# DOUBLE SINGULAR INTEGRALS: INTERPOLATION AND CORRECTION 

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## §0. Introduction

0.1. Interpolation. Let $(\Omega, \mu)$ be a measure space. Consider an operator $Q$ that acts in all spaces $L^{p}(\mu)$ with $1<p<\infty$ and is a projection in each of these spaces. Put $X_{p}=\left\{f \in L^{p}(\mu): Q f=f\right\}$. If a meaning is lent to the expression $Q f$ also for $f \in L^{1}(\mu)$ or $f \in L^{\infty}(\mu)$ (surely, in interesting cases the function $Q f$ may fall out of these "extreme" spaces - for instance, it may be a distribution), then the definition of $X_{p}$ makes sense also for $p=1, \infty$, and we may ask about the extent to which the scale $X_{p}$, including both endpoints (or only one of them), inherits properties of the scale $L^{p}$.

Among other things, we shall be interested in the possibility of interpolating between spaces $X_{p}$ in accordance with the formulas inherent for the scale $L^{p}$. We deal with the real interpolation only; moreover, in this context we are interested mostly in the basic notion of $K$-closedness. If we manage to verify the relation $X_{p}=\left(X_{1}+X_{\infty}\right) \cap L^{p}$, $1<p<\infty$ (in the specific examples treated below this formula will always be true), then $K$-closedness (if it happens to occur) implies that the interpolation formulas of the real method are inherited indeed by the scale $X_{p}$.
Definition. A subcouple $\left(F_{0}, F_{1}\right)$ of an interpolation couple $\left(E_{0}, E_{1}\right)$ is said to be $K$ closed if for every $f \in F_{0}+F_{1}$ and every decomposition $f=e_{0}+e_{1}$ of $f$ with $e_{0} \in E_{0}$, $e_{1} \in E_{1}$ there exists another decomposition $f=f_{0}+f_{1}$, where $f_{0} \in F_{0}, f_{1} \in F_{1}$ and $\left\|f_{i}\right\| \leq C\left\|e_{i}\right\|, i=0,1$ (the constant $C$ does not depend on the vectors involved).
0.2. Singular integrals. So, we want to provide some conditions that ensure the $K$ closedness of the couple $\left(X_{p_{0}}, X_{p_{1}}\right)$ in $\left(L^{p_{0}}, L^{p_{1}}\right)$ if $p_{0}<p_{1}$ and at least one of the exponents takes the extreme value 1 or $\infty$. It is fairly well known that this is true indeed if $Q$ is a Calderón-Zygmund singular integral operator (CZO); see, e.g., the survey paper [11]. The theory presented in [11] easily extends to spaces of vector-valued functions. We give a summary of results in the vector setting.
0.2 .1 . It would be natural to take a so-called "homogeneous metric space" for the role of the basic measure space (see, e.g., [3]); however, for simplicity we assume that $\Omega$ is a disjoint union of finitely many copies of $\mathbb{R}^{n}$ or of $\mathbb{T}^{n}$ and $\mu$ is Lebesgue measure on this union. For short, the $L^{p}$-space of functions with values in a Banach space $X$ is often denoted simply by $L^{p}(X)$, without any mention of the set $\Omega$ and the measure $\mu$.
0.2.2. Let $A$ and $B$ be Banach spaces (for simplicity we assume them to be reflexive). An operator $T$ is called a Calderón-Zygmund operator if
(a) $T$ acts from $L^{p}(A)$ to $L^{p}(B)$ for some $p \in(1, \infty)$;

[^0](b) $T$ has a kernel $K(\cdot, \cdot)$ with values in the space $\mathcal{L}(A, B)$ of bounded linear operators from $A$ to $B$; this kernel is related to $T$ by the formula
$$
(T f)(s)=\int_{\Omega} K(s, t) f(t) d \mu(t)
$$
valid for a.e. $s$ off the support of $f, f \in L^{p}(A)$;
(c) $K$ possesses some smoothness.

Different and nonequivalent forms of condition (c) can be found in the literature. The following three versions suffice for our purposes:
(c1) there is a constant $C$ such that for every ball $D$ in $\Omega$ we have

$$
\int_{s \notin 5 D}\left\|\left(K\left(s, t_{1}\right)-K\left(s, t_{2}\right)\right) a\right\| d \mu(s) \leq C\|a\|, \quad t_{1}, t_{2} \in D, \quad a \in A
$$

(here $5 D$ is the ball of the same center as $D$ but of radius five times the radius of $D$ );
(c2) the same, but with the inequality

$$
\int_{s \notin 5 D}\left\|K\left(s, t_{1}\right)-K\left(s, t_{2}\right)\right\|_{\mathcal{L}(A, B)} d \mu(s) \leq C, \quad t_{1}, t_{2} \in D
$$

(c3) the same, but with the inequality

$$
\left\|K\left(s, t_{1}\right)-K\left(s, t_{2}\right)\right\|_{\mathcal{L}(A, B)} \leq \frac{C\left|t_{1}-t_{2}\right|^{\alpha}}{\left|s-t_{1}\right|^{n+\alpha}}, \quad s \notin 5 D, \quad t_{1}, t_{2} \in D
$$

for some $\alpha>0$ and $C>0$ independent of $t_{1}, t_{2}$, and $s$ (here $n$ is the dimension of the space $\mathbb{R}^{n}$ or $\mathbb{T}^{n}$ involved in the definition of the basic measure space).

The inequalities in (c1)-(c3) become stronger consecutively. In many cases, the weakest of them suffices; however, for weighted estimates (c3) is required. Usually (c1) and (c2) are called Hörmander conditions.
0.2.3. Suppose $A=B$ and $Q$ is a projection in $L^{p}(A)$ that is a Calderón-Zygmund operator. It is well known that, automatically, $Q$ takes $L^{r}(A)$ into itself boundedly for $1<r \leq p$, and it maps $L^{1}(A)$ to $L^{1, \infty}(A)$, so that the definition

$$
\begin{equation*}
X_{r, Q}=X_{r}=\left\{f \in L^{r}(A): Q f=f\right\} \tag{1}
\end{equation*}
$$

makes sense for $1 \leq r \leq p$. If $Q^{*}$ is also a Calderón-Zygmund operator (now it is natural to take $p^{\prime}$ in the role of the parameter in condition (a)), then $Q$ acts in $L^{r}(A)$ for $p \leq r<\infty$, and for these $r$ the space $X_{r, Q}$ is also defined by (1). In this case it is natural to put

$$
X_{\infty}=X_{\infty, Q}=\left(X_{1, I-Q^{*}}\right)^{\perp}
$$

If $f \in L^{\infty}(A) \cap L^{s}(A)$ for some $s<\infty$, then the relation $f \in X_{\infty, Q}$ is equivalent to the identity $Q f=f$ (so, the heuristic discussion in Subsection 0.1 is not too deceiving at this point).
0.2.4. K-closedness theorem. I. If $Q$ is a Calderón-Zygmund operator, then the couple $\left(X_{r_{1}}, X_{r_{2}}\right)$ is $K$-closed in $\left(L^{r_{1}}(A), L^{r_{2}}(A)\right)$ for $1 \leq r_{1} \leq r_{2} \leq p$.
II. If $Q^{*}$ is a Calderón-Zygmund operator, then the couple $\left(X_{r_{1}}, X_{r_{2}}\right)$ is $K$-closed in $\left(L^{r_{1}}(A), L^{r_{2}}(A)\right)$ for $p \leq r_{1} \leq r_{2} \leq \infty$.
III. If both $Q$ and $Q^{*}$ are Calderón-Zygmund operators, then the couple $\left(X_{r_{1}}, X_{r_{2}}\right)$ is $K$-closed in $\left(L^{r_{1}}(A), L^{r_{2}}(A)\right)$ for $1 \leq r_{1} \leq r_{2} \leq \infty$.

We shall outline the proof later (see Subsection 0.7).
0.3. Correction. Let $T$ be an arbitrary linear operator acting boundedly from $L^{p}(A, \mu)$ to $L^{p}(B, \mu)$ for $1<p<\infty$ (as before, $A$ and $B$ are reflexive spaces). As in Subsection 0.1, assume that a meaning can be lent to the symbol $T f$ also for $f$ in the "extreme" spaces $L^{1}(A, \mu)$ and $L^{\infty}(A, \mu)$. Put

$$
\mathfrak{X}_{p} \stackrel{\text { def }}{=}\left\{f \in L^{p}(A): T f \in L^{p}(B)\right\}, \quad 1 \leq p \leq \infty .
$$

Clearly, $\mathfrak{X}_{p}=L^{p}(A)$ for $1<p<\infty$. For us, the extreme point $p=\infty$ will be of much greater interest than $p=1$. Specifically, we want to know whether an arbitrary function belonging to $L^{\infty}(A)$ (or at least to $L^{\infty}(A) \cap L^{1}(A)$ ) can be corrected up to a function in $\mathfrak{X}_{\infty}$ by a change on a set of small measure.

If $T^{*}$ is a Calderón-Zygmund operator, the scalar version of the next theorem was proved in 9 . The vector case does not differ substantially.

Correction Theorem. Suppose $T^{*}$ is a Calderón-Zygmund operator and the norm in $A$ is strictly convex. Let $f \in L^{\infty}(A) \cap L^{1}(A)$, let $\|f\|_{\infty} \leq 1$, and let $0<\varepsilon<1$. Then there exists a scalar measurable function $\varphi$ such that $0 \leq \varphi \leq 1$ a.e., $\mu\{\varphi \neq 1\} \leq \varepsilon\|f\|_{L^{1}(A)}$, and

$$
\|T(\varphi f)\|_{\infty} \leq C\left(1+\log \frac{1}{\varepsilon}\right)
$$

where $C$ depends only on $T$.
In other words, the correction is done by multiplication by a function all values of which lie between 0 and 1 and differ from 1 on a set of small measure. Moreover, the norm of the corrected function in $\mathfrak{X}_{\infty}$ jumps only quite moderately, as compared to the $L^{\infty}$-norm of $f$ (note that a control on $\|T f\|_{\infty}$ is in fact equivalent here to a control on $\left.\|f\|_{\mathfrak{X}_{\infty}}\right)$.
0.3.1. It should be mentioned that the spaces $\mathfrak{X}_{p}$ can be interpreted as spaces $X_{p}$ treated in Subsection 0.1 or Subsection 0.2 .3 . For this, we introduce a projection $Q$ in $L^{p}(A \oplus B)=L^{p}(A) \oplus L^{p}(B)$ by the formula $Q(f, g)=(f, T f)$. Clearly, the image of $Q$ identifies with the set of pairs of the form $(\varphi, T \varphi)$ with $\varphi \in L^{p}(A)$.

So, for the spaces $\mathfrak{X}_{p}$, the interpolation problem described in preceding subsections also makes sense. Below we shall see that this problem is intimately related to the correction problem.
0.4. Double singular integrals. We do not attempt to give a general definition here; instead, we discuss a model example. For the two-dimensional torus $\mathbb{T}^{2}$ with Lebesgue measure, consider the projection $Q$ defined by

$$
Q f=\sum_{k, l \geq 0} \hat{f}(k, l) z_{1}^{k} z_{2}^{l}
$$

This is not a Calderón-Zygmund operator (for example, this follows from the fact that, as $p \rightarrow \infty$, its norm on $L^{p}\left(\mathbb{T}^{2}\right)$ grows as $p^{2}$ rather than as $\left.p\right)$. However, it is the tensor product of two Riesz projections acting in different variables: $Q=\mathbb{P} \otimes \mathbb{P}$, where

$$
\mathbb{P} g=\sum_{n \geq 0} \hat{g}(n) z^{n}
$$

for functions $g$ on the unit circle. Since $\mathbb{P}$ is a CZO (we recall that the latter abbreviation stands for "Calderón-Zygmund operator"), it seems appropriate to call $Q$ a "double singular integral".

The spaces $X_{p}$ generated by $Q$ are none other than the Hardy classes $H^{p}\left(\mathbb{T}^{2}\right)$. The following theorem was proved in [14].

Theorem on the two-dimensional torus. For arbitrary exponents $p_{1}$, $p_{2}$ with $0<$ $p_{1}<p_{2} \leq \infty$, the couple $\left(H^{p_{1}}\left(\mathbb{T}^{2}\right), H^{p_{2}}\left(\mathbb{T}^{2}\right)\right)$ is $K$-closed in $\left(L^{p_{1}}\left(\mathbb{T}^{2}\right), L^{p_{2}}\left(\mathbb{T}^{2}\right)\right)$.

Under the restriction $p_{2}<\infty$, the theorem had been known prior to [14] and for the tori of arbitrary dimension; see [19]. It is still unclear whether infinite exponents are admissible even in dimension 3 .

For the two-dimensional torus, the argument in [14] that made it possible to cover the case of $p_{2}=\infty$ involved a fairly specific complex analysis trick. Therefore, that argument is not quite suitable for generalization. Roughly speaking, it is only hoped that these techniques may extend to the situations where one of the two CZO's that generate a double singular integral is somewhat similar to the Riesz projection.

Our main purpose in this paper is to show that some important operators of the onedimensional Fourier analysis (on the line and the circle) can be interpreted as double singular integrals precisely of this kind. Unlike the torus $\mathbb{T}^{2}$, no separation of variables occurs here; however, again the $L^{p}$-boundedness of the operators in question for $p \in$ $(1, \infty)$ reduces to the boundedness of the composition of two CZO's and some simple transformations. We consider two examples. The first is the Hardy-Littlewood square function. In the case of the circle (we fix it for definiteness) it is introduced as follows:

$$
\sigma f=\left(\sum_{k \geq 0}\left(\left|\sum_{2^{k}-1 \leq n<2^{k+1}-1} \hat{f}(n) z^{n}\right|^{2}+\left|\sum_{-2^{k+1}<n \leq-2^{k}} \hat{f}(n) z^{n}\right|^{2}\right)\right)^{1 / 2}
$$

The second example is the projection $\mathbb{Q}$ defined by the formula

$$
\begin{equation*}
\mathbb{Q} f=\sum_{k \geq 0} \sum_{2^{2 k}-1 \leq n<2^{2 k+1}-1} \hat{f}(n) z^{n} \tag{2}
\end{equation*}
$$

(The analogs of $\sigma$ and $\mathbb{Q}$ for the line $\mathbb{R}$ can easily be written out. The entire material presented below can be rephrased for the case of the line.)

