( \phi : M \to V/\Gamma \cong \mathbb{R}^n/\mathbb{Z}^n \) be a homotopy equivalence; we may assume that \( \phi \) is simplicial. Since \( \phi \) induces an isomorphism between fundamental groups, the lifting function \( \tilde{\phi} : \tilde{M} \to V \) commutes with \( \Gamma \).

Since the functions \( \tilde{\phi} \) and \( \mathcal{E} \) commute with \( \Gamma \), we have \( \| \tilde{\phi} - \mathcal{E} \| \leq \text{const} \). Thus, in the definition of \( \tilde{R} \) we can replace \( \mathcal{E} \) by \( \tilde{\phi} \). Since the differential \( d\tilde{\phi} \) is defined almost everywhere on \( T\tilde{M} \) and is \( \Gamma \)-invariant, it is the lift of some measurable function \( \omega : TM \to V \). For a geodesic \( \gamma \) in \( M \) and its lifting \( \tilde{\gamma} \), we have

\[
\tilde{\phi}(\tilde{\gamma}(T)) - \tilde{\phi}(\tilde{\gamma}(0)) = \int_0^T \omega(\gamma') = \int_0^T \omega(\gamma').
\]

Thus, \( R(v) \) is equal to the average of \( \omega \) along \( \gamma \). The Birkhoff ergodic theorem shows that \( R(v) \) is defined for almost all \( v \in SM \). \( \square \)

**Lemma 4.** Let \( L : V \to R \) be a linear function with \( \| L \| = 1 \). Recall that \( \tilde{v}_L \) denotes the gradient field of \( B_L \). Then

\[
\lim_{T \to -\infty} \frac{1}{T} \int_0^T \langle \gamma', \tilde{v}_L \rangle = L(\gamma(\gamma))
\]

if both sides are well defined.

**Proof.** Since \( B_L \circ \gamma \) is Lipschitz, the Newton–Leibniz formula yields

\[
\int_0^T \langle \gamma', \tilde{v}_L \rangle = \int_0^T (B_L \circ \gamma)' = B_L(\gamma(T)) - B_L(\gamma(0)).
\]

Since the function \( B_L(x) - L(\mathcal{E}(x)) \) is bounded on the fundamental domain and periodic, it is bounded. This implies that \( B_L(\gamma(T)) - B_L(\gamma(0)) \) differs from \( L(\mathcal{E}(\gamma(T))) - L(\mathcal{E}(\gamma(0))) \) by a constant. So, we have

\[
\lim_{T \to -\infty} \frac{1}{T} \int_0^T \langle \gamma', \tilde{v}_L \rangle = \lim_{T \to -\infty} \frac{1}{T} \int_0^T \left( \frac{\mathcal{E}(\gamma(T)) - \mathcal{E}(\gamma(0))}{T} \right) = L(\gamma(\gamma)). \quad \square
\]

Let \( F \) denote the unit sphere of the norm \( \| \cdot \| \), and let \( m \) be the measure on \( F \) that is the image of \( \mu_L \) under \( R : SM \to F \).

**Lemma 5.** If \( L : V \to R \) is a linear function with \( \| L \| = 1 \), then

\[
\int_F L^2 dm \leq \frac{1}{n}.
\]

