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A LOCAL TWO-RADII THEOREM ON THE SPHERE

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Abstract. Various classes of functions with vanishing integrals over all balls of a fixed radius on the sphere $\mathbb{S}^n$ are studied. For such functions, uniqueness theorems are proved, and representations in the form of series in special functions are obtained. These results made it possible to completely resolve the problem concerning the existence of a nonzero function with vanishing integrals over all balls on $\mathbb{S}^n$ the radii of which belong to a given two-element set.

§1. Introduction

Let X be a Riemannian two-point homogeneous manifold (see [1, Chapter 1]), let G be the isometry group of X, and let dx be Riemannian measure on X. We consider a collection $\mathcal{F} = \{E_i\}_{i=1}^k$ of compact subsets in X of positive measure. For every open set $\mathcal{U} \subset X$ such that every set $G_i = \{g \in G : g^{-1}E_i \subset \mathcal{U}\}$, $i = 1, \dots, k$, is nonempty, the *Pompeiu transformation* $\mathcal{P}_{\mathcal{F};\mathcal{U}}$ maps the set $C(\mathcal{U})$ into the direct product $C(G_1) \times \dots \times C(G_k)$ as follows: $\mathcal{P}_{\mathcal{F};\mathcal{U}} f = (f_1, \dots, f_k)$, where $f_i(g) = \int_{g^{-1}E_i} f(x)\,dx$, $\quad g \in G_i$, $\quad i = 1, \dots, k$. If $\mathcal{U} = X$, we call $\mathcal{P}_{\mathcal{F};\mathcal{U}}$ the *global Pompeiu transformation* and denote it by $\mathcal{P}_{\mathcal{F}}$.

For given $\mathcal{F}$ and $\mathcal{U}$, the following questions arise.

Problem (see [2])**.** Is the transformation $\mathcal{P}_{\mathcal{F};\mathcal{U}}$ injective? If not, what is its kernel?

A set $E \subset X$ for which $\mathcal{P}_E$ is injective is called a Pompeiu set.

The injectivity of the Pompeiu transformation and related questions for some X, $\mathcal{F}$, and $\mathcal{U}$ were studied in many papers (see the surveys [2]–[5]). It turned out that there is a qualitative difference between the results for noncompact spaces X and their analogs for compact spaces. Here, we consider some of them.

Suppose X is a noncompact two-point homogeneous space, i.e., X is isometric either to the Euclidean space $\mathbb{R}^n$, or to one of the hyperbolic spaces $\mathbb{H}^n(\mathbb{R})$, $\mathbb{H}^n(\mathbb{C})$, $\mathbb{H}^n(\mathbb{H})$, or $\mathbb{H}^{16}(\mathbb{C}\mathrm{ay})$ (see [1, Chapter 1, §4, Subsection 3]. Then every ball in X is not a Pompeiu set. This fact follows easily from the mean-value theorem for eigenfunctions of the Laplace–Beltrami operator on X (see [1, Chapter 4, Proposition 2.4]) (for Euclidean spaces, this fact was first mentioned in the paper [6]). Some necessary and sufficient conditions for the injectivity of $\mathcal{P}_{\mathcal{F}}$ for the family $\mathcal{F} = \{\overline{B}_{r_1}, \overline{B}_{r_2}\}$ (where the symbol B_{r_i}, $i = 1, 2$, stands for the open ball of radius r_i and centered at the origin of X, and $\overline{B}_{r_i}$ is the closure of B_{r_i}) were obtained in [7]. The results of this type are called "two-radii theorems". Very general conditions ensuring the injectivity of $\mathcal{P}_E$ were obtained by Williams [8] for $X = \mathbb{R}^n$, and by Berenstein and Shahshahani [9] for hyperbolic spaces. In these papers, it was proved that if a bounded open set E with connected complement and with Lipschitz

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boundary is not a Pompeiu set, then the boundary of E is a real-analytic submanifold of X. In particular, every polygon in $\mathbb{R}^2$ is a Pompeiu set.

Some radically new effects arise when we pass to compact two-point homogeneous spaces X (which means that X is isometric to the sphere $\mathbb{S}^n$ or to one of the projective spaces $\mathbb{P}^n(\mathbb{R})$, $\mathbb{P}^n(\mathbb{C})$, $\mathbb{P}^n(\mathbb{H})$, or $\mathbb{P}^{16}(\mathbb{C}\mathrm{ay})$; see [1, Chapter 1, §4, Subsection 2]). In this case, the transformation $\mathcal{P}_{\{\overline{B}_{r_i}\}_{i=1}^k}$ $(k \in \{1,2,\dots\})$ is injective if and only if the simultaneous equations $\varphi_m^{(\alpha+1,\beta+1)}(r_i) = 0$, where $i = 1,\dots,k$ and $m \in \{1,2,\dots\}$, are unsolvable. Here α and β are the parameters of X, and

$$\varphi_m^{(\alpha+1,\beta+1)}(t) = F\left(-m, m+\alpha+\beta+3; \alpha+2; \sin^2\left(\frac{\pi t}{2a}\right)\right),$$

where F is the hypergeometric function and a is the diameter of X (see [7, Theorem 4]). This implies that the set of all r such that $\mathcal{P}_{\overline{B}_r}$ is injective, as well as the complement of this set, is dense in $[0,a]$. Moreover, in [10, 11] there are examples of regular polygons on $\mathbb{S}^2$ that are not Pompeiu sets.

The study of the kernel of the local Pompeiu transformation (we mean the case where $\mathcal{U} \neq X$) is a considerably more difficult problem. This is related to the violation of the group structure acting on the set of solutions of the equations $\mathcal{P}_{\mathcal{F}} f = 0$. So far, these issues were investigated only in the case of Euclidean and hyperbolic spaces (see [2]–[5] and [12]–[16] and the references therein).

In the present paper, we begin to study the injectivity of the local Pompeiu transformation on the sphere $\mathbb{S}^n$. A considerable development of the methods suggested in [12]–[16] allows us to find the precise conditions for the injectivity of $\mathcal{P}_{\mathcal{F};\mathcal{U}}$ for $\mathcal{F} = \{\overline{B}_{r_1}, \overline{B}_{r_2}\}$ and $\mathcal{U} = B_R$ (see Theorem 2.1 below). Uniqueness theorems for various classes of functions with zero spherical mean play a key role here (see §8). We note that in the proof of uniqueness theorems we cannot use the well-known Titchmarsh theorem on convolutions (see [17, Appendix 7, §12]), which was used earlier in such situations in the case of noncompact two-point homogeneous spaces (cf. [14, the proof of Theorem 4]). Furthermore, in the spherical case, additional difficulties arise in the construction of functions f of finite smoothness satisfying the conditions $\mathcal{P}_{\overline{B}_r;B_R} f = 0$ and $f|_{B_r} = 0$ (such functions play an important role in the study of injectivity of the transformation $\mathcal{P}_{\{\overline{B}_{r_1},\overline{B}_{r_2}\};B_R}$ for $R = r_1 + r_2$; see §11). To overcome these difficulties, we need new ideas related to the distribution of the zeros ν of the function $\varphi_{\nu-1}^{(\frac{n}{2},\frac{n}{2})}(r)$ in the set of positive integers (see Lemma 5.3 in §5 and also §8). Also, we note that the method of proof of Theorem 2.1 makes it possible to obtain some other definitive results on spherical means on a sphere (see §9).

§2. Statement of the main result

Let $n \geq 2$. We denote by $\mathbb{S}^n$ the standard unit sphere in $\mathbb{R}^{n+1}$ with intrinsic metric d and surface measure $d\xi$. Let $B_R = \{\xi \in \mathbb{S}^n : d(o,\xi) < R\}$ be the open geodesic ball of radius R with center at $o = (0,\dots,0,1) \in \mathbb{S}^n$, and let $L_{\mathrm{loc}}(B_R)$ be the set of functions locally integrable on B_R with respect to the measure $d\xi$. Observe that $B_\pi = \mathbb{S}^n \backslash \{(0,\dots,0,-1)\}$, and that the ball B_R coincides with $\mathbb{S}^n$ if $R > \pi$. Throughout, we assume that $r, r_1, r_2 \in (0,\pi)$, $r_1 \neq r_2$, and $\max(r, r_1, r_2) < R$. We denote by $V_r(B_R)$ the class of functions $f \in L_{\mathrm{loc}}(B_R)$ with vanishing integrals over all closed geodesic balls in B_R of radius r. For a nonnegative integer s or for $s = \infty$, we put $V_r^s(B_R) = V_r(B_R) \cap C^s(B_R)$ and $V_{r_1,r_2}^s(B_R) = V_{r_1,r_2}(B_R) \cap C^s(B_R)$, where $V_{r_1,r_2}(B_R) = V_{r_1}(B_R) \cap V_{r_2}(B_R)$.

Let $N(r)$ be the set of all zeros ν of the equation $\varphi_{\nu-1}^{(\frac{n}{2},\frac{n}{2})}(r) = 0$ in the interval $((1-n)/2, +\infty)$, and let $N(r_1,r_2) = N(r_1) \cap N(r_2)$. A detailed information on the

structure of $N(r)$ is contained in §5. We denote by Ω the set of all pairs (r_1, r_2) with the following property: for every $q > 0$, there exist points $\alpha \in N(r_1)$ and $\beta \in N(r_2)$ such that $|\alpha - \beta| < (\alpha + \beta)^{-q}$. We list some properties of the sets $N(r_1, r_2)$ and Ω (for the proofs, see §10):

a) for each $r_1 \in (0, \pi)$, the set $\{r_2 \in (0, \pi) : N(r_1, r_2) \neq \varnothing\}$ is countable and dense in $(0, \pi)$;

b) for each $r_1 \in (0, \pi)$, the intersection of the set $\{r_2 \in (0, \pi) : (r_1, r_2) \in \Omega\}$ with an arbitrary interval $(a, b) \subset (0, \pi)$ is uncountable;

c) for each $r_1 \in (0, \pi)$, the set $\{r_2 \in (0, \pi) : (r_1, r_2) \in \Omega\}$ has zero Lebesgue measure on $(0, \pi)$;

d) if $(r_1, r_2) \in \Omega$ and $N(r_1, r_2) = \varnothing$, then the number r_1/r_2 is irrational.

Now, we state the main result of the present paper.

Theorem 2.1. *Suppose that* $r_1, r_2 \in (0, \pi)$, $r_1 \neq r_2$, *and* $\max(r_1, r_2) < R \leq \pi$. *Then*:

1) *if* $f \in V_{r_1,r_2}(B_R)$, $r_1 + r_2 < R$, *and* $N(r_1, r_2) = \varnothing$, *then* $f = 0$;

2) *if* $f \in V^{\infty}_{r_1,r_2}(B_R)$, $r_1 + r_2 = R$, *and* $N(r_1, r_2) = \varnothing$, *then* $f = 0$;

3) *if* $f \in V_{r_1,r_2}(B_R)$, $r_1 + r_2 = R$, $N(r_1, r_2) = \varnothing$, *and* $(r_1, r_2) \in \Omega$, *then* $f = 0$;

4) *if* $r_1 + r_2 = R$ *and* $(r_1, r_2) \notin \Omega$, *then for each integer* $q \geq 0$ *there exists a nonzero function* $f \in V^q_{r_1,r_2}(B_R)$;

5) *if* $r_1 + r_2 > R$, *then there exists a nonzero function* $f \in V^{\infty}_{r_1,r_2}(B_R)$;

6) *if* $N(r_1, r_2) \neq \varnothing$, *then on* B_π *there exists a nonzero real-analytic function* $f \in V_{r_1,r_2}(B_\pi)$.

The properties of the sets $N(r_1, r_2)$ and Ω listed above show that all situations described in statements 1)–6) of Theorem 2.1 are realized for appropriate $r_1, r_2 \in (0, \pi)$.

For the first time, necessary and sufficient conditions of the injectivity of the global Pompeiu transformation $\mathcal{P}_{\{\overline{B}_{r_i}\}_{i=1}^k}$ $(k \in \{1, 2, \dots\})$ on a sphere were found in [10, 18]. Earlier, local versions of the two-radii theorem were known only for Euclidean and hyperbolic spaces; see [2]–[5] and [12, 13, 15].

In §§3–5 of the present paper, we give information necessary in what follows and present some auxiliary statements related to the Legendre functions. In §§6–9, we study various properties of the functions of class $V_r(B_R)$. In particular, in §9 we solve the problem mentioned above for $X = \mathbb{S}^n$, $\mathcal{F} = \{\overline{B}_r\}$, and $\mathcal{U} = B_R$. In §10, we study the structure of the sets $N(r_1, r_2)$ and Ω. The proof of the main result is given in §11.

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§3. Basic notation

As usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{Q}$ denote the set of positive integers, integers, nonnegative integers, and rational numbers, respectively. For basic multivalued functions, we choose the following branches in $\mathbb{C}\backslash(-\infty, 0]$: $\ln z = \ln|z| + i \arg z$, $\arg z \in (-\pi, \pi)$; $z^\nu = e^{\nu \ln z}$.

For $\xi = (\xi_1, \dots, \xi_{n+1}) \in \mathbb{S}^n$, we denote $\xi' = (\xi_1, \dots, \xi_n) \neq 0$, and $\sigma = \xi'/|\xi'| \in \mathbb{S}^{n-1}$. Let $\theta_1, \dots, \theta_n$ be the spherical coordinates of ξ $(0 \leq \theta_1 \leq 2\pi,\ 0 \leq \theta_k \leq \pi,\ k \neq 1$, and $\xi_1 = \sin\theta_n \cdots \sin\theta_1,\ \xi_2 = \sin\theta_n \cdots \sin\theta_2 \cos\theta_1, \dots, \xi_{n+1} = \cos\theta_n)$. The symbol $\mathcal{H}_k$ will stand for the space of spherical harmonics of degree k on $\mathbb{S}^{n-1}$, regarded as a subspace of $L^2(\mathbb{S}^{n-1})$ (see [19, Chapter 4, §2]). Let a_k be the dimension of $\mathcal{H}_k$, and let $\{Y_l^{(k)}\}$, $1 \leq l \leq a_k$, be a fixed orthonormal basis in $\mathcal{H}_k$. For each function $f(\xi) = f(\sigma \sin\theta_n, \cos\theta_n) \in L_{\mathrm{loc}}(B_R)$, we have the Fourier series

$$f(\xi) \sim \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} f_{k,l}(\theta_n) Y_l^{(k)}(\sigma), \quad \theta_n \in (0, R), \tag{3.1}$$

where

$$f_{k,l}(\theta_n) = \int_{\mathbb{S}^{n-1}} f(\zeta \sin\theta_n, \cos\theta_n)\overline{Y_l^{(k)}(\zeta)}\, d\zeta \tag{3.2}$$

($d\zeta$ is the surface measure on $\mathbb{S}^{n-1}$).