The similarity between the operators $\sigma, \mathbb{Q}$, etc. and the double Riesz projection for the two-dimensional torus was known previously to a certain extent - see [8].
0.5. Statements. In the interpolation theorem for the Hardy-Littlewood square function we do not resort literally or explicitly to the construction in Subsection 0.3.1. Moreover, we present a stronger result than is suggested by that construction. However, the material of Subsection 0.3 .1 should be kept in mind.

For a function $f$ on the unit circle, we put

$$
M_{k} f= \begin{cases}\sum_{2^{k-1}-1 \leq n<2^{k}-1} \hat{f}(n) z^{n}, & k \geq 1 \\ \sum_{2^{-k} \leq n<2^{-k+1}} \hat{f}(n) z^{n}, & k<0 .\end{cases}
$$

Since $\sigma f=\left(\sum_{k}\left|M_{k} f\right|^{2}\right)^{1 / 2}$, we can consider the linear mapping $f \mapsto\left\{M_{k} f\right\}_{k \in \mathbb{Z}}$ instead of $\sigma$. This linear mapping takes $f$ to an $l^{2}$-valued function. Next, in the space $L^{p}\left(l^{2}\right)=$ $\left\{\left\{f_{j}\right\}_{j \in \mathbb{Z}}:\left(\sum_{j}\left|f_{j}\right|^{2}\right)^{1 / 2} \in L^{p}(\mathbb{T})\right\}$, we single out the subspaces $\mathcal{Y}_{p}=\left\{\left\{f_{j}\right\}_{j \in \mathbb{Z}}: M_{s} f_{0}=\right.$ $f_{s}$ for $\left.s \neq 0\right\}$ and $\mathcal{Y}_{p, A}=\left\{\left\{f_{j}\right\}_{j \in \mathbb{Z}} \in \mathcal{Y}_{p}: f_{0} \in H^{p}\right\}$. (Thus, $f_{j}=0$ for $j<0$ if $\left\{f_{j}\right\} \in \mathcal{Y}_{p, A}$. The spaces $\mathcal{Y}_{p, A}$ are suitable for the study of the operator $\sigma$ on the classes $H^{p}$.)

Theorem 1 (square function, interpolation). For every $r_{1}, r_{2} \in[1, \infty]$ the couples $\left(\mathcal{Y}_{r_{1}}, \mathcal{Y}_{r_{2}}\right)$ and $\left(\mathcal{Y}_{r_{1}, A}, \mathcal{Y}_{r_{2}, A}\right)$ are $K$-closed in $\left(L^{r_{1}}\left(l^{2}\right), L^{r_{2}}\left(l^{2}\right)\right)$.

Generically, a precise analog of the statement in Subsection 0.3 should fail for a "double singular integral". The first power of the logarithm in the estimate for the corrected function is no longer expected to suffice (this is related to the fact that the $L^{p}$-norm of the operator in question may grow faster than $p$ as $p \rightarrow \infty$ ). Consequently, a correction
cannot always be done by multiplication by a function with values between 0 and 1 (otherwise a logarithmic estimate would have been restored by iteration). The authors were not able to learn whether it is possible to ensure that the $L^{\infty}$-norm of the corrected function remain bounded as $\varepsilon \rightarrow 0$ or the corrected function vanish where the original one does (these features were present in Subsection 0.3). However, the following statement is true for $\sigma$. (Note that, by some peculiar features of $\sigma$, there is a related statement with the first power of the logarithmic factor in place of the second; see the comments to Theorem $2^{\prime}$ in $\S 1$.)

Theorem 2 (square function, correction). Suppose $f \in L^{\infty}(\mathbb{T}),\|f\|_{\infty} \leq 1$, and $0<\varepsilon<$ 1. Denote by $f^{*}$ the Hardy-Littlewood maximal function for $f$. Then for every positive $\alpha<1$ there is a function $g$ with $|g|, \sigma g \leq C_{\alpha}\left(1+\log \frac{1}{\varepsilon}\right)^{2}\left(f^{*}\right)^{\alpha}$ and $m\{f \neq g\} \leq \varepsilon\|f\|_{1}$.

The pointwise control of $g$ in terms of $f^{*}$ makes Theorem 2 somewhat similar to the statement in Subsection 0.3. It should be noted that, moreover, $\sigma g$ in Theorem 2 is also controlled pointwise in terms of $f^{*}$, which is not the case for $T(\varphi f)$ in the statement in Subsection 0.3 (see, however, the remark at the end of $\S 3$ ).

Finally, we put $Z_{p}=\left\{f \in L^{p}(\mathbb{T}): f=\mathbb{Q} f\right\}$, where the projection $\mathbb{Q}$ is given by formula (2) and $1 \leq p \leq \infty$.

Theorem 3. If $1 \leq p_{1} \leq p_{2} \leq \infty$, then the couple $\left(Z_{p_{1}}, Z_{p_{2}}\right)$ is $K$-closed in $\left(L^{p_{1}}(\mathbb{T})\right.$, $\left.L^{p_{2}}(\mathbb{T})\right)$.

Theorems 1 and 3 are proved in a similar way, but the latter is slightly harder. In order to prove Theorem 2, we shall need a weighted analog of a partial case of Theorem 1. Surely, Muckenhoupt $A_{p}$-weights will be involved, but the "right" way to incorporate them is not immediate in the present context. We formulate a statement about weighted estimates that plays an auxiliary role for us. It seems to be new (and then interesting by itself), in spite of the fact that the weighted estimates' territory is explored quite thoroughly.

Theorem 4. Let $a$ and $w$ be two weights with $w \in A_{1}$ and $a \in A_{\infty}$. Suppose that both $T$ and $T^{*}$ are CZO 's (the smoothness condition for the kernel in the form (c3) is assumed). Put $u=\frac{a}{w}$ and define an operator $R$ by the formula $R f=u^{-1} T(u f)$. Then there exists $p_{0} \in(1, \infty), p_{0}=p_{0}(w, a)$, such that $R$ is bounded on $L^{t}(a)$ for $1<t \leq p_{0}$ and is of weak type $(1,1)$ relative to the weight $a$ :

$$
\int_{\{|R f|>\lambda\}} a \leq \frac{C}{\lambda} \int|f| a, \quad \lambda>0, \quad C=C(w, a)
$$

Corollary. Under the assumptions of the theorem, we have

$$
\|R\|_{L^{t}(a) \rightarrow L^{t}(a)}=O\left(\frac{1}{t-1}\right), \quad t \rightarrow 1
$$

(the constant in " $O$ " depends only on $w$ and a).
The corollary follows in a standard way by interpolation between continuity in $L^{p_{0}}(a)$ and weak type $(1,1)$ with weight $a$. Vector-valued singular integrals are admitted in Theorem 4, though this is not reflected notationally. Probably, it should be mentioned that if we would have talked about $L^{1}$-boundedness, we would have dealt with a usual density change, so that $R$ and $T$ would have been simultaneously bounded or not on $L^{1}(a)$ and $L^{1}(w)$, respectively. However, in our case $T$ is only of weak type $(1,1)$ on $L^{1}(w)$, and this property is unstable under a density change in general. The essence of Theorem 4 is in the statement that it is stable if $a \in A_{\infty}$.
0.6. Other square functions. For diversity, we leave the circle for a while and pass to $\mathbb{R}^{n}$. For a function $f$ on $\mathbb{R}^{n}$, let $\tilde{f}$ denote its harmonic extension to the upper halfspace $\mathbb{R}_{+}^{n+1}$ (i.e., the Poisson integral of $f$ ). By "other" square functions we mean the Littlewood-Paley function $g$, the area integral $S$, and the function $g_{\lambda}^{*}$. These three operators are defined by the formulas

$$
\begin{aligned}
g(f)(x) & \left.=\left(\int_{0}^{\infty}|\nabla \tilde{f}(x, y)|^{2} y d y\right)\right)^{1 / 2}, \quad x \in \mathbb{R}^{n} \\
S(f)(x) & \left.=\left(\int_{\Gamma(x)}|\nabla \tilde{f}(x, y)|^{2} y^{1-n} d y d t\right)\right)^{1 / 2}, \quad x \in \mathbb{R}^{n} ; \\
g_{\lambda}^{*}(f)(x) & =\left(\int_{0}^{\infty} \int_{t \in \mathbb{R}^{n}}\left(\frac{y}{|t|+y}\right)^{\lambda n}|\nabla \tilde{f}(x-t, t)|^{2} y^{1-n} d t d y\right)^{1 / 2}, \quad x \in \mathbb{R}^{n} .
\end{aligned}
$$

In the second formula, $\Gamma(x)=\left\{(t, y) \in \mathbb{R}_{+}^{n+1}:|x-t| \leq y\right\}$. The well-known pointwise inequalities $g(f) \leq S f \leq C_{\lambda} g_{\lambda(f)}^{*}$ (see, e.g., [16, 4]) show the relationship between interpolation or correction problems associated with these functions.

It is well known that the operators $g$ and $S$ can be interpreted as the results of pointwise evaluation of the norm for certain Calderón-Zygmund operators from $L^{2}\left(\mathbb{R}^{n}\right)$ into $L^{2}\left(\mathcal{H}, \mathbb{R}^{n}\right)$ for an appropriate Hilbert space $\mathcal{H}$; see [16, 4]. Consequently, the stuff presented in Subsections 0.2-0.3 applies to these operators; concretization of the theorems cited there is left to the reader. It is equally well known that, largely, $g_{\lambda}^{*}$ cannot be interpreted in terms of CZO's. However, in the recent paper [6] some results related in part to those treated here were obtained (with the help of the Brownian motion) for the analog of $g_{2}^{*}$ in the case of the disk, i.e., for the operator

$$
\begin{equation*}
G(f)(\alpha)=\left(\frac{1}{\pi} \int_{\mathbb{D}}|\nabla \tilde{f}(z)|^{2} \frac{1-|z|^{2}}{\left|e^{i \alpha}-z\right|^{2}} \log \frac{1}{|z|} d A(z)\right)^{1 / 2}, \quad \alpha \in \mathbb{T} \tag{3}
\end{equation*}
$$

(here $\tilde{f}$ stands for the Poisson integral of $f$ in the disk, and $A$ denotes planar Lebesgue measure).

We wish to explain that, despite all, a singular integral is present in this particular case, so that the methods outlined in Subsections $0.2-0.3$ are also applicable. We show this by the example of an interpolation theorem; see [6, Theorem 4.5]. After elimination of the probabilistic part, we obtain the following statement 1

Proposition 1. Suppose $1<p<\infty, f \in H^{p}(\mathbb{T})$, and $\lambda>0$. Then there exists a function $g \in H^{\infty}$ such that $\|g\|_{\infty} \leq \lambda,\|G(g)\|_{\infty} \leq A \lambda$, and $\|f-g\|_{1} \leq A \lambda^{1-p}\|f\|_{p}^{p}$.

A momentary reflection shows that the operator involved in this situation is $f \mapsto$ $G(\mathbb{P} f), f \in L^{p}(\mathbb{T})$, where $\mathbb{P}$ is the Riesz projection, $\mathbb{P} f=\sum_{n \geq 0} \hat{f}(n) z^{n}$. Thus, it might seem that a double singular integral can appear here (namely, the Riesz projection followed by a linearization of $G$ ). However, in fact we deal with a single CZO. Indeed, $\widetilde{\mathbb{P} f}(z)=\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f\left(e^{i \theta}\right)}{1-z e^{-i \theta}} d \theta$. Since, practically, the complex variable derivative $\frac{d}{d z}[(\mathbb{P} f)(z)]$ occurs in the formula for $G(\mathbb{P} f)$, we arrive at the square function (here $z=r e^{i s}$ )

$$
\mathcal{G} f(\alpha)=\left(\int_{0}^{1} \int_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi} \frac{e^{-i \theta}}{\left(1-r e^{i(s-\theta)}\right)^{2}} f\left(e^{i \theta}\right) d \theta\right|^{2} \frac{\left(1-r^{2}\right) r}{\left|1-r e^{i(s-\alpha)}\right|^{2}} \log \frac{1}{r} d s d r\right)^{1 / 2}
$$

[^1]or, after the change of variables $s-\alpha=t$ in the integral in $s$, at the square function
\[

$$
\begin{equation*}
\mathcal{G} f(\alpha)=\left(\int_{0}^{1} \int_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi} \frac{e^{-i \theta}}{\left(1-r e^{i(t+\alpha-\theta)}\right)^{2}} f\left(e^{i \theta}\right) d \theta\right|^{2} \frac{\left(1-r^{2}\right) r}{\left|1-r e^{i t}\right|^{2}} \log \frac{1}{r} d t d r\right)^{1 / 2} \tag{4}
\end{equation*}
$$

\]

Let $\mathcal{H}$ be the Hilbert space of functions on the circle that are square integrable with the weight given in polar coordinates by

$$
(r, t) \mapsto \frac{r\left(1-r^{2}\right)}{\left|1-r e^{i t}\right|} \log \frac{1}{r}
$$

Formula (4) is linked to the linear operator $T: L^{2}(\mathbb{T}) \rightarrow L^{2}(\mathcal{H}, \mathbb{T})$, the $\mathcal{L}(\mathbb{C}, \mathcal{H})$-valued kernel of which is given by $K(\alpha, \theta)(r, t)=\frac{e^{-i \theta}}{\left(1-r e^{i(t+\alpha-\theta)}\right)^{2}}$. The operator $T$ is $L^{2}$-bounded indeed; see, e.g., [16, 4].

Lemma. $T^{*}$ is a Calderón-Zygmund operator (the Hörmander condition is fulfilled in the form (c1)).

