Алгебра и анализ
Том. 16 (2004), вып. 3

St. Petersburg Math. J.
Vol. 16 (2005), No. 3, Pages 561–581
S 1061-0022(05)00865-4
Article electronically published on May 2, 2005

ON THE SPECTRUM OF THE WANNIER–STARK OPERATOR

A. A. POZHARSKIĬ

Basic results

We consider the one-dimensional Schrödinger equation

$$-\psi'' - Fx\psi + p(x)\psi = E\psi \tag{1}$$

on the semiaxis $0 \le x < +\infty$. Here the potential $p(x)$ is a periodic real function, $p(x+1) = p(x)$, satisfying the condition $p(x) \in L_1[0,1]$. We assume that the constant F is positive and that $\int_0^1 p(x)\,dx = 0$; the latter condition can always be ensured by a shift of the spectral parameter E.

We introduce the operator

$$H = -\frac{d^2}{dx^2} - Fx + p(x)$$

in $L_2(\mathbb{R}_+)$ with the domain

$$\operatorname{dom}(H) = \left\{ \varphi : \begin{matrix} \varphi' \text{ is absolutely continuous, } \varphi(0) = 0, \operatorname{supp}\varphi \text{ is} \\ \text{bounded, and } -\varphi'' + p\varphi \in L_2(\mathbb{R}_+) \end{matrix} \right\}.$$

Remark. The function $-\varphi'' + p(x)\varphi$ belongs to $L_2(\mathbb{R}_+)$ if $\varphi \in \operatorname{dom}(H)$, but φ'' and $p\varphi$ separately may fail to belong to $L_2(\mathbb{R}_+)$. However, they belong to $L_1(\mathbb{R}_+)$.

The operator H is essentially selfadjoint. This fact is proved in §5. We denote by H_d the closure of H.

In the case where the potential $p(x)$ is smooth, equation (1) and the operator H_d have been studied thoroughly; see [AGZ, AZ, Ba, B2, BD1, BD2, NN]. In the case of a nonsmooth potential $p(x)$, only few results are known (see [B1, DSS, E, P]).

We introduce the notation

$$r(l) = e^{-i\frac{3\pi}{4}} \frac{1}{\sqrt{2F}} \int_0^1 p(t) e^{-2i\pi lt}\,dt, \quad \omega(l) = \int_0^1 \int_0^t p(y)p(t)e^{2i\pi lt}\,dy\,dt. \tag{2}$$

In [P], the following Theorem 1 was proved.

Assumption (A). $\sum_{l=1}^{\infty} |r(l)| l^{-\frac{1}{2}} < \infty$, $\sum_{l=1}^{\infty} |\omega(l)| l^{-1} < \infty$.

Assumption (B). $\sum_{l=1}^{\infty} |r(l)|^2 l^{-\frac{1}{2}} < \infty$, $\sum_{l=1}^{\infty} |r(l)| l^{-\frac{3}{4}} < \infty$, $\sum_{l=1}^{\infty} |\omega(l)| l^{-1} < \infty$.

Theorem 1. i) *If the potential $p(x)$ satisfies Assumption* (A), *then the spectrum of the operator H_d is absolutely continuous and fills the real axis.*

ii) *If the potential $p(x)$ satisfies Assumption* (B), *then the absolutely continuous spectrum of H_d fills the real axis.*

In the present paper, we study the absolutely continuous spectrum of H_d and generalize the results of Theorem 1. Our main result in this paper is Theorem 2 below.

2000 *Mathematics Subject Classification.* Primary 34L40.

Key words and phrases. Schrödinger operator, essentially selfadjoint operator, continuous spectrum.

Assumption (C). $\sum_{l=1}^{\infty}|r(l)|l^{-1}<\infty$, $\sum_{l=1}^{\infty}|\omega(l)|l^{-1}<\infty$.

Remark. If on the interval $[0,1]$ the potential $p(x)$ is given by the equation $|p(x)| = C_1|x-x_0|^{-1+\varepsilon}$ for $\varepsilon>0$, then Assumption (C) is valid.

Theorem 2. *If $p(x)$ satisfies Assumption* (C), *then the absolutely continuous spectrum of H_d fills the real axis.*

The proof of this theorem is based on the results obtained in [P]. We state two results of [P] that we need in the sequel (Theorems 3 and 4).

Theorem 3. *Let $p(x)$ be a periodic potential absolutely integrable on the period and satisfying $\int_0^1 p(t)\,dt=0$. Suppose $F>0$ and $E\in\mathbb{R}$. Then an arbitrary solution of equation* (1) *has the following asymptotic expansion as $x\to+\infty$:*

$$\psi(x)=\frac{s_l}{\sqrt[4]{Fx}}e^{i\frac{2}{3F}(E+Fx)^{3/2}}+\frac{t_l}{\sqrt[4]{Fx}}e^{-i\frac{2}{3F}(E+Fx)^{3/2}}+O\Big(\frac{\|\mathbf{s}_l\|}{\sqrt{x}}\Big),\quad x\in I_l.$$

Here $I_l=(\tilde n_{l-1},\tilde n_l)\setminus(\tilde i_{l-1}\cup\tilde i_l)$, $\tilde n_l=((\pi l)^2-E)/F$, and $\tilde i_l=\{n:|n-\tilde n_l|=O(l^{2/3})\}$. On the adjacent intervals I_l and I_{l+1}, the coefficients s_l, t_l and s_{l+1}, t_{l+1} are related by the transformation

$$\mathbf{t}_{l+1}=\widetilde W_l\mathbf{t}_l,\quad \mathbf{t}_l=\begin{pmatrix}s_l\\ t_l\end{pmatrix},\quad \widetilde W_l=\widetilde W_l(E), \tag{3}$$

where

$$\widetilde W_l=e^{-i\pi l\frac{E}{F}\sigma_3}S_le^{i\pi l\frac{E}{F}\sigma_3},\quad \sigma_1\overline{S}_l\sigma_1=S_l,\quad S_l=S_l(E). \tag{4}$$

The Pauli matrices σ_1 and σ_3 are given by $\sigma_1=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $\sigma_3=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. For large l, the matrix S_l has the asymptotics

$$S_l=\begin{pmatrix}1+q(l)l^{-1}+O(l^{-7/6}) & r(l)l^{-1/2}e^{\frac{2i}{3F}(\pi l)^3}+O(l^{-1}d(l))\\ \bar r(l)l^{-1/2}e^{-\frac{2i}{3F}(\pi l)^3}+O(l^{-1}d(l)) & 1+q(l)l^{-1}+O(l^{-7/6})\end{pmatrix},$$

where

$$q(l)=\frac12|r(l)|^2,\quad d(l)=|r(l)|+|\omega(l)|+O(l^{-1/3}),$$

and r and ω are the functions defined in (2).

Furthermore, we have

$$\int_{\tilde n_l}^{\tilde n_{l+1}}|\psi(x)|^2\,dx=\frac{2\pi}{F}\|\mathbf{t}_l\|^2(1+o(1)).$$

We note that, for the first time, a similar result was proved by Buslaev in [B1] for the equation

$$-\psi''-Fx\psi+V\sum_{n=-\infty}^{+\infty}\delta(x-n)\psi=E\psi.$$

Theorem 3 is a generalization of the results of [B1], and its proof is based on the ideas developed in [B1].

Definition 1. A solution $\psi^u(x)$ of equation (1) is said to be *subordinate* at $+\infty$ if for each solution $\psi(x)$ linearly independent of $\psi^u(x)$ we have

$$\lim_{x\to+\infty}\left(\int_0^x|\psi^u(t)|^2\,dt\Big/\int_0^x|\psi(t)|^2\,dt\right)=0.$$

Definition 2. A solution $\mathbf{t}_l^u$ of system (3) is said to be *subordinate* if for each solution $\mathbf{t}_l$ linearly independent of $\mathbf{t}_l^u$ we have

$$\lim_{L\to\infty}\left(\sum_{l=1}^{L}\|\mathbf{t}_l^u\|^2\Big/\sum_{l=1}^{L}\|\mathbf{t}_l\|^2\right)=0.$$

Theorem 4. *Suppose the assumptions of Theorem* 3 *are satisfied. Then equation* (1) *has a subordinate solution if and only if this is true for system* (3).

Now, we state yet another result to be used in the sequel. The following theorem was proved in [GP].

Theorem 5. *If equation* (1) *has no subordinate solutions for almost any* $E \in \mathbb{R}$ *with respect to Lebesgue measure*, *then the absolutely continuous spectrum of* H_d *fills the real axis.*

The change of variables $\mathbf{s}_l = \exp(i\pi l\frac{E}{F}\sigma_3)\mathbf{t}_l$ reduces (3) to the system

$$\mathbf{s}_{l+1} = e^{i\pi\frac{E}{F}\sigma_3}S_l\mathbf{s}_l. \tag{5}$$

Moreover,

$$\|\mathbf{s}_l\| = \|\mathbf{t}_l\|. \tag{6}$$

In what follows, we study system (5) rather than (3).

Lemma 1. *Suppose that the conditions of Theorem* 3 *are fulfilled*, *and that*, *for almost all* $E \in \mathbb{R}$ *with respect to Lebesgue measure*, *system* (5) *has no subordinate solutions. Then the absolutely continuous spectrum of* H_d *fills the real axis.*

Proof. By (6), systems (3) and (5) admit subordinate solutions simultaneously. Consequently, for almost all $E \in \mathbb{R}$ with respect to Lebesgue measure, system (3) has no subordinate solutions. Now, the claim follows from Theorems 4 and 5. ☐

Thus, the proof of Theorem 2 reduces to the study of asymptotic properties of system (5). Observe that this system is similar to the Schrödinger equation with slowly decaying potential (the matrix S_l tends to the identity matrix as $l \to \infty$). Recently, for the Schrödinger operator with slowly decaying potential, a simple method of localization of the absolutely continuous spectrum was proposed in [DK]. This method is based on the use of certain spectral identities, which have recently received the name of BFZ (Buslaev–Faddeev–Zakharov) identities; see [BF]. In the case of system (5), we follow the same idea. Some difficulties in the study of system (5) are due to the fact that the matrix S_l may depend on the spectral parameter E. It is convenient to begin with a simpler system. Consider the two-dimensional recurrence system

$$\mathbf{s}_{l+1} = z^{\sigma_3}W_l\mathbf{s}_l, \quad l = 0, 1, \dots . \tag{7}$$

Here $\mathbf{s}_l \in \mathbb{C}^2$, $z \in \mathbb{C}\setminus\{0\}$, and the W_l are complex matrices of size 2×2 that satisfy the conditions

$$\sigma_1\overline{W_l}\sigma_1 = W_l, \quad \det W_l = 1 \tag{8}$$

and do not depend on z. Conditions (8) imply that the matrices W_l can be represented in the form

$$W_l = \begin{pmatrix} w_l & v_l \\ \bar v_l & \bar w_l \end{pmatrix}, \quad |w_l|^2 - |v_l|^2 = 1. \tag{9}$$

The following theorem is the main result concerning system (7) (we note that the matrices S_l may depend on the spectral parameter E, while the matrices W_l are independent of z).