Equality occurs if and only if \( \langle \tilde{v}_L, w \rangle = L(\tilde{R}(w)) \) for almost every \( w \in \tilde{S}M \).
Proof. Consider the average of $\langle v_L, \cdot \rangle$ along geodesics. By Lemma 3, we have
$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \gamma', v_L \rangle = L \circ R.$$ By the Schwartz inequality,
$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \gamma', v_L \rangle^2 \geq (L \circ R)^2.$$ Since $R$ is constant on every trajectory of the geodesic flow, we have
$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle v_L, w \rangle^2 = (L \circ R)^2 + \lim_{T \to \infty} \frac{1}{T} \int_0^T \langle v_L, \cdot \rangle - L \circ R\rangle^2.$$ Integrating and using the Birkhoff ergodic theorem, we obtain
$$\int_{SM} \langle v_L, \cdot \rangle^2 d\mu_L = \int_{SM} (L \circ R)^2 d\mu_L + \int_{SM} \langle (v_L, \cdot) - L \circ R\rangle^2 d\mu_L.$$ From the inequality $|v_L| < 1$ it follows that $\int_{SM} \langle v_L, \cdot \rangle^2 d\mu_L \leq 1/n$. Consequently,
$$\int F L^2 dm = \int_{SM} (L \circ R)^2 d\mu_L \leq \frac{1}{n} - \int_{SM} \langle (v_L, \cdot) - L \circ R\rangle^2 d\mu_L.$$ The integral on the right is nonnegative, and it vanishes if and only if $\langle v_L, w \rangle = L(R(w))$ for almost every $w \in SM$. The lemma is proved. \qed

We use the following known result (for its proof, see, e.g., [BG]).

Lemma 6. Let $(V, \| \cdot \|)$ be an $n$-dimensional Banach space, let $F$ be the unit sphere of the norm $\| \cdot \|$, and let $F^*$ be the set of linear functions $L$ such that $\|L\| = 1$. Then there exists an (“inscribed”) quadratic form $Q : V \to \mathbb{R}$ representable as a finite sum
$$Q = \sum a_i L_i^2, \quad L_i \in F^*, \quad a_i > 0, \quad \sum a_i = n,$$
and such that $Q(x) \geq \|x\|^2$ for every $x \in V$. In particular, $Q$ is positive.

Remark 1. The unit ball of $Q$ is the ellipsoid of maximal volume inscribed in $F$.

Let $Q = \sum a_i L_i^2$ be the corresponding (inscribed) quadratic form for the stable norm $\| \cdot \|$ associated with $\bar{p}$. We denote by $B_i$ the functions constructed as in Lemma 2 for the linear functions $L_i$, and by $\tilde{v}_i$ their gradients.

Lemma 7. For all $i$, we have
$$\langle \tilde{v}_i, w \rangle = L_i(R(w))$$
for almost every $w \in SM$.

Proof. Applying Lemma 5 to $L_i$, we obtain
$$\int F Q dm = \sum a_i \int F L_i^2 dm \leq \frac{1}{n} \sum a_i = 1.$$ But $Q|_F \geq 1$ on $F$. Therefore, $\int F Q dm = 1$, so that $\int F L_i^2 dm = \frac{1}{n}$ for every $i$. By Lemma 5 it follows that $\langle \tilde{v}_i, w \rangle = L_i(R(w))$ for almost every $w \in SM$. \qed

The lemma just proved implies that (1) is true almost everywhere for almost every trajectory of the geodesic flow; this means that for almost every $w \in SM$, if $\gamma$ is a geodesic with $\gamma'(0) = w$, then the function $(\tilde{v}_i, \gamma') = (B_i \circ \gamma')$ is defined almost everywhere. Moreover it is equal to the constant $L_i(R(\gamma))$. Since this function is Lipschitz, it is linear. Thus,
$$\langle B_i \circ \gamma', t \rangle \equiv L_i(R(\gamma)), \quad t \in \mathbb{R},$$ (2)
Since $Q|_F \geq 1$, the relation $\int_F Q \, dm = 1$ implies that $m$-almost everywhere on $F$ we have $Q = 1$. By the definition of $m$, this means that

$Q(R(w)) = 1$

for almost all $w \in S\tilde{M}$. Since $Q$ is nondegenerate, there is no loss of generality in assuming that $L_1, \ldots, L_n$ are linearly independent.

Consider the map

$\tilde{f} = (\tilde{B}_1, \ldots, \tilde{B}_n) : \tilde{M} \to \mathbb{R}^n$.

We endow $\mathbb{R}^n$ with the Euclidean structure corresponding to the quadratic form $Q$ under the isomorphism

$I = (L_1, \ldots, L_n) : V \to \mathbb{R}^n$.