If the lemma is proved, Proposition 1 follows easily. Indeed, let $W_{p}$ be the space of functions $f \in H^{p}(\mathbb{T})$ satisfying $\mathcal{G} f \in L^{p}(\mathbb{T})$ (with the natural norm). Then, at least for $2 \leq p<\infty$, we have $W_{p}=H^{p}$. Proposition 1 claims that $H^{p} \subset\left(H^{1}, W_{\infty}\right)_{\theta, \infty}$, where $p^{-1}=(1-\theta) 1^{-1}+\theta(\infty)^{-1}=1-\theta$. Since $T^{*}$ is a CZO, Statement II in Subsection 0.2.4 (in combination with the idea outlined in Subsection 0.3.1) shows that $H^{r}=\left(H^{2}, W_{\infty}\right)_{\eta, r}$ for $2<r<\infty$, where $r^{-1}=(1-\eta) 2^{-1}+\eta(\infty)^{-1}$ (interpolation "runs as in the $L^{p}$ scale"). It is well known that real interpolation between $H^{1}$ and $H^{s}, 1 \leq s \leq \infty$, also "runs as in the $L^{p}$-scale". Taking $s \in(2, \infty)$ (so that the intervals $[1, s]$ and $[2, \infty)$ overlap) and applying Wolff's theorem [18], we see that the scale consisting of the spaces $H^{p}$ for $1 \leq p<\infty$ and $W_{\infty}$ at the point $p=\infty$ also admits $L^{p}$-type interpolation. In particular, $H^{p}=\left(H^{1}, W_{\infty}\right)_{\theta, p} \subset\left(H^{1}, W_{\infty}\right)_{\theta, \infty}$ for $1<p<\infty$, as required.

Since the Poisson kernel is a difference of two Cauchy kernels, Lemma 2 also implies a statement in the spirit of Subsection 0.3 about correction up to a function $f \in L^{\infty}(\mathbb{T})$ with $G(f)$ (see (3)) uniformly bounded.

Note that the proof of Statement II in Subsection 0.2.4 involves the Hahn-Banach theorem (via the duality lemma in Subsection 0.7 .2 below). So, in the above argument, the required function $g$ arises eventually by separation of convex sets. We emphasize that the paper [6] was written in a different methodological context. In distinction to our approach, the main purposes claimed in [6] were shelving linear duality, giving explicit formulas (even though in terms of the Brownian motion) for splitting a function into a "large" and a "small" part, etc.

Proof of the lemma. Let $I$ be an arc of the circle, and let $5 I$ be the arc with the same center and of length five times that of $I$. Suppose $\alpha_{1}, \alpha_{2} \in I$. We must verify the inequality

$$
\begin{aligned}
\int_{\theta \notin 5 I} & \left|\int_{0}^{1} \int_{-\pi}^{\pi}\left(K\left(\alpha_{1}, \theta\right)(r, t)-K\left(\alpha_{2}, \theta\right)(r, t)\right) h(r, t) \frac{\left(1-r^{2}\right) r}{\left|1-r e^{i t}\right|^{2}} \log \frac{1}{r} d t d r\right| d \theta \\
& \leq C\|h\|_{\mathcal{H}}
\end{aligned}
$$

for every $h \in \mathcal{H}$ with a constant $C$ independent of $h$. Bringing the modulus within the integral sign, changing the order of integration, and passing to the supremum over $h \in \mathcal{H}$ with $\|h\|_{\mathcal{H}} \leq 1$, we see that it suffices to verify the inequality

$$
\left(\int_{0}^{1} \int_{-\pi}^{\pi}\left(\int_{\theta \notin 5 I}\left|K\left(\alpha_{1}, \theta\right)(r, t)-K\left(\alpha_{2}, \theta\right)(r, t)\right| d \theta\right)^{2} \frac{\left(1-r^{2}\right) r}{\left|1-r e^{i t}\right|^{2}} \log \frac{1}{r} d t d r\right)^{1 / 2} \leq C
$$

It is quite easy to see that the domain $|r| \leq 1 / 2$ gives a bounded contribution to this integral. Thus, in what follows we restrict ourselves to the domain $1 / 2 \leq r \leq 1$. Then $\log \frac{1}{r} \asymp 1-r$, and we must prove that

$$
\begin{equation*}
\int_{\frac{1}{2}}^{1} \int_{-\pi}^{\pi}\left(\int_{\theta \notin 5 I}\left|K\left(\alpha_{1}, \theta\right)(r, t)-K\left(\alpha_{2}, \theta\right)(r, t)\right| d \theta\right)^{2} \frac{(1-r)^{2}}{\left|1-r e^{i t}\right|^{2}} d t d r \leq C^{2} \tag{5}
\end{equation*}
$$

Next, in (5) we split the domain of integration over $r$ and $t$ into three parts:

$$
\begin{aligned}
& \Omega_{1}=\{(r, t):|t| \leq|I|\}, \\
& \Omega_{2}=\{(r, t):|t|>|I|, 1-r>|I|\}, \\
& \Omega_{3}=\{(r, t):|t|>|I|, 1-r \leq|I|\} .
\end{aligned}
$$

In order to estimate the contribution of $\Omega_{3}$, we extend the integration in $\theta$ to the entire circle (i.e., to the interval $(-\pi, \pi)$ ), replace the modulus of the difference of the kernels by the sum of their moduli, and use the equivalence $\left|1-r e^{i \xi}\right|^{2}=(1-r)^{2}+2 r(1-\cos \xi) \asymp$ $(1-r+|\xi|)^{2}$ for $r \in\left[\frac{1}{2}, 1\right], \xi \in[-\pi, \pi]$. It follows that the integral in $\theta$ is dominated by

$$
C \int_{0}^{\pi} \frac{d \xi}{(1-r+\xi)^{2}} \leq C^{\prime} \frac{1}{1-r}
$$

whence the integral over $\Omega_{3}$ is dominated by $\int_{1-|I|}^{1} \int_{|t| \geq|I|} \frac{d t}{t^{2}} d r \asymp$ const.
When estimating the contributions of the domains $\bar{\Omega}_{1}$ and $\Omega_{2}$, we write

$$
K\left(\alpha_{2}, \theta\right)(r, t)-K\left(\alpha_{1}, \theta\right)(r, t)=\int_{\alpha_{1}}^{\alpha_{2}} \frac{d}{d \alpha} K(\alpha, \theta)(r, t) d \alpha
$$

The derivative under the integral sign does not exceed the quantity

$$
\begin{equation*}
\frac{\text { const }}{\left|1-r e^{i(t+\alpha-\theta)}\right|^{3}} \tag{6}
\end{equation*}
$$

Here $\alpha \in I$. If we are in the domain $\Omega_{1}$, then the argument of the imaginary exponential in (6) is bounded away from zero by the quantity $3|I|$. This leads to the following majorant for the integral over $\Omega_{1}$ :

$$
C \int_{\frac{1}{2}}^{1} \int_{|t| \leq|I|}\left[\left|\alpha_{1}-\alpha_{2}\right| \int_{|\xi| \geq 3|I|} \frac{d \xi}{(1-r+|\xi|)^{3}}\right]^{2} \frac{(1-r)^{2}}{(1-r+|t|)^{2}} d t d r
$$

We observe that

$$
\int_{|t| \leq|I|} \frac{d t}{(1-r+|t|)^{2}}=2\left(\frac{1}{1-r}-\frac{1}{1-r+|I|}\right)=2 \frac{|I|}{(1-r)(1-r+|I|)}
$$

and

$$
\int_{|\xi| \geq|I|} \frac{d \xi}{(1-r+|\xi|)^{3}} \leq C \frac{1}{(1-r+|I|)^{2}}
$$

Since also $\left|\alpha_{1}-\alpha_{2}\right| \leq|I|$, after substituting $s=1-r$ we are left with estimating the quantity

$$
|I|^{3} \int_{0}^{1 / 2} \frac{s d s}{(s+|I|)^{5}}
$$

from above. But

$$
\int_{0}^{1 / 2} \frac{s d s}{(s+|I|)^{5}} \asymp \int_{0}^{1 / 2} \frac{d s^{2}}{\left(s^{2}+|I|^{2}\right)^{5 / 2}} \leq\left. C \frac{1}{\left(\sigma+|I|^{2}\right)^{3 / 2}}\right|_{\sigma=(1 / 2)^{2}} ^{\sigma=0} \leq C \frac{1}{|I|^{3}}
$$

so that integration over $\Omega_{1}$ also yields a bounded contribution to (5).
It remains to treat the domain $\Omega_{2}$. Nothing definite can be said now about the argument of the imaginary exponential in (6), so that we again extend integration over
$\theta$ in (5) to the entire circle. Taking (6) into account, we arrive at the following majorant for the integral over $\Omega_{2}$ (we again substitute $s=1-r$ ):

$$
\begin{aligned}
& C \int_{s>|I|} \int_{|t|>|I|}\left[\left|\alpha_{1}-\alpha_{2}\right| \int_{0}^{\pi} \frac{d \xi}{(s+\xi)^{3}}\right]^{2} \frac{s^{2}}{(s+|t|)^{2}} d t d s \\
& \quad \leq C|I|^{2} \int_{s \geq|I|} \int_{|t|>|I|} \frac{d s d t}{s^{2}(s+|t|)^{2}} \leq C^{\prime}
\end{aligned}
$$

This finishes the proof of inequality (5) and, with it, of the lemma.
0.7. It is useful to briefly outline the arguments leading to the $K$-closedness theorem in Subsection 0.2 .4 because many of the details will be needed in the proofs of Theorems 1, 2 , and 3 .
0.7.1. Statement I in Subsection 0.2 .4 is proved first. The argument is due to Bourgain [1]. Let $1<r \leq p$. Then $Q$ is bounded on $L^{r}$, so that it suffices to prove that the couple ( $X_{1}, X_{r}$ ) is $K$-closed in $\left(L^{1}, L^{r}\right)$.

Suppose $f \in X_{1}+X_{r}$ and $f=g+h$ with $g \in L^{1}, h \in L^{r}$. We put $a=\|g\|_{1}$, $b=\|g\|_{r}$ and apply the Calderón-Zygmund decomposition procedure with parameter $\lambda$ : $g=g_{0}+g_{1}$, where $\left\|g_{0}(\cdot)\right\|_{A} \leq \lambda$ a.e., $\int\left\|g_{0}(\cdot)\right\|_{A} \leq C a$, and there exists a set $\Omega$ such that $|\Omega| \leq c \lambda^{-1} a$; moreover, for every Calderón-Zygmund operator $T$ we have

$$
\begin{equation*}
\int_{\Omega^{c}}\left\|T g_{1}(\cdot)\right\|_{A} \leq C^{\prime} a \tag{7}
\end{equation*}
$$

( $C^{\prime}$ depends only on $T$ ). See [16] (and also $\S 2$ below, where a more involved setting is treated). The required decomposition is $f=Q g_{1}+Q\left(g_{0}+h\right)$. The parameter $\lambda$ is chosen in such a way that the norm of $g_{0}$ in $L^{r}$ is of order $b$ :

$$
\int\|g(\cdot)\|_{A}^{r} \leq \lambda^{r-1} \int\|g(\cdot)\|_{A}=\lambda^{r-1} a
$$

so that it suffices to put $\lambda=\left(b^{r} a^{-1}\right)^{\frac{1}{r-1}}$. Then $\left\|Q\left(g_{0}+h\right)\right\|_{L^{r}(A)} \leq c b$, and it remains to show that $\int\left\|Q g_{1}(\cdot)\right\|_{A} \leq c a$. The integral over $\Omega^{c}$ is indeed dominated by ca; see (7). In order to estimate $\int_{\Omega}\|Q g(\cdot)\|_{A}$, we write $0=(I-Q) g_{1}+(I-Q)\left(g_{0}+h\right)$, whence

$$
\left\|Q g_{1}(\cdot)\right\|_{A} \leq\left\|g_{1}(\cdot)\right\|_{A}+\left\|(I-Q)\left(g_{0}+h\right)(\cdot)\right\|_{A}
$$

It suffices to integrate this inequality over $\Omega$ and employ the Hölder inequality to estimate the second summand:

$$
\int_{\Omega} \cdots \leq\left\|(I-Q)\left(g_{0}+h\right)\right\|_{r}|\Omega|^{1 / r^{\prime}}
$$

The remaining calculations are left to the reader.
0.7.2. Statement II in Subsection 0.2.4 follows from Statement I by duality. Namely, the following lemma is true.

Duality Lemma. Let $(X, Y)$ be an interpolation couple, and let $X \cap Y$ be dense both in $X$ and in $Y$. A subcouple $(E, F)$ is $K$-closed in $(X, Y)$ if and only if the couple $\left(E^{\perp}, F^{\perp}\right)$ of annihilators is $K$-closed in $\left(X^{*}, Y^{*}\right)$.

See [15] or the survey [11]. We note that the annihilator of $X_{r, Q}$ is $X_{r^{\prime}, I-Q^{*}}$ and that $Q^{*}$ and $I-Q^{*}$ can be Calderón-Zygmund operators only simultaneously.
0.7.3. Gluing scales. In Subsection 0.6 we already resorted to Wolff's theorem on gluing interpolation scales. A similar statement is true for $K$-closedness (i.e., not merely for interpolation formulas). The preservation of $K$-closedness under gluing scales was first established in [14, Theorem 2]; see also the survey [11] (and [12, Proposition 5] for a slight refinement).

Under the conditions of Statement III in Subsection 0.2.4, the $K$-closedness of the couple $\left(X_{1}, X_{\infty}\right)$ turns out to be a formal consequence of the $K$-closedness of the couples $\left(X_{1}, X_{r}\right)$ and $\left(X_{s}, X_{\infty}\right)$ with $s<r$ (in order that "gluing" be possible, the intervals $[1, r)$ and $(s, \infty]$ must overlap). We do not give a precise statement of the abstract result, though we shall resort to such gluing in what follows.
0.8. Theorem 1 is proved in $\S \S 1$ and 3 , Theorem 4 in $\S 2$, and Theorem 3 in $\S 5$. A stronger version of the correction theorem (i.e., of Theorem 2) is formulated in $\S 1$ and proved in $\S 4$. In the appendix at the end of the paper we collect some information about singular integrals.