Theorem 6. *Suppose the matrices W_l are representable in the form* (9). *We assume that the series*

$$\sum_{l=0}^{\infty} |v_l|^2 \tag{10}$$

converges and the limit

$$\lim_{l\to\infty} W_l = I, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{11}$$

exists. Then, for almost all $\phi \in [0, 2\pi)$ with respect to Lebesgue measure, system (7) *has no subordinate solutions for $z = e^{i\phi}$.*

Remark. We note that condition (11) can be lifted. To verify this, it suffices to make the change of variables $\mathbf{s}_l = e^{i\sum_{k=0}^{l-1} \arg w_l \sigma_3}\mathbf{t}_l$ in system (7).

The proof of Theorem 6 is based on the study of the Weyl function $m(z)$ (the definition of it is given in Theorem 7). We consider a specific basis of solutions of (7). Let $\theta(z)$ and $\varphi(z)$ be solutions of (7) such that $\theta_0(z) = \binom{1}{0}$ and $\varphi_0(z) = \binom{0}{1}$. In what follows, by a solution $\theta(z)$ of system (7) we mean a sequence $\{\theta_l(z)\}_{l=0}^{\infty}$ satisfying (7).

Theorem 7. *If the matrices W_l are representable in the form* (9), *then there exists a Weyl function $m(z)$ analytic in $|z| < 1$ and such that*

$$f(z) = \theta(z) + m(z)\varphi(z) \in l^2(\mathbb{Z}_+, \mathbb{C}^2), \quad 0 < |z| < 1.$$

Furthermore, $|m(z)| < 1$ for $|z| < 1$.

Theorem 7 is proved in §1.

Corollary 1. *The Weyl function $m(z)$ has radial boundary values*[1] *$m(e^{i\phi})$, where $\phi \in [0, 2\pi)$, almost everywhere with respect to Lebesgue measure.*

Proof. The inequality $|m(z)| < 1$ for $|z| < 1$ shows that $m(z)$ belongs to the Hardy class $H_\infty(|z| \le 1)$. This implies the required statement (see [Z]). □

The following lemma (to be proved in §2), plays a key role in the proof of Theorem 6.

Lemma 2. *If the series* (10) *converges, then*

$$-\int_0^{2\pi} \ln\bigl(1 - |m(e^{i\phi})|^2\bigr)\, d\phi \le 4\pi \sum_{l=0}^{\infty} \ln |w_l|.$$

Remark. The series $\sum_{l=0}^{\infty} \ln |w_l|$ and $\sum_{l=0}^{\infty} |v_l|^2$ converge simultaneously.

By this lemma, the Weyl function satisfies the estimate

$$|m(e^{i\phi})| < 1 \tag{12}$$

for almost all ϕ. Estimate (12) implies that system (7) has no subordinate solutions at ϕ (this fact can be proved by the methods of [GP]), and now the statement of Theorem 6 follows directly. However, the proof of this fact is relatively long, and we prove a simpler but slightly weaker statement. More precisely, we prove Theorem 6 for a set of ϕ that is narrower than the set of all ϕ for which estimate (12) is valid, but, certainly, is a set of full measure.

Theorem 6 follows easily from the next two lemmas. The main ideas of their proofs are borrowed from [LS]. We note that the proof of Lemma 3 is based on the results of Lemma 2. The proofs of Lemmas 3 and 4 are given in §3.

---

[1]The radial boundary values are defined by $m(e^{i\phi}) = \lim_{r\to 1-0} m(re^{i\phi})$.

We represent system (7) in the matrix form:

$$F_{l+1} = e^{i\phi\sigma_3} W_l F_l, \quad F_0 = I.$$

Lemma 3. *Suppose the series* (10) *converges and*

$$\|W_l - I\| \le 1/8, \quad l = 0, 1, \dots . \tag{13}$$

Let a_l be a sequence of nonnegative numbers such that the series $\sum_{l\ge 0} a_l$ is convergent. Then, for almost all $\phi \in [0, 2\pi)$ with respect to Lebesgue measure, there exists a constant $C_9(\phi) < \infty$ such that

$$\liminf_{L\to\infty} \frac{1}{L} \sum_{l=0}^{L} \|F_l\|^2 \le C_9(\phi), \quad \sum_{l=0}^{\infty} a_l \|F_l\| \, \|F_{l+1}\| \le C_9(\phi).$$

Lemma 4. *Let B_l be a matrix solution of the system*

$$B_{l+1} = K_l B_l, \quad B_0 = I, \tag{14}$$

where the matrix K_l satisfies $\det K_l = 1$*. If*

$$\liminf_{L\to\infty} \frac{1}{L} \sum_{l=0}^{L} \|B_l\|^2 < \infty,$$

then system (14) *has no subordinate solutions.*

We note that the matrices K_l may depend on the spectral parameter.

Now we show how Theorem 6 can be derived from Lemmas 3 and 4. Since the series (10) converges, there is a constant N such that condition (13) is fulfilled for all $l \ge N$. Consider the auxiliary system obtained from the given system by replacing the first N matrices W_l by the identity matrix. Obviously, the behavior of a subordinate solution does not depend on the behavior of the matrices W_l for finite l (e.g., for $l \le N$); therefore, the two systems in question have subordinate solutions simultaneously. We apply Lemma 3 and then Lemma 4 to the auxiliary system, putting $K_l = e^{i\phi\sigma_3} W_l$. As a result, we obtain the required statement.

Now, we proceed to system (5). The method used in the proof of Theorem 6 cannot be applied to system (5) because, in general, the matrices S_l may depend on the spectral parameter E. Therefore, we are forced to perform the construction in two steps. For this, we represent the matrices S_l in the form $S_l = W_l + V_l$, where the matrices W_l satisfy the conditions of Theorem 6 and, in particular, do not depend on the spectral parameter E, and the matrices V_l decay sufficiently fast (more precisely, we require the convergence of the series $\sum_{l\ge 0} \|V_l\|$), but may depend on E. From Theorem 6 it follows that the unperturbed system

$$\mathbf{s}_l = e^{i\pi \frac{E}{F} \sigma_3} W_l \mathbf{s}_l \tag{15}$$

has no subordinate solutions for almost all $E \in \mathbb{R}$.

Next, using the fact that the perturbation V_l is small, we can compare the solutions of systems (5) and (15), thus arriving at Theorem 2. More precisely, we prove that, for almost all ϕ, the solutions of systems (5) and (15) are related by a bounded matrix transformation U_l (see Theorem 9). In §4, we give a rigorous proof of Theorem 2. Now, we outline the proof. We use the following notation: F_l is a matrix solution of the unperturbed system (15), B_l is a matrix solution of the perturbed system (5) with the matrices $S_l = W_l + V_l$, and the matrix solutions B_l and F_l are related to each other by the matrix transformation U_l:

$$F_{l+1} = e^{i\phi\sigma_3} W_l F_l \to \text{the Weyl function } m(z) \text{ (Theorem 7)}$$
$$\to |m(e^{i\varphi})| < 1 \text{ for a.e. } \varphi \text{ (Lemma 2)} \to \text{Lemma 3}$$
$$B_{l+1} = K_l B_l \longrightarrow \text{Lemma 4}$$

$$F_{l+1} = e^{i\phi\sigma_3} W_l F_l$$

$\sum_{l\ge 0} |v_l|^2 < \infty$ $\downarrow$ Lemma 3 $\qquad$ $\sum_{l\ge 0} |v_l|^2 < \infty$, $\sum_{l\ge 0} \|V_l\| < \infty$ $\downarrow$ Lemma 3

$$\liminf_{L\to\infty} L^{-1} \sum_{l=0}^{L} \|F_l\|^2 \le C(\phi) < \infty \text{ for a.e. } \phi \in [0, 2\pi)$$

$$\sum_{l=0}^{\infty} \|V_l\|\, \|F_l\|\, \|F_{l+1}\| \le C(\phi) < \infty \text{ for a.e. } \phi \in [0, 2\pi)$$

$\downarrow$ see Theorem 9

$$B_l = F_l U_l,\ \|U_l\| \le C(\phi) < \infty$$

$$\liminf_{L\to\infty} L^{-1} \sum_{l=0}^{L} \|B_l\|^2 \le C(\phi) < \infty \text{ for a.e. } \phi \in [0, 2\pi)$$

$\downarrow$ Lemma 4

System (5) has no subordinate solutions for a.e. $\phi \in [0, 2\pi)$

$\downarrow$ Lemma 1

Theorem 2

We note that the basic constructions are taken from the corresponding parts of the theory of the one-dimensional Schrödinger equation

$$-\psi'' + q(x)\psi = E\psi. \tag{16}$$

For basic notions concerning the Weyl function for (16), see [L]. The link between the boundary values of the Weyl function and the existence of a subordinate solution of (16) was established in [GP]. The idea to use the BFZ formulas in the study of the spectral properties of (16) is borrowed from [DK]. In the proofs of Lemmas 3 and 4, we used some ideas from [LS].

The present paper consists of five sections. In §1, we prove Theorem 7; §2 contains the proof of Lemma 2. Lemmas 3 and 4 are proved in §3, and in §4 we prove Theorem 2. Finally, in §5, we prove that the operator H is essentially selfadjoint.

Acknowledgements. The author wants to thank Buslaev for numerous discussions and information about the literature on the subject under consideration.

§1. The Weyl function

Our goal in the present section is to construct the Weyl function $m(z)$ and to prove Theorem 7.