Let $SO(n+1)$ be the rotation group of $\mathbb{R}^{n+1}$, and let $SO(n) = \{\tau \in SO(n+1) : \tau o = o\}$. For the normalized Haar measure $d\tau$ on $SO(n)$, let $T^k(\tau)$ be the restriction of the quasiregular representation of $SO(n)$ to the space $\mathcal{H}_k$ (see [20, Chapter 9, §2, Subsection 7]), and let $\{t^k_{l,p}\}$, $1 \le l, p \le a_k$, be the matrix of $T^k(\tau)$ in the basis $\{Y_l^{(k)}\}$, i.e.,

$$\big(T^k(\tau)Y_l^{(k)}\big)(\zeta) = Y_l^{(k)}(\tau^{-1}\zeta) = \sum_{p=1}^{a_k} t^k_{l,p}(\tau)Y_p^{(k)}(\zeta), \quad \tau \in SO(n),\ \ \zeta \in \mathbb{S}^{n-1}.$$

In particular, for $k = 0$ we have $a_0 = 1$, and $Y_1^{(0)}(\zeta) = \omega_{n-1}^{-1/2}$, $t^0_{1,1}(\tau) = 1$ for all $\zeta \in \mathbb{S}^{n-1}$ and $\tau \in SO(n)$, where ω_{n-1} is the area of the sphere $\mathbb{S}^{n-1}$. For $n = 2$ and $k \ge 1$, it is convenient to use the following basis in $\mathcal{H}_k$: $Y_1^{(k)}(\zeta) = \zeta^k/\sqrt{2\pi}$, $Y_2^{(k)}(\zeta) = \overline{\zeta}^k/\sqrt{2\pi}$ (in this case, $a_k = 2$ for all $k \ge 1$). If τ is the rotation through the angle θ in $\mathbb{R}^2$, then for this basis we have $t^k_{1,1}(\tau) = \overline{t^k_{2,2}(\tau)} = e^{-ik\theta}$ and $t^k_{1,2}(\tau) = t^k_{2,1}(\tau) = 0$. For the terms of the series (3.1), formula (3.2) implies the relation

$$f_{k,l}(\theta_n)Y_l^{(k)}(\sigma) = \int_{SO(2)} f(\tau^{-1}\xi)\overline{t^k_{l,l}(\tau)}\, d\tau. \tag{3.3}$$

For $n \ge 3$, the irreducibility of $T^k(\tau)$ (see [20, Chapter 9, §2, Subsection 10]) implies that the following relation is valid for all $1 \le l, p \le a_k$:

$$f_{k,l}(\theta_n)Y_p^{(k)}(\sigma) = a_k \int_{SO(n)} f(\tau^{-1}\xi)\overline{t^k_{l,p}(\tau)}\, d\tau \tag{3.4}$$

(see [21, the proof of formula (6)]). Formulas (3.3) and (3.4) show that, for all $f \in C^s(B_R)$ ($0 < R \le \pi$), the terms of (3.1) can be defined at $\xi = o$ so that they become functions of class $C^s(B_R)$. In what follows, we assume that all functions in $C^s(B_R\backslash\{o\})$ admitting such continuation to $\xi = o$ are defined at this point as described above.

Let L be the Laplace–Beltrami operator on $\mathbb{S}^n$. The operator L is invariant under the action of $SO(n+1)$ and coincides with the restriction of the spherical part of the Laplace operator Δ in $\mathbb{R}^{n+1}$ to $\mathbb{S}^n$. In the spherical coordinates, the operator L has the form

$$\begin{aligned} L = {} & \frac{1}{\sin^{n-1}\theta_n}\frac{\partial}{\partial\theta_n}\sin^{n-1}\theta_n\frac{\partial}{\partial\theta_n} \\ & + \frac{1}{\sin^2\theta_n\sin^{n-2}\theta_{n-1}}\frac{\partial}{\partial\theta_{n-1}}\sin^{n-2}\theta_{n-1}\frac{\partial}{\partial\theta_{n-1}} + \cdots \\ & + \frac{1}{\sin^2\theta_n\cdots\sin^2\theta_2}\frac{\partial^2}{\partial\theta_1^2} \end{aligned}$$

(see [20, Chapter 9, §5, Subsection 1]). Moreover, the operator L is symmetric, i.e.,

$$\int_{\mathbb{S}^n} f_1(\xi)(Lf_2)(\xi)\, d\xi = \int_{\mathbb{S}^n} (Lf_1)(\xi)f_2(\xi)\, d\xi, \quad f_i \in C^\infty(\mathbb{S}^n),\ \ i = 1, 2.$$

For $k \in \mathbb{Z}_+$ we put $L_k = L + k(n+k-1)\,\mathrm{Id}$, where Id is the identity operator. For each $m \in \mathbb{Z}$, we consider the differential operator d_m defined on the space $C^1(0,\pi)$ as follows:

$(d_m u)(\theta) = \sin^m\theta \frac{d}{d\theta}(u(\theta)(\sin\theta)^{-m})$, $u \in C^1(0,\pi)$. A simple calculation shows that if $f \in C^2(B_R)$ has the form $f(\xi) = u(\theta_n)Y_l^{(k)}(\sigma)$, then

$$\begin{aligned}(L_{k-1}f)(\xi) &= (d_{k-1}d_{2-n-k}u)(\theta_n)Y_l^{(k)}(\sigma),\\ (L_k f)(\xi) &= (d_{1-k-n}d_k u)(\theta_n)Y_l^{(k)}(\sigma).\end{aligned} \tag{3.5}$$

In what follows, we use the standard notation P_ν^μ and Q_ν^μ for the Legendre functions of the first and second kind on the interval $(-1,1)$ (see [22, Subsection 3.4, formulas (6) and (10)]. For $\nu \in \mathbb{C}$ and $k \in \mathbb{Z}_+$, we put

$$\psi_{\nu,k}(\theta) = (\sin\theta)^{1-\frac{n}{2}} P_{\nu+\frac{n}{2}-1}^{-\frac{n}{2}-k+1}(\cos\theta), \quad \theta \in [0,\pi). \tag{3.6}$$

Note that

$$\psi_{\nu,k}(\theta) = 2^{1-k-\frac{n}{2}}\Big(\Gamma\Big(\frac{n}{2}+k\Big)\Big)^{-1}(\sin\theta)^k F\Big(\nu+n+k-1, k-\nu; \frac{n}{2}+k; \sin^2\frac{\theta}{2}\Big), \tag{3.7}$$

where Γ is the gamma-function (see [22, Subsection 3.5, formula (8)]. Moreover, the Mehler–Dirichlet formula yields

$$\psi_{\nu,k}(\theta) = \frac{(\sin\theta)^{2-n-k}}{\sqrt{2\pi}\Gamma\left(k+\frac{n-1}{2}\right)}\int_{-\theta}^{\theta}(\cos t - \cos\theta)^{\frac{n-3}{2}+k} e^{i\left(\nu+\frac{n-1}{2}\right)t}\,dt \tag{3.8}$$

(see [22, Subsection 3.7, formula (27)]). From (3.7) with $k=1$ we see that

$$N(r) = M(r) + (1-n)/2, \tag{3.9}$$

where $M(r)$ is the set of all positive zeros λ of the function $s_r(\lambda)=\psi_{\lambda+\frac{1-n}{2},1}(r)$. We also need the following relations:

$$d_k\psi_{\nu,k} = (k-\nu)(k+\nu+n-1)\psi_{\nu,k+1}, \quad d_{1-n-k}\psi_{\nu,k+1} = \psi_{\nu,k}, \tag{3.10}$$

$$(L+\nu(\nu+n-1)\,\mathrm{Id})\left(\Psi_\nu^{k,l}(\xi)\right) = 0, \tag{3.11}$$

where $\Psi_\nu^{k,l}(\xi) = \psi_{\nu,k}(\theta_n)Y_l^{(k)}(\sigma)$ and $\xi \in B_\pi$. To prove (3.10), it suffices to use (3.6) and the recurrence relations for the Legendre functions (see [22, Subsection 3.8, formulas (15), (17), and (19)]. Equation (3.11) follows from (3.10) and (3.5).

§4. Auxiliary statements about the functions $\psi_{\nu,k}$

Lemma 4.1. *Suppose $\alpha,\beta \in \mathbb{C}$ and $\theta \in [0,\pi)$. Then*

$$\begin{aligned}(\alpha-\beta)(\alpha+\beta+n-1)&\int_0^\theta \psi_{\alpha,1}(t)\psi_{\beta,1}(t)\sin^{n-1}t\,dt\\ &= \sin^{n-1}\theta\left(\psi_{\alpha,1}(\theta)\psi'_{\beta,1}(\theta) - \psi_{\beta,1}(\theta)\psi'_{\alpha,1}(\theta)\right).\end{aligned} \tag{4.1}$$

Proof. Relations (3.5) and (3.11) with $k=1$ imply $d_0 d_{1-n}\psi_{\nu,1} = \nu(1-n-\nu)\psi_{\nu,1}$. Putting $\nu=\alpha,\beta$, we obtain $(\alpha-\beta)(\alpha+\beta+n-1)\psi_{\alpha,1}\psi_{\beta,1} = d_{1-n}(\psi_{\alpha,1}\psi'_{\beta,1} - \psi_{\beta,1}\psi'_{\alpha,1})$, and the lemma follows. □

Lemma 4.2. *Let $0 < \alpha < \beta < \pi$, and let $\varepsilon \in (0, \pi)$. Then, for $r \in [\alpha, \beta]$, if $\lambda \to \infty$ so that $|\arg \lambda| \le \pi - \varepsilon$, then*

$$\begin{aligned}\sqrt{\frac{\pi}{2}}&(\sin r)^{\frac{n-1}{2}} s_r(\lambda)\\ &= \frac{\cos\left(\lambda r - \frac{\pi}{4}(n+1)\right)}{\lambda^{\frac{n+1}{2}}} - \frac{(n^2-1)}{8}\cot r \frac{\sin(\lambda r - \frac{\pi}{4}(n+1))}{\lambda^{\frac{n+3}{2}}} + O\left(\frac{e^{r|\operatorname{Im}\lambda|}}{|\lambda|^{\frac{n+5}{2}}}\right),\end{aligned} \tag{4.2}$$

$$\begin{aligned}\sqrt{\frac{\pi}{2}}&(\sin r)^{\frac{n-1}{2}} s_r'(\lambda)\\ &= \frac{-r\sin(\lambda r - \frac{\pi}{4}(n+1))}{\lambda^{\frac{n+1}{2}}} + O\left(\frac{e^{r|\operatorname{Im}\lambda|}}{|\lambda|^{\frac{n+3}{2}}}\right),\end{aligned} \tag{4.3}$$

where the constants involved in the O signs depend only on α, β, n, and ε.

Proof. This follows from the relation

$$s_r(\lambda) = \frac{1}{\sqrt{2\pi}} \frac{(\sin r)^{1-n}}{\Gamma\left(\frac{n+1}{2}\right)} \int_{-r}^{r} (\cos t - \cos r)^{\frac{n-1}{2}} e^{i\lambda t}\, dt \tag{4.4}$$

(see (3.8)) and the asymptotic expansion of Fourier integrals (see [23, Chapter 2, the proof of Theorem 10.2]). □

Lemma 4.3. *Suppose $\xi = (\xi_1, \dots, \xi_{n+1}) \in \mathbb{S}^n$, $\xi_{n+1} > 0$. Then*

$$\int_{\mathbb{S}^{n-1}} (\xi_{n+1} + i(\zeta, \xi'))^{\nu} Y_l^{(k)}(\zeta)\, d\zeta = (2\pi)^{\frac{n}{2}} i^k \frac{\Gamma(\nu+1)}{\Gamma(\nu-k+1)} \Psi_{\nu}^{k,l}(\xi), \tag{4.5}$$

where (ζ, ξ') is the Euclidean scalar product of the vectors $\zeta, \xi' \in \mathbb{R}^n$.

Proof. Using formula (23) in [22, Subsection 3.7], we obtain

$$\begin{aligned}\psi_{\nu,k}(\theta_n) &= \frac{\sin^k \theta_n}{\sqrt{\pi} 2^{\frac{n}{2}+k-1} \Gamma\left(k + \frac{n-1}{2}\right)}\\ &\quad \times \int_0^{\pi} (\cos\theta_n + i\sin\theta_n \cos t)^{\nu-k} (\sin t)^{n+2k-2}\, dt.\end{aligned} \tag{4.6}$$

Integrating by parts and arguing by induction on k, from (4.6) we deduce the relation

$$\begin{aligned}\int_0^{\pi} &(\cos\theta_n + i\sin\theta_n\cos t)^{\nu} C_k^{\frac{n-2}{2}}(\cos t)(\sin t)^{n-2}\, dt\\ &= \frac{\pi}{2^{\frac{n}{2}-2}} \frac{i^k}{k!} \frac{\Gamma(n+k-2)}{\Gamma\left(\frac{n-2}{2}\right)} \frac{\Gamma(\nu+1)}{\Gamma(\nu-k+1)} \psi_{\nu,k}(\theta_n),\end{aligned}$$

where $C_k^{\frac{n-2}{2}}$ is the Gegenbauer polynomial of degree k and of order $\frac{n-2}{2}$ (see [22, Subsection 3.15, formulas (3) and (10)]). Recalling the Funk–Hecke theorem (see [24, Subsection 11.4]), we arrive at (4.5). □

Corollary 4.1. *Let $\theta \in [0, \pi)$. Then, as $\nu \to +\infty$, we have*

$$\max_{t \in [0,\theta]} \left| \frac{d^s \psi_{\nu,k}(t)}{dt^s} \right| = O(\nu^{s-k}) \tag{4.7}$$

with the constant independent of ν.