## §1. Hardy-Littlewood SQuare function: interpolation

1.1. In view of the ideas outlined in Subsection 0.7.3, it suffices to prove Theorem 1 in two cases: (a) $1=r_{1}<r_{2}<\infty$ and (b) $1<r_{1}<r_{2}=\infty$. First, we treat the simpler case (a).

Lemma 1. For $1<s<\infty$ the couples $\left(\mathcal{Y}_{1}, \mathcal{Y}_{s}\right)$ and $\left(\mathcal{Y}_{1, A}, \mathcal{Y}_{s, A}\right)$ are $K$-closed in $\left(L^{1}\left(l^{2}\right)\right.$, $\left.L^{s}\left(l^{2}\right)\right)$.

Proof. The arguments for the scales $\mathcal{Y}_{s}$ and $\mathcal{Y}_{s, A}$ are quite similar, so we only consider the second case (also, a formal reduction of the first case to the second is possible). Let $F=\left\{f_{j}\right\}_{j \in \mathbb{Z}} \in L^{1}\left(l^{2}\right)$ with $f_{j}=0$ for $j<0$ and $f_{0} \in H^{1}$, and let $M_{s} f_{0}=f_{s}$ for $s \geq 1$. Suppose $f_{j}=g_{j}+h_{j}$, where $G=\left\{g_{j}\right\}_{j \in \mathbb{Z}} \in L^{1}\left(l^{2}\right)$ and $H=\left\{h_{j}\right\}_{j \in \mathbb{Z}} \in L^{s}\left(l^{2}\right)$. We denote by $a$ and $b$ the norms of the sequences $G$ and $H$ in their respective spaces.

We observe that it suffices to prove the following: for positive $j$ there exist representations $f_{j}=\alpha_{j}+\beta_{j}$ such that the Fourier transforms of $\alpha_{j}$ and $\beta_{j}$ are supported on the interval $I_{j}=\left[2^{j-1}-1,2^{j-1}-1\right)$ and

$$
\left\|\left(\sum_{j \geq 1}\left|\alpha_{j}\right|^{2}\right)^{1 / 2}\right\|_{1} \leq C a, \quad\left\|\left(\sum_{j \geq 1}\left|\beta_{j}\right|^{2}\right)^{1 / 2}\right\|_{s} \leq C b
$$

Indeed, suppose this is done. Then for $j<0$ we simply replace $g_{j}$ and $h_{j}$ by zeros, and for $j=0$ we write $f_{0}=\alpha_{0}+\beta_{0}$, where $\alpha_{0}=\sum_{j \geq 1} \alpha_{j}$ and $\beta_{0}=\sum_{j \geq 1} \beta_{j}$. The functions $\alpha_{0}$ and $\beta_{0}$ belong, respectively, to $H^{1}$ and $H^{s}$ and have norms of order $a$ and $b$ there - see Appendix, Statements II and III. Clearly, the relations $M_{s} \alpha_{0}=\alpha_{s}$ and $M_{s} \beta_{0}=\beta_{s}$ are fulfilled for $s \geq 1$.

The proof of the above claim is not difficult modulo some known facts (similar arguments can be found in [10]). It may be assumed from the outset that the functions $g_{j}$ and $h_{j}$ have no spectrum to the left of the point $2^{j-1}-1$. For instance, the latter is a consequence of the results of [13] concerning interpolation for Hardy spaces of vectorvalued functions (see [11] for simpler proofs). Another way to see this is to apply the $K$-closedness theorem in Subsection 0.2.4: take the Riesz projection on $L^{r}\left(l^{2}\right)$ for the role of $Q$, passing beforehand to the functions $\bar{z}^{2^{j-1}-1} f_{j}$.

Consider the functions $\sigma_{n}$ and $\tau_{n}$ whose Fourier transforms coincide with the restrictions to $\mathbb{Z}$ of the piecewise-linear functions with the graphs shown in Figure 1.



Figure 1.

Here the points $a_{n}$ and $b_{n}$ split the segment $\left[2^{n-1}-1,2^{n-1}-1\right]$ into 3 parts of equal length, and $c_{n}$ is the middle of the segment $\left[2^{n-2}-1,2^{n-1}-1\right]$. For $j \geq 1$ we have

$$
\begin{align*}
f_{j} & =f_{j} * \sigma_{j}+f_{j} * \tau_{j} \\
f_{j} * \sigma_{j} & =g_{j} * \sigma_{j}+h_{j} * \sigma_{j}  \tag{8}\\
f_{j} * \tau_{j} & =g_{j} * \tau_{j}+h_{j} * \tau_{j} \tag{9}
\end{align*}
$$

Since the operators $x=\left\{x_{j}\right\} \mapsto\left\{x_{j} * \sigma_{j}\right\}$ and $x \mapsto\left\{x_{j} * \tau_{j}\right\}$ are CZO's in spaces of $l^{2}$-valued functions (see the Appendix), the summands in (8) and (9) are of the right order (among other things, we use the fact that these operators take boundedly $H^{1}\left(l^{2}\right)$ into itself):

$$
\begin{array}{ll}
\left\|\left(\sum\left|g_{j} * \sigma_{j}\right|^{2}\right)^{1 / 2}\right\|_{1}, \quad\left\|\left(\sum\left|g_{j} * \tau_{j}\right|\right)^{1 / 2}\right\|_{1} \leq C a \\
\left\|\left(\sum\left|h_{j} * \sigma_{j}\right|^{2}\right)^{1 / 2}\right\|_{s}, \quad\left\|\left(\sum\left|g_{j} * \tau_{j}\right|\right)^{1 / 2}\right\|_{s} \leq C b
\end{array}
$$

In (9), we replace the summands on the right by functions with no spectrum to the right of $2^{n}-2$ with preservation of estimates (again by $K$-closedness in the scale $H^{p}\left(l^{2}\right)$ ), and then convolve the resulting decomposition with the function the graph of the Fourier transform of which is depicted in Figure 2.


Figure 2.

This yields the identity $f_{j} * \tau_{j}=\varphi_{j}+\psi_{j}$, where (see the Appendix) $\left\|\left(\sum\left|\varphi_{j}\right|^{2}\right)^{1 / 2}\right\|_{1} \leq$ $C a,\left\|\left(\sum\left|\psi_{j}\right|^{2}\right)^{1 / 2}\right\|_{s} \leq C b$, and, moreover, the functions $\varphi_{j}$ and $\psi_{j}$ have spectrum only on the interval $\left[2^{j-1}-1,2^{j}-2\right]$. Adding this identity to (8), we prove the claim.
1.2. $K$-closedness with weight, and a refinement of the correction theorem. In order to finish the proof of Theorem 1 , now we need to verify that the couples $\left(\mathcal{Y}_{s}, \mathcal{Y}_{\infty}\right)$ and $\left(\mathcal{Y}_{s, A}, \mathcal{Y}_{\infty, A}\right)$ are $K$-closed in $\left(L^{s}\left(l^{2}\right), L^{\infty}\left(l^{2}\right)\right)$ for $s>1$. Again, we restrict ourselves to "analytic" spaces $\mathcal{Y}_{s, A}$.

In order to prove the correction theorem, a weighted analog of the $K$-closedness statement will be needed. Concerning the Muckenhoupt conditions $A_{p}$, we refer the reader to [5, 2]. Often we denote by $A_{p}$ also the class of all weights satisfying condition $A_{p}$. We remind the reader that $A_{\infty}=\bigcup_{p<\infty} A_{p}$ and that any $A_{\infty}$-weight $a$ satisfies the reverse Hölder inequality: there exists $q=q(a)>1$ such that $\left(|I|^{-1} \int_{I} a^{q}\right)^{1 / q} \leq C_{q}(a)|I|^{-1} \int_{I} a$ for any ball $I$ of the basic measure space (for any arc $I$ in the case of the unit circle). The logarithmic convexity of $L^{r}$-norms implies that then we also have $|I|^{-1} \int_{I} a \leq$ $C_{\alpha, a}\left(|I|^{-1} \int_{I} a^{\alpha}\right)^{1 / \alpha}$ for $\alpha<1$. We mention the recent paper [17], where various parameters related to these features were calculated exactly in terms of the $A_{\infty}$ (and $A_{p}$ ) constants in the one-dimensional situation.

We shall also need the P. Jones factorization theorem: $a \in A_{p}$ if and only if $a=a_{1} a_{2}^{1-p}$ with $a_{1}, a_{2} \in A_{1}$; see [5].

Lemma 2. Suppose $w \in A_{1}, a \in A_{\infty}$. There exists a number $r=r(w, a)>1$ such that $a, a w^{-s} \in A_{s}$ for $s \geq r$ (equivalently, $a^{-\frac{1}{s-1}},\left(a w^{-s}\right)^{-\frac{1}{s-1}}=\frac{w^{s^{\prime}}}{a^{s^{\prime}-1}} \in A_{s^{\prime}}$ for $1<s^{\prime} \leq r^{\prime}$; incidentally, the latter inclusion is trivially true also for $s^{\prime}=1$ ).

Proof. Since $a \in A_{\infty}$, we have $a \in A_{p}$ for some $p<\infty$, and then $a=a_{1} a_{2}^{1-p}$ with some $a_{1}, a_{2} \in A_{1}$. Condition $A_{s}$ (to be verified) for the weight $a w^{-s}$ looks like this:

$$
\begin{equation*}
\left(\frac{1}{|I|} \int_{I} \frac{a_{1}}{a_{2}^{p-1}} \frac{1}{w^{s}}\right)\left(\frac{1}{|I|} \int_{I} \frac{a_{2}^{\frac{p-1}{s-1}}}{a_{1}^{\frac{1}{s-1}}} w^{\frac{s}{s-1}}\right)^{s-1} \leq C \tag{10}
\end{equation*}
$$

where $C$ does not depend on $I$. Condition $A_{1}$ for $a_{1}, a_{2}$, and $w$ implies

$$
\frac{1}{a_{j}} \leq c \frac{|I|}{\int_{I} a_{j}}, \quad j=1,2 ; \quad \frac{1}{w} \leq c \frac{|I|}{\int_{I} w} \text { on } I .
$$

These estimates show that the left-hand side of (10) does not exceed the quantity

$$
\left(\frac{|I|}{\int_{I} a_{2}}\right)^{p-1}\left(\frac{|I|}{\int_{I} w}\right)^{s}\left(\frac{1}{|I|} \int_{I} a_{2}^{\frac{p-1}{s-1}} w^{\frac{s}{s-1}}\right)^{s-1}
$$

To the third factor, we apply the Hölder inequality with the exponents $\gamma=\frac{s-1}{p-1}(\gamma>1$ if $s>p$ ) and $\gamma^{\prime}=\frac{s-1}{s-p}$. This shows that it suffices to estimate the expression

$$
\left(\left[\frac{1}{|I|} \int_{I} w\right]^{-1}\left[\frac{1}{|I|} \int_{I} w^{\frac{s}{s-p}}\right]^{\frac{s-p}{s}}\right)^{s}
$$

from above by a constant independent of $I$. But this estimate is none other than the reverse Hölder inequality for $w$ : it is true if $\frac{s}{s-p} \leq q(w)$, i.e., $s \geq q(w)^{\prime} p$. Since $s>p$ under this condition a fortiori, we can take $r(w, a)=q(w)^{\prime} p$.

Now we state a weighted result on $K$-closedness in a neighborhood of infinity. The definition of the weighted spaces $\mathcal{Y}_{s, A}(u)$ of $l^{2}$-valued functions is clear without explanations if we know what the (scalar) spaces $L^{p}(u)$ are. We simply indicate the norms in
the latter spaces: $\|f\|_{p, u}=\left(\int|f|^{p} u\right)^{1 / p}$ for $p<\infty$, but $\|f\|_{\infty, u}=\left\|f u^{-1}\right\|_{\infty}$. (Thus, the weighted scale has a "discontinuity" at $\infty$; however, these definitions are convenient.)

Lemma 3. Suppose $w$ and $a$ are two weights on the circle, $w \in A_{1}, a \in A_{\infty}$. Put $v_{s}=$ $a w^{-s}$ for $s \geq r(w, a)$. Then the couple $\left(\mathcal{Y}_{s, A}\left(v_{s}\right), \mathcal{Y}_{\infty, A}(w)\right)$ is $K$-closed in $\left(L^{s}\left(l^{2}, v_{s}\right)\right.$, $L^{\infty}\left(l^{2}, w\right)$ ) (the corresponding constant depends only on $s$, the $A_{1}$-constant for $w$, the number $p$ for which $a \in A_{p}$, and the $A_{p}$-constant for $\left.a\right)$.

If the two weights are identically equal to 1 , then the proof of Lemma 3 will work for $s>1$ (consequently, taken together, Lemmas 3 and 1 yield Theorem 1). The proof of Lemma 3 will be postponed until $\S 3$ because Theorem 4 must be verified first (this will be done in $\S 2$ ). A precise analog of Lemma 3 is valid also for the spaces $\mathcal{Y}_{s-}=\left\{f \in \mathcal{Y}_{s}\right.$ : $(I-\mathbb{P}) f=f\}$ (it suffices to consider the transformation $f \mapsto \overline{f(\bar{z})}$ ). In $\S 4$ we shall deduce the following refinement of Theorem 2 by using Lemma 3 . Put $\mathbb{P}_{-}=I-\mathbb{P}, \mathbb{P}_{+}=\mathbb{P}$.

Theorem $2^{\prime}$. Suppose $a$ and $w$ are two weights on the unit circle, $w \in A_{1}, a \in A_{\infty}$, and $s \geq r(w, a)$. If $f$ is a measurable function, $|f| \leq w$, and $0<\varepsilon<1$, then there exists a function $g$ with

$$
\int_{\{f \neq g\}} a \leq \varepsilon \int_{\mathbb{T}}\left|\frac{f}{w}\right|^{s} a \quad \text { and } \quad|g|,\left|\mathbb{P}_{ \pm g} g\right|, \sigma\left(\mathbb{P}_{ \pm} g\right) \leq C(w, a, s)\left(1+\log \frac{1}{\varepsilon}\right)^{2} w
$$

Theorem $2^{\prime}$ suggests how the weight $v_{s}$ arose in Lemma 3: $\|f\|_{s, v_{s}}=\left(\int\left|\frac{f}{w}\right|^{s} a\right)^{1 / s}$. We can tell immediately that the proof of Theorem $2^{\prime}$ consists in successive "careful splitting" of the given function in the scale determined by $w$ (i.e., $K$-closedness is employed instead of crude truncation; see $\S 4$ for the details).