Let $g(z)$ be an arbitrary solution of system (7). We have

$$\begin{pmatrix} z^{-1}\alpha_{l+1} \\ z\beta_{l+1} \end{pmatrix} = \begin{pmatrix} w_l\alpha_l + v_l\beta_l \\ \bar{w}_l\beta_l + \bar{v}_l\alpha_l \end{pmatrix}, \quad \text{where } \begin{pmatrix} \alpha_l \\ \beta_l \end{pmatrix} \overset{\text{def}}{=} g_l.$$

Consequently,

$$\begin{aligned} |z|^{-2}|\alpha_{l+1}|^2 - |z|^2|\beta_{l+1}|^2 &= |w_l\alpha_l + v_l\beta_l|^2 - |\bar{w}_l\beta_l + \bar{v}_l\alpha_l|^2 \\ &= (|w_l|^2 - |v_l|^2)|\alpha_l|^2 - (|w_l|^2 - |v_l|^2)|\beta_l|^2 = |\alpha_l|^2 - |\beta_l|^2. \end{aligned}$$

As a result, we obtain the identity

$$|z|^{-2}|\alpha_{l+1}|^2 - |z|^2|\beta_{l+1}|^2 = |\alpha_l|^2 - |\beta_l|^2. \tag{1.1}$$

Lemma 5. *For each solution $g(z)$ of system* (7), *we have*

$$(1-|z|^2)\sum_{k=1}^{l}\Big[|z|^{-2}|\alpha_k|^2 + |\beta_k|^2\Big] = |\alpha_0|^2 - |\beta_0|^2 - |\alpha_l|^2 + |\beta_l|^2.$$

For simplicity, we introduce the notation

$$\mathrm{Sm}_l(g(z)) \overset{\text{def}}{=} (1-|z|^2)\sum_{k=1}^{l}\Big[|z|^{-2}|\alpha_k|^2 + |\beta_k|^2\Big], \quad \mathrm{Sm}_0(g(z)) \overset{\text{def}}{=} 0.$$

Proof. Using (1.1), we obtain

$$\begin{aligned}|\alpha_0|^2 - |\beta_0|^2 - |\alpha_l|^2 + |\beta_l|^2 &= (|\alpha_0|^2 - |\alpha_l|^2) - (|\beta_0|^2 - |\beta_l|^2)\\ &= \sum_{k=1}^{l}(|\alpha_{k-1}|^2 - |\alpha_k|^2) - \sum_{k=1}^{l}(|\beta_{k-1}|^2 - |\beta_k|^2)\\ &= \sum_{k=1}^{l}(|\alpha_{k-1}|^2 - |\beta_{k-1}|^2) - \sum_{k=1}^{l}(|\alpha_k|^2 - |\beta_k|^2)\\ &= \sum_{k=1}^{l}(|z|^{-2}|\alpha_k|^2 - |z|^2|\beta_k|^2) - \sum_{k=1}^{l}(|\alpha_k|^2 - |\beta_k|^2) = \mathrm{Sm}_l(g(z)).\end{aligned}$$

The lemma is proved. □

We denote by $\|\cdot\|$ the norm of a vector in $\mathbb{C}^2$.

Lemma 6. *For every solution $g(z)$ of system* (7), *the following estimate is valid for* $0 < |z| < 1$:

$$\frac{|z|^2}{1-|z|^2}\,\mathrm{Sm}_l(g(z)) \le \sum_{k=1}^{l}\|g_k(z)\|^2 \le \frac{1}{1-|z|^2}\,\mathrm{Sm}_l(g(z)).$$

Proof. This follows from the obvious inequalities

$$\sum_{k=1}^{l}(|\alpha_k|^2 + |z|^2|\beta_k|^2) \le \sum_{k=1}^{l}(|\alpha_k|^2 + |\beta_k|^2) \le \sum_{k=1}^{l}(|z|^{-2}|\alpha_k|^2 + |\beta_k|^2). \qquad \square$$

Consider the solutions $\theta(z)$ and $\varphi(z)$ of system (7) that satisfy the initial conditions $\theta_0(z) = \binom{1}{0}$ and $\varphi_0(z) = \binom{0}{1}$ (see the preceding section). For the coordinates of $\theta_l(z)$ and $\varphi_l(z)$, we use the notation

$$\theta_l(z) = \begin{pmatrix} s_l \\ t_l \end{pmatrix}, \quad \varphi_l(z) = \begin{pmatrix} e_l \\ h_l \end{pmatrix}.$$

To construct the Weyl function $m(z)$, we consider the sequence of vectors $f_l(z) = \theta_l(z) + m(z)\varphi_l(z)$. The next step reduces to the search for a function $m(z)$ such that the element $f(z) = \{f_l(z)\}_{l=0}^{\infty}$ belongs to the space $l^2(\mathbb{Z}_+, \mathbb{C}^2)$.

Applying Lemma 5 to $f(z)$, we obtain

$$\mathrm{Sm}_l(f(z)) = 1 - |m(z)|^2 + |t_l + m(z)h_l|^2 - |s_l + m(z)e_l|^2. \tag{1.2}$$

Lemma 6 yields the following estimates for $0 < |z| < 1$:

$$\frac{|z|^2}{1-|z|^2}\,\mathrm{Sm}_l(f(z)) \le \sum_{k=1}^{l}\|f_k\|^2 \le \frac{1}{1-|z|^2}\,\mathrm{Sm}_l(f(z)). \tag{1.3}$$

Thus, the relation $f(z) \in l^2(\mathbb{Z}_+, \mathbb{C}^2)$ is equivalent to the inequality $\mathrm{Sm}_l(f(z)) \le C_1(z) < \infty$ for $0 < |z| < 1$, with some $C_1(z)$ independent of l.

We construct a function $m(z)$ satisfying the condition

$$|t_l + m(z)h_l|^2 - |s_l + m(z)e_l|^2 \le 0, \quad l = 0, 1, \dots . \tag{1.4}$$

It is easily seen that condition (1.4) implies the estimate $\mathrm{Sm}_l(f(z)) \le 1$. Therefore, by (1.3), $f(z) \in l^2(\mathbb{Z}_+, \mathbb{C}^2)$. We consider the collection of sets

$$D_l(z) = \{\zeta : |t_l(z) + \zeta h_l(z)|^2 - |s_l(z) + \zeta e_l(z)|^2 \le 0\}, \quad l = 0, 1, \dots ;$$

obviously, these sets are closed. We shall drop z in the notation $D_l(z)$ if this does not lead to confusion.

Lemma 7. *We have $D_{l+1} \subset D_l \subset \{\zeta : |\zeta| < 1\}$ for $l = 1, 2, \dots$.*

Proof. We prove the inclusions $D_{l+1} \subset D_l$ by contradiction. Assume that there exists l and a point ζ_0 such that $\zeta_0 \in D_{l+1} \setminus D_l$. Then

$$\begin{aligned} \mathrm{Sm}_{l+1}(f(z)) &= 1 - |\zeta_0|^2 + |t_{l+1} + \zeta_0 h_{l+1}|^2 - |s_{l+1} + \zeta_0 e_{l+1}|^2 \le 1 - |\zeta_0|^2 \\ &< 1 - |\zeta_0|^2 + |t_l + \zeta_0 h_l|^2 - |s_l + \zeta_0 e_l|^2 = \mathrm{Sm}_l(f(z)). \end{aligned}$$

Thus, $\mathrm{Sm}_{l+1}(f(z)) < \mathrm{Sm}_l(f(z))$. However, by definition, the sequence $\mathrm{Sm}_l(f(z))$ is strictly monotone increasing. This contradiction proves the required inclusions.

The inclusions $D_l \subset \{\zeta : |\zeta| < 1\}$ are also proved by contradiction. We assume that there exists $l \ge 1$ and a point ζ_0 such that $\zeta_0 \in D_l \setminus \{\zeta : |\zeta| < 1\}$. Then

$$0 < \mathrm{Sm}_l(f(z)) = 1 - |\zeta_0|^2 + |t_l + \zeta_0 h_l|^2 - |s_l + \zeta_0 e_l|^2 \le 1 - |\zeta_0|^2 \le 0,$$

a contradiction. □

Lemma 8. *Let f^1 and f^2 be solutions of system (7) belonging to $l^2(\mathbb{Z}_+, \mathbb{C}^2)$. Then f^1 and f^2 are linearly dependent.*

Proof. The matrix (f_l^1, f_l^2) formed by the vectors f_l^1 and f_l^2 satisfies the relation

$$(f_{l+1}^1, f_{l+1}^2) = z^{\sigma_3} W_l (f_l^1, f_l^2).$$

The condition $\det(z^{\sigma_3} W_l) = 1$ implies

$$\det(f_{l+1}^1, f_{l+1}^2) = \det(z^{\sigma_3} W_l) \det(f_l^1, f_l^2) = \det(f_l^1, f_l^2) = C_2 \ne C_2(l). \tag{1.5}$$

Since $f^j \in l^2(\mathbb{Z}_+, \mathbb{C}^2)$, $j = 1, 2$, the limits $\lim_{l\to\infty} f_l^j$ exist and are equal to 0. Using (1.5), we obtain $\det(f_l^1, f_l^2) = 0$, which implies that the vectors f^1 and f^2 are linearly dependent. □

Lemma 9. *The set $\bigcap_{l\ge0} D_l$ is nonempty and consists of a unique point.*

Proof. The existence of a point $\zeta_0 \in \bigcap_{l\ge0} D_l$ follows from the fact that the sets D_l are closed and the space of complex numbers is complete. To prove uniqueness, we argue by contradiction. Let $\zeta_j \in \bigcap_{l\ge0} D_l$, $j = 0, 1$, and let $\zeta_0 \ne \zeta_1$. Then $f^j = \theta + \zeta_j \varphi \in l^2(\mathbb{Z}_+, \mathbb{C}^2)$ for $j = 0, 1$ and, by Lemma 8, the f^j are linearly dependent. Putting $l = 0$, we see that $\zeta_0 = \zeta_1$. □

We represent condition (1.4) in the form $|t_l(z) + m(z)h_l(z)|/|s_l(z) + m(z)e_l(z)| \le 1$ and consider the mapping

$$w = \frac{t_l(z) + m(z)h_l(z)}{s_l(z) + m(z)e_l(z)}.$$

The relation $s_l h_l - t_l e_l = \det(\theta_l, \varphi_l) = 1$ implies the existence of the inverse mapping, which is given by

$$m(z,l,w) = -\frac{t_l(z) - s_l(z)w}{h_l(z) - e_l(z)w}. \tag{1.6}$$

Consequently,

$$D_l(z) = \bigcup_{|w|\le 1} \{m(z,l,w)\}.$$

Therefore, Lemmas 7 and 9 imply that the limit $\lim_{l\to\infty} m(z,l,w)$ exists and is independent of w. We denote this limit by $m(z)$.