The proof follows from Lemma 4.3 and formulas (3.8), (4.2), and (3.10).

§5. Properties of the set $N(r)$

Let $r \in (0,\pi)$. Relation (4.4) shows that s_r is an even entire function of λ. Moreover, we have $s_r(\overline{\lambda}) = \overline{s_r(\lambda)}$; in particular, s_r is real-valued on $\mathbb{R}^1$. Also from (4.4), we see that $s_r > 0$ on the imaginary axis. Next, s_r tends to 0 as $\lambda \to \infty$ along the real axis and is a function of exponential type r. Invoking Hadamard's theorem [17, Chapter 1, Theorem 13], we conclude that the set $N(r)$ is infinite (see (3.9)).

Lemma 5.1. *Let $r \in (0,\pi)$. Then*: 1) *all zeros ν of the function $\psi_{\nu,1}(r)$ are real, simple, and symmetric with respect to the point $\nu = (1-n)/2$; and* 2) *if $\nu \in [-n,1]$, then $\psi_{\nu,1}(r) > 0$.*

Proof. Let $s_r(\lambda) = 0$. We prove that $\lambda \in \mathbb{R}^1$ and $s_r'(\lambda) \neq 0$. Suppose $\lambda \notin \mathbb{R}^1$. Then $\lambda^2 \neq \overline{\lambda}^2$ because $i\lambda \notin \mathbb{R}^1$. Putting $\alpha = \overline{\beta} = \lambda + \frac{1-n}{2}$ and $\theta = r$ in (4.1) and recalling that $s_r(\overline{\lambda}) = 0$, we obtain

$$\int_0^r (\sin t)^{n-1} |s_t(\lambda)|^2 \, dt = 0, \tag{5.1}$$

which is impossible. Now, we assume that $s_r'(\lambda) = 0$. Putting $\alpha = \lambda + \frac{1-n}{2}$ and $\theta = r$ in (4.1), and letting $\beta \to \alpha$, we obtain (5.1) once again. Thus, all zeros of s_r are real, simple, and symmetric with respect to $\lambda = 0$. This and the definition of s_r imply statement 1). Statement 2) follows from formula (3.7) and the definition of the hypergeometric function. Lemma 5.1 is proved. □

From equation (3.9) and Lemma 5.1 it follows that the set $N(r)$ has the form $N(r) = \left\{\lambda_j + \frac{1-n}{2}\right\}_{j\in\mathbb{N}}$, where $\lambda_j = \lambda_j(r)$ is the sequence of all positive zeros of s_r enumerated in ascending order, and $\lambda_1 > \frac{1+n}{2}$.

Lemma 5.2. *Let $0 < \alpha < \beta < \pi$ and $r \in [\alpha,\beta]$. Then*

$$r\lambda_j = \pi\Big(\frac{n+3}{4} + j + q(r,n)\Big) - \frac{n^2-1}{8\lambda_j}\cot r + O(\lambda_j^{-3}), \tag{5.2}$$

where $q(r,n)$ is an integer independent of j, and the constant involved in the O sign depends only on α,β, and n. In particular, for every $\varepsilon > 0$ we have

$$\sum_{j=1}^{\infty} \lambda_j^{-1-\varepsilon} < \infty. \tag{5.3}$$

Proof. For $r \in [\alpha,\beta]$, the number of zeros of s_r in the interval $[-a,a]$ does not exceed $c_1(1+a)$, where $c_1 > 0$ depends only on α,β, and n (see [17, formula (1.27)] and (4.2)). Combined with (4.2) and (4.3), this implies the existence of $q(r,n) \in \mathbb{Z}$ such that

$$r\lambda_j = \pi\Big(\frac{n+3}{4} + j + q(r,n)\Big) + \varepsilon_j(r,n), \quad j \in \mathbb{N}, \tag{5.4}$$

where $|\varepsilon_j(r,n)| \le c_2/\lambda_j$, $j \in \mathbb{N}$, and the constant $c_2 > 0$ depends only on α,β, and n. Putting $\lambda = \lambda_j$ in (4.2) and using (5.4), we obtain $\varepsilon_j(r,n) = (1-n^2)(\cot r)/(8\lambda_j) + O(\lambda_j^{-3})$. Using (5.4) once again, we finish the proof. □

We denote by $[t]$ the integral part of $t \in \mathbb{R}^1$.

Lemma 5.3. *For $s \in \mathbb{N}\backslash\{1\}$, let $N_s(r) = \{2,3,\dots,s\} \cap N(r)$, and let* $\operatorname{card} N_s(r)$ *be the number of elements of $N_s(r)$. Then*: 1) *if $r \neq \pi/2$, then* $\operatorname{card} N_s(r) \le \left[\frac{s}{3}\right] + 1$; *and* 2) *if $r = \pi/2$, then* $\operatorname{card} N_s(r) = \left[\frac{s}{2}\right]$.

Proof. Let $r \neq \pi/2$. We assume that $m \in \mathbb{N}$ and $\psi_{m,1}(r) = 0$. Since the zeros of the Gegenbauer polynomial $C_{m-1}^{\frac{n+1}{2}}$ are simple, we have $\psi_{m+1,1}(r) \neq 0$ (see [22, Subsection 3.8, formula (19)]). Recalling the formula $(2\nu+1)tP_\nu^\mu(t) = (\nu-\mu+1)P_{\nu+1}^\mu(t)+(\nu+\mu)P_{\nu-1}^\mu(t)$ (see [22, Subsection 3.8]), we conclude that each triple of consecutive positive integers contains at most one element of $N(r)$. This proves statement 1). Statement 2) follows from the fact that $N(\pi/2) = \{2j\}_{j=1}^\infty$ (see [22, Subsection 3.4, formula (20)]). □

Lemma 5.4. *For $u \in L[0,r]$, let $v(z) = \int_0^r (\sin\theta)^{n-1} u(\theta)\psi_{z,1}(\theta)\,d\theta$. If $v(\nu) = 0$ for all $\nu \in N(r)$, then $u = 0$.*

Proof. By (3.8) and the assumptions of the lemma, we have

$$v(z) = \frac{1}{\sqrt{2\pi}\Gamma\left(\frac{n+1}{2}\right)} \int_0^r u(\theta)K(z,\theta)\,d\theta,$$

where

$$K(z,\theta) = \int_{-\theta}^{\theta} e^{i\left(z+\frac{n-1}{2}\right)t} g(\theta,t)\,dt, \quad g(\theta,t) = (\cos t - \cos\theta)^{\frac{n-1}{2}}. \tag{5.5}$$

Integrating (5.5) by parts for $z \neq \frac{1-n}{2}$, we obtain

$$K(z,\theta) = \left(\frac{i}{z+\frac{n-1}{2}}\right)^{\left[\frac{n}{2}\right]} \int_{-\theta}^{\theta} e^{i\left(z+\frac{n-1}{2}\right)t} \left(\frac{d}{dt}\right)^{\left[\frac{n}{2}\right]} (g(\theta,t))\,dt.$$

Consequently, $|v(z)| \leq c(1+|z|)^{-\left[\frac{n}{2}\right]} e^{r|\operatorname{Im} z|}$, where c is independent of z. Consider the function $\varphi(z) = v(z)/\psi_{z,1}(r)$. Since $\psi_{z,1}(\theta) = \psi_{-z-n+1,1}(\theta)$ (see [22, Subsection 3.4, formula (7)], our assumptions and Lemma 5.1 imply that φ is an entire function of at most first order and $\varphi(z) = \varphi(-z-n+1)$, $z \in \mathbb{C}$. Furthermore, the estimate obtained for v and (4.2) imply that $\varphi(z) = O(|z|)$ as $z \to \infty$ along the lines $\operatorname{Im} z = \pm \operatorname{Re} z$. This fact and the Phragmèn–Lindelöf principle show that φ is a polynomial of degree at most one. Since φ is even with respect to $(1-n)/2$, we have $\varphi(z) = \varphi(0)$ for all z. This means (see (5.5) and (3.8)) that $\varphi(0)(\sin r)^{1-n}K(z,r) = \int_{-r}^r e^{i\left(z+\frac{n-1}{2}\right)t} \int_{|t|}^r u(\theta)g(\theta,t)\,d\theta dt$. Then $\varphi(0)(\sin r)^{1-n}g(r,t) = \int_t^r u(\theta)g(\theta,t)\,d\theta$ for all $t \in [0,r]$. Letting $t \to r$, from the latter equation we deduce that $\varphi(0) = 0$ and that $\int_t^r u(\theta)g(\theta,t)\,d\theta = 0$ on $[0,r]$. By [1, Chapter 1, §2, proof of Theorem 2.6], this implies that $u = 0$. □

Lemma 5.5. *Let $\delta(\alpha,\beta) = \int_0^r \psi_{\alpha,1}(t)\psi_{\beta,1}(t)(\sin t)^{n-1}\,dt$, $\alpha,\beta \in N(r)$. Then $\delta(\alpha,\beta) = 0$ for $\alpha \neq \beta$ and $\delta(\alpha,\alpha)\alpha^{n+2} > c$, where the constant $c > 0$ is independent of α.*

Proof. For $\alpha \neq \beta$, the claim follows from Lemma 4.1. Note that it is suffices to prove the required inequality for $\delta(\alpha,\alpha)$ for sufficiently large $\alpha \in N(r)$. Let $\alpha > \frac{\pi}{4r} + \frac{1-n}{2}$. By (3.8), we have

$$\begin{aligned}
\delta(\alpha,\alpha) &= \frac{2}{\pi\Gamma^2\left(\frac{n+1}{2}\right)} \int_0^r (\sin\theta)^{1-n} \left(\int_0^\theta g(\theta,t)\cos\left(\left(\alpha+\frac{n-1}{2}\right)t\right)\,dt\right)^2 d\theta \\
&\geq \frac{2}{\pi\Gamma^2\left(\frac{n+1}{2}\right)} \int_0^{\frac{\pi}{4(\alpha+\frac{n-1}{2})}} (\sin\theta)^{1-n} \left(\int_0^\theta g(\theta,t)\cos\left(\left(\alpha+\frac{n-1}{2}\right)t\right)\,dt\right)^2 d\theta \\
&\geq \frac{1}{\pi\Gamma^2(\frac{n+1}{2})} \int_0^{\frac{\pi}{4(\alpha+\frac{n-1}{2})}} (\sin\theta)^{1-n} \left(\int_{\theta/2}^\theta g(\theta,t)\,dt\right)^2 d\theta.
\end{aligned}$$

Now, the required estimate for $\delta(\alpha,\alpha)$ follows from the definition of $g(\theta,t)$. □

§6. Elementary properties of the class V_r

Lemma 6.1. *Let $f \in V_r^s(B_R)$. Then $f_{k,l}(\theta_n)Y_p^{(k)}(\sigma) \in V_r^s(B_R)$ for all $k \in \mathbb{Z}_+$ and $1 \le l, p \le a_k$. A similar statement is valid also for $f \in V_r(B_R)$.*

To prove this lemma, it suffices to notice that $SO(n) = \{\tau \in SO(n+1) : \tau o = o\}$, and to use formulas (3.3) and (3.4).

Lemma 6.2. *Suppose $s \in \mathbb{N}$, $f \in V_r^s(B_R)$, and $\mathring{f}(\theta_1, \dots, \theta_n) = f(\sin\theta_n \cdots \sin\theta_1, \dots, \cos\theta_n)$. Then $-\sin\theta_{n-1}\cot\theta_n \frac{\partial \mathring{f}}{\partial \theta_{n-1}} + \cos\theta_{n-1}\frac{\partial \mathring{f}}{\partial \theta_n} \in V_r^{s-1}(B_R)$.*

Proof. Let $\tau \in SO(n+1)$, and let $\tau\overline{B}_r \subset B_R$. We denote by $\mathfrak{a}_t$ the rotation of $\mathbb{R}^{n+1}$ through the angle t in the plane (x_{n+1}, x_n). For sufficiently small $|t|$ we have $\int_{\tau B_r} F(\mathfrak{a}_t\xi)\,d\xi = 0$, where $F(x) = f\left(\frac{x}{|x|}\right)$. Differentiating with respect to t and putting $t = 0$, we obtain $\int_{\tau B_r} h(\xi)\,d\xi = 0$, where $h(\xi) = \xi_{n+1}\frac{\partial F}{\partial x_n}(\xi) - \xi_n \frac{\partial F}{\partial x_{n+1}}(\xi)$, $\xi \in B_R$. Since $\mathring{h}(\theta_1, \dots, \theta_n) = -\sin\theta_{n-1}\cot\theta_n \frac{\partial \mathring{f}}{\partial \theta_{n-1}} + \cos\theta_{n-1}\frac{\partial \mathring{f}}{\partial \theta_n}$, we arrive at the statement of Lemma 6.2. □

Lemma 6.3. *Suppose that $s \in \mathbb{N}$ and $u(\theta_n)Y(\sigma) \in V_r^s(B_R)$ for some $Y \in \mathcal{H}_k$. Then:* a) *$(d_k u)(\theta_n)Y_l^{(k+1)}(\sigma) \in V_r^{s-1}(B_R)$ for all $l = 1, \dots, a_k$;* b) *$(d_{2-k-n}u)(\theta_n)Y_l^{(k-1)}(\sigma) \in V_r^{s-1}(B_R)$ for all $l = 1, \dots, a_k$ if $k \in \mathbb{N}$; and* c) *if $s \ge 2$, then $L\left(u(\theta_n)Y(\sigma)\right) \in V_r^{s-2}(B_R)$.*