Surely, in applications it is natural to start with $f$ and to choose appropriate $w$ and $a$ afterwards. For instance, let $|f| \leq 1$. Taking $w=a \equiv 1$, we obtain a version incomparable with Theorem 2. (Incidentally, in this case, i.e., if the two weights are identically equal to 1 , the factor $\left(1+\log \frac{1}{\varepsilon}\right)^{2}$ can be replaced with $1+\log \frac{1}{\varepsilon}-$ see $\S 4$ for the explanation. We push aside the question as to whether this carries over to more general weights.) Another option is in taking the weight $a$ arbitrarily and putting $w=\left(f^{*}\right)^{\gamma}$, $0<\gamma<1$ (we recall that $f^{*}$ is the Hardy-Littlewood maximal function). Then $w \in A_{1}$; see [5]. Next, $\int\left|\frac{f}{w}\right|^{s} a \leq \int|f|^{s(1-\gamma)} a \leq \int|f| a$ if $s \geq(1-\gamma)^{-1}$. Putting $a \equiv 1$, we arrive at Theorem 2. Apparently, the possibility of measuring the set of difference between the original and corrected functions in terms of an arbitrary $A_{\infty}$-weight may also be of some use.

Remark. It should be noted that the correction theorem cited in Subsection 0.3 also admits a weighted version: if $w \in A_{1}$ and $a \in A_{\infty}$, then for $|f| \leq w$ and arbitrary $\varepsilon \in(0,1)$ there exists a function $\varphi$ such that $0 \leq \varphi \leq 1, \int_{\{\varphi \neq 1\}} a \leq \varepsilon \int\left|\frac{f}{w}\right| a$, and $\|T(\varphi f)\|_{\infty} \leq C(T, w, a)\left(1+\log \frac{1}{\varepsilon}\right) w$. This statement follows from the results of 9$]$ with the help of Theorem 4.

## §2. Calderón-Zygmund decomposition and weak type $(1,1)$

In this section we prove Theorem 4. All constructions are applicable to CZO's on vector-valued functions, but this will not be reflected notationally (however, the reader may interpret the modulus sign as the norm calculated pointwise).

The $L^{t}(a)$-boundedness of the operator $R$ in Lemma 4 for $1<t \leq r(w, a)^{\prime}$ follows from Lemma 2:

$$
\int|R f|^{t} a=\int|T(u f)|^{t} \frac{w^{t}}{a^{t-1}}
$$

and the weight $\frac{w^{t}}{a^{t-1}}$ satisfies $A_{t}$ for the $t$ in question.

To show that $R$ is of weak type $(1,1)$ on $L^{1}(a)$, we resort to the Calderón-Zygmund procedure in a somewhat nonstandard form. The same procedure will be used once again in the proof of Lemma 3. Recall that $u=\frac{a}{w}$.

Let $f \in L^{1}(a)$; then $f=G u^{-1}$, where $G \in L^{1}(w)$ and $\|G\|_{1, w}=\|f\|_{1, a}$. The weight $a$ satisfies a doubling condition (as any $A_{\infty}$-weight does), and so the Calderón-Zygmund construction applies to the function $f=G u^{-1}$ with weight $a$. For every $\lambda>0$ this construction yields a family $\mathcal{A}$ of essentially disjoint cubes $I$ (of arcs in the case of the circle) such that

$$
\begin{equation*}
\lambda \leq \frac{1}{\int_{I} a} \int_{I}\left|G u^{-1}\right| a \leq C \lambda, \quad I \in \mathcal{A} \tag{11}
\end{equation*}
$$

and $\left|G u^{-1}\right| \leq \lambda$ almost everywhere off $\bigcup_{I \in \mathcal{A}} I$.
We need a lemma, which can also be formulated as follows: $w \in A_{1}(u)$.
Lemma 4. For every cube I we have

$$
\frac{1}{\int_{I} u} \int_{I} u w \leq C \operatorname{ess} \inf _{I} w
$$

where $C$ does not depend on $I$.
Proof. Using the reverse Hölder inequality for $a$ followed by the Cauchy inequality and condition $A_{1}$ for $w$, we see that

$$
\begin{aligned}
& \frac{1}{|I|} \int_{I} u w=\frac{1}{|I|} \int_{I} a \leq C\left(\frac{1}{|I|} \int_{I} a^{1 / 2}\right)^{2} \leq C\left(\frac{1}{|I|} \int_{I} \frac{a}{w}\right)\left(\frac{1}{|I|} \int_{I} w\right) \\
& \quad \leq C^{\prime} \frac{1}{|I|} \int_{I} u \cdot \operatorname{essinf}_{I} w
\end{aligned}
$$

Inequality (11) and Lemma 4 imply

$$
\begin{equation*}
C \lambda \geq \frac{1}{\int_{I} a} \int_{I}|G| w \geq c \frac{1}{\int_{I} a} \frac{\int_{I} u w}{\int_{I} u} \int_{I}|G|=c \frac{1}{\int_{I} u} \int_{I}|G|, \quad I \in \mathcal{A} . \tag{12}
\end{equation*}
$$

Let $5 I$ be the cube with the same center as $I$ but of diameter five times that of $I$. We put $\Omega=\bigcup_{I \in \mathcal{A}} 5 I$ and define

$$
G^{0}(t)= \begin{cases}G(t) & \text { if } t \notin \bigcup_{I \in \mathcal{A}} I  \tag{13}\\ \frac{u}{\int_{I} u} \int_{I} G & \text { if } t \in I \in \mathcal{A}\end{cases}
$$

$G^{1}=G-G^{0}$. We mention several properties of the objects constructed above.
$1^{\circ}\left|G^{0}\right| \leq C \lambda u($ see (12)).
$2^{\circ}$ If $I \in \mathcal{A}$, then by Lemma 4 (more precisely, by an inequality in the chain (12)), we have

$$
\int_{I}\left|G^{0}\right| w \leq \frac{\int_{I} u w}{\int_{I} u} \int_{I}|G| \leq C^{\prime} \int_{I}|G| w
$$

(and then also $\left.\int_{I}\left|G^{1}\right| w \leq C^{\prime \prime} \int_{I}|G| w\right)$; consequently,

$$
\begin{aligned}
\left\|G^{0}\right\|_{L^{1}(w)} & \leq C\|G\|_{L^{1}(w)}
\end{aligned}=C\|f\|_{L^{1}(a)}, ~=~\left(G^{1}\left\|_{L^{1}(w)} \leq C\right\| G\left\|_{L^{1}(w)}=C\right\| f \|_{L^{1}(a)} .\right.
$$

$3^{\circ} \int_{I} G_{1}=\int_{I} G-\int_{I}\left[\frac{u}{\int_{I} u} \int_{I} G\right]=0$ for $I \in \mathcal{A}$.
$4^{\circ} \int_{I} a \leq \frac{1}{\lambda} \int_{I}|G| w$ for $I \in \mathcal{A}$ (see (11)), whence $\int_{\Omega} a \leq \frac{C}{\lambda} \int|G| w$ by the doubling condition for $a$.

After this, the proof of Lemma 4 finishes in a nearly standard way. Indeed,

$$
\begin{align*}
& \left\|\frac{1}{u} T G^{0}\right\|_{L^{t}(a)}=\left\|T G^{0}\right\|_{L^{t}\left(\frac{w^{t}}{a^{t-1}}\right)} \leq C\left\|G^{0}\right\|_{L^{t}\left(\frac{w^{t}}{a^{t-1}}\right)} \\
& \quad=C\left\|\frac{1}{u} G^{0}\right\|_{L^{t}(a)} \leq C^{\prime}\left(\lambda^{t-1}\|f\|_{L^{1}(a)}\right)^{1 / t} \tag{14}
\end{align*}
$$

by $1^{\circ}$ (the constant $C^{\prime}$ depends on $t$ only, but now we fix some particular $\left.t \in\left(1, r(w, a)^{\prime}\right)\right)$. Therefore,

$$
\begin{equation*}
\int_{\left\{\left|\frac{1}{u} T G^{0}\right|>\lambda\right\}} a \leq C^{\prime} \lambda^{-1} \int|f| a . \tag{15}
\end{equation*}
$$

Next, let $J=\int_{\Omega^{c}}\left|\frac{1}{u} T G^{1}\right| a$. By $3^{\circ}$, for $x \in \Omega^{c}$ we have

$$
T G^{1}(x)=\sum_{I \in \mathcal{A}} \int_{I}\left[K(x, y)-K\left(x, y_{I}\right)\right] G_{1}(y)
$$

where $y_{I}$ is an arbitrary fixed point of $I$ and $K$ is the kernel of $T$. Condition (c3) on $K$ implies the following standard estimate ( $w^{*}$ is the Hardy-Littlewood maximal function for $w$ ):

$$
\begin{align*}
J & \leq C \sum_{I \in \mathcal{A}} \int_{x \notin 5 I} \frac{(\operatorname{diam} I)^{\alpha}}{\left|x-y_{I}\right|^{n+\alpha}} w(x) d x \int_{I}\left|G^{1}\right| \leq C^{\prime} \sum_{I \in \mathcal{A}} w^{*}\left(y_{I}\right) \int_{I}\left|G^{1}\right|  \tag{16}\\
& \leq C^{\prime \prime} \sum_{I} \int_{I}\left|G^{1}\right| w \leq C\|f\|_{1, a}
\end{align*}
$$

(we have used the condition $A_{1}$ for $w$ and property $2^{\circ}$ ). Now, (15), (16), and $4^{\circ}$ imply the theorem because $\left\{\left|\frac{1}{u} T G\right|>2 \lambda\right\} \subset\left\{\left|\frac{1}{u} T G^{0}\right|>\lambda\right\} \cup \Omega \cup\left(\Omega^{c} \cap\left\{\left|\frac{1}{u} T G^{1}\right|>\lambda\right\}\right)$.
Corollary. If $w \in A_{1}, a \in A_{\infty}$, and an operator $\Gamma$ has the property that both $\Gamma$ and $\Gamma^{*}$ are CZO's (the smoothness condition for the kernel is fulfilled in the form (c3)), then

$$
\|\Gamma\|_{L^{s}\left(a w^{-s}\right)}=O(s)
$$

for large s (the constant in the estimate depends only on a and w; spaces of vector-valued functions are admissible).

This is a dual form of the corollary to Theorem 4 in Subsection 0.5.
Remark. If $T$ is a projection, then the Calderón-Zygmund decomposition described above makes it possible to do the construction of Subsection 0.7.1 for weighted spaces. Some details will change (see $\S 3$ for a similar argument in a more involved case). This leads to the following statement: the couple $\left(X_{1, w}, X_{t, v}\right)$ is $K$-closed in $\left(L^{1}(w), L^{t}(v)\right)$ if $w \in A_{1}$, $v=\frac{w^{t}}{a^{t-1}}$ with $a \in A_{\infty}$, and $1<t \leq r(w, a)^{\prime}$.

It is not clear whether it is possible to prove weighted $K$-closedness under the only assumption $w \in A_{1}, v \in A_{t}$. In any case, this cannot be done by the method outlined above. For instance, if $w \in A_{1}, v \equiv 1$, and $t=2$, this method would require that $w^{2} \in A_{\infty}$, which, surely, may fail (the square of an $A_{1}$-weight may even fail to be locally integrable).

## §3. Proof of Lemma 3

By the duality lemma in Subsection 0.7.2, Lemma 3 reduces to the claim that the couple of annihilators is $K$-closed. Always (even if a weight is involved), we introduce duality with the help of the nonweighted form $\langle f, g\rangle=\int f \bar{g}$ (the sesquilinear version is convenient because the operators $M_{j}$ become "selfadjoint"). We note that, under
this duality and with our definition of $L^{\infty}(\varkappa)$ (see the paragraph before Lemma 3), for every weight $\varkappa$ we have $L^{1}(\varkappa)^{*}=L^{\infty}(\varkappa)$. Presently, it is natural to agree that the symbol $L^{s}\left(l^{2}, \varkappa\right)\left(\varkappa\right.$ is a weight) stands for the space of one-sided sequences $\left\{f_{j}\right\}_{j \geq 0}$, because if $\left\{g_{j}\right\} \in \mathcal{Y}_{s, A}(\varkappa)$, then $g_{j}=0$ for $j<0$. The annihilator of $\mathcal{Y}_{p, A}(\varkappa)$ is then described by the condition $M_{j} f_{0}=-M_{j} f_{j}, j \geq 1$ (i.e., $f_{j}$ and $f_{0}$ must have opposite Fourier coefficients with indices in $\left[2^{j-1}-1,2^{j}-1\right)$ ). Next, $L^{\infty}\left(l^{2}, w\right)=L^{1}\left(l^{2}, w\right)^{*}$ and $L^{s}\left(l^{2}, v_{s}\right)=L^{t}\left(l^{2}, b_{t}\right)^{*}$, where $t=s^{\prime}$ and $b_{t}=\frac{w^{t}}{a^{t-1}}$. So, we must prove the following statement.

Lemma 3 (dualized). Suppose $\left\{f_{j}\right\}_{j \geq 0} \in L^{1}\left(l^{2}, w\right)+L^{t}\left(l^{2}, b_{t}\right)$, and $M_{j} f_{j}=-M_{j} f_{0}$ for $j \geq 1$. Suppose also that $f_{j}=g_{j}+h_{j}$, where $\left\|\left\{g_{j}\right\}\right\|_{L^{1}\left(l^{2}, w\right)} \leq \xi,\left\|\left\{h_{j}\right\}\right\|_{L^{t}\left(l^{2}, a\right)} \leq \eta$. Then the functions $g_{j}$ and $h_{j}$ can be modified to satisfy the relations $M_{j} g_{j}=-M_{j} g_{0}$ and $M_{j} h_{j}=-M_{j} h_{0}$ for $j \geq 1$ with preservation of the identity $f_{j}=g_{j}+h_{j}$ and of the above estimates, up to a multiplicative constant.