Lemma 10. *For $0 < |z| < 1$, the function $m(z)$ is analytic and satisfies $|m(z)| < 1$.*

Proof. First, we prove that for all l the function $m(z,l,w)$ is continuous in two variables (z,w) in the region $\{z : 0 < |z| < 1\} \times \{w : |w| \le 1\}$ and analytic in z in the region $\{z : 0 < |z| < 1\}$. Indeed, relation (1.1) implies the inequality $|h_l(z)| > |e_l(z)|$, which, in turn, implies the estimate

$$|h_l(z) - e_l(z)w| > 0, \quad (z,w) \in \{z : 0 < |z| < 1\} \times \{w : |w| \le 1\}, \tag{1.7}$$

valid for all l. The definition of $\theta_l(z)$ and $\varphi_l(z)$ shows that the functions $t_l(z)$, $s_l(z)$, $h_l(z)$, and $e_l(z)$ are analytic in the region $0 < |z| < 1$. Taking (1.6) and (1.7) into account, we obtain the required statement.

Now, we prove that the function $m(z)$ is continuous in the region $0 < |z| < 1$. For this, we fix a point z_0 and an arbitrary $\varepsilon > 0$. By Lemmas 7 and 9, there exists l_0 such that $\operatorname{diam} D_{l_0}(z_0) < \varepsilon/2$. The continuity of $m(z, l_0, w)$ in the variables (z,w) implies the existence of a number $\delta > 0$ such that

$$\operatorname{diam} \bigcup_{|z-z_0|<\delta} D_{l_0}(z) < \varepsilon. \tag{1.8}$$

This fact can be proved by contradiction. Indeed, assume that there exist sequences z_j^1, w_j^1 and z_j^2, w_j^2 such that $\lim_{j\to\infty} z_j^1 = \lim_{j\to\infty} z_j^2 = z_0$ and

$$|m(z_j^1, l_0, w_j^1) - m(z_j^2, l_0, w_j^2)| \ge \varepsilon. \tag{1.9}$$

Since the w_j^1 belong to the compact set $\{w : |w| \le 1\}$, there exists a subsequence $w_{j_l}^1$ converging to some w^1. Similarly, there exists a subsequence $w_{j_{l_k}}^2$ of $w_{j_l}^2$ converging to some w^2. For simplicity, we preserve the notation w_i^1 and w_i^2 for the subsequences $w_{j_{l_i}}^1$ and $w_{j_{l_i}}^2$. Since $m(z, l_0, w)$ is continuous, there exists $\delta > 0$ such that $|m(z_j^k, l_0, w_j^k) - m(z_0, l_0, w^k)| < \varepsilon/4$ for $|z_j^k - z^0| + |w_j^k - w^k| < \delta$, where $k = 1, 2$. Moreover, since $\operatorname{diam} D_{l_0}(z_0) < \varepsilon/2$, we have $|m(z_0, l_0, w^1) - m(z_0, l_0, w^2)| < \varepsilon/2$. If $|z_j^k - z^0| + |w_j^k - w^k| < \delta$, then

$$\begin{aligned}
&|m(z_j^1, l_0, w_j^1) - m(z_j^2, l_0, w_j^2)| \\
&\quad \le |m(z_j^1, l_0, w_j^1) - m(z_0, l_0, w^1)| \\
&\qquad + |m(z_0, l_0, w^1) - m(z_0, l_0, w^2)| + |m(z_0, l_0, w^2) - m(z_j^2, l_0, w_j^2)| \\
&\quad < \varepsilon,
\end{aligned}$$

which contradicts (1.9).

Lemmas 7 and 9 show that $m(z) \in D_{l_0}(z)$. Therefore,

$$m(z) \in \bigcup_{|z-z_0|<\delta} D_{l_0}(z), \quad z \in \{z : |z - z_0| < \delta\}.$$

Now, recalling (1.8), we see that $|m(z) - m(z_0)| < \varepsilon$ for $|z - z_0| < \delta$. Thus, $m(z)$ is a continuous function for $0 < |z| < 1$.

We fix an arbitrary w (say, $w = 0$) and prove that the limit $m(z) = \lim_{l\to\infty} m(z, l, 0)$ is attained uniformly on every compact subset $K \subset \{z : 0 < |z| < 1\}$. Assume the contrary. Then there exist $\varepsilon > 0$, a sequence $l_j \to \infty$, and points z_j in K such that

$$|m(z_j) - m(z_j, l_j, 0)| > \varepsilon. \tag{1.10}$$

Since K is compact, there is a subsequence z_{j_k} that converges to some $z^1 \in K$. Since $m(z)$ is continuous, for some $\delta_1 > 0$ we have

$$|m(z) - m(z^1)| < \varepsilon/2 \quad \text{for } |z - z^1| < \delta_1. \tag{1.11}$$

As above, there exists l^1 such that $\operatorname{diam} D_{l^1}(z^1) < \varepsilon/4$, and $\operatorname{diam} \bigcup_{|z-z^1|<\delta_2} D_{l^1}(z) < \varepsilon/2$ for some $\delta_2 > 0$. Since the sequence z_{j_k} converges to z^1, there exists C_3 such that $|z_{j_k} - z^1| < \min(\delta_1, \delta_2)$ for $k \geq C_3$. Now, let $k \geq C_3$; then

$$m(z_{j_k}, l_{j_k}, 0) \in D_{l_{j_k}}(z_{j_k}) \subset \bigcup_{|z-z^1|<\delta_2} D_{l_{j_k}}(z) \subset \bigcup_{|z-z^1|<\delta_2} D_{l^1}(z).$$

Consequently,

$$|m(z^1) - m(z_{j_k}, l_{j_k}, 0)| < \varepsilon/2 \quad \text{for } k \geq C_3. \tag{1.12}$$

Combining (1.11) and (1.12), we obtain

$$|m(z_{j_k}) - m(z_{j_k}, l_{j_k}, 0)| \leq |m(z_{j_k}) - m(z^1)| + |m(z^1) - m(z_{j_k}, l_{j_k}, 0)| \leq \varepsilon,$$

which contradicts (1.10), thus proving the required uniformity.

Since $m(z, l, 0)$ is analytic and the limit $m(z) = \lim_{l\to\infty} m(z, l, 0)$ is uniform, we conclude that $m(z)$ is analytic for $0 < |z| < 1$. The estimate $|m(z)| < 1$ follows from Lemma 7. □

Proof of Theorem 7. Since the function $m(z)$ is analytic and bounded in the annulus $0 < |z| < 1$ (see Lemma 10), it has a removable singularity at $z = 0$. Consequently, $m(z)$ admits analytic continuation to the region $|z| < 1$. The maximum principle implies that $|m(z)| < 1$ everywhere in the disk $|z| < 1$. Since $m(z)$ satisfies (1.4) for all l, Lemmas 5 and 6 imply that $f(z) \in l^2(\mathbb{Z}_+, \mathbb{C}^2)$ for $0 < |z| < 1$. The theorem is proved. □

§2. A BFZ-type spectral identity

In the present section, we assume that $W_l \equiv I$ for $l \geq l_0$, with some l_0. For such W_l, there is a solution h of system (7) satisfying the condition $h_l = \binom{z^l}{0}$ for $l \geq l_0$. We use the notation

$$\begin{pmatrix} a(z) \\ b(z) \end{pmatrix} = h_0(z), \quad \begin{pmatrix} \alpha_l(z) \\ \beta_l(z) \end{pmatrix} = h_l(z).$$

Lemma 11. *The function $a(z)$ is analytic and does not vanish in the region* $0 < |z| \leq 1$.

Proof. The analyticity of $a(z)$ follows from that of the matrix $(z^{\sigma_3} W_l)^{-1}$.

Assume that there is z_0 such that $a(z_0) = 0$ and $0 < |z_0| \leq 1$. Then, by (1.1), we have $|\alpha_l(z_0)| < |\beta_l(z_0)|$ for all l. Putting $l = l_0$ in this inequality, we arrive at a contradiction: $|z_0|^{l_0} < 0$. □

Lemma 12. *The function $a(z)$ admits analytic continuation to the disk $\{z : |z| \leq 1\}$,* $a(0) = \prod_{l\geq 0} \bar{w}_l$.

Proof. We prove the following estimates by induction:

$$\alpha_l(z) = z^l\Big(\prod_{k\ge l}\bar w_k + O(z^2)\Big),\ \ \beta_l(z) = O(z^l),\quad l\ge 0. \tag{2.1}$$

For $l = l_0$, the claim follows from the relations $\alpha_l(z) = z^l$ and $\beta_l = 0$, $l \ge l_0$. Now, suppose (2.1) is fulfilled for $l \ge p$. We prove that (2.1) is valid for $l = p-1$. Indeed,

$$\begin{aligned}\begin{pmatrix}\alpha_{p-1}(z)\\ \beta_{p-1}(z)\end{pmatrix} &= W_{p-1}^{-1}z^{-\sigma_3}\begin{pmatrix}\alpha_p(z)\\ \beta_p(z)\end{pmatrix} = \begin{pmatrix} z^{-1}\bar w_{p-1} & -zv_{p-1}\\ -z^{-1}\bar v_{p-1} & zw_{p-1}\end{pmatrix}\begin{pmatrix}\alpha_p(z)\\ \beta_p(z)\end{pmatrix}\\ &= \begin{pmatrix} z^{p-1}\bar w_{p-1}\big(\prod_{k\ge p}\bar w_k + O(z^2)\big) - zv_{p-1}O(z^p)\\ -z^{-1}\bar v_{p-1}O(z^p) + zw_{p-1}O(z^p)\end{pmatrix}\\ &= \begin{pmatrix} z^{p-1}\big(\prod_{k\ge p}\bar w_k + O(z^2)\big)\\ O(z^{p-1})\end{pmatrix}.\end{aligned}$$

Thus, estimates (2.1) are proved.