Proof. Since $\sin^k\theta_{n-1}\cdots\sin^k\theta_2 e^{ik\theta_1} \in \mathcal{H}_k$ (see [20, Chapter 9, §3, Subsection 6]), we have $u(\theta_n)\sin^k\theta_{n-1}\cdots\sin^k\theta_2 e^{ik\theta_1} \in V_r^s(B_R)$ by Lemma 6.1. Then, by Lemma 6.2,

$$(d_k u)(\theta_n)\cos\theta_{n-1}\sin^k\theta_{n-1}\cdots\sin^k\theta_2 e^{ik\theta_1} \in V_r^{s-1}(B_R).$$

Since $\cos\theta_{n-1}\sin^k\theta_{n-1}\cdots\sin^k\theta_2 e^{ik\theta_1} \in \mathcal{H}_{k+1}$ (see [20, Chapter 9, §3, Subsection 6]), we obtain statement a) from Lemma 6.1. Now we prove statement b). By Lemma 6.1, $u(\theta_n)C_k^{\frac{n-2}{2}}(\cos\theta_{n-1}) \in V_r^s(B_R)$. Applying Lemma 6.2 to this function and using formulas (4) and (10) in [20, Chapter 9, §3, Subsection 2], we see that

$$(d_{2-k-n}u)(\theta_n)\cos\theta_{n-1}C_k^{\frac{n-2}{2}}(\cos\theta_{n-1}) - (k+1)u(\theta_n)\cot\theta_n C_{k+1}^{\frac{n-2}{2}}(\cos\theta_{n-1}) \in V_r^{s-1}(B_R).$$

Since $C_{k+1}^{\frac{n-2}{2}}(\cos\theta_{n-1}) \in \mathcal{H}_{k+1}$ and $\cos\theta_{n-1}C_k^{\frac{n-2}{2}}(\cos\theta_{n-1}) \in \mathcal{H}_{k-1} + \mathcal{H}_{k+1}$ (see [20, Chapter 9, §2, Subsection 3]), we can deduce statement b) from Lemma 6.1. Statement c) is proved by successive application of a) and b) to the function $u(\theta_n)Y(\sigma)$ (see (3.5)). □

Corollary 6.1. *If $u(\theta_n)Y(\sigma) \in V_r^1(B_R)$ for some $Y \in \mathcal{H}_1$, then $u(r) = 0$.*

Proof. Statement b) of Lemma 6.3 yields

$$\sin^{1-n}\theta_n \frac{d}{d\theta_n}\left(u(\theta_n)\sin^{n-1}\theta_n\right) \in V_r^0(B_R).$$

Integrating this function over the ball B_r, we obtain $u(r) = 0$. □

§7. Examples of functions of class V_r

Let $f(\xi) = u(\theta_n)$ be a function belonging to $L(\mathbb{S}^n)$ and such that its support $\operatorname{supp} f$ lies in the ball $\overline{B}_\varepsilon$ for some $\varepsilon \in (0, \pi)$. For $\nu \in \mathbb{C}$, we put

$$\widetilde{f}(\nu) = \int_{\mathbb{S}^n} f(\xi)\psi_{\nu,0}(\arccos\xi_{n+1})\,d\xi = \omega_{n-1}\int_0^\varepsilon u(t)\psi_{\nu,0}(t)\sin^{n-1}t\,dt.$$

Then (3.8) implies that $\widetilde{f}$ is an entire function of exponential type not exceeding ε. Furthermore, if $f \in C^\infty(\mathbb{S}^n)$, then the fact that the operator L is symmetric and formula (3.11) show that $\widetilde{f}(\nu) = O(\nu^{-c})$ as $\nu \to +\infty$, for every fixed $c > 0$.

Lemma 7.1. *Suppose* $\Phi(\xi) = \varphi(\theta_n) \in L(\mathbb{S}^n)$ *and* $\operatorname{supp}\Phi \subset B_\varepsilon$ *for some* $\varepsilon \in (0,\pi)$. *If* $\tau \in SO(n+1)$, $f \in C^2(\tau B_\varepsilon)$, *and* $Lf = \nu(1-n-\nu)f$ *in* τB_ε, *then*

$$\int_{B_\varepsilon} f(\tau\xi)\Phi(\xi)\,d\xi = 2^{\frac{n}{2}-1}\Gamma\Big(\frac{n}{2}\Big) f(\tau o)\widetilde{\Phi}(\nu). \tag{7.1}$$

In particular,

$$\int_{\tau B_\varepsilon} f(\xi)\,d\xi = (2\pi)^{\frac{n}{2}}(\sin\varepsilon)^{n-1}\psi_{\nu,1}(\varepsilon) f(\tau o). \tag{7.2}$$

Proof. Let $F(\xi) = f(\tau\xi)$, $\xi \in B_\varepsilon$. Then $LF = \nu(1-n-\nu)F$, and therefore F is real analytic in B_ε (see [1, Chapter 4, §2, the proof of Proposition 2.2]). Next,

$$\int_{B_\varepsilon} F(\xi)\Phi(\xi)\,d\xi = \omega_{n-1}\int_0^\varepsilon (\sin\rho)^{n-1}\varphi(\rho)\int_{S_\rho} F(\xi)\,d\mu(\xi)\,d\rho, \tag{7.3}$$

where $S_\rho = \{\xi \in \mathbb{S}^n : d(0,\xi) = \rho\}$, and $d\mu(\xi)$ is the induced measure on S_ρ normalized by the condition $\int_{S_\rho} d\mu(\xi) = 1$. By Pizzetti's formula (see [7]), we have

$$\begin{aligned}
&\int_{S_\rho} F(\xi)\,d\mu(\xi)\\
&\quad= \frac{\Gamma(\frac{n}{2})}{(\cos\frac{\rho}{2})^{n-2}}\sum_{m=0}^{\infty}\Big(\sin\frac{\rho}{2}\Big)^{2m}\\
&\qquad\times\frac{\Big(\Big(L-\frac{(n-2)n}{4}\operatorname{Id}\Big)\cdots\Big(L-\frac{(n+2m-2)(n-2m)}{4}\operatorname{Id}\Big)F\Big)(o)}{m!\Gamma\left(\frac{n}{2}+m\right)}\\
&\quad= \frac{\Gamma(\frac{n}{2})}{(\cos\frac{\rho}{2})^{n-2}}\sum_{m=0}^{\infty}\Big(\sin\frac{\rho}{2}\Big)^{2m}\\
&\qquad\times\frac{\Big(\nu(1-n-\nu)-\frac{(n-2)n}{4}\Big)\cdots\Big(\nu(1-n-\nu)-\frac{(n+2m-2)(n-2m)}{4}\Big)}{m!\Gamma\left(\frac{n}{2}+m\right)} f(\tau o)\\
&\quad= \frac{f(\tau o)}{\big(\cos\frac{\rho}{2}\big)^{n-2}} F\left(-\nu-\frac{n}{2}+1,\nu+\frac{n}{2};\frac{n}{2};\sin^2\frac{\rho}{2}\right) = 2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)\psi_{\nu,0}(\rho) f(\tau o)
\end{aligned}$$

(see [22, Subsection 3.5, formula (9)]. Invoking (7.3), we obtain (7.1). Relation (7.2) follows from (7.1) and (3.10). □

Corollary 7.1. *If* $Y \in \mathcal{H}_k$ *and* $\nu \in N(r)$, *then* $\psi_{\nu,k}(\theta_n)Y(\sigma) \in V_r^\infty(B_\pi)$.

The assertion follows from (7.2) and (3.11).

Lemma 7.2. *Suppose* $R \le \pi$, $f \in L_{\mathrm{loc}}(B_R)$, *and each coefficient of the series* (3.1) *is of the form*

$$f_{k,l}(\theta_n) = \sum_{\nu\in N(r)} c_{\nu,k,l}\psi_{\nu,k}(\theta_n), \tag{7.4}$$

where $c_{\nu,k,l} \in \mathbb{C}$ *and* $\sum_{\nu\in N(r)} |c_{\nu,k,l}|\nu^{-k} < \infty$. *Then* $f \in V_r(B_R)$.

Proof. Estimate (4.7) shows that the series (7.4) converges uniformly on every compact subset of B_π. By the corollary to Lemma 7.1, we have $f_{k,l}(\theta_n)Y_p^{(k)}(\sigma) \in V_r(B_R)$. This means (see (3.3) and (3.4)) that $I_{k,l}(\theta_n)Y_p^{(k)}(\sigma) = 0$ in B_{R-r}, where $I(\eta) = \int_{d(\eta,\xi)\le r} f(\xi)\,d\xi$, $\eta \in B_{R-r}$. Thus, $I = 0$ and $f \in V_r(B_R)$. □

§8. Uniqueness theorems

The results of this section substantially generalize the well-known John theorem for functions with zero spherical mean on $\mathbb{R}^n$ (see, e.g., [25]).

Theorem 8.1. 1) *If $f \in V_r^\infty(B_R)$ and $f = 0$ in B_r, then $f = 0$ in B_R.*

2) *If $f \in V_r(B_R)$ and $f = 0$ in $B_{r+\varepsilon}$ for some $\varepsilon \in (0, R-r)$, then $f = 0$ in B_R.*

3) *Let $R \le \pi$; then* a) *for each $q \in \mathbb{Z}_+$ there exists a nonzero function $f \in V_r^q(B_R)$ vanishing in B_r;* b) *for each $\varepsilon \in (0, r)$ there exists a nonzero function $f \in V_r^\infty(B_R)$ vanishing in $B_{r-\varepsilon}$.*

We note that if $R > \pi$ (i.e., $B_R = \mathbb{S}^n$), then, in general, statements 3a) and 3b) are false (see [7, Theorem 4] and also §1 of the present paper).

Analogs of Theorem 8.1 for noncompact two-point homogeneous spaces were obtained in [14, 15]. The method used in these papers was based on the Titchmarsh theorem on convolutions and cannot be applied to the sphere $\mathbb{S}^n$.

To prove Theorem 8.1, we need two auxiliary statements.

Lemma 8.1. *Suppose $R \le 2r$, $f(\xi) = u(\theta_n) \in V_r^\infty(B_R)$, and $f = 0$ in B_r. Then $f = 0$ in B_R.*

Proof. Without loss of generality, we may assume that $R \le \pi$. Let $0 < \varepsilon < R - r$. We consider a function w_ε satisfying the following conditions: 1) $w_\varepsilon \in C^\infty[0,\pi]$; 2) $w_\varepsilon = 1$ on $[0, R-\varepsilon]$ and $w_\varepsilon = 0$ on $[R - \frac{\varepsilon}{2}, \pi]$. For $\theta \in [0,\pi]$, we put $\Phi(\theta) = u(\theta) w_\varepsilon(\theta)$, where $u = 0$ on $[R, \pi]$. Then $\Phi \in C^\infty[0,\pi]$ and $\Phi(\theta_n) = \sum_{j=0}^\infty b_j(\Phi) C_j^{\frac{n-1}{2}}(\cos\theta_n)$, where

$$b_j(\Phi) = \frac{2^{n-3}(2j+n-1)j!\Gamma^2\left(\frac{n-1}{2}\right)}{\pi\Gamma(n+j-1)} \int_0^\pi \Phi(\theta) C_j^{\frac{n-1}{2}}(\cos\theta)\sin^{n-1}\theta\, d\theta \tag{8.1}$$

(see [20, Chapter 9, §3, Subsection 4]). For every fixed $c > 0$ we have $b_j(\Phi) = O(j^{-c})$ as $j \to +\infty$. We shall use the mapping $\mathfrak{a}_t$ constructed in the proof of Lemma 6.2. By assumption, we have $\int_{B_r} F(\mathfrak{a}_t\xi)\, d\xi = 0$, $\quad 0 \le t < R - r - \varepsilon$, where $F(\xi) = \Phi(\cos^{-1}\xi_{n+1})$. Passing to the spherical coordinates in this integral, we obtain

$$\begin{aligned} &\sum_{j=0}^\infty b_j(\Phi) \\ &\quad \times \int_0^r \int_0^{2\pi} \int_0^\pi \cdots \int_0^\pi C_j^{\frac{n-1}{2}}(\cos\theta_n\cos t + \sin\theta_n \sin t \cos\theta_{n-1}) \\ &\qquad\qquad \times \sin^{n-1}\theta_n \sin^{n-2}\theta_{n-1}\cdots\sin\theta_2\, d\theta_1\cdots d\theta_n = 0, \\ &\qquad\qquad\qquad\qquad 0 \le t < R - r - \varepsilon. \end{aligned} \tag{8.2}$$

Taking into account that $\int_0^\pi C_m^{\frac{n-2}{2}}(\cos\theta_{n-1})\sin^{n-2}\theta_{n-1}\, d\theta_{n-1} = 0$, $m \in \mathbb{N}$, and

$$j\int_0^r C_j^{\frac{n-1}{2}}(\cos\theta_n)\sin^{n-1}\theta_n\, d\theta_n = (n-1)\frac{\Gamma(j+n-1)}{\Gamma(j+n)}(\sin r)^n C_{j-1}^{\frac{n+1}{2}}(\cos r), \quad j \in \mathbb{N}$$

(see [20, Chapter 9, §3, Subsection 4 and §4, Subsection 8, equation (7)]), and using the addition formula for the Gegenbauer polynomials, we obtain from (8.2) that

$$\frac{b_0(\Phi)}{\Gamma(n)(\sin r)^n}\int_0^r (\sin\theta)^{n-1}\, d\theta + \sum_{j=1}^\infty \frac{b_j(\Phi)\Gamma(j)}{\Gamma(j+n)} C_{j-1}^{\frac{n+1}{2}}(\cos r) C_j^{\frac{n-1}{2}}(\cos t) = 0,$$
$$0 \le t < R - r - \varepsilon.$$