Proof. (Compare with the proof of the theorem about the two-dimensional torus in [14].) As in Theorem 4, we put $u=\frac{a}{w}$. Denote by $G$ the $l^{2}$-valued function $\zeta \mapsto\left\{g_{j}(\zeta)\right\}$ and construct a Calderón-Zygmund decomposition for $G u^{-1}$ and the weight $a$ as in $\S 2$ (the parameter $\lambda$ will be chosen shortly). We use the formula $G=G^{0}+G^{1}$ (see (12)); now the summands are $l^{2}$-valued functions: $G^{0}=\left\{g_{j}^{0}\right\}_{j \geq 0}, G^{1}=\left\{g_{j}^{1}\right\}_{j \geq 0}$. Thus,

$$
f_{j}=g_{j}^{1}+\left(g_{j}^{0}+h_{j}\right), \quad j \geq 0 ; \quad M_{j} f_{j}=-M_{j} f_{0}, \quad j \geq 1
$$

Consider again the functions $\sigma_{j}$ and $\tau_{j}$, the graphs of the Fourier transforms of which are depicted in Figure 1. We note that $\widehat{\sigma}_{j}+\widehat{\tau}_{j}=1$ on $\left[2^{j-1}-1,2^{j}-1\right]$. Clearly, for $j \geq 1$ we have

$$
\begin{aligned}
x_{j} & \stackrel{\text { def }}{=} \sigma_{j} *\left(f_{j}+f_{0}\right)=\sigma_{j} *\left(g_{j}^{1}+g_{0}^{1}\right)+\sigma_{j} *\left(g_{j}^{0}+g_{0}^{0}+h_{j}+h_{0}\right), \\
y_{j} & \stackrel{\text { def }}{=} \tau_{j} *\left(f_{j}+f_{0}\right)=\tau_{j} *\left(g_{j}^{1}+g_{0}^{1}\right)+\tau_{j} *\left(g_{j}^{0}+g_{0}^{0}+h_{j}+h_{0}\right)
\end{aligned}
$$

Since $M_{j}\left(f_{j}+f_{0}\right)=0$, we see that $x_{j}$ has no spectrum (nonstrictly) to the right of $2^{j-1}$, and $y_{j}$ has no spectrum (strictly) to the left of $2^{j}-1$; in particular, $M_{j} x_{j}=M_{j} y_{j}=0$.

Consider two functions $\Phi$ and $\Psi$, where the former is antianalytic and the latter is analytic. (We mean that, in fact, $\Phi$ and $\Psi$ are defined on the circle, but the former has no spectrum in $\mathbb{N}$, and the latter has no spectrum in $-\mathbb{N}$.) A specific choice of these functions will be postponed for a while. We introduce the corrections

$$
\begin{aligned}
\varphi_{j} & =\Phi x_{j}-\sigma_{j} *\left(g_{j}^{0}+g_{0}^{0}+h_{j}+h_{0}\right) \\
\psi_{j} & =\Psi y_{j}-\tau_{j} *\left(g_{j}^{0}+g_{0}^{0}+h_{j}+h_{0}\right)
\end{aligned}
$$

Since $\Phi x_{j}$ has no spectrum to the left of $2^{j-1}-1$, for $j \geq 1$ we obtain $M_{j} \varphi_{j}=$ $-M_{j}\left(\sigma_{j} *\left(g_{j}^{0}+g_{0}^{0}+h_{j}+h_{0}\right)\right)=M_{j}\left(\sigma_{j} *\left(g_{j}^{1}+g_{0}^{1}\right)\right) ;$ similarly, $M_{j} \psi_{j}=M_{j}\left(\tau_{j} *\left(g_{j}^{2}+g_{0}^{1}\right)\right)$. Therefore, $M_{j}\left(\varphi_{j}+\psi_{j}\right)=M_{j}\left(g_{j}^{1}+g_{0}^{1}\right), j \geq 1$. Now we modify the original decomposition of $f_{j}$ in the following way:

$$
f_{0}=g_{0}^{1}+\left(g_{0}^{0}+h_{0}\right)
$$

and

$$
f_{j}=\left(g_{j}^{1}-\varphi_{j}-\psi_{j}\right)+\left(g_{j}^{0}+h_{j}+\varphi_{j}+\psi_{j}\right)
$$

for $j \geq 1$. From the preceding discussion it is clear that the spectral condition $M_{j} g_{0}^{1}=$ $-M_{j}\left(g_{j}^{1}-\varphi_{j}-\psi_{j}\right), j \geq 1$ is fulfilled (and, with it, also a similar condition for the second summands), so that it remains to ensure the required estimates. For this, in particular, we must specify the functions $\Phi$ and $\Psi$ and the parameter $\lambda$. We split the arguments into several steps.

In the proof, we shall repeatedly use the operators $\gamma \mapsto\left\{\sigma_{j} * \gamma\right\},\{\gamma\} \mapsto\left\{\tau_{j} * \gamma\right\}$ and $\left\{\gamma_{j}\right\} \mapsto\left\{\sigma_{j} * \gamma_{j}\right\},\left\{\gamma_{j}\right\} \mapsto\left\{\tau_{j} * \gamma_{j}\right\}$ (which map scalar functions to vector-valued or vector-valued functions to vector-valued). These operators and their conjugates are CZO's (see the Appendix).
(i) We have $\left\|\left(\sum_{j \geq 0}\left|g_{j}^{1}\right|^{2}\right)^{1 / 2}\right\|_{L^{1}(w)} \leq C \xi$ by property $2^{\circ}$ in $\S 2$.
(ii) Properties $1^{\circ}$ and $2^{\circ}$ in the same section imply

$$
\begin{aligned}
& \left\|\left(\sum_{j \geq 0}\left|g_{j}^{0}\right|^{2}\right)^{1 / 2}\right\|_{L^{t}\left(b_{t}\right)} \\
& \quad \leq C\left(\int_{\mathbb{T}} \lambda^{t-1} u^{t-1}\left(\sum_{j \geq 0}\left|g_{j}^{0}\right|^{2}\right)^{1 / 2} \frac{w^{t}}{a^{t-1}}\right)^{1 / t}=C \lambda^{\frac{t-1}{t}}\left(\int_{\mathbb{T}}\left(\sum_{j \geq 0}\left|g_{j}^{0}\right|^{2}\right)^{1 / 2} w\right)^{1 / t} \\
& \quad \leq C^{\prime} \lambda^{\frac{t-1}{t}} \xi^{1 / t}
\end{aligned}
$$

Therefore, if $\lambda^{\frac{t-1}{t}} \xi^{\frac{1}{t}}=\eta$, i.e., $\lambda=\eta^{s} \xi^{\frac{1}{1-t}}\left(s=t^{\prime}\right)$, then the sequence $\left\{g_{j}^{0}\right\}_{j \geq 0}$ has norm of order $\eta$ in $L^{t}\left(b_{t}\right)$ (i.e., roughly the same as the sequence $\left\{h_{j}\right\}_{j \geq 0}$ ). Thus, we have fixed the parameter $\lambda$.
(iii) It remains to choose $\Phi$ and $\Psi$ in such a way that $\left\|\left\{\varphi_{j}\right\}\right\|_{L^{1}\left(l^{2}, w\right)},\left\|\left\{\psi_{j}\right\}\right\|_{L^{1}\left(l^{2}, w\right)} \leq$ $C \xi$ and, at the same time, $\left\|\left\{\varphi_{j}\right\}\right\|_{L^{t}\left(l^{2}, b_{t}\right)},\left\|\left\{\psi_{j}\right\}\right\|_{L^{t}\left(l^{2}, b_{t}\right)} \leq C \eta$. We only show how to choose $\Psi$ (the choice of $\Phi$ is similar). Put

$$
\begin{aligned}
& \alpha=\max \left\{1, u^{-1} \xi^{\frac{1}{t-1}} \eta^{-s}\left(\sum_{j \geq 1}\left|\tau_{j} *\left(g_{j}^{1}+g_{0}^{1}\right)\right|^{2}\right)^{1 / 2}\right\} \\
& \Psi=\exp (-\log \alpha-i \mathcal{H}(\log \alpha))
\end{aligned}
$$

where $\mathcal{H}$ stands for the harmonic conjugation operator. We have

$$
\begin{equation*}
\psi_{j}=\Psi \cdot\left(\tau_{j} *\left(g_{j}^{1}+g_{0}^{1}\right)\right)-(1-\Psi)\left(\tau_{j} *\left(g_{j}^{0}+g_{0}^{0}+h_{j}+h_{0}\right)\right) \tag{17}
\end{equation*}
$$

We begin with estimating the norm $\left\|\left\{\psi_{j}\right\}\right\|_{L^{t}\left(l^{2}, b_{t}\right)}$. Since $|\Psi| \leq 1$, the $L^{t}\left(l^{2}, b_{t}\right)$-norm of the sequence (with index $j$ ) of the second summands in (17) is dominated by $C \eta$. By the definition of $\Psi$, we have

$$
\begin{aligned}
& W \stackrel{\text { def }}{=}\left(\sum_{j \geq 1}\left|\Psi\left(\tau_{j} *\left(g_{j}^{1}+g_{0}^{1}\right)\right)\right|^{2}\right)^{1 / 2} \\
& \quad \leq \min \left\{\eta^{s} \xi^{\frac{1}{1-t}} u,\left(\sum_{j \geq 1}\left|\tau_{j} *\left(g_{j}^{1}+g_{0}^{1}\right)\right|^{2}\right)^{1 / 2}\right\} .
\end{aligned}
$$

We introduce the notation $|e|_{a}=\int_{e} a$. Since the operators $\gamma \mapsto\left\{\tau_{j} * \gamma\right\}$ and $\left\{\gamma_{j}\right\} \mapsto$ $\left\{\tau_{j} * \gamma_{j}\right\}$ are CZO's together with their conjugates, Theorem 4 implies

$$
\rho(\omega) \stackrel{\text { def }}{=}\left|\left\{\frac{1}{u} W>\omega\right\}\right|_{a} \leq \frac{c \xi}{\omega}
$$

for $\omega \leq \eta^{s} \xi^{\frac{1}{1-t}}$, and otherwise $\rho(\omega)=0$. Therefore,

$$
\begin{aligned}
\|W\|_{L^{t}\left(b_{t}\right)}^{t} & =\left\|\frac{1}{u} W\right\|_{L^{t}(a)}^{t}=t \int_{0}^{\eta^{s} \xi^{1 /(1-t)}} \omega^{t-1} \rho(\omega) d \omega \leq C \frac{t}{t-1} \xi\left(\eta^{s} \xi^{\frac{1}{1-t}}\right)^{t-1} \\
& \leq C^{\prime} \eta^{t}
\end{aligned}
$$

as required.

It remains to estimate the norm $\left\|\left\{\psi_{j}\right\}\right\|_{L^{1}\left(l^{2}, w\right)}$. By (17) (and the definition of the set $\Omega$ in $\S 2$ ), we see that

$$
\begin{aligned}
& \int\left(\sum\left|\psi_{j}\right|^{2}\right)^{1 / 2} w \\
& \quad \leq \int_{\mathbb{T} \backslash \Omega}\left(\sum\left|\tau_{j} *\left(g_{j}^{1}+g_{0}^{1}\right)\right|^{2}\right)^{1 / 2} w+\eta^{s} \xi^{\frac{1}{1-t}} \int_{\Omega} u w \\
& \quad \quad+\|1-\Psi\|_{L^{s}(a)}\left\|\left(\sum\left|\tau_{j} *\left(g_{j}^{0}+g_{0}^{0}+h_{j}+h_{0}\right)\right|^{2}\right)^{1 / 2}\right\|_{L^{t}\left(b_{t}\right)}
\end{aligned}
$$

Since $u w=a$, from (16) and property $4^{\circ}$ in $\S 2$ we deduce that the first two summands on the right-hand side of this inequality are of order $\xi$. In the third summand, the second factor is of order $\eta$ (because $b_{t} \in A_{t}$ ). Consequently,

$$
\int\left(\sum\left|\psi_{j}\right|^{2}\right)^{1 / 2} w \leq C\left(\xi+\|1-\Psi\|_{L^{s}(a)} \eta\right)
$$

Next, $\|1-\Psi\|_{L^{s}(a)} \leq C\left(\|\log x\|_{L^{s}(a)}+\|\mathcal{H}(\log \alpha)\|_{L^{s}(a)}\right)$. Since $s \geq r(w, a)$, we have $a \in A_{s}$ (see Lemma 2); therefore $\|1-\Psi\|_{L^{s}(a)} \leq C^{\prime}\|\log \alpha\|_{L^{s}(a)}$. Putting

$$
\left.E=\left\{\sum_{j \geq 1}\left|\tau_{j} *\left(g_{j}^{1}+g_{0}^{1}\right)\right|^{2}\right)^{1 / 2} u^{-1} \geq \eta^{s} \xi^{\frac{1}{1-t}}\right\}
$$

we observe that $\log \alpha=0$ on the complement of $E$. For the distribution function $\pi(\omega)=\int_{\{\zeta \in E:|\log \alpha(\zeta)|>\omega\}} a$, we have

$$
\begin{aligned}
\pi(\omega) & \leq\left|E \cap\left\{\left(\sum_{j \geq 1}\left|\tau_{j} *\left(g_{j}^{1}+g_{0}^{1}\right)\right|^{2}\right)^{1 / 2} u^{-1}>\eta^{s} \xi^{\frac{1}{1-t}} e^{\omega}\right\}\right|_{a} \\
& \leq C \xi\left(\eta^{-s} \xi^{\frac{1}{t-1}}\right) e^{-\omega}=C\left(\frac{\xi}{\eta}\right)^{s} e^{-\omega}
\end{aligned}
$$

(we have used Theorem 4). Now,

$$
\|1-\Psi\|_{L^{s}(a)}^{s} \leq C \int_{E}(\log \alpha)^{s} a \leq s C \int_{0}^{\infty} \omega^{s-1} \pi(\omega) d \omega \leq C^{\prime}\left(\frac{\xi}{\eta}\right)^{s} \int_{0}^{\infty} \omega^{s-1} e^{-\omega} d \omega
$$

which implies the required estimate for $\int\left(\sum\left|\psi_{j}\right|^{2}\right)^{1 / 2} w$.