Putting $l = 0$ in the first relation in (2.1), we obtain

$$a(z) = \prod_{l\ge 0}\bar w_l + O(z^2),$$

which implies the lemma. □

Lemma 13. *The function $a(z)$ satisfies the identity*

$$\int_0^{2\pi}\ln|a(re^{i\phi})|^2\,d\phi = 4\pi\sum_{l\ge 0}\ln|w_l|$$

for all $r\le 1$.

Proof. By Lemmas 11 and 12, the function $\ln a(z)$ is analytic in the disk $\{z : |z|\le 1\}$. The function $\ln|a(z)|$ is harmonic in the same region, being the real part of an analytic function. Applying the mean value theorem, we obtain

$$\int_0^{2\pi}\ln|a(re^{i\phi})|^2\,d\phi = 2\pi\ln|a(0)|^2 = 4\pi\sum_{l\ge 0}\ln|w_l|. \qquad \square$$

Lemma 14. *Suppose that there exists l_0 such that $W_l\equiv I$ for $l\ge l_0$. Then the Weyl function of system* (7) *satisfies the inequality*

$$-\int_0^{2\pi}\ln\big(1-|m(re^{i\phi})|^2\big)\,d\phi \le 4\pi\sum_{l\ge 0}\ln|w_l|$$

for all $r\le 1$.

Proof. By Lemma 8, the solutions $f(z)$ and $h(z)$ are linearly dependent. Putting $l = 0$, we see that the vectors $\binom{1}{m(z)}$ and $\binom{a(z)}{b(z)}$ are also linearly dependent. This implies that $a(z)$, $b(z)$, and $m(z)$ are related by the formula

$$m(z) = b(z)/a(z).$$

Applying (1.1) to $h_l(z)$, we obtain

$$\begin{aligned}|a(z)|^2 - |b(z)|^2 &= |z|^{-2}(|\alpha_1|^2 - |z|^4|\beta_1|^2)\\ &\ge |z|^{-2}(|\alpha_1|^2 - |\beta_1|^2) \ge |z|^{-2l_0}(|\alpha_{l_0}|^2 - |\beta_{l_0}|^2) = 1\end{aligned}$$

for $0 < |z|\le 1$. Thus,

$$|a(z)|^2 - |b(z)|^2 \ge 1,\quad 0<|z|\le 1. \tag{2.2}$$

By continuity, inequality (2.2) extends to the entire disk $|z| \le 1$. Hence,

$$\big(1 - |m(z)|^2\big)^{-1} \le |a(z)|^2.$$

To conclude the proof, it remains to use Lemma 13. □

In $l^2(\mathbb{Z}_+, \mathbb{C}^2)$ we consider the operator

$$Mg_l = g_{l+1} - z^{\sigma_3} W_l g_l, \quad 0 < |z| < 1, \;\; l = 0, 1, \dots, \tag{2.3}$$

with domain

$$\operatorname{dom}(M) = \Big\{ g : g_0 = \begin{pmatrix} 0 \\ \cdot \end{pmatrix}, \; g \in l^2(\mathbb{Z}_+, \mathbb{C}^2) \Big\}.$$

We recall that $\theta(z)$, $\varphi(z)$, and $f(z)$ are the vector-valued functions defined in Theorem 7. The kernel of the operator M is trivial. Indeed, let g be a nonzero solution of the equation $Mg = 0$. Since $g \in \operatorname{dom}(M)$, we have $g = C_4 \varphi$ on one hand, and $g = C_5 f$ on the other. As a result, we see that the vectors φ and f are proportional, which contradicts the definition of f. Since the kernel of M is trivial, the operator M is invertible. The inverse is given by the formula

$$\{M^{-1} q\}_l \equiv M^{-1} q_l = \sum_{k \ge 0} G(l, k) q_k, \quad l = 0, 1, \dots, \tag{2.4}$$

where

$$G(l,k) = \begin{cases} f_l \varphi_{k+1}^t J & \text{if } l \ge k+1, \\ \varphi_l f_{k+1}^t J & \text{if } l \le k, \end{cases} \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.5}$$

Here $\cdot^t$ stands for transposition. The domain of $\operatorname{dom}(M^{-1})$ is $l^2(\mathbb{Z}_+, \mathbb{C}^2)$.

Consider the set of matrices W_l satisfying (9) and assume that the series (10) converges. Let $m(z)$ be the Weyl function for (7), and let M be the operator constructed starting with W_l. Now, for $k \ge 1$, we consider the system (7) determined by the following set of matrices:

$$\begin{cases} W_l^k = W_l & \text{if } l \le k-1, \\ W_l^k = I & \text{if } l \ge k. \end{cases}$$

Let $m_k(z)$ and M^k denote the Weyl function and the operator of the resulting system, respectively.

Lemma 15. *Suppose the limit* (11) *exists. Then, as $k \to \infty$, the Weyl function $m_k(z)$ tends to $m(z)$ uniformly on every compact subset of $\{z : 0 < |z| < 1\}$.*

Proof. We have the resolvent identity

$$M^{-1} - M_k^{-1} = M^{-1}(M_k - M) M_k^{-1}. \tag{2.6}$$

For $k \ge 1$, we have the well-defined vector-valued function

$$\theta_1(z) = z^{\sigma_3} W_0^k \theta_0 = z^{\sigma_3} W_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Consider the following element q of the space $\operatorname{dom}(M^{-1}) = \operatorname{dom}(M_k^{-1})$:

$$q = \{q_l\}_{l=0}^{\infty}, \quad q_l = \begin{cases} \theta_1(z) & \text{if } l = 0, \\ 0 & \text{if } l \ge 1. \end{cases}$$

The following relations are valid:

$$\{M_k^{-1}q\}_l = G(l,0),\ \ \theta_1(z) = \begin{cases} m_k(z)\binom{0}{1} & \text{if } l=0, \\ f_l^k(z) & \text{if } l \geq 1, \end{cases} \quad \{M^{-1}q\}_0 = m(z)\begin{pmatrix}0\\1\end{pmatrix},$$

$$\{M^{-1}(M_k - M)M_k^{-1}q\}_0 = \begin{pmatrix}0\\1\end{pmatrix}\sum_{l\geq k}(f_{l+1}(z))^t J z^{\sigma_3}(I - W_l)f_l^k(z).$$

Now, we calculate the values of both sides of (2.6) at q. We have

$$m(z) - m_k(z) = \sum_{l\geq k}(f_{l+1}(z))^t J z^{\sigma_3}(I - W_l)f_l^k(z),$$

whence

$$\begin{aligned} |m(z) - m_k(z)| &\leq \sum_{l\geq k}\|z^{\sigma_3}\|\,\|W_l - I\|\|f_{l+1}(z)\|\,\|f_l^k(z)\| \\ &\leq \|z^{\sigma_3}\|\Big(\sum_{l\geq 1}\|f_l(z)\|^2\Big)^{1/2}\Big(\sum_{l\geq 1}\|f_l^k(z)\|^2\Big)^{1/2}\sup_{l\geq k}\|W_l - I\|. \end{aligned} \tag{2.7}$$

We fix an arbitrary compact set $K \subset \{z : 0 < |z| < 1\}$ and put $r = \min_{z\in K}|z| > 0$ and $R = \max_{z\in K}|z| < 1$. Lemma 3 and relations (1.2)–(1.4) imply the inequalities

$$\sum_{l\geq 1}\|f_l(z)\|^2 \leq \frac{1}{1-R^2}, \quad \sum_{l\geq 1}\|f_l^k(z)\|^2 \leq \frac{1}{1-R^2}.$$

Obviously, $\|z^{\sigma_3}\| = 1/r$. Substituting this in (2.7), we obtain

$$|m(z) - m_k(z)| \leq \frac{1}{r(1-R^2)}\sup_{l\geq k}\|W_l - I\|, \quad z \in K,$$

which implies the lemma. □

Proof of Lemma 2. By Lemma 15 and Theorem 7, we can pass to the limit under the integral sign,

$$-\lim_{k\to\infty}\int_0^{2\pi}\ln\big(1 - |m_k(re^{i\phi})|^2\big)\,d\phi = -\int_0^{2\pi}\ln\big(1 - |m(re^{i\phi})|^2\big)\,d\phi \tag{2.8}$$

for $0 < r < 1$. Applying Lemma 14 to the left-hand side of (2.8), we obtain

$$-\int_0^{2\pi}\ln\big(1 - |m(re^{i\phi})|^2\big)\,d\phi \leq 4\pi\sum_{l\geq 0}\ln|w_l|. \tag{2.9}$$

It remains to observe that, by Corollary 1, we can apply the Fatou lemma to inequality (2.9) when passing to the limit as $r \to 1-0$. This yields the required estimate. □

§3. Some estimates

In the present section, we prove Lemmas 3 and 4, which are needed in the proof of Theorem 2. These results are not really new; their analogs can be found in [LS].

Proof of Lemma 3. We consider the operator M given by formula (2.3), together with the operator M_0 given by (2.3) with $W_l \equiv I$. These operators are boundedly invertible (see (2.4)). We use the resolvent identity

$$M^{-1} - M_0^{-1} = M_0^{-1}(M_0 - M)M^{-1}.$$

Consequently,

$$\big(I + M_0^{-1}z^{\sigma_3}(I - \widehat{W})\big)M^{-1} = M_0^{-1}, \tag{3.1}$$

where $\widehat{W}$ is the operator of multiplication by W_l.