Consequently (see [20, Chapter 9, §3, Subsection 2, formula (4)] and (8.1)),

$$\sum_{j=1}^{\infty} \frac{j!(j-1)!\left(j+\frac{n-1}{2}\right)}{\Gamma(j+n-1)\Gamma(j+n)} C_{j-1}^{\frac{n+1}{2}}(\cos r) C_{j-1}^{\frac{n+1}{2}}(\cos t) \times \int_0^{\pi} \Phi(\theta) C_j^{\frac{n-1}{2}}(\cos\theta)\sin^{n-1}\theta\, d\theta = 0, \quad 0 \le t < R-r-\varepsilon. \tag{8.3}$$

Consider the function H defined as follows:

$$H(\theta) = \frac{\pi(\cos\theta - \cos t\cos r)}{2^{n+1}\left(\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\right)^2} \frac{\left(\sin\frac{r-t+\theta}{2}\sin\frac{\theta-r+t}{2}\sin\frac{\theta+t+r}{2}\sin\frac{t+r-\theta}{2}\right)^{\frac{n-3}{2}}}{(\sin\theta)^{n-2}(\sin t\sin r)^n}$$

if $|t-r| \le \theta \le t+r$, and $H(\theta) = 0$ if $\theta \in [0,\pi]\backslash[|t-r|, t+r]$. By [20, Chapter 9, §4, Subsection 3, formula (6)] and (8.1), we have

$$b_j(H) = \frac{j!(j-1)!\left(j+\frac{n-1}{2}\right)}{\Gamma(j+n-1)\Gamma(j+n)} C_{j-1}^{\frac{n+1}{2}}(\cos r) C_{j-1}^{\frac{n+1}{2}}(\cos t), \quad j \in \mathbb{N},$$

and $b_0(H) = 0$. Since $\Phi = 0$ on $[0, r]$ and $R \le 2r$, formula (8.3) implies that

$$\int_r^{r+t} \Phi(\theta) H(\theta)\sin^{n-1}\theta\, d\theta = 0, \quad 0 \le t < R-r-\varepsilon. \tag{8.4}$$

We represent (8.4) in the form

$$\int_{\cos t}^{\cos r} \Phi(\arccos x)(x - \cos r\cos(t-r))\left((\cos(t-2r)-x)(x-\cos t)\right)^{\frac{n-3}{2}} dx = 0, \quad r \le t < R-\varepsilon. \tag{8.5}$$

Integrating (8.5) by parts, we obtain

$$\int_{\cos t}^{\cos r} h_1(x)\left((x-\cos t)(\cos(t-2r)-x)\right)^{\frac{n-1}{2}} dx = 0, \quad r \le t < R-\varepsilon, \tag{8.6}$$

where $h_1(x) = \left(\Phi(\cos^{-1} x)\right)'$. By (8.6),

$$\int_r^t h_2(x)\Big(\sin\frac{x+t}{2}\sin\frac{t-x-2r}{2}\left(\cos(t-r)-\cos(x-r)\right)\Big)^{\frac{n-1}{2}} dx = 0, \quad r \le t < R-\varepsilon,$$

where $h_2(x) = h_1(\cos x)\sin x$. Consequently,

$$\int_t^1 h_3(x)(x-t)^{\frac{n-1}{2}} g_1(x,t)\, dx = 0, \quad \cos(R-r-\varepsilon) < t \le 1, \tag{8.7}$$

where $h_3(x) = \frac{h_2(r+\cos^{-1} x)}{\sqrt{1-x^2}}$, $g_1(x,t) = \left(t + \sqrt{1-x^2}\sin 2r - x\cos 2r\right)^{\frac{n-1}{2}}$. Suppose that $\cos(R-r-\varepsilon) < y \le 1$. We multiply (8.7) by $(t-y)^{\frac{n-1}{2}}$ and integrate with respect to t from y to 1. Changing the order of integration, we obtain

$$\int_y^1 h_3(x)\int_y^x \left((x-t)(t-y)\right)^{\frac{n-1}{2}} g_1(x,t)\, dt dx = 0, \quad \cos(R-r-\varepsilon) < y \le 1.$$

The change of variables $(x-y)z = x+y-2t$ in the inner integral yields

$$\int_y^1 h_3(x)(x-y)^n g_2(x,y)\, dx = 0, \quad \cos(R-r-\varepsilon) < y \le 1, \tag{8.8}$$

where $g_2(x,y) = \int_{-1}^{1}(1-z^2)^{\frac{n-1}{2}} g_1\big(x, \frac{x+y-(x-y)z}{2}\big)\,dz$. Differentiating $n+1$ times with respect to y, we deduce from (8.8) that $h_3(y) + \int_y^1 h_3(x)\mathcal{K}(x,y)\,dx = 0$, $\cos(R-r-\varepsilon) < y \le 1$, where $\mathcal{K}(x,y) = \frac{\frac{\partial^{n+1}}{\partial y^{n+1}}((x-y)^n g_2(x,y))}{(-1)^{n+1}n!g_2(y,y)}$. Thus, h_3 is a solution of the homogeneous integral Volterra equation of the second kind with the bounded kernel $\mathcal{K}(x,y)$. This means (see, e.g., [27, Chapter 9, §2, Subsection 5]) that $h_3 = 0$ on $(\cos(R-r-\varepsilon), 1)$, and the lemma follows. □

Lemma 8.2. *Let $\tau \in SO(n+1)$. Then there exist rotations $\mathcal{T}_1$ and $\mathcal{T}_2 \in SO(n+1)$ with the following properties*: 1) $\mathcal{T}_1 o = o$; 2) $\mathcal{T}_2$ *is a rotation in the plane* (x_{n+1}, x_n); 3) $\tau B_r = \mathcal{T}_1\mathcal{T}_2 B_r$ *for all* $r \in (0,\pi)$.

Proof. Let θ_j^k, where $1 \le k \le n$ and $1 \le j \le k$, be the Euler angles of the rotation τ (see [20, Chapter 9, §1, Subsection 3]). Then $\tau = \tau^{(n)}\cdots\tau^{(1)}$, where $\tau^{(k)} = \tau_1(\theta_1^k)\cdots\tau_k(\theta_k^k)$, and $\tau_j(\theta)$ is the rotation through the angle θ in the plane (x_{j+1}, x_j). Putting

$$\mathcal{T}_1 = \tau_1(\theta_1^n)\cdots\tau_{n-1}(\theta_{n-1}^n), \quad \mathcal{T}_2 = \tau_n(\theta_n^n),$$

we prove the lemma. □

Now, we proceed to the proof of Theorem 8.1.

Proof of statement 1). First, let $r < R \le 2r$. Our assumptions, Lemma 6.1, and formulas (3.3) and (3.4) imply that $f_{k,l}(\theta_n)Y_l^{(k)}(\sigma) \in V_r^\infty(B_R)$ and $f_{k,l}(\theta_n)Y_l^{(k)}(\sigma) = 0$ in B_r. Using Lemma 8.1 and statement b) of Lemma 6.3, we obtain $f_{k,l}(\theta_n)Y_l^{(k)}(\sigma) = 0$ in B_R. Consequently, $f = 0$ in B_R.

Next, suppose that statement 1) is valid for a radius $R \le (m+1)r$, where m is a fixed positive integer. We prove statement 1) for $R \in ((m+1)r, (m+2)r]$. Consider the function $g_\tau(\xi) = f(\tau\xi)$, $\xi \in B_{R-mr}$, where $\tau \in SO(n+1)$ and $\tau B_{R-mr} \subset B_R$. Since $g_\tau \in V_r^\infty(B_{R-mr})$ and $\tau B_r \subset B_{(m+1)r}$ (see Lemma 8.2), we have $g_\tau = 0$ in B_{R-mr} by induction. Since τ is arbitrary, this means that $f = 0$ in B_R. Thus, statement 1) is proved. □

Proof of statement 2). Let Φ_δ, $\delta \in (0,\varepsilon)$, be a function with the following properties: 1) $\Phi_\delta \in C^\infty(\mathbb{S}^n)$ and $\operatorname{supp}\Phi_\delta \subset B_\delta$; 2) Φ_δ is of the form $\Phi_\delta(\xi) = \varphi_\delta(\theta_n)$; 3) $\Phi_\delta \ge 0$ and $\int_{\mathbb{S}^n}\Phi_\delta(\xi)\,d\xi = 1$. We consider the convolution

$$\begin{aligned} f_\delta(\mathcal{T}o) = (f * \Phi_\delta)(\mathcal{T}o) &= \int_{SO(n+1)} f(\tau o)\Phi_\delta(\tau^{-1}\mathcal{T}o)\,d\tau \\ &= \omega_n^{-1}\int_{\mathbb{S}^n} f(\xi)\Phi_\delta(\mathcal{T}^{-1}\xi)\,d\xi, \end{aligned}$$

where $\mathcal{T} \in SO(n+1)$ and $\mathcal{T}o \in B_{R-\delta}$. Then $f_\delta \in V_r^\infty(B_{R-\delta})$, $f_\delta = 0$ in $B_{r+\varepsilon-\delta}$, and the f_δ converge uniformly to f on the compact subsets of B_R as $\delta \to 0$. Referring to statement 1), we conclude that $f = 0$ in B_R. □

Proof of statement 3a). First, we assume that $r \ne \pi/2$. Suppose $q, s \in \mathbb{Z}_+$, $s \ge 3(n+q+8)$, $p = s - \left[\frac{s}{3}\right] - 4$, and $\mathbb{D} = d_0 d_{1-n}$. Since $\operatorname{card}\mathcal{N}_s(r) \ge p+2$, where $\mathcal{N}_s(r) = \{2,3,\dots,s\}\backslash N_s(r)$ (see Lemma 5.3), there exist constants γ_m, $m \in \mathcal{N}_s(r)$, such that not all of them are zero and the function $u(\theta) = \sum_{m\in\mathcal{N}_s(r)}\gamma_m\psi_{m,1}(\theta)$ satisfies $(\mathbb{D}^j u)(r) = 0$, $j = 0,1,\dots,p$. For $\nu \in N(r)$, we put

$$c_\nu(u) = (\delta(\nu,\nu))^{-1}\int_0^r u(\theta)\psi_{\nu,1}(\theta)\sin^{n-1}\theta\,d\theta. \tag{8.9}$$

Integrating (8.9) by parts and using (3.10), (3.11), and the corollary to Lemma 6.3, we obtain $c_\nu(u) = (\nu(1-n-\nu))^{-p-1}c_\nu(\mathbb{D}^{p+1}u)$. By (4.7) and Lemma 5.5, we have $c_\nu(u) = O\left(\nu^{n-1-2p}\right)$ as $\nu \to +\infty$. Thus, the function $w(\theta) = \sum_{\nu\in N(r)} c_\nu(u)\psi_{\nu,1}(\theta)$ is continuous on $[0,r]$ and $c_\nu(w) = c_\nu(u)$ for all $\nu \in N(r)$ (see (4.7), (5.3), and Lemma 5.5). From Lemma 5.4 and (8.9) it follows that $w = u$ on $[0,r]$, whence $d_s \cdots d_1 w = 0$ on $[0,r]$ (see (3.10)). This means that $\sum_{\nu\in N(r)} \alpha_\nu \psi_{\nu,s+1}(\theta) = 0$ on $[0,r]$, where $\alpha_\nu = c_\nu(u)\prod_{j=0}^{s-1}(1-\nu+j)(\nu+n+j)$ (see (3.10), (4.7), and (5.3)). Consequently, the function

$$f(\xi) = \sum_{\nu\in N(r)} \alpha_\nu \psi_{\nu,s+1}(\theta_n) Y_1^{(s+1)}(\sigma) \tag{8.10}$$

belongs to $V_r^{s-2\left[\frac{s}{3}\right]-n-8}(B_R)$ and is zero on B_r (see (4.7) and (5.3)). We assume that $f = 0$ in B_R, fix $\varepsilon \in (0, R-r)$, and consider an arbitrary function $\Phi(\xi) = \varphi(\theta_n) \in C^\infty(\mathbb{S}^n)$ with support in B_ε. Then Lemma 7.1 and formulas (3.11) and (8.10) imply $\sum_{\nu\in N(r)} \alpha_\nu \widetilde{\Phi}(\nu)\psi_{\nu,s+1}(\theta) = 0$ on $[0,r]$. Since for any fixed $c > 0$ we have $\widetilde{\Phi}(\nu) = O(\nu^{-c})$ as $\nu \to +\infty$ (see the beginning of §7), relations (3.10), (4.7), (5.3) and Lemma 5.5 imply that $\alpha_\nu \widetilde{\Phi}(\nu) = 0$ for all $\nu \in N(r)$. Since Φ is arbitrary, all α_ν are equal to 0, whence $c_\nu(u) = 0$ for $\nu \in N(r)\backslash\{2,3,\dots,s\}$. Since $u = w$ on $[0,r]$, the definition of u and w implies the existence of constants $\beta_2, \beta_3, \dots, \beta_s$ such that not all of them are zero and $\sum_{m=2}^{s} \beta_m \psi_{m,1}(\theta) = 0$ on $[0,r]$. This contradicts (3.7), i.e., the function f satisfies all the required conditions for $r \neq \pi/2$.