## §4. Hardy-Littlewood square function: correction

In a standard way (see, e.g., [7, 9]), Theorem 1 is deduced from a special decomposition of a function into a big and a small part. We are going to obtain such a decomposition. Fix $s \geq r(w, a)$, and let $p \in[s,+\infty)$. We put $v_{p}=a w^{-p}$. The corollary at the end of $\S 2$ implies the estimates

$$
\begin{equation*}
\left(\int_{\mathbb{T}}\left|\mathbb{P}_{ \pm} f\right| v_{p}\right)^{1 / p} \leq C(a, w, s) \cdot p \cdot\left(\int_{\mathbb{T}}|f|^{p} v_{p}\right)^{1 / p}, \quad f \in L^{p}\left(v_{p}\right) \tag{18}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\int_{\mathbb{T}}|\sigma f|^{p} v_{p}\right)^{1 / p} \leq C(a, w, s) \cdot p^{2} \cdot\left(\int_{\mathbb{T}}|f|^{p} v_{p}\right)^{1 / p}, \quad f \in L^{p}\left(v_{p}\right) \tag{19}
\end{equation*}
$$

In (19), the factor $p^{2}$ arises because the proof of the boundedness of $\sigma$ on $L^{p}\left(v_{p}\right)$ involves two CZO's. The first takes scalar functions to $l^{2}$-valued functions and is of the form $f \mapsto\left\{f * \varphi_{j}\right\}$ (see the Appendix); here for instance, $\varphi_{j}=\sigma_{j}+\tau_{j}$ (see Figure 1 ) - we need the relation $\widehat{\varphi}_{j}=1$ on $\left[2^{n-1}-1,2^{n}-1\right.$ ) (also, the functions $\varphi_{j}$ must be introduced for the negative part of the spectrum; surely, this is done in an obvious way). The second
operator $\left\{f_{j}\right\} \mapsto\left\{g_{j}\right\}$ is expressed in terms of the Riesz projection on $l^{2}$-valued functions. For $j \geq 1$, this operator acts like this:

$$
g_{j}=z^{2^{j-1}-1} \mathbb{P}_{+}\left(\bar{z}^{2^{j-1}-1} f_{j}\right)-z^{2^{j}-1} \mathbb{P}_{+}\left(\bar{z}^{2^{j}-1} f_{j}\right)
$$

Again, we omit an obvious formula for negative $j 2$
Thus, suppose $f \in L^{\infty}(w)$ and $\|f\|_{\infty, w} \leq 1$. We observe that $\sigma\left(\mathbb{P}_{ \pm} f\right) \leq \sigma f$, so that (18) and (19) imply

$$
\left\|\mathbb{P}_{ \pm} f\right\|_{p, v_{p}},\left\|\sigma\left(\mathbb{P}_{ \pm} f\right)\right\|_{p, v_{p}} \leq C p^{2}\left(\int\left|\frac{f}{w}\right|^{p} a\right)^{1 / p} \quad \text { for } p \geq s
$$

Thus, the norms of the sequences

$$
\begin{aligned}
F^{\prime} & =\left(\ldots, 0,0, \mathbb{P}_{+} f, M_{1} f, M_{2} f, \ldots\right) \\
F^{\prime \prime} & =\left(\ldots, M_{-2} f, M_{-1} f, \mathbb{P}_{-} f, 0,0, \ldots\right)
\end{aligned}
$$

in $L^{p}\left(l^{2}(\mathbb{Z}), v_{p}\right)$ do not exceed $C=C(w, a, s) p^{2}$ for $p \geq s$. We "cut" these sequences ( $l^{2}$-valued functions) at the level $\lambda w$ ( $\lambda$ is a numerical parameter):

$$
F^{\prime}=\chi_{\left\{\left\|F^{\prime}(\cdot)\right\|_{l^{2}} \leq \lambda w(\cdot)\right\}} F^{\prime}+G^{\prime}=\varphi+\psi
$$

(and similarly for $F^{\prime \prime}$ ). Obviously, $\|\varphi\|_{L^{\infty}\left(l^{2}, w\right)} \leq \lambda$, and

$$
\begin{aligned}
& \|\psi\|_{L^{s}\left(l^{2}, v_{s}\right)}=\left(\int_{\left\|F^{\prime}(\cdot)\right\|_{l^{2}}>\lambda w(\cdot)}\left\|F^{\prime}(\cdot)\right\|_{l^{2}}^{s} w^{-s} a\right)^{1 / s} \\
& \quad \leq \lambda^{-\frac{p-s}{s}}\left\|F^{\prime}\right\|_{L^{p}\left(l^{2}, v_{p}\right)}^{p / s} \leq \lambda^{1-\frac{p}{s}}\left(C p^{2}\right)^{p / s}\left(\int\left|\frac{f}{w}\right|^{p} a\right)^{1 / s} \\
& \quad \leq \lambda^{1-\frac{p}{s}}\left(C p^{2}\right)^{p / s}\|f\|_{L^{s}\left(v_{s}\right)}
\end{aligned}
$$

(at the last step we have used the estimate $|f| \leq w$, whence $|f / w|^{p} \leq|f / w|^{s}$ ).
The same estimates are true for the analogous decomposition of $F^{\prime \prime}$. Correcting these decompositions in accordance with Lemma 3 and its analog for the scale $\mathcal{Y}_{p,-}$, we arrive at the following conclusion: for every $\lambda>0$ and every $p>s$ we can write $f=g+h$, where

$$
\begin{equation*}
|g|,\left|\mathbb{P}_{ \pm} g\right|, \sigma(g) \leq R \lambda w ; \quad \int\left|\frac{h}{w}\right|^{s} a \leq \lambda^{s-p}\left(R p^{2}\right)^{p} \int\left|\frac{f}{w}\right|^{s} a \tag{20}
\end{equation*}
$$

with some constant $R=R(w, a, s)$. (In (20), only the terms with index 0 in the modified sequences $\varphi$ and $\psi$ and in their analogs for $F^{\prime \prime}$ are taken into account - the other terms may now be forgotten.)

The coefficient of the integral on the right in (20) attains its minimum at $p=$ $e^{-1}(\lambda / R)^{1 / 2}$. This value must be greater than or equal to $s$, which implies the restriction $\lambda \geq e^{2} R s^{2}$. With this $p$, the coefficient in question is equal to $\lambda^{s} \exp \left[-2 e^{-1}\left(\frac{\lambda}{R}\right)^{1 / 2}\right]$. Replacing $R$ in the exponent by a slightly larger constant, we can change the factor $\lambda^{s}$ for a constant $B=B(w, a, s)$. Increasing $B$ if necessary, we can drop the restriction $\lambda \geq R e^{2} s^{2}$ : for small $\lambda$ we shall be able to put simply $g=0, h=f$. Finally, a homogeneity transformation allows us to consider arbitrary functions $f \in L^{\infty}(w)$ (and not merely those satisfying $|f| \leq w)$. As a result, we obtain the following statement.

[^2]Lemma 4 (Decomposition Lemma). Let $w \in A_{1}, a \in A_{\infty}$, and let $s \geq r(w, a)$. Suppose $f \in L^{\infty}(w)$ and $|f| \leq \rho w(\rho$ is a number). Then for every $\lambda>0$ there is a representation $f=g+h$ with

$$
|g|,\left|\mathbb{P}_{ \pm} g\right|, \sigma g \leq R \lambda w ; \quad \int\left|\frac{h}{w}\right|^{s} a \leq \Omega\left(\frac{\lambda}{\rho}\right) \int\left|\frac{f}{w}\right|^{s} a .
$$

Here $\Omega(t)=B \exp \left[-\left(\frac{t}{D}\right)^{1 / 2}\right], t>0$, and the constants $R, B$, and $D$ depend on $w, a$, and $s$ only.

Now we prove Theorem 2'. Fix $\eta \geq 1$. Suppose $f \in L^{\infty}(w),\|f\|_{\infty, w} \leq 1$. We put $f_{0}=f$. Recall the notation $|e|_{a}=\int_{e} a$. We construct functions $f_{j}, g_{j}$, and $\varphi_{j}(j \geq 1)$ in such a way that the following relations are fulfilled for $j \geq 0$ :

$$
\begin{gather*}
f_{j}=g_{j+1}+f_{j+1}+\varphi_{j} ;  \tag{21}\\
\left|g_{j+1}\right|,\left|\mathbb{P}_{ \pm} g_{j+1}\right|, \sigma g_{j+1} \leq R \eta 2^{-j} w ;  \tag{22}\\
\left|f_{j}\right| \leq 4^{-j} w, \quad\left|\operatorname{supp} \varphi_{j+1}\right|_{a} \leq 4^{s(j+1)} \Omega\left(\eta 2^{j}\right) \int\left|\frac{f_{j}}{w}\right|^{s} a ;  \tag{23}\\
\int\left|\frac{f_{j+1}}{w}\right|^{s} a \leq \Omega\left(\eta 2^{j}\right) \int\left|\frac{f_{j}}{w}\right|^{s} a . \tag{24}
\end{gather*}
$$

Indeed, suppose $f_{j}$ is constructed (note that $f_{0}$ is with us from the outset). We apply Lemma 4 to $f_{j}$ with $\rho=4^{-j}$ (see the first inequality in (23)) and $\lambda=\eta 2^{-j}$. The result is $f_{j}=g_{j+1}+h$, where $g_{j+1}$ satisfies (22) and $h$ satisfies the inequality

$$
\begin{equation*}
\int\left|\frac{h}{w}\right|^{s} a \leq \Omega\left(\eta 2^{j}\right) \int\left|\frac{f_{j}}{w}\right|^{s} a . \tag{25}
\end{equation*}
$$

Next, we put $f_{j+1}=h \chi_{\left\{|h| \leq 4^{-j-1} w\right\}}, \varphi_{j+1}=h-f_{j+1}$. Then (23) and (24) are immediate consequences of (25) and these definitions, and the induction is complete.

From (24) it easily follows that $\int\left|f_{j}\right|^{s} w^{-s} a \leq C \int|f|^{s} w^{-s} a$, where $C$ does not depend on $j$ and $\eta$; hence, the second inequality in (23) yields the following estimate for the measure of the set $E=\bigcup_{j \geq 0} \operatorname{supp} \varphi_{j+1}$ :

$$
|E|_{a} \leq\left(\sum_{j \geq 0} 4^{s(j+1)} \Omega\left(\eta 2^{j}\right)\right) \int\left|\frac{f}{w}\right|^{s} a
$$

It is easily seen that the coefficient of the integral on the right is dominated by $d_{1} e^{-\left(\frac{\eta}{d_{2}}\right)^{1 / 2}}$, where the constants $d_{1}$ and $d_{2}$ depend only on $w, a$, and $s$. On the other hand, putting $g=\sum_{j \geq 0} g_{j+1}$, we readily deduce that $\{f \neq g\} \subset E$ (see (21)) and that $|g|,\left|\mathbb{P}_{ \pm} g\right|$, $\sigma g \leq R \eta w$ (see (22)). This proves Theorem $2^{\prime}$.

## §5. Proof of Theorem 3

As in the proof of Theorem 1, in Theorem 3 it suffices to consider the cases of $p_{1}=1$, $p_{2}<\infty$ and $p_{1}>1, p_{2}=\infty$ separately (see Subsection 0.7.3). The first case is simple. Indeed, we must modify in a proper way the decomposition $f=g+h$, where $f$ has spectrum only on the union of the intervals $I_{k}=\left[2^{2 k}-1,2^{2 k+1}-1\right), k \geq 0$, and $g$ and $h$ are arbitrary measurable functions with $g \in L^{1}, h \in L^{p_{2}}$. $K$-closedness in the $H^{p}$-scale allows us to assume from the outset that $g \in H^{1}, h \in H^{p_{2}}$. Next, let $\varphi_{k}=\sigma_{2 k+1}+\tau_{2 k+1}$ (see Figure $1 ; \widehat{\varphi}$ is identically equal to 1 on $I_{k}$ and to 0 on $I_{j}$ for $j \neq k$ ). Since the operator $x \mapsto\left\{x * \varphi_{k}\right\}$ is a CZO (see the Appendix), which maps, in particular, $H^{1}$ to $H^{1}\left(l^{2}\right)$, we arrive at the identities $f * \varphi_{k}=g * \varphi_{k}+h * \varphi_{k}$. By Lemma 1 , they can be converted to $f * \varphi_{k}=g_{k}+h_{k}$, where supp $\hat{g}_{k}, \operatorname{supp} \hat{h}_{k} \subset I_{k}$ and
$\left\|\left(\sum\left|g_{k}\right|^{2}\right)^{1 / 2}\right\|_{1} \leq C\|g\|_{1},\left\|\left(\sum\left|h_{k}\right|^{2}\right)^{1 / 2}\right\|_{p_{2}} \leq C\|h\|_{p_{2}}$. It remains to refer to the fact that the operator $\left\{y_{k}\right\} \mapsto \sum_{k} y_{k} * \varphi_{k}$ maps boundedly $H^{1}\left(l^{2}\right)$ to $H^{1}$ and $H^{p_{2}}\left(l^{2}\right)$ to $H^{p_{2}}$.

Now, we verify that the couple $\left(Z_{p}, Z_{\infty}\right)$ is $K$-closed in $\left(L^{p}(\mathbb{T}), L^{\infty}(\mathbb{T})\right)$ for $1<p<\infty$. Applying the duality lemma (Subsection 0.7.2), we reduce the matter to the annihilators, which consist of the functions with spectrum only in $\mathbb{Z} \backslash \bigcup_{k=0}^{\infty} I_{k}=\Lambda$. So, let $f$ be a function with spectrum in $\Lambda$ and represented as $f=g+h$ with $g \in L^{1}(\mathbb{T}), h \in L^{t}(\mathbb{T})$ $\left(t=p^{\prime}\right)$. We must modify this decomposition so that the Fourier transforms of the summands become supported on $\Lambda$ and the norms grow unessentially. Let $\|g\|_{1}=\xi$, $\|h\|_{t}=\eta$.