Let $G_0(l,k)$ be the Green function for M_0 (see (2.5)), and let φ^0 and f^0 be the corresponding solutions of system (7) with $W_l \equiv I$ (see §1). For $0<|z|<1$, simple calculations show that

$$G_0(l,k) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^{l-k-1} \qquad \text{if } l \geq k+1,$$
$$G_0(l,k) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} z^{k-l+1} \quad \text{if } l \leq k.$$

In particular,

$$\|G_0(l,k)\| \leq |z|^{|l-k-1|}, \quad \|G_0(l,k) z^{\sigma_3}\| \leq |z|^{|l-k|}.$$

Invoking Schur's criterion, we obtain

$$\|M_0^{-1} z^{\sigma_3}\| \leq \sup_{l\geq 0} \sum_{k=0}^{\infty} |z|^{|l-k|} \leq \sum_{k\in\mathbb{Z}} |z|^{|k|} \leq \frac{2}{1-|z|}, \quad \|M_0^{-1}\| \leq \frac{2}{1-|z|}. \tag{3.2}$$

Estimates (13) and (3.2) imply that

$$\begin{aligned} \|M_0^{-1} z^{\sigma_3}(I-\widehat{W})\| &\leq \|M_0^{-1} z^{\sigma_3}\| \, \|(I-\widehat{W})\| \\ &\leq \frac{1}{4(1-|z|)} \leq \frac{1}{2} \quad \text{for } 0<|z|\leq \frac{1}{2}. \end{aligned}$$

Consequently, for $0<|z|\leq 1/2$ the operator $I+M_0^{-1} z^{\sigma_3}(I-\widehat{W})$ is boundedly invertible. By (3.1) and (3.2),

$$\begin{aligned} \|M^{-1}\| &\leq \left\| \left(I + M_0^{-1} z^{\sigma_3}(I-\widehat{W})\right)^{-1} \right\| \|M_0^{-1}\| \leq 2\|M_0^{-1}\| \\ &\leq \frac{4}{1-|z|} \leq 8 \quad \text{for } 0<|z|\leq \frac{1}{2}. \end{aligned}$$

The norm of the matrix $G(l,l-1,z)$ can be estimated in terms of the norm of the operator M^{-1}. Indeed,

$$\begin{aligned} \|G(l,l-1)\|^2 &\leq \sum_{k=0}^{\infty} \|G(k,l-1)\|^2 = \sum_{k=0}^{\infty} \left\| \sum_{n=0}^{\infty} G(k,n)\delta_{l-1}^{n} \right\|^2 \\ &\leq \sup_{\|q\|=1} \sum_{k=0}^{\infty} \left\| \sum_{n=0}^{\infty} G(k,n) q \right\|^2 = \sup_{\|q\|=1} \|M^{-1}q\|^2 = \|M^{-1}\|^2. \end{aligned}$$

Consequently,

$$\|G(l,l-1,z)\| \leq \|M^{-1}\| \leq 8 \quad \text{for } 0<|z|\leq \frac{1}{2}. \tag{3.3}$$

The functions $f(z)$ and $\varphi(z)$ are analytic in the annulus $0<|z|<1$. Therefore, representation (2.5) for the matrix $G(l,l-1,z)$ implies that this matrix is analytic in the same region. By (3.3), the matrix $G(l,l-1,z)$ admits analytic continuation to the disk $|z|<1$. By the mean-value theorem,

$$\int_0^{2\pi} G(l,l-1,re^{i\phi})\, d\phi = 2\pi G(l,l-1,0), \quad r<1. \tag{3.4}$$

By Corollary 1 and Fatou's lemma, in (3.4) we can pass to the limit as $r\to 1$, which yields

$$\left\| \int_0^{2\pi} G(l,l-1,re^{i\phi})\, d\phi \right\| \leq 16\pi. \tag{3.5}$$

From property (8) it follows that $\theta = \sigma_1\bar{\varphi}$ for $z = e^{i\phi}$. We denote by φ_l^1 and φ_l^2 the components of the vector φ_l. Applying (2.5), we obtain

$$\begin{aligned} G(l,l-1,e^{i\phi}) &= (\sigma_1\bar{\varphi}_l + m(e^{i\phi})\varphi_l)\varphi_l^t J \\ &= \left(\begin{pmatrix} \varphi_l^1\bar{\varphi}_l^2 & |\varphi_l^2|^2 \\ |\varphi_l^1|^2 & \varphi_l^2\bar{\varphi}_l^1 \end{pmatrix} + m(e^{i\phi})\begin{pmatrix} (\varphi_l^1)^2 & \varphi_l^1\varphi_l^2 \\ \varphi_l^1\varphi_l^2 & (\varphi_l^2)^2 \end{pmatrix}\right)J \\ &= \begin{pmatrix} |\varphi_l^2|^2 + m(e^{i\phi})\varphi_l^1\varphi_l^2 & -\varphi_l^1\bar{\varphi}_l^2 - m(e^{i\phi})(\varphi_l^1)^2 \\ \varphi_l^2\bar{\varphi}_l^1 + m(e^{i\phi})(\varphi_l^2)^2 & -|\varphi_l^1|^2 - m(e^{i\phi})\varphi_l^1\varphi_l^2 \end{pmatrix}. \end{aligned} \tag{3.6}$$

We subtract the diagonal entries in (3.6) and use estimate (3.5). As a result, we see that

$$\int_0^{2\pi} \big(1 - |m(e^{i\phi})|\big)\|\varphi_l(e^{i\phi})\|^2\,d\phi \le 32\pi. \tag{3.7}$$

For $z = e^{i\phi}$ we have $F_l = (\sigma_1\bar{\varphi}_l, \varphi_l)$. Now, estimate (3.7) can be transformed to

$$\int_0^{2\pi} \big(1 - |m(e^{i\phi})|\big)\|F_l(e^{i\phi})\|^2\,d\phi \le 64\pi, \tag{3.8}$$

which allows us to write

$$\begin{aligned} &\int_0^{2\pi} \big(1 - |m(e^{i\phi})|\big)\sum_{l=0}^{L} a_l\|F_l\|\,\|F_{l+1}\|\,d\phi \\ &\quad\le \frac{1}{2}\int_0^{2\pi} \big(1 - |m(e^{i\phi})|\big)\sum_{l=0}^{L} a_l\|F_l\|^2\,d\phi + \frac{1}{2}\int_0^{2\pi} \big(1 - |m(e^{i\phi})|\big)\sum_{l=0}^{L} a_l\|F_{l+1}\|^2\,d\phi \\ &\quad\le 64\pi\sum_{l=0}^{\infty} a_l. \end{aligned}$$

Applying Beppo Levi's theorem and recalling that $|m(e^{i\phi})| < 1$ for almost all $\phi \in [0, 2\pi)$ (see Lemma 2), we conclude that the series $\sum_{l=0}^{\infty} a_l\|F_l\|\,\|F_{l+1}\|$ converges almost everywhere with respect to Lebesgue measure.

The second estimate in Lemma 3 follows from the inequalities

$$\begin{aligned} &\int_0^{2\pi} \big(1 - |m(e^{i\phi})|\big)\liminf_{L\to\infty}\frac{1}{L}\sum_{l=0}^{L}\|F_l\|^2\,d\phi \\ &\qquad\le \liminf_{L\to\infty}\frac{1}{L}\sum_{l=0}^{L}\int_0^{2\pi} \big(1 - |m(e^{i\phi})|\big)\|F_l\|^2\,d\phi \le 64\pi \end{aligned}$$

and the estimate $|m(e^{i\phi})| < 1$, which is valid for almost all $\phi \in [0, 2\pi)$ (see Lemma 2). □

Proof of Lemma 4. Let u and v be two linearly independent solutions of system (14). To simplify calculation, we assume that $\det(u_0, v_0) = 1$. This can always be ensured by multiplying one of the solutions by an appropriate constant. Since $\det K_l = 1$, we have

$$\det(u_l, v_l) = \det(u_0, v_0) = 1,$$

whence

$$1 \le \|u_l\|\,\|v_l\|.$$

Summing from 0 to L, we obtain

$$1 \le \left(\frac{1}{L}\sum_{l=0}^{L}\|u_l\|\,\|v_l\|\right)^2 \le \left(\frac{1}{L}\sum_{l=0}^{L}\|u_l\|^2\right)\left(\frac{1}{L}\sum_{l=0}^{L}\|v_l\|^2\right).$$

Since $u_l = B_l u_0$, we have

$$\frac{\sum_{l=0}^{L}\|u_l\|^2}{\sum_{l=0}^{L}\|v_l\|^2} = \frac{1}{L}\frac{\sum_{l=0}^{L}\|u_l\|^2}{\frac{1}{L}\sum_{l=0}^{L}\|v_l\|^2} \le \left(\frac{1}{L}\sum_{l=0}^{L}\|u_l\|^2\right)^2 \le \|u_0\|^4\left(\frac{1}{L}\sum_{l=0}^{L}\|B_l\|^2\right)^2. \tag{3.9}$$

Now, let v_l be a subordinate solution. By the assumptions of the lemma,

$$\liminf_{L\to\infty}\frac{1}{L}\sum_{l=0}^{L}\|B_l\|^2 < \infty.$$

Therefore, by (3.9), there exists a subsequence L_j such that

$$\frac{\sum_{l=0}^{L_j}\|u_l\|^2}{\sum_{l=0}^{L_j}\|v_l\|^2} \le C_{10} < \infty, \quad j = 1, 2, \dots,$$

which contradicts the definition of a subordinate solution. □

§4. Proof of Theorem 2

The main difficulty arising in the proof of Theorem 2 is that the matrices S_l occurring in system (5) may depend on the spectral parameter E, and Theorem 6 cannot be applied to system (5) directly. Nevertheless, we know that, for large l, the matrices S_l are independent of E. Therefore, from (5) we can extract a new, unperturbed system (15) satisfying the assumptions of Theorem 6. Moreover, the unperturbed system (15) differs from (5) by a small correction, which may depend on E. Since this correction is small, we can prove that the solutions of the two systems are close to each other, and this will allow us to apply the results of Theorem 6 to system (5).

First, we choose W_l in (7) so as to be able to apply Theorem 6 and Lemma 3 to system (7). Since the Fourier coefficients $r(l)$ are bounded, we have $\lim_{l\to\infty} r(l)l^{-1/2} = 0$. Therefore, there exists a constant L_1 such that

$$|r(l)|l^{-1/2} \le 1/16, \quad l \ge L_1.$$

Now, we consider system (5) with matrices

$$\begin{aligned} &W_l = I \quad \text{for } 0 \le l < L_1,\\ &W_l = \begin{pmatrix} \sqrt{1+|r(l)|^2 l^{-1}} & r(l)l^{-1/2}e^{\frac{2i}{3F}(\pi l)^3} \\ \bar r(l)l^{-1/2}e^{-\frac{2i}{3F}(\pi l)^3} & \sqrt{1+|r(l)|^2 l^{-1}} \end{pmatrix}, \quad \text{where } l \ge L_1. \end{aligned} \tag{4.1}$$

It is easily seen that

$$\|W_l - I\| \le 1/8 \quad \text{for } l = 0, 1, \dots .$$

We rewrite systems (5) and (15) in the matrix form:

$$B_{l+1} = e^{i\phi\sigma_3}(W_l + V_l)B_l, \;\; B_0 = I, \quad \phi = \pi\frac{E}{F}, \tag{4.2}$$

$$F_{l+1} = e^{i\phi\sigma_3}W_l F_l, \quad F_0 = I. \tag{4.3}$$

Here the matrices S_l are represented as $S_l = W_l + V_l$, where the W_l are defined in (4.1).