Now, suppose $r = \pi/2$, $q \in \mathbb{Z}_+$, and $s \geq 3+q$. Let $\gamma_1, \dots, \gamma_{s-1}$ be constants such that not all of them are zero and

$$\sum_{m=1}^{s-1} \gamma_m \Big(\frac{d}{dt}\Big)^j \left(P_{2m+\frac{n}{2}-1}^{-\frac{n}{2}-2s+1}(t)\right)\Big|_{t=0} = 0, \quad j = 0, \dots, s-3.$$

Then the function

$$h(\xi) = \sum_{m=1}^{s-1} \gamma_m (1-\xi_{n+1}^2)^{\frac{1}{2}-\frac{n}{4}} P_{2m+\frac{n}{2}-1}^{-\frac{n}{2}-2s+1}(|\xi_{n+1}|) C_{2s}^{\frac{n-2}{2}}\Big(\xi_n / \sqrt{1-\xi_{n+1}^2}\Big)$$

belongs to $C^{s-3}(\mathbb{S}^n)$ and is even on $\mathbb{S}^n$, and

$$\mathring{h}(\theta_1, \dots, \theta_n) = \sum_{m=1}^{s-1} \gamma_m \psi_{2m,2s}(\theta_n) C_{2s}^{\frac{n-2}{2}}(\cos\theta_{n-1})$$

for $\theta_n \in [0, \pi/2]$. Since $N(\pi/2) = \{2j\}_{j=1}^\infty$, the corollary to Lemma 7.1 implies $h \in V_{\pi/2}^{s-3}(\mathbb{S}^n)$. Therefore, the function

$$f(\xi) = h(\xi) - \sum_{m=1}^{s-1} \gamma_m (1-\xi_{n+1}^2)^{\frac{1}{2}-\frac{n}{4}} P_{2m+\frac{n}{2}-1}^{-\frac{n}{2}-2s+1}(\xi_{n+1}) C_{2s}^{\frac{n-2}{2}}\Big(\xi_n / \sqrt{1-\xi_{n+1}^2}\Big)$$

belongs to $V_{\pi/2}^{s-3}(B_R)$ and vanishes in $B_{\pi/2}$. Suppose $f = 0$ in B_R. Then

$$\sum_{m=1}^{s-1} \gamma_m \left(P_{2m+\frac{n}{2}-1}^{-\frac{n}{2}-2s+1}(t) - P_{2m+\frac{n}{2}-1}^{-\frac{n}{2}-2s+1}(-t)\right) = 0$$

on $(0, -\cos R)$. By [22, Subsection 3.4, formula (14) and Subsection 3.5, formula (3)], we have

$$\sum_{m=1}^{s-1} \gamma_m \sum_{j=0}^{s-m-1} \alpha_{j,s,m} \cos((2s-2m-2j-1)\theta) = 0, \quad \theta \in (\pi - R, \pi/2),$$

where $\alpha_{j,s,m} > 0$. Since this contradicts the choice of $\gamma_1, \dots, \gamma_{s-1}$, the function f satisfies all conditions of statement 3) for $r = \pi/2$. □

Proof of statement 3b). Suppose $\varepsilon \in (0, r)$, $u \in C^\infty[0, r]$, and $\operatorname{supp} u = [r - \varepsilon, r - \varepsilon/2]$. The proof of statement 3a) shows that if $0 \le \theta \le r$, then $u(\theta) = \sum_{\nu \in N(r)} u_\nu \psi_{\nu,1}(\theta)$, where $u_\nu \in \mathbb{C}$, and for any fixed $c > 0$ we have $u_\nu = O(\nu^{-c})$ as $\nu \to +\infty$. Then the function $f(\xi) = \sum_{\nu \in N(r)} u_\nu \psi_{\nu,1}(\theta_n) Y_1^{(1)}(\sigma)$, $\xi \in B_R$, satisfies all the requirements of statement 3b) (see (4.7), (5.3), and Lemma 7.2). Thus, Theorem 8.1 is proved completely. □

The first statement of Theorem 8.1 admits the following refinement.

Theorem 8.2. *Let $f \in V_r^s(B_R)$, and let $f = 0$ in B_r. Then $f_{k,l}(\theta_n) = 0$ in B_R for all $0 \le k \le s$ and $1 \le l \le a_k$.*

For the proof of Theorem 8.2 we need the following lemma.

Lemma 8.3. *Suppose $R \le \pi$ and $f \in C^\infty(B_R)$. Then $f \in V_r(B_R)$ if and only if relation* (7.4) *is valid for all $k \in \mathbb{Z}_+$ and $1 \le l \le a_k$, and $c_{\nu,k,l} = O(\nu^{-c})$ as $\nu \to +\infty$ for any fixed $c > 0$.*

Proof. The "only if" part. By Lemma 6.1, $f_{k,l}(\theta_n) Y_l^{(k)}(\sigma) \in V_r^\infty(B_R)$ for all $k \in \mathbb{Z}_+$ and $1 \le l \le a_k$. First, we consider the case where $k = 1$. We put $u(\theta_n) = f_{1,l}(\theta_n)$ and $c_{\nu,1,l} = c_\nu(u)$, where $c_\nu(u)$ is defined by (8.9). The same argument as in the proof of statement 3a) of Theorem 8.1 shows that $c_{\nu,1,l} = O(\nu^{-c})$ for every $c > 0$ as $\nu \to +\infty$, and the series on the right-hand side of (7.4) with $k = 1$ converges uniformly on the compact subsets of B_R. By Lemmas 5.4 and 5.5, the sum of this series is equal to $u(\theta_n)$ in B_r. Using (4.7), (5.3), and Lemma 7.2, from the first statement of Theorem 8.1 we deduce relation (7.4) for $k = 1$ on the entire B_R. For $k > 1$, formula (7.4) is obtained by induction on k with the help of (3.10) and statement b) of Lemma 6.3. Similarly, for $k = 0$, the required expansion follows from the case of $k = 1$, formula (3.10), and statement a) of Lemma 6.3.

The "if" part of Lemma 8.3 follows from estimate (5.3) and Lemma 7.2. □

Proof of Theorem 8.2. By Lemma 6.1 and formulas (3.3) and (3.4), from the assumptions of Theorem 8.2 it follows that $f_{k,l}(\theta_n) Y_l^{(k)}(\sigma) \in V_r^s(B_R)$ and $f_{k,l}(\theta_n) = 0$ in B_r. Let $k = 0$. Using the standard smoothing procedure (see the proof of the second statement of Theorem 8.1), relation (3.10), and Lemmas 5.5 and 8.3, we obtain $f_{0,1}(\theta_n) = \sum_{\nu \in N(r)} c_{\nu,0,1} \psi_{\nu,0}(\theta_n)$, where

$$c_{\nu,0,1} = (\nu(\nu + n - 1)\delta(\nu, \nu))^{-1} \int_0^r f_{0,1}(\theta) \psi_{\nu,0}(\theta) \sin^{n-1} \theta \, d\theta,$$

and the series converges to $f_{0,1}(\theta_n)$ in the space $\mathcal{D}'(B_R)$ of distributions. Hence, $f_{0,1}(\theta_n) = 0$ in B_R. For $0 < k \le s$, the claim follows by induction on k with the help of statement b) of Lemma 6.3. □

§9. Description of the functions of class V_r

Theorem 9.1. *Suppose $R \le \pi$ and $f \in L_{\mathrm{loc}}(B_R)$. Then $f \in V_r(B_R)$ if and only if the following relation is valid for all $k \in \mathbb{Z}_+$ and $1 \le l \le a_k$:*

$$f_{k,l}(\theta_n) Y_l^{(k)}(\sigma) = \sum_{\nu \in N(r)} c_{\nu,k,l} \Psi_\nu^{k,l}(\xi), \tag{9.1}$$

where the series converges in the space $\mathcal{D}'(B_R)$.

For the proof of Theorem 9.1 we need three lemmas.

Lemma 9.1. *Suppose $R \le \pi$, $s \ge n+5$, and $f \in V_r^s(B_R)$. If $|k-1| \le s-n-5$ and $1 \le l \le a_k$, then (7.4) is true with $c_{\nu,k,l} \in \mathbb{C}$, and $c_{\nu,k,l} = O(\nu^{n+k+2-s})$ as $\nu \to +\infty$.*

Proof. Repeating the same argument as in the proof of Lemma 8.3 and using Theorem 8.2 in place of Theorem 8.1, we obtain the required statement. □

Let χ_r denote the characteristic function (indicator) of the ball B_r.

Lemma 9.2. *Suppose that $f \in L_{\mathrm{loc}}(B_R)$, $f(\xi) = f_{k,l}(\theta_n)Y_l^{(k)}(\sigma)$ in B_R, and $F(\xi) = (f * \chi_r)(\xi)$, $\xi \in B_{R-r}$. Then $F(\xi) = F_{k,l}(\theta_n)Y_l^{(k)}(\sigma)$ in B_{R-r}.*

Proof. The claim is obtained from formulas (3.3) and (3.4) by simple manipulations. □

Lemma 9.3. *If $R \le \pi$ and $f(\xi) = u(\theta_n)Y_l^{(k)}(\sigma) \in V_r(B_R)$, then for each $s \in \mathbb{N}$ there exists a function Φ with the following properties: 1) $\Phi \in V_r^s(B_R)$; 2) Φ is of the form $\Phi(\xi) = \varphi(\theta_n)Y_l^{(k)}(\sigma)$; 3) $\mathcal{L}^{s+[(n+k)/2]+1}\Phi = f$ in $\mathcal{D}'(B_R)$, where $\mathcal{L} = L_k$ if $\psi_{k,1}(r) \ne 0$, and $\mathcal{L} = L_{k-1}$ if $\psi_{k,1}(r) = 0$.*

Proof. First, let $\psi_{k,1}(r) \ne 0$. We put $u_1(\theta) = u(\theta)$. By the Fubini theorem, we have

$$\int_0^{R-\varepsilon} |u_1(t)| \sin^{n-1} t\, dt < \infty \tag{9.2}$$

for every $\varepsilon \in (0, R)$. We consider the sequence of functions $u_m(t)$ defined as follows: if $1 \le m \le [(n+k)/2]$, then

$$u_{m+1}(\theta) = \sin^k \theta \int_{r/4}^{\theta} (\sin\alpha)^{1-2k-n} \int_0^{\alpha} (\sin t)^{n+k-1} u_m(t)\, dt d\alpha, \tag{9.3}$$

and if $m \ge [(n+k)/2] + 1$, then

$$u_{m+1}(\theta) = \sin^k \theta \int_0^{\theta} (\sin\alpha)^{1-2k-n} \int_0^{\alpha} (\sin t)^{n+k-1} u_m(t)\, dt\, d\alpha. \tag{9.4}$$

For $2 \le m \le [(n+k)/2] + 1$, the following inequalities are easily deduced from (9.2) and (9.3) by induction on m:

$$|u_m(\theta)| \le c_1\theta^k |\ln\theta| + c_2\theta^{2m-n-2}, \quad \theta \in (0, R-\varepsilon), \tag{9.5}$$

where the constants c_1 and c_2 are independent of θ. Similarly, identity (9.4) and estimate (9.5) with $m = [(n+k)/2] + 1$ show that if $m \ge [(n+k)/2] + 2$ and $0 \le j \le 2m - 2\,[(n+k)/2] - 4$, then

$$\Big|\Big(\frac{d}{d\theta}\Big)^{2m-2[(n+k)/2]-4-j}(u_m(\theta))\Big| \le c_3\theta^{k+j+1}, \quad \theta \in (0, R-\varepsilon), \tag{9.6}$$

and the constant c_3 is independent of θ. We put $f_m(\xi) = u_m(\theta_n)Y_l^{(k)}(\sigma)$, $m \in \mathbb{N}$. The functions f_m have the following properties (see (9.5), (9.6), (3.5)): 1) $f_m \in L_{\mathrm{loc}}(B_R)$, $m \in \mathbb{N}$; 2) if $m \ge [(n+k)/2] + 2$, then $f_m \in C^{m-[(n+k)/2]-2}(B_R)$; 3) $\mathcal{L}f_{m+1} = f_m$ in $\mathcal{D}'(B_R)$ for $m \in \mathbb{N}$. Since $f_1 \in V_r(B_R)$, property 3) implies that $\mathcal{L}(f_2 * \chi_r) = 0$ in $\mathcal{D}'(B_{R-r})$. Since the operator $\mathcal{L}$ is elliptic and the function $f_2 * \chi_r$ is continuous, we have $f_2 * \chi_r \in C^\infty(B_{R-r})$. Now, using (3.5), (3.11), and Lemma 9.2, we see that $(f_2 * \chi_r)(\xi) = c_4\psi_{k,k}(\theta_n)Y_l^{(k)}(\sigma)$, where $c_4 \in \mathbb{C}$. Therefore, the function $f_{1,1}(\xi) = f_2(\xi) - c_4\sqrt{\pi}\big(2^{\frac{n}{2}-1}\Gamma\left(\frac{n+1}{2}\right)\psi_{k,1}(r)\sin^{n-1} r\big)^{-1}\psi_{k,k}(\theta_n)Y_l^{(k)}(\sigma)$ belongs to $V_r(B_R) \cap C(B_R\backslash\{o\})$ (see Lemma 7.1 and (3.11)). Here $f_{1,1}(\xi)$ is of the form $u_{1,1}(\theta_n)Y_l^{(k)}(\sigma)$ and $\mathcal{L}f_{1,1} = f_1$ in $\mathcal{D}'(B_R)$. Repeating the same argument with $f_{1,1}$ in place of f and so on, we construct a sequence of functions $f_{m,1}$ of the form $u_{m,1}(\theta_n)Y_l^{(k)}(\sigma)$ and such that $f_{m,1} \in V_r(B_R) \cap$

$C(B_R\backslash\{o\})$ and $\mathcal{L}^m(f_{m,1}) = f_1$ in $\mathcal{D}'(B_R)$. The properties 2) and 3) of the functions f_m and the fact that the operator $\mathcal{L}^m$ is elliptic imply that $f_{m,1} \in C^{m-[(n+k)/2]-1}(B_R)$ for $m \geq [(n+k)/2]+1$. Putting $\Phi = f_{m,1}$, where $m = s + [(n+k)/2] + 1$, we complete the proof of Lemma 9.3 in the case where $\psi_{k,1}(r) \neq 0$. If $\psi_{k,1}(r) = 0$, then $\psi_{k-1,1}(r) \neq 0$ (see the proof of Lemma 5.3), and the claim is obtained similarly (see (3.5)). □

Proof of Theorem 9.1. Let $f \in V_r(B_R)$. By Lemma 6.1, we have $f_{k,l}(\theta_n)Y_l^{(k)}(\sigma) \in V_r(B_R)$. Put $s = |k-1| + n + 5$. By Lemma 9.3, there exists a function $\varphi(\theta_n)Y_l^{(k)}(\sigma) \in V_r^s(B_R)$ such that

$$\mathcal{L}^{s+[(n+k)/2]+1}\big(\varphi(\theta_n)Y_l^{(k)}(\sigma)\big) = f_{k,l}(\theta_n)Y_l^{(k)}(\sigma) \tag{9.7}$$

in $\mathcal{D}'(B_R)$. By Lemma 9.1,

$$\varphi(\theta_n)Y_l^{(k)}(\sigma) = \sum_{\nu\in N(r)} \gamma_{\nu,k,l}\Psi_\nu^{k,l}(\xi),$$

where $\gamma_{\nu,k,l} = O\left(\nu^{n+k+2-s}\right)$ as $\nu \to +\infty$. Since $s-n-k \geq 4$, this series converges uniformly on the compact subsets of B_R (see (4.7) and (5.3)). Invoking (9.7), we obtain the expansion (9.1) with the series converging in $\mathcal{D}'(B_R)$ and with $c_{\nu,k,l} = O\left(\nu^{s+2[(n+k)/2]+n+k+4}\right)$ as $\nu \to +\infty$ (see (3.11)).