For almost every geodesic $\gamma : \mathbb{R} \to \tilde{M}$ we obtain

$(\tilde{f} \circ \gamma)'(t) = (L_1(\tilde{R}(\gamma)), \ldots, L_n(\tilde{R}(\gamma))) = I(\tilde{R}(\gamma))$.

Since for almost every geodesic $\gamma$ the vector $I(\tilde{R}(\gamma))$ is a unit vector with respect to the new Euclidean structure, the image $\tilde{f}(\gamma)$ is a straight line with constant unit velocity.

Now we prove Proposition 1.

**Proof.** Since $\tilde{f}$ commutes with the group $\Gamma$ of integral translations on $\tilde{M}$ and $\mathbb{R}^n$, $\tilde{f}$ induces a map $f : M \to T^n$, where $T^n$ is a flat torus. The homomorphism of fundamental groups induced by $f$ is an isomorphism, which implies statement (3) of Proposition 1.

The map $f$ is Lipschitz because so is $\tilde{f}$.

Recall that $M'$ denotes the complement of the $(n-2)$-skeleton of $M$.

We show that $f|_{M'} \to T^n$ is a local isometry. Consider a convex neighborhood $U \in M'$ and fix two points $x, y \in U$. For any neighborhoods $U_x, U_y \subset U$ of $x$ and $y$, let $V(U_x, U_y)$ be the set of initial velocity vectors of all shortest paths starting in $U_x$ and ending in $U_y$.

Since for almost every geodesic $\gamma : [a, b] \to M$ the image $f \circ \gamma$ is a straight line with a constant unit speed and $\mu_L V(U_x, U_y) > 0$, there exist two points $x' \in U_x$ and $y' \in U_y$ such that $f$ preserves the distance between them. Since $U_x$ and $U_y$ are arbitrary and $f$ is continuous, $f$ preserves the distance between $x$ and $y$. Thus, $f|_U$ preserves distances.

Since $M'$ and $T^n$ are $n$-dimensional manifolds, and $f|_{M'}$ preserves the distances, for any $x \in M'$ the image of some neighborhood of $x$ is a neighborhood of $f(x)$, and we see that $f$ is an open map.

$\square$

**§4. $f$ is an Isometry**

The following Lemma 8 is an obvious consequence of Proposition 1.

**Lemma 8.** $f|_{M'}$ preserves the lengths of curves.

**Lemma 9.** The map $f|_{M'} : M' \to f(M')$ is bijective, and $f : M \to T^n$ is surjective. As a consequence (because $f|_{M'}$ is a local isometry), the map $(f|_{M'})^{-1}$ is well defined, is continuous, and preserves the lengths of curves.

**Proof.** Recall that $M$ is homotopy equivalent to an $n$-dimensional torus. Consequently, the $n$-homology group of $M$ is isomorphic to $\mathbb{Z}$. We fix an isomorphism between $H_n(T^n)$ and $\mathbb{Z}$ and choose a generator of $H_n(M)$. The induced homomorphism $f_* : H_n(M) \to H_n(T^n) = \mathbb{Z}$ takes the generator of $H_n(M)$ to some integer; this integer is called the degree of $f$. We show that the degree of $f$ is $\pm 1$. Since the universal covering space of $M$ is contractible, the induced homomorphism $f_*$ determines the homotopy type of $f$. Proposition 3 shows that $f_*$ is an isomorphism; then $f$ is a homotopy equivalence. Thus, the degree of $f$ is $\pm 1$. 


The choice of generators of the homology group fixes orientations of the manifolds \( M' \subset M \) and \( T^n \). We define the degree of \( f \) at \( x \in M' \) to be equal to 1 if \( d_uf \) preserves the orientations of the tangent spaces at \( x \), and to \(-1 \) if \( d_uf \) reverses the orientations. Suppose \( y \in T^n \) is a regular point, i.e., the preimage \( f^{-1}(y) = x_1, \ldots, x_l \) is contained in \( M' \). As in the case of Riemannian manifolds, it can be proved that the degree of \( f \) is the sum of the degrees of \( f \) at the points \( x_1, \ldots, x_l \). Hence, \( f \) is surjective.