Theorem 9. *If Assumption* (C) *is fulfilled, then system* (4.2) *has no subordinate solutions for almost any* $\phi \in [0, 2\pi)$ *with respect to Lebesgue measure.*

Proof. By Assumption (C), the series

$$\sum_{l=0}^{\infty}\|V_l\| = \sum_{l=0}^{\infty}\|S_l - W_l\|$$

converges. The matrices W_l satisfy $\det W_l = 1$. Therefore, $\det F_l = 1$ for all $l \geq 0$. It follows that the F_l are invertible, and $\|F_l\| = \|F_l^{-1}\|$ for all $l \geq 0$. Using the substitution

$$B_l = F_l U_l, \quad U_0 = I,$$

we reshape (4.2) to

$$U_{l+1} = \big(I + F_{l+1}^{-1} e^{i\phi\sigma_3} V_l F_l\big) U_l.$$

To system (4.3), we can apply Lemma 3 with $a_l = \|V_l\|$. By that lemma, for almost all $\phi \in [0, 2\pi)$ we have

$$\|U_l\| \leq \prod_{k=0}^{l-1} \big(I + \|F_{l+1}^{-1}\|\,\|V_l\|\,\|F_l\|\big) \leq \exp\frac{3}{2}\big(I + \|V_l\|\,\|F_{l+1}\|\,\|F_l\|\big) \leq e^{\frac{3}{2}C_9(\phi)}.$$

Again by Lemma 3,

$$\liminf_{L\to\infty} \frac{1}{L}\sum_{l=0}^{L} \|B_l\|^2 \leq \liminf_{L\to\infty} \frac{1}{L}\sum_{l=0}^{L} \|F_l\|^2 \|U_l\|^2$$
$$\leq e^{3C_9(\phi)} \liminf_{L\to\infty} \frac{1}{L}\sum_{l=0}^{L} \|F_l\|^2 \leq C_9(\phi) e^{3C_9(\phi)} < \infty$$

for almost all $\phi \in [0, 2\pi)$. Now, we apply Lemma 4 to system (4.2), putting $K_l = e^{i\pi\frac{E}{F}\sigma_3} S_l$. As a result, we arrive at the required statement. □

Proof of Theorem 2. Theorem 9 implies that system (5) has no subordinate solutions for almost any $E \in \mathbb{R}$ with respect to Lebesgue measure. Now, Theorem 2 is a consequence of Lemma 1. □

§5. Application

This section is devoted to the proof of the fact that the operator

$$H = -\frac{d^2}{dx^2} - Fx + p(x) \quad \text{in } L_2(\mathbb{R}_+)$$

with domain

$$\operatorname{dom}(H) = \left\{ \varphi : \begin{array}{l} \varphi' \text{ is } \textit{absolutely continuous},\ \varphi(0) = 0,\ \operatorname{supp}(\varphi) \text{ is} \\ \textit{bounded}, \text{ and } -\varphi'' + p\varphi \in L_2(\mathbb{R}_+) \end{array} \right\}$$

is essentially selfadjoint. The fact that the domain dom(H) is dense in $L_2(\mathbb{R}_+)$ follows from [K, Chapter VI, §4, Theorem 4.2]. It can easily be checked that the operator H is symmetric. Therefore, to prove that H is essentially selfadjoint, it suffices to verify that H^* is symmetric.

We introduce the operator $h = -\frac{d^2}{dx^2} - Fx + p(x)$ with domain $\operatorname{dom}(h) = \{\varphi : \varphi' \text{ is absolutely continuous}\}$ and with values in $L_{1,\mathrm{loc}}(\mathbb{R}_+)$.

Lemma 16. *Let* $v \in \operatorname{dom}(H^*)$. *Then* v' *is absolutely continuous*, $hv \in L_2(\mathbb{R}_+)$, *and* $v(0) = 0$.

Proof. The definition of H^* implies that for some $w \in L_2(\mathbb{R}_+)$ we have

$$\int_0^\infty hu(x)\bar{v}(x)\,dx = \int_0^\infty u(x)\bar{w}(x)\,dx \tag{5.1}$$

for all $u \in \operatorname{dom}(H)$. Consider the equation $h\psi = w$ with initial conditions $\psi(0) = 0$ and $\psi'(0) = 0$. Reducing this equation to an integral one and applying Picard's method, it is not hard to show that it is uniquely solvable. Moreover, from the integral representation it follows that ψ' is absolutely continuous. We note that, in general, the function ψ does not belong to $L_2(\mathbb{R}_+)$.

Let $\operatorname{supp}(u) \subset [0,l]$. Using (5.1), we obtain

$$\begin{aligned}\int_0^\infty hu(x)\bar v(x)\,dx &= \int_0^l u(x)h\bar\psi(x)\,dx\\ &= \int_0^l u(x)\big(-\bar\psi''(x) - Fx\bar\psi(x) + p(x)\bar\psi(x)\big)\,dx\\ &= -u(0)\bar\psi'(0) + u'(0)\bar\psi(0) + \int_0^l hu(x)\bar\psi(x)\,dx = \int_0^\infty hu(x)\bar\psi(x)\,dx.\end{aligned}$$

Thus, for all $u \in \operatorname{dom}(H)$ we have

$$\int_0^\infty hu(x)(\bar v(x) - \bar\psi(x))\,dx = 0. \tag{5.2}$$

The operator H_{al} in $L_2[0,l]$ given by the formula $H_{al}\varphi = h\varphi$ on the domain

$$\operatorname{dom}(H_{al}) = \left\{\varphi : \begin{matrix}\varphi' \text{ is absolutely continuous, and } h\varphi \in\\ L_2[0,l],\ \varphi(0) = \varphi'(l) = 0\end{matrix}\right\}$$

is selfadjoint (see [K, Chapter VI, §4, Theorem 4.2]). Therefore, the kernel of $H_{al}^* - i$ is zero. Consequently, there exists $\psi_a \in \operatorname{dom}(h)$ such that $h\psi = i\psi$, $\psi_a(0) = 0$, and $\psi_a'(l) = 1$.

We consider the family of operators H_l in $L_2[0,l]$ with domain

$$\operatorname{dom}(H_l) = \left\{\varphi : \begin{matrix}\varphi' \text{ is absolutely continuous, } h\varphi \in L_2[0,l],\\ \text{and } \varphi(0) = \varphi(l) = \varphi'(l) = 0\end{matrix}\right\}$$

that are given by the rule $H_l\varphi = h\varphi$ for $\varphi \in \operatorname{dom}(H_l)$. It is clear that $\operatorname{dom}(H_{al}) = \operatorname{dom}(H_l)\dot+\{C\psi_a\}$. Since $H_l \subset H_{al} = H_{al}^* \subset H_l^*$, we see that the operator H_l is closed and that the kernel of $H_l^* - i$ is one-dimensional. Moreover, it is easy to show that $\psi_a \in \ker(H_l^* - i)$, whence $\ker(H_l^* - i) = \{C\psi_a\}$. The domain of the adjoint operator looks like this:

$$\operatorname{dom}(H_l^*) = \operatorname{dom}(H_l)\dot+\ker(H_l^* - i)\dot+\ker(H_l^* + i) = \operatorname{dom}(H_l)\dot+\{C\psi_a\}\dot+\{C\bar\psi_a\}$$

(see [BS]). Consequently,

$$\operatorname{dom}(H_l^*) \subset \big\{\varphi : \varphi' \text{ is absolutely continuous, } h\varphi \in L_2[0,l]\big\} \equiv \Omega_l. \tag{5.3}$$

Identity (5.2) is valid for the functions u in $\operatorname{dom}(H_l)$ extended by zero to the region $x \ge l$. Therefore, $v - \psi \in \operatorname{dom}(H_l^*) \subset \Omega_l$. It follows that $v \in \Omega_l$. Since this is true for all $l > 0$, v' is absolutely continuous on the semiaxis.

Since $v \in \Omega_l$, from (5.1) we deduce that

$$\int_0^\infty u(x)\big(h\bar v(x) - \bar w(x)\big)\,dx = u'(0)\bar v(0), \quad u \in \operatorname{dom}(H).$$

Consequently, $hv(x) = w(x) \in L_2[0,\infty)$ and $v(0) = 0$. □

Lemma 17. *For every $\varepsilon > 0$, there exists $N > 0$ such that $p(x) = p_\varepsilon(x) + p_0(x)$,*

$$\int_0^1 |p_\varepsilon(x)|\,dx \le \varepsilon, \quad |p_0| \le N.$$

Proof. This follows from the fact that $p(x)$ is absolutely integrable. □

Lemma 18. *If u is a continuously differentiable function, then*

$$\int_0^\zeta |p_\varepsilon(x)||u(x)|^2\,dx \le 4\varepsilon\int_0^\zeta |u'(x)|^2 + |u(x)|^2\,dx, \quad \zeta > 0.$$

Proof. For each $\alpha \in [1,2]$, we have

$$\begin{aligned}\int_n^{n+\alpha} &|p_\varepsilon(x)||u(x)|^2\,dx \\ &\le \max_{n\le x\le n+\alpha}|u(x)|^2\int_n^{n+\alpha}|p_\varepsilon(x)|\,dx \le 2\varepsilon\max_{n\le x\le n+\alpha}|u(x)|^2 \\ &\le 4\varepsilon\int_n^{n+\alpha}|u'(x)|^2+|u(x)|^2\,dx\end{aligned}$$