Now, we prove the converse statement. Since the convolution $\varphi \mapsto \varphi * \chi_r$ is continuous on $\mathcal{D}'(B_R)$, the corollary to Lemma 7.1 and (9.1) imply $f_{k,l}(\theta_n)Y_l^{(k)}(\sigma) \in V_r(B_R)$. The same argument as in the proof of Lemma 7.2 yields $f \in V_r(B_R)$. Thus, Theorem 9.1 is proved. □

§10. The structure of the set Ω

Lemma 10.1. *Let $r_1, r_2 \in (0,\pi)$ and $r_1 \neq r_2$. If $\{\alpha_m\}_{m=1}^\infty \in N(r_1)$ is a monotone increasing sequence such that $|\psi_{\alpha_m,1}(r_2)| < (1+\alpha_m)^{-(n+7)/2}$, then* $\inf\left\{\frac{\alpha_{m+1}}{\alpha_m} : m \in \mathbb{N}\right\} > 1$.

Proof. Let $\beta_m = \alpha_m + (n-1)/2$. Lemma 5.2, formulas (4.2), and the estimate for $\psi_{\alpha_m,1}(r_2)$ imply the relation

$$r_i\beta_m = \pi\Big(\frac{n+3}{4} + n_{m,i}\Big) + \frac{1-n^2}{8\beta_m}\cot r_i + O(\beta_m^{-3}), \quad i = 1,2, \tag{10.1}$$

where $n_{m,i} \in \mathbb{Z}$ and the constant involved in the O sign is independent of m. Consequently,

$$\frac{r_1}{r_2} - \frac{n+3+4n_{m,1}}{n+3+4n_{m,2}} = \frac{c(r_1,r_2)}{\beta_m(n+3+4n_{m,2})} + O(\beta_m^{-4}), \tag{10.2}$$

where $c(r_1,r_2) = \frac{(1-n^2)(r_2\cot r_1 - r_1\cot r_2)}{(2\pi r_2)}$. Suppose $\inf\{\alpha_{m+1}/\alpha_m : m \in \mathbb{N}\} = 1$, and let $\{m_l\}_{l=1}^\infty$ be a monotone increasing sequence of positive integers such that $\lim_{l\to\infty}\frac{\alpha_{m_l+1}}{\alpha_{m_l}} = 1$. Recalling (10.1) and using (10.2) with $m = m_l, m_l+1$, we obtain

$$\begin{aligned}(n+3+4n_{m_l,1})(n+3+4n_{m_l+1,2}) &- (n+3+4n_{m_l+1,1})(n+3+4n_{m_l,2})\\ &= c(r_1,r_2)\Big(\frac{(n+3+4n_{m_l,2})}{\beta_{m_l+1}} - \frac{(n+3+4n_{m_l+1,2})}{\beta_{m_l}}\Big) + O(\beta_{m_l}^{-2}).\end{aligned}$$

Since $c(r_1,r_2) \neq 0$ and $(n+3+4n_{m_l,2})/\beta_{m_l+1} - (n+3+4n_{m_l+1,2})/\beta_{m_l} \to 0$ as $l \to \infty$, the above relation implies that $(n+3+4n_{m_l,2})/\beta_{m_l+1} - (n+3+4n_{m_l+1,2})/\beta_{m_l} = O\left(\beta_{m_l}^{-2}\right)$. Consequently, $(n_{m_l+1,2} - n_{m_l,2})(4(n_{m_l,2} + n_{m_l+1,2}) + 2n + 6) = O(1)$ (see (10.1)). This means that $n_{m_l+1,2} = n_{m_l,2}$ for all sufficiently large $l \in \mathbb{N}$. In particular, $\lim_{l\to\infty}(\beta_{m_l+1} - \beta_{m_l}) = 0$ (see (10.1)), which contradicts (5.2). This proves Lemma 10.1. □

We proceed to the study of the properties of Ω.

Lemma 10.2. *Let $r_1, r_2 \in (0,\pi)$. Then $(r_1,r_2) \in \Omega$ if and only if for each $m \in \mathbb{N}$ there exists $\alpha_m \in N(r_1)$ such that $|\psi_{\alpha_m,1}(r_2)| < (1+\alpha_m)^{-m}$.*

Proof. Assuming that for each $m \in \mathbb{N}$ there exists $\alpha_m \in N(r_1)$ satisfying the hypothesis of the lemma, we prove that $(r_1,r_2) \in \Omega$. Let $p > (n+7)/2$. Without loss of generality, we may assume that the sequence $\{\alpha_m\}_{m=1}^{\infty}$ is monotone increasing and $|\psi_{\alpha_m,1}(r_2)| < (1+\alpha_m)^{-p}$ for $m \in \mathbb{N}$ (otherwise, $N(r_1,r_2) \neq \varnothing$, whence $(r_1,r_2) \in \Omega$). Then, by (10.1) and relation (5.2) with $r = r_2$, we have $\alpha_m - \gamma_m \to 0$ as $m \to \infty$, where γ_m is the element of $N(r_2)$ closest to α_m (such γ_m is uniquely determined for sufficiently large m; see (5.2)). Applying the mean-value theorem to the function $u(t) = \psi_{t,1}(r_2)$ on the interval with endpoints α_m and γ_m and using (4.3) and (5.2) with $r = r_2$, we obtain $|\alpha_m - \gamma_m| < c(1+\alpha_m)^{-p+(n+1)/2}$, where c does not depend on m. Thus, $(r_1,r_2) \in \Omega$.

To prove the converse statement, we assume that for every $q > 0$ there exist points $\alpha \in N(r_1)$ and $\beta \in N(r_2)$ such that $|\alpha-\beta| < (\alpha+\beta)^{-q}$. Then $\psi_{\alpha,1}(r_2) = \psi_{\alpha,1}(r_2) - \psi_{\beta,1}(r_2)$. Now, as above, we can apply the mean-value theorem and (4.3) with $r = r_2$ to obtain the required statement. Lemma 10.2 is proved. □

Corollary 10.1. *Suppose $(r_1,r_2) \in \Omega$ and $N(r_1,r_2) = \varnothing$. Then $r_1/r_2 \notin \mathbb{Q}$.*

Proof. The assumption and Lemma 10.2 imply that there exist a monotone increasing sequence $\{\alpha_m\}_{m=1}^{\infty}$ satisfying the hypothesis of Lemma 10.1. The proof of Lemma 10.1 shows that r_1/r_2 admits fast approximation by rational fractions (see (10.2)). Therefore, $r_1/r_2 \notin \mathbb{Q}$. □

Lemma 10.3. *For each $r_1 \in (0,\pi)$, the set $A = \{r_2 \in (0,\pi) : N(r_1,r_2) \neq \varnothing\}$ is countable and dense in $(0,\pi)$.*

Proof. For a fixed $\lambda > (1-n)/2$, the set $\{r \in (0,\pi) : \psi_{\lambda,1}(r) = 0\}$ has no finite accumulation points; therefore, A is at most countable. Consider an arbitrary interval $(a,b) \subset (0,\pi)$. Let $\alpha \in (a,b)$ be such that $\cos\left(\frac{\pi\alpha(n+3)}{4r_1} - \frac{\pi(n+1)}{4}\right) \neq 0$, and let $n_{m,1}$ and $n_{m,2}$ be sequences of positive integers such that $n_{m,1} \to \infty$, $n_{m,2} \to \infty$, and $n_{m,1}/n_{m,2} \to \alpha/r_1$ as $m \to \infty$. From Lemma 5.2 it follows that (5.2) is valid for $r = r_1$ and $j = n_{m,2} - q(r_1,n)$. Let $\rho_{m,1} = r_1 n_{m,1}/n_{m,2}$ and $\rho_{m,2} = r_1(n_{m,1}+1)/n_{m,2}$. Using (4.2), we see that, for sufficiently large m, the numbers $s_{\rho_{m,1}}(\lambda_j(r_1))$ and $s_{\rho_{m,2}}(\lambda_j(r_1))$ have opposite signs, whence $s_{r_2}(\lambda_j(r_1)) = 0$ for some $r_2 \in [\rho_{m,1},\rho_{m,2}] \subset (a,b)$. Thus, $(a,b) \cap A \neq \varnothing$, and Lemma 10.3 is proved. □

Lemma 10.4. *For every $r_1 \in (0,\pi)$, the intersection of the set $\{r_2 \in (0,\pi) : (r_1,r_2) \in \Omega\}$ with an arbitrary interval $(a,b) \subset (0,\pi)$ is uncountable.*

Proof. Let $\varkappa_m$ be the smallest element of $N(r_1,\alpha_m)$, where $\{\alpha_m\}_{m=1}^{\infty}$ is a sequence of pairwise distinct numbers such that $\{r_2 \in (a,b) : N(r_1,r_2) \neq \varnothing\} = \{\alpha_m\}_{m=1}^{\infty}$ (see Lemma 10.3). Then $\varkappa_m \to \infty$ as $m \to \infty$, because otherwise the function $\psi_{\nu,1}(r)$ has infinitely many zeros $r \in (a,b)$ for some $\nu \in N(r_1)$. Therefore, since the sequence $\{\alpha_m\}_{m=1}^{\infty}$ is dense on (a,b), we see that the set A_1 of all t such that there exists a subsequence $\{m_l\}_{l=1}^{\infty} \in \mathbb{N}$ with $|t-\alpha_{m_l}| < e^{-\varkappa_{m_l}}$, $l \in \mathbb{N}$, is uncountable. Since $A_1 \subset \{r_2 \in [a,b] : (r_1,r_2) \in \Omega\}$ (see (4.7) and Lemma 10.2), the required statement is proved. □

Lemma 10.5. *For each $r_1 \in (0,\pi)$, the set $\{r_2 \in (0,\pi) : (r_1,r_2) \in \Omega\}$ has nonzero Lebesgue measure on $(0,\pi)$.*

Proof. Suppose r_1 is fixed, $[a,b] \subset (0,\pi)$, $r_2 \in [a,b]$, and $(r_1,r_2) \in \Omega$. Then there exists a subsequence $\{j_m\}_{m=1}^{\infty}$ of positive integers depending, possibly, on r_2 and such that,

for $j \in \{j_m\}_{m=1}^{\infty}$, we have $r_2\lambda_j(r_1) = \pi\big(\frac{n+3}{4} + n_j(r_2)\big) + (1-n^2)(\cot r_2)/(8\lambda_j(r_1)) + O(\lambda_j^{-3}(r_1))$, with $n_j(r_2) \in \mathbb{Z}$ and with constant depending only on r_1, a, b, and n (see Lemma 10.2 and the proof of Lemma 5.2). In particular, if, for $r_2 = \rho_1$ and $\rho_2 \in [a,b]$, the above relation is valid for the same $\lambda_j(r_1)$ and $n_j(\rho_1) = n_j(\rho_2)$, then $\rho_1 - \rho_2 = O(\lambda_j^{-4}(r_1))$. This means that for each $q \in \mathbb{N}$ the set $A_2 = \{r_2 \in [a,b] : (r_1, r_2) \in \Omega\}$ can be covered by intervals with the sum of lengths estimated by $O\big(\sum_{j=q}^{\infty} \lambda_j^{-3}(r_1)\big)$, i.e., the Lebesgue measure of A_2 is zero (see (5.3)). Since a and b are arbitrary, the lemma is proved □

§11. Proof of the main result

First, we note that the first statement of Theorem 2.1 can easily be deduced from the second statement with the help of the standard smoothing procedure (see the proof of Theorem 8.1, item 2)).