Since \( M \) is a pseudomanifold that is homotopy equivalent to an \( n \)-dimensional torus, the space \( M' \) is connected. Indeed, assume the contrary; then the group \( H_n(M, \mathbb{Z}_2) \) contains two nonzero elements. Since \( f|_{M'} \) is a local isometry, it preserves the orientation of tangent spaces everywhere, or it reverses these orientations. Consequently, the degree of \( f \) is constant at the points \( x_1, \ldots, x_l \). Since the degree of \( f \) is 1, this means that each regular point has a unique preimage. By the definition of a regular point, it follows that all points having two or more preimages are contained in \( f^{-1}(f(M' \setminus M')) \). We put \( J = f^{-1}(f(M \setminus M')) \). Observe that the dimension of \( J \) does not exceed \( n - 2 \).

Suppose that \( f|_{M'} \) is not injective. Let \( y \in f(M') \) be a point with more than one preimage in \( M' \), and let \( x_1, x_2 \) be two such preimages. Let \( D_{r_0}(x_1), D_{r_0}(x_2) \) be balls centered at \( x_1 \) and \( x_2 \) and such that the restriction of \( f \) to these balls is an isometry. Since the dimension of \( J \) is at most \( n - 2 \), there exists a point \( x_3 \in D_{r_0}(x_1) \setminus M' \setminus J \). The image of this point coincides with an image of some point contained in \( D_{r_0}(x_2) \), which contradicts the fact that \( f \) is injective on \( M \setminus J \) \( (x_3 \in M \setminus J) \).

We complete the proof of Theorem 1 by the following statement.

**Lemma 10.** The map \( f : M \to T^n \) is an isometry.

**Proof.** We show that \( f \) is noncontracting and nonexpanding. Every path in \( M \) can be approximated by a piecewise differentiable path of almost the same length. We can move each of the corresponding pieces to the interior of an appropriate \( n \)-simplex, leaving the endpoints fixed and almost length preserving.

The map \( f \) preserves the lengths of these pieces (see Lemma 8). Therefore, the map is nonexpanding.

Now we show that \( f \) is noncontracting. Let \( x, y \in M \) be arbitrary points. Given \( \varepsilon > 0 \), we let \( x', y' \in M' \) be points such that \( \rho(x, x') < \varepsilon \) and \( \rho(y, y') < \varepsilon \). Since \( f \) is nonexpanding, we have \(|(f(x), f(x'))| < \varepsilon \) and \(|(f(y), f(y'))| < \varepsilon \), where \(|(\cdot, \cdot)|\) denotes the metric on the flat torus.

Since \( f \) is Lipschitz and surjective, the Hausdorff dimension of the set \( T^n \setminus f(M') \) does not exceed \( n - 2 \). Therefore, the shortest path \([f(x'), f(y')]| \in T^n \) can be approximated by a path in \( f(M') \) with almost the same length and the same endpoints. Let \( s : [a, b] \to f(M') \) be a path that joins \( f(x') \) and \( f(y') \) and such that the length of \( s \) differs from \(|f(x'), f(y')|\) by less than \( \varepsilon \). Since \((f|_{M'})^{-1} \) preserves distances, the length of the path \( s \circ (f|_{M'})^{-1} : [a, b] \to M' \), which joins \( x' \) and \( y' \), differs from \(|f(x'), f(y')|\) by less than \( \varepsilon \). Thus,

\[
\rho(x, y) < \rho(x', y') + 2\varepsilon < |f(x'), f(y')| + 3\varepsilon < |f(x), f(y)| + 5\varepsilon.
\]

Therefore, \( f \) is noncontracting.

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