(the latter estimate is based on the fact that the embedding $H^1[0,1]\subset C[0,1]$ is continuous). Next,

$$\begin{aligned}\int_0^\zeta |p_\varepsilon(x)||u(x)|^2\,dx &= \sum_{n=0}^{[\zeta]-2}\int_n^{n+1}|p_\varepsilon(x)||u(x)|^2\,dx+\int_{[\zeta]-1}^{\zeta}|p_\varepsilon(x)||u(x)|^2\,dx \\ &\le \sum_{n=0}^{[\zeta]-2}4\varepsilon\int_n^{n+1}|u'(x)|^2+|u(x)|^2\,dx+4\varepsilon\int_{[\zeta]-1}^{\zeta}|u'(x)|^2+|u(x)|^2\,dx \\ &= 4\varepsilon\int_0^\zeta|u'(x)|^2+|u(x)|^2\,dx.\end{aligned}$$

□

Lemma 19. *Let $u\in\operatorname{dom}(H^*)$. Then for all but finitely many $n\in\mathbb{N}$ there exists $\zeta_n\in[n,n+1)$ such that $|\operatorname{Re}(\bar u(\zeta_n)u'(\zeta_n))|\le 1$.*

Proof. Assume the contrary. Then for infinitely many intervals we have

$$|\operatorname{Re}(u(x)\bar u'(x))|>1,\quad x\in[n,n+1).$$

For definiteness, assume that $\operatorname{Re}(u(x)\bar u'(x))>1$ (the case of $\operatorname{Re}(u(x)\bar u'(x))<-1$ is analyzed similarly). Then

$$\begin{aligned}|u(y)|^2-|u(n)|^2 &= \int_n^y(|u(x)|^2)'\,dx=\int_n^y u(x)\bar u'(x)+\bar u(x)u'(x)\,dx \\ &= \int_n^y 2\operatorname{Re}(\bar u(x)u'(x))\,dx\ge 2(y-n),\end{aligned}$$

whence

$$|u(y)|^2\ge 2(y-n),\quad \int_n^{n+1}|u(y)|^2\,dy\ge 1.$$

Since $u\in L_2(\mathbb{R}_+)$, the latter inequality cannot be true for infinitely many n. □

Lemma 20. *If $u\in\operatorname{dom}(H^*)$, then*

$$\int_0^R|u'(x)|^2\,dx\le A(u)R+B(u).$$

Proof. By Lemma 16, we have

$$\begin{aligned}\int_0^{\zeta_n}\bar u(x)hu(x)\,dx &= \int_0^{\zeta_n}\bar u(x)\big(-u''(x)-Fxu(x)+p(x)u(x)\big)\,dx \\ &= \int_0^{\zeta_n}|u'(x)|^2\,dx+\int_0^{\zeta_n}p(x)|u(x)|^2\,dx-\int_0^{\zeta_n}Fx|u(x)|^2\,dx-\bar u(\zeta_n)u'(\zeta_n).\end{aligned}$$

Let $\|u\|$ denote the norm of u in $L_2(\mathbb{R}_+)$. By Lemmas 16–19, we have the estimate

$$\begin{aligned}
&\int_0^{\zeta_n} |u'(x)|^2\,dx \\
&\quad\le \int_0^{\zeta_n} Fx|u(x)|^2\,dx + \int_0^{\zeta_n} |p(x)||u(x)|^2\,dx + \operatorname{Re}(\bar{u}(\zeta_n)u'(\zeta_n)) \\
&\qquad + \int_0^{\zeta_n} |u(x)|\,|hu(x)|\,dx \\
&\quad\le F\zeta_n\|u\|^2 + \int_0^{\zeta_n} |p(x)||u(x)|^2\,dx + 1 + C \\
&\quad\le F\zeta_n\|u\|^2 + 1 + C + \int_0^{\zeta_n} |p_\varepsilon(x)||u(x)|^2\,dx + \int_0^{\zeta_n} |p_0(x)||u(x)|^2\,dx \\
&\quad\le (F\zeta_n + N)\|u\|^2 + 1 + C + 4\varepsilon\int_0^{\zeta_n} |u'(x)|^2 + |u(x)|^2\,dx.
\end{aligned}$$

Choosing $\varepsilon = 1/8$, we obtain

$$\int_0^{\zeta_n} |u'(x)|^2\,dx \le (2F\zeta_n + 2N + 1)\|u\|^2 + 2 + 2C,$$

and the claim follows. □

Lemma 21. *The operator H^* is symmetric.*

Proof. Let $u, v \in \operatorname{dom}(H^*)$. Then

$$\int_0^R hu\bar{v}\,dx - \int_0^R uh\bar{v}\,dx = -u'(R)\bar{v}(R) + u(R)\bar{v}'(R).$$

By Lemma 16, the left-hand side has a finite limit as $R \to \infty$; we denote this limit by C_{uv}. To prove the lemma, it suffices to show that $C_{uv} = 0$. Using Lemma 20, we can write

$$\int_0^R \big(-u'(x)\bar{v}(x) + u(x)\bar{v}'(x)\big)\,dx = C_{uv}R + o(R), \tag{5.4}$$

and

$$\begin{aligned}
\left|\int_0^R \big(-u'(x)\bar{v}(x) + u(x)\bar{v}'(x)\big)\,dx\right| &\le \int_0^R |u'(x)||v(x)| + |u(x)||v'(x)|\,dx \\
&\le \left(\int_0^R |u'(x)|^2\,dx\right)^{1/2}\|v\| + \left(\int_0^R |v'(x)|^2\,dx\right)^{1/2}\|u\| \\
&\le \|v\|\sqrt{A_uR + B_u} + \|u\|\sqrt{A_vR + B_v} \le A_{uv}\sqrt{R} + B_{uv}
\end{aligned}$$

for some $A_{uv} > 0$ and $B_{uv} > 0$. Comparing the latter inequality with (5.4), we see that $C_{uv} = 0$. The lemma is proved. □

References

[AGZ] J. Avron, L. Gunter, and J. Zak, *Energy uncertainty in "Stark ladder"*, Solid State Comm. **16** (1975), no. 2, 189–191.

[AZ] J. Avron and J. Zak, *Instability of the continuous spectrum: the N-band Stark ladder*, J. Math. Phys. **18** (1977), no. 5, 918–921. MR0438948 (55:11851)

[B1] V. S. Buslaev, *Kronig–Penney electron in a homogeneous electric field*, Differential Operators and Spectral Theory, Amer. Math. Soc. Transl. (2), vol. 189, Amer. Math. Soc., Providence, RI, 1999, pp. 45–57. MR1730502 (2001b:81039)

[DK] P. Deift and R. Killip, *On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials*, Comm. Math. Phys. **203** (1999), no. 2, 341–347. MR1697600 (2000c:34223)

[DSS] F. Delyon, B. Simon, and B. Souillard, *From power pure point to continuous spectrum in disordered systems*, Ann. Inst. H. Poincaré Phys. Théor. **42** (1985), no. 3, 283–309. MR0797277 (87d:35098)

[E] P. Exner, *The absence of the absolutely continuous spectrum for δ' Wannier–Stark ladders*, J. Math. Phys. **36** (1995), no. 9, 4561–4570. MR1347099 (96g:81050)

[GP] D. J. Gilbert and D. B. Pearson, *On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators*, J. Math. Anal. Appl. **128** (1987), no. 1, 30–56. MR0915965 (89a:34033)

[K] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1995. MR1335452 (96a:47025)

[LS] Y. Last and B. Simon, *Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators*, Invent. Math. **135** (1999), no. 2, 329–367. MR1666767 (2000f:47060)

[NN] A. Nenciu and G. Nenciu, *Dynamics of Bloch electrons in external electric fields.* I. *Bounds for interband transitions and effective Wannier Hamiltonians*, J. Phys. A **14** (1981), no. 10, 2817–2827. MR0629327 (82j:81069)

[BS] M. Sh. Birman and M. Z. Solomyak, *Spectral theory of selfadjoint operators in Hilbert space*, Leningrad. Gos. Univ., Leningrad, 1980; English transl., D. Reidel Publishing Co., Dordrecht, 1987. MR0609148 (82k:47001); MR1192782 (93g:47001)

[B2] V. S. Buslaev, *Adiabatic perturbation of a periodic potential*, Teoret. Mat. Fiz. **58** (1984), no. 2, 233–243; English transl., Theoret. and Math. Phys. **58** (1984), no. 2, 153–159. MR0743409 (85i:34032)

[Ba] M. V. Buslaeva, *The one-dimensional Schrödinger operator with accelerating potential*, Funktsional. Anal. i Prilozhen. **18** (1984), no. 1, 65–66; English transl., Functional Anal. Appl. **18** (1984), no. 1, 53–55. MR0739094 (86f:81026)

[BD1] V. S. Buslaev and L. A. Dmitrieva, *Adiabatic perturbation of a periodic potential.* II, Teoret. Mat. Fiz. **73** (1987), no. 3, 430–442; English transl., Theoret. and Math. Phys. **73** (1987), no. 3, 1320–1329. MR0939788 (89c:34058)

[BD2] ———, *A Bloch electron in an external field*, Algebra i Analiz **1** (1989), no. 2, 1–29; English transl., Leningrad Math. J. **1** (1990), no. 2, 287–320. MR1025153 (91g:81162)

[BF] V. S. Buslaev and L. D. Faddeev, *Formulas for traces for a singular Sturm–Liouville differential operator*, Dokl. Akad. Nauk SSSR **132** (1960), no. 1, 13–16; English transl., Soviet Math. Dokl. **1** (1960), 451–454. MR0120417 (22:11171)

[Z] A. Zygmund, *Trigonometric series*, Cambridge Univ. Press, Cambridge, 1988. MR0933759 (89c:42001)

[L] B. M. Levitan, *Inverse Sturm–Liouville problems*, "Nauka", Moscow, 1984; English transl., VSP, Zeist, 1987. MR0771843 (86d:34002); MR0933088 (89b:34001)

[P] A. A. Pozharskiĭ, *Wannier–Stark type operators with singular potentials*, Algebra i Analiz **14** (2002), no. 1, 158–193; English transl., St. Petersburg Math. J. **14** (2003), no. 1, 119–145. MR1893324 (2003e:81051)

Department of Physics, St. Petersburg State University, Ulyanovskaya 1, Petrodvorets, St. Petersburg 198504, Russia

E-mail address: `lehman@sbor.net`

Received 10/AUG/2003

Translated by B. M. BEKKER