Proof of statement 2). Suppose $f \in V^{\infty}_{r_1,r_2}(B_R)$, $r_1 + r_2 = R$, and $N(r_1, r_2) = \varnothing$. By Lemma 8.3 with $r = r_1$, we have relation (7.4), where $c_{\nu,k,l} = O(\nu^{-c})$ as $\nu \to +\infty$ and $c > 0$ is arbitrary and fixed. Since $f_{k,l}(\theta_n)Y_l^{(k)}(\sigma) \in V^{\infty}_{r_2}(B_R)$ (see Lemma 6.1), from (7.4), (3.11), and (7.2) it follows that $\sum_{\nu\in N(r_1)} c_{\nu,k,l}\psi_{\nu,1}(r_2)\psi_{\nu,k}(\theta) = 0$ for $\theta \in [0, r_1]$. Since $N(r_1, r_2) = \varnothing$, we have $c_{\nu,k,l} = 0$ for all ν, k, and l (see (3.10) and Lemma 5.5). Thus, all $f_{k,l}(\theta_n)$ vanish in B_R, which implies statement 2). □

Proof of statement 3). Suppose $f \in V_{r_1,r_2}(B_R)$, $r_1 + r_2 = R$, $N(r_1, r_2) = \varnothing$, and $(r_1, r_2) \in \Omega$. It suffices to prove that $f_{k,l}(\theta_n) = 0$ in B_R for all $r \in \mathbb{Z}_+$ and all $1 \le l \le a_k$. We put $s = |k-1| + n + 5$ and consider the function $\Phi(\xi) = \varphi(\theta_n)Y_l^{(k)}(\sigma) \in V^s_{r_1}(B_R)$ such that $(\mathcal{L}^p\Phi)(\xi) = f_{k,l}(\theta_n)Y_l^{(k)}(\sigma)$ in $\mathcal{D}'(B_R)$ for $p = s + [(n+k)/2] + 1$ (see Lemmas 6.1 and 9.3). By Lemma 9.1, we have

$$\Phi(\xi) = \sum_{\nu\in N(r_1)} c_{\nu,k,l}\Psi_\nu^{k,l}(\xi), \quad \xi \in B_R, \tag{11.1}$$

where $c_{\nu,k,l} = O(\nu^{n+k+2-s})$ as $\nu \to +\infty$. By (11.1) and (7.2),

$$(\Phi * \chi_{r_2})(\xi) = \frac{2^{\frac{n}{2}-1}}{\sqrt{\pi}}\Gamma\Big(\frac{n+1}{2}\Big)(\sin r_2)^{n-1}H_1(\theta_n)Y_l^{(k)}(\sigma),$$

where $\xi \in B_{r_1}$ and $H_1(\theta_n) = \sum_{\nu\in N(r_1)} c_{\nu,k,l}\psi_{\nu,1}(r_2)\psi_{\nu,k}(\theta_n)$. Since the operator $\mathcal{L}$ is invariant under the action of $SO(n+1)$, by the latter identity and the definition of Φ we obtain

$$\mathcal{L}^p\big(H_1(\theta_n)Y_l^{(k)}(\sigma)\big) = 0. \tag{11.2}$$

Since the operator $\mathcal{L}^p$ is elliptic, we have $H_1(\theta_n)Y_l^{(k)}(\sigma) \in C^\infty(B_{r_1})$, and the function H_1 can be extended uniquely to a function of class $C^\infty[0,\pi)$ (see (3.5)). For $t \in (0,\pi)$, we put $H_2(t) = (d_{-n}d_{-n-1}\cdots d_{-n-k+2}H_1)(t)$ if $k \ge 2$, $H_2(t) = H_1(t)$ if $k = 1$, and $H_2(t) = (d_0H_1)(t)$ if $k = 0$. By (3.10) and (4.7), we obtain $H_2(t) = \sum_{\nu\in N(r_1)} d_{\nu,k,l}\psi_{\nu,1}(t)$, $t \in [0, r_1]$, where $d_{\nu,k,l} = c_{\nu,k,l}\psi_{\nu,1}(r_2)$ if $k \ge 1$, and $d_{\nu,0,1} = \nu(1-n-\nu)c_{\nu,0,1}\psi_{\nu,1}(r_2)$. The estimate for $c_{\nu,k,l}$ shows that

$$d_{\nu,k,l} = O\Big(\frac{|\psi_{\nu,1}(r_2)|}{\nu^2}\Big), \quad \nu \to +\infty. \tag{11.3}$$

Repeating the same argument as in the proof of statement 3a) of Theorem 8.1, we see that the following relation is valid for all $m \in \mathbb{Z}_+$:

$$d_{\nu,k,l} = (\nu(1-n-\nu))^{-m} c_\nu (\mathbb{D}^m H_2) + \frac{(\sin r_1)^{n-1}\psi_{\nu,0}(r_1)}{\delta(\nu,\nu)} \sum_{j=0}^{m-1} \frac{\left(\mathbb{D}^j H_2\right)(r_1)}{(\nu(1-n-\nu))^{j+1}}, \tag{11.4}$$

where $\mathbb{D} = d_0 d_{1-n}$ (for the definition of c_ν, see (8.9)). By (3.8), (4.2), and (4.4), we have the estimate $|\psi_{\nu,0}(r_1)| > c\nu^{(1-n)/2}$ for all sufficiently large $\nu \in N(r_1)$, where the constant $c > 0$ is independent of ν. We assume that at least one of the numbers $\left(\mathbb{D}^j H_2\right)(r_1)$, $j \in \mathbb{Z}_+$, is nonzero. Then, by (11.4) and (4.7), $|d_{\nu,k,l}| > \nu^{-\alpha}$ for some fixed $\alpha > 0$ and all sufficiently large $\nu \in N(r_1)$. Since $(r_1, r_2) \in \Omega$, we have $N(r_1, r_2) = \varnothing$, which contradicts (11.3) (see Lemma 10.2). Thus, $\left(\mathbb{D}^j H_2\right)(r_1) = 0$ for all $j \in \mathbb{Z}_+$, and (11.4) implies that for any fixed $\beta > 0$ we have $d_{\nu,k,l} = O(\nu^{-\beta})$ as $\nu \to +\infty$. Then, by (3.11) and (11.2), $\sum_{\nu \in N(r_1)} b_{\nu,k} c_{\nu,k,l} \psi_{\nu,1}(r_2)\psi_{\nu,k}(\theta) = 0$ for $\theta \in [0, r_1]$, where $b_{\nu,k} = ((k-\nu)(k+\nu+n-1))^p$ if $\psi_{k,1}(r_1) \neq 0$ and $b_{\nu,k} = ((k-\nu-1)(k+\nu+n-2))^p$ if $\psi_{k,1}(r_1) = 0$. Since $N(r_1, r_2) = \varnothing$, all $c_{\nu,k,l}$ vanish (see (3.10) and Lemma 5.5). This fact and (11.1) show that $\Phi = 0$ in B_R. Since all $f_{k,l}(\theta_n)$ vanish in B_R, statement 3) is proved. □

Proof of statement 4). Suppose $r_1 + r_2 = R$ and $(r_1, r_2) \notin \Omega$. Without loss of generality, we may assume that $r_1 \neq \pi/2$. By Lemma 10.2, there exists a constant $\alpha > 0$ such that $|\psi_{\nu,1}(r_2)|^{-1} \leq (\nu+1)^\alpha$ for all $\nu \in N(r_1)$. Let $s, q \in \mathbb{Z}_+$ be such that $s \geq 3(\alpha+n+q+8)$. Fixing s, we consider the function f defined by formula (8.10) with $r = r_1$. From the proof of statement 3) of Theorem 8.1 it follows that $f = 0$ in B_{r_1}, and the coefficients α_ν in (8.10) satisfy $\alpha_\nu = O\left(\nu^{2[s/3]+n+7}\right)$ as $\nu \to +\infty$. For $\xi \in B_R$, we put

$$f_1(\xi) = \sum_{\nu \in N(r_1)} \alpha_\nu (\psi_{\nu,1}(r_2))^{-1} \psi_{\nu,s+1}(\theta_n) Y_1^{(s+1)}(\sigma). \tag{11.5}$$

Since $s \geq 3(\alpha+n+q+8)$, the estimates for α_ν and $|\psi_{\nu,1}(r_2)|^{-1}$ show that $f_1 \in C^q(B_R)$ (see (4.5), (4.7), and (5.3)). Moreover, $f_1 \in V_{r_1}(B_R)$ by Lemma 7.2. Using (7.2) with $r = r_2$ and (11.5), (8.10), we see that $f_1 \in V_{r_2}(B_R)$, because $f = 0$ in B_{r_1}. By the same argument as in the proof of statement 3a) of Theorem 8.1, from (11.5) we deduce that f_1 is nonzero in B_R. Thus, the function f_1 satisfies all the requirements of Statement 4). □

The following lemma is needed in the proof of Statement 5).

Lemma 11.1. *Let $\{\alpha_m\}_{m=1}^\infty$ be a monotone increasing sequence of positive numbers such that $\alpha_{m+1} - \alpha_m \to \infty$ as $m \to \infty$. Then for every $\varepsilon \in (0, \pi)$ there exists a nonzero function $\Phi(\xi) = \varphi(\theta_n) \in C^\infty(\mathbb{S}^n)$ with support in B_ε and such that $\widetilde{\Phi}(\alpha_m) = 0$ for all m.*

Proof. We may assume that $\varepsilon \in (0, \pi/2)$. From [26, Lemma 1], it follows that there exists a nonzero even function $h \in C^\infty(\mathbb{R}^1)$ such that $\operatorname{supp} h \subset \{t \in \mathbb{R}^1 : \varepsilon/3 \leq |t| \leq \varepsilon\}$ and $\int_{-\varepsilon}^{\varepsilon} e^{i(\alpha_m + (n-1)/2)t} h(t)\, dt = 0$ for $m \in \mathbb{N}$. Let $\psi \in C^\infty[0,1]$ be a function satisfying

$$h(t) = \int_{|t|}^{\pi/2} \psi(\cos\theta)(\cos t - \cos\theta)^{(n-3)/2} \sin\theta\, d\theta, \quad |t| \leq \pi/2$$

(this equation reduces easily to an Abel integral equation). We put $\varphi(\theta) = \psi(\cos\theta)$ if $\theta \in [0, \pi/2]$ and $\varphi = 0$ on $[\pi/2, \pi]$. Then the function $\Phi(\xi) = \varphi(\cos^{-1}\xi_{n+1})$ satisfies all the requirements of Lemma 11.1 (for the definition of $\widetilde{\Phi}$, see §7 and (3.8)). □

Proof of statement 5). First, we consider the case where $N(r_1, r_2) = \varnothing$. Let $r_1 + r_2 > R$, and let $\varepsilon = \min\{(r_1 + r_2 - R)/4, R - r_2\}$. We denote by $\mathcal{A}$ the set of all numbers $\nu \in N(r_2)$ such that $|\psi_{\nu,1}(r_1)| < (\nu + 1)^{-(n+7)/2}$. By Lemma 10.1, either the set $\mathcal{A}$ is empty, or it is lacunary in the sense of Hadamard. Then there exists a nonzero function $\Phi(\xi) = \varphi(\theta_n) \in C^\infty(\mathbb{S}^n)$ such that $\operatorname{supp}\Phi \subset B_\varepsilon$ and $\widetilde{\Phi}(\nu) = 0$ for all $\nu \in \mathcal{A}$ (see Lemma 11.1). We choose $j \in \mathbb{N}$ such that

$$\int_0^\pi \varphi(\theta) C_j^{\frac{n-1}{2}}(\cos\theta)(\sin\theta)^{n-1}\,d\theta \neq 0. \tag{11.6}$$

Let $u_1(\theta_n)$ be a nonzero function of class $C^\infty(\overline{B}_{r_2})$ and with support contained in the set $B_{r_2-2\varepsilon}\backslash\overline{B}_{R-r_1+\varepsilon}$. We put $u_2(\theta) = u_1(\theta)\cos^k\theta$, where $k \in \mathbb{Z}_+$ and

$$\int_0^{r_2} u_2(\theta) C_{j-1}^{\frac{n+1}{2}}(\cos\theta)(\sin\theta)^n\,d\theta \neq 0. \tag{11.7}$$

The proof of statement 3) of Theorem 8.1 shows that $u_2(\theta) = \sum_{\nu\in N(r_2)} c_\nu \psi_{\nu,1}(\theta)$ for $0 \le \theta \le r_2$, where $c_\nu \in \mathbb{C}$, and for any fixed $c > 0$ we have $c_\nu = O(\nu^{-c})$ as $\nu \to +\infty$. In particular, if $v_1(\xi) = (d_{1-n}u_2)(\theta_n)$, then

$$v_1(\xi) = \sum_{\nu\in N(r_2)} c_\nu \psi_{\nu,0}(\theta_n), \quad \xi \in \overline{B}_{r_2} \tag{11.8}$$

(see (3.10)). By formula (11.8), the function v_1 extends to a function of class $C^\infty(B_R)$ (see (4.5), (4.7), and (5.3)). The definition of convolution and the location of the supports of v_1 and Φ show that $v_1 * \Phi = 0$ in B_{R-r_1}. Therefore, relations (11.8) and (7.2) imply that the function

$$f(\xi) = \sum_{\nu\in N(r_2)\backslash\mathcal{A}} c_\nu \widetilde{\Phi}(\nu)(\psi_{\nu,1}(r_1))^{-1}\Psi_\nu^{0,1}(\xi) \tag{11.9}$$

belongs to $V^\infty_{r_1,r_2}(B_R)$ (see (4.5), (4.7), and Lemma 7.2). Suppose $f = 0$ on B_R. Using (11.9), (3.10), and Lemma 5.5, we see that $c_\nu\widetilde{\Phi}(\nu) = 0$ for all $\nu \in N(r_2)$. Thus, $v_1 * \Phi = 0$ in $B_{R-\varepsilon}$ (see (11.8) and (7.2)). Consequently, $v_2 * \Phi = 0$ on $\mathbb{S}^n$, where $v_2 = v_1$ in B_{r_2} and $v_2 = 0$ in $\mathbb{S}^n\backslash B_{r_2}$. Therefore (see the proof of Lemma 7.1),

$$\begin{aligned} 0 &= \int_{\mathbb{S}^n} (v_2 * \Phi)(\xi) C_j^{\frac{n-1}{2}}(\xi_{n+1})\,d\xi \\ &= \frac{\Gamma(n-1)\Gamma(j+1)}{\omega_n\Gamma(j+n-1)} \int_{\mathbb{S}^n} v_2(\xi) C_j^{\frac{n-1}{2}}(\xi_{n+1})\,d\xi \int_{\mathbb{S}^n} \Phi(\xi) C_j^{\frac{n-1}{2}}(\xi_{n+1})\,d\xi, \end{aligned}$$

which contradicts (11.6) and (11.7). Thus, if $N(r_1, r_2) = \varnothing$, then the function f satisfies all the requirements of Statement 5).

Now, suppose $N(r_1, r_2) \neq \varnothing$. Then, for each $\nu \in N(r_1, r_2)$, the function $\Psi_\nu^{0,1}(\xi)$ is real-analytic in B_π and belongs to $V_{r_1,r_2}(B_\pi)$ (see the corollary to Lemma 7.1). This implies both statement 5) in the case where $N(r_1, r_2) \neq \varnothing$ and statement 6). This completes the proof of Theorem 2.1. □

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