The main calculations will be done with square functions in a manner similar to that in $\S 3$ - but now without weights. Consider the functions $\sigma_{2 k}$ and $\tau_{2 k}$ (see Figure 1; the sum of their Fourier transforms is equal to 1 on the interval $I_{k}$ where $f$ has no spectrum). We apply the Calderón-Zygmund procedure to $g$ (this time without weights) with parameter $\lambda=\eta^{t^{\prime}} \xi^{\frac{1}{1-t}}$ (the same value was fixed in $\S 3$ ): $g=g_{0}+g_{1},\left|g_{0}\right| \leq C \lambda$, and so on. Recalling the notation $M_{k}$ introduced in Subsection 0.5, we write down the following identities for the parts of $g_{1}$ and $g_{0}+h$ "to be suppressed":

$$
\begin{equation*}
M_{2 k} g_{1}=M_{2 k}\left(\sigma_{2 k} * g_{1}+\tau_{2 k} * g_{1}\right) ; \quad M_{2 k}\left(g_{0}+h\right)=M_{2 k}\left(\sigma_{2 k} *\left(g_{0}+h\right)+\tau_{2 k} *\left(g_{0}+h\right)\right) \tag{26}
\end{equation*}
$$

Next,

$$
\begin{align*}
\sigma_{2 k} * f & =\sigma_{2 k} * g_{1}+\sigma_{2 k} *\left(g_{0}+h\right),  \tag{27}\\
\tau_{2 k} * f & =\tau_{2 k} * g_{1}+\tau_{2 k} *\left(g_{0}+h\right) \tag{28}
\end{align*}
$$

Clearly, the function on the right in (27) has no spectrum to the right of $2^{2 k-1}-1$, and the function on the right in (28) has no spectrum to the left of $2^{2 k}-1$.

The new decomposition of $f$ we look for will be of the form $f=\left[g_{1}-\gamma\right]+\left[g_{0}+h+\gamma\right]$, where the correction term $\gamma$ is to be chosen. This will be done in two steps. First, we choose an antianalytic function $\Phi$ and an analytic function $\Psi$ in a special way, and put

$$
\begin{aligned}
\varphi_{k} & =\Phi \cdot\left(\sigma_{2 k} * f\right)-\sigma_{2 k} *\left(g_{0}+h\right) \\
\psi_{k} & =\Psi \cdot\left(\tau_{2 k} * f\right)-\tau_{2 k} *\left(g_{0}+h\right)
\end{aligned}
$$

We observe that, on $I_{k}$, the Fourier transforms of $\varphi_{k}$ and $\psi_{k}$ coincide with those of $\sigma_{2 k} * g_{1}$ and $\tau_{2 k} * g_{1}$, respectively. Indeed, since $\Phi$ is antianalytic and $\sigma_{2 k} * f$ has no spectrum to the right of $I_{k}$, neither has $\Phi \cdot\left(\sigma_{2 k} * f\right)$, and so on. The functions $\Phi$ and $\Psi$ will be chosen in such a way that

$$
\begin{align*}
& \left\|\left(\sum\left|\varphi_{k}\right|^{2}\right)^{1 / 2}\right\|_{1}, \quad\left\|\left(\sum\left|\psi_{k}\right|\right)^{1 / 2}\right\|_{1} \leq C \xi  \tag{29}\\
& \left\|\left(\sum\left|\varphi_{k}\right|^{2}\right)^{1 / 2}\right\|_{t}, \quad\left\|\left(\sum\left|\psi_{k}\right|^{2}\right)^{1 / 2}\right\|_{t} \leq C \eta
\end{align*}
$$

We postpone the choice of $\Phi$ and $\Psi$ and the proof of (29) for a while to describe the next step of the construction. Consider the convolutions $\widetilde{\varphi}_{k}=\varphi_{k} *\left(\sigma_{2 k}+\tau_{2 k}\right)$ and $\widetilde{\psi}_{k}=\psi_{k} *\left(\sigma_{2 k}+\tau_{2 k}\right)$ and observe that among the intervals $I_{j}$ there is only one on which their Fourier transforms can differ from 0, namely, $I_{k}$. Next,

$$
M_{2 k} \widetilde{\varphi}_{k}=M_{2 k} \varphi_{k}=M_{2 k}\left(\sigma_{2 k} * g_{1}\right) \quad \text { and } \quad M_{2 k} \widetilde{\psi}_{k}=M_{2 k} \psi_{k}=M_{2 k}\left(\tau_{2 k} * g_{1}\right)
$$

If the functions $\varphi=\sum_{k \geq 0} \widetilde{\varphi}_{k}$ and $\psi=\sum_{k \geq 0} \widetilde{\psi}_{k}$ make sense, these identities and (26) show that $g_{1}-(\varphi+\psi)$ and $g_{0}+h+\varphi+\psi$ have no spectrum on the intervals $I_{j}$, that is, $\varphi+\psi$ can be tested for the role of $\gamma$.

The functions $\varphi$ and $\psi$ are well defined indeed and satisfy the inequalities $\|\varphi\|_{1}$, $\|\psi\|_{1} \leq C^{\prime} \xi,\|\varphi\|_{t},\|\psi\|_{t} \leq C^{\prime} \eta$ (so that the summands $g_{1}-\gamma$ and $g_{0}+h+\gamma$ satisfy not only the spectral condition but also the metric condition: we remind the reader that
$\left\|g_{1}\right\|_{1} \sim \xi,\left\|g_{0}\right\|_{t} \sim \eta$, see $\left.\S 3\right)$. This is a consequence of (29) and the facts presented in the Appendix. To obtain $L^{1}$-estimates, we need the information pertaining to $H^{1}$. We leave the easy details to the reader.

It remains to choose $\Phi$ and $\Psi$ and to prove estimates (29). We shall be busy with $\Psi$ and the inequalities involving $\psi_{k}$ (the choice of $\Phi$ is similar). By analogy with $\S 3$, we put

$$
\begin{aligned}
& \alpha=\max \left\{1, \xi^{\frac{1}{t-1}} \eta^{-t^{\prime}}\left(\sum_{j \geq 1}\left|\tau_{2 k} * g_{1}\right|^{2}\right)^{1 / 2}\right\} \\
& \Psi=\exp (-\log \alpha-i \mathcal{H}(\log \alpha))
\end{aligned}
$$

where $\mathcal{H}$ is the harmonic conjugation operator. The analog of (17) looks like this:

$$
\psi_{k}=\Psi \cdot\left(\tau_{2 k} * g_{1}\right)-(1-\Psi)\left(\tau_{2 k} *\left(g_{0}+h\right)\right)
$$

The further calculations copy those in $\S 3$ after formula (17) with some simplifications (now the weights are identically equal to 1 ). For instance, in order to estimate the distribution function $\rho$, now we use the weak type $(1,1)$ inequality for the $\mathrm{CZO} x \mapsto$ $\left\{x * \tau_{2 j}\right\}$, which takes scalar functions to $l^{2}$-valued, etc. The reader will easily trace the details himself.

## Appendix

The information presented in this Appendix is well known; however (as usually happens), it is difficult to find it in the literature precisely in this form. We collect it only for references.

Let $\left\{\varphi_{n}\right\}$ be a sequence of trigonometric polynomials with the following properties: (i) $\left|\widehat{\varphi}_{n}\right| \leq C$ for all $n$; (ii) the function $\widehat{\varphi}_{n}$ vanishes off the interval $\left[-C 2^{n}, C 2^{n}\right]$; (iii) $\widehat{\varphi}_{n}=F_{n} \mid \mathbb{Z}$, where $F_{n}$ is a piecewise linear function with at most $C$ intervals of linearity and satisfying the inequality $\left|F_{n}^{\prime}\right| \leq C 2^{-n}$.
I. Let $T_{1}: L^{2}\left(l^{2}\right) \rightarrow L^{2}\left(l^{2}\right)$ be the operator given by the formula $T_{1}\left\{x_{n}\right\}=\left\{\varphi_{n} * x_{n}\right\}$. Then $T$ is a Calderón-Zygmund operator.
II. Suppose that the $\varphi_{n}$ satisfy the following additional condition: (iv) card\{n : $\left.\widehat{\varphi}_{n}(k) \neq 0\right\} \leq C$ for all $k$. Then the operators $T_{2}: L^{2} \rightarrow L^{2}\left(l^{2}\right)$ and $T_{3}: L^{2}\left(l^{2}\right) \rightarrow L^{2}$ given by the formulas $T_{2} x=\left\{x * \varphi_{n}\right\}$ and $T_{3}\left\{x_{n}\right\}=\sum x_{n} * \varphi_{n}$ are Calderón-Zygmund operators.
III. In particular, $T_{1}, T_{2}$, and $T_{3}$ map, respectively, $H^{1}\left(l^{2}\right)$ to $H^{1}\left(l^{2}\right), H^{1}$ to $H^{1}\left(l^{2}\right)$, and $H^{1}\left(l^{2}\right)$ to $H^{1}$.
IV. The smoothness condition for the kernel is fulfilled in all cases in the form (c3) (see Subsection 0.2.2); therefore, standard weighted estimates (for Muckenhoupt weights) are true for $T_{1}, T_{2}$, and $T_{3}$.

We say some words about the proof. The $L^{2}$-estimates are clear for the three operators, so it suffices to check (c3). The assumption that the functions $\widehat{\varphi}_{n}$ are piecewise-linear allows us to do this in a particularly simple way. (As a matter of fact, we have chosen the above version precisely for this reason, though, in principle, the conditions on the $\widehat{\varphi}_{n}$ can be relaxed considerably.) Indeed, the second difference $\widehat{\varphi}_{n}(k+2)-2 \widehat{\varphi}(k+1)+$ $\widehat{\varphi}(k)$ may happen to be nonzero at $C$ values of $k$ at the greatest, it does not exceed $C^{\prime} 2^{-n}$ at these $k$ 's, and all of them lie on the interval $\left[-C^{\prime \prime} 2^{n}, C^{\prime \prime} 2^{n}\right]$. Consequently, $\left\|(z-1)^{2} \varphi_{n}(z)\right\|_{\infty} \leq C 2^{-n}$ (we have dominated the $L^{\infty}$-norm by the sum of the moduli of the Fourier coefficients). The same estimate via Fourier coefficients yields $\left\|\varphi_{n}\right\|_{\infty} \leq C 2^{n}$.

Let $I$ be an arc centered at 1 on the circle, and let $5 I$ be the arc with the same center but 5 times as long. Suppose $\zeta \in I, \eta \notin 5 I$. We have $\left|\varphi_{n}(\eta \bar{\zeta})-\varphi_{n}(\eta)\right| \leq$ $C|\zeta-1| 2^{n}\left\|\varphi_{n}\right\|_{\infty} \leq C|\zeta-1| 2^{2 n}$ by the Bernstein inequality. Employing also the estimate
$\left|\varphi_{n}(z)\right| \leq \frac{C 2^{-n}}{|z-1|^{2}}$, we arrive at

$$
\left|\varphi_{n}(\eta \bar{\zeta})-\varphi_{n}(\eta)\right| \leq\left\{\begin{array}{l}
C|\zeta-1| 2^{2 n}  \tag{30}\\
\frac{C 2^{-n}}{|\eta-\zeta|^{2}}
\end{array}\right.
$$

In order to verify condition (c3) for $T_{2}$ and $T_{3}$, we must estimate the quantity $A(\eta, \zeta)=$ $\left(\sum_{n}\left|\varphi_{n}(\eta \bar{\zeta})-\varphi_{n}(\eta)\right|^{2}\right)^{1 / 2}$ for $\eta$ and $\zeta$ as indicated (for $T_{1}$ the situation is even easier: it suffices to estimate the upper bound in place of the quadratic sum). Splitting the sum into two parts and using the upper (respectively, the lower) line in (30) to estimate the first (the second) sum, we obtain

$$
\begin{aligned}
A(\eta, \zeta)= & \left(\sum_{n: 2^{n}<|\zeta-1|^{-1 / 3}|\eta-\zeta|^{-2 / 3}}\left|\varphi_{n}(\eta \bar{\zeta})-\varphi_{n}(\eta)\right|^{2}\right. \\
& \left.+\sum_{n: 2^{n} \geq|\zeta-1|^{-1 / 3}|\eta-\zeta|^{-2 / 3}}\left|\varphi_{n}(\eta \bar{\zeta})-\varphi_{n}(\eta)\right|^{2}\right)^{1 / 2} \\
\leq & \left(|\zeta-1|^{-4 / 3}|\eta-\zeta|^{-8 / 3}|\zeta-1|^{2}+|\zeta-1|^{+2 / 3}|\eta-\zeta|^{+4 / 3}|\eta-\zeta|^{-4}\right)^{1 / 2} \\
\leq & C \frac{|\zeta-1|^{1 / 3}}{|\eta-\zeta|^{1+1 / 3}}
\end{aligned}
$$

Since the kernel is translation invariant, this means that (c3) is fulfilled with $\alpha=1 / 3$.

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[^1]:    ${ }^{1}$ In [6], an explicit Brownian motion formula for $g$ was written out. Here we restrict ourselves to mere existence.

[^2]:    ${ }^{2}$ However, if there are no weights, a better estimate is true: $\left(\int_{\mathbb{T}}|\sigma f|^{p}\right)^{1 / p} \leq C p\left(\int_{\mathbb{T}}|f|^{p}\right)^{1 / p}, p \geq 2$. Indeed, this follows by duality from the fact that the constant in the inequality $\|f\|_{q} \leq c\|\sigma f\|_{q}$ does not grow as $q \rightarrow 1$, which can be deduced from Statement III in the Appendix. This leads to $1+\log \frac{1}{\varepsilon}$ instead of $\left(1+\log \frac{1}{\varepsilon}\right)^{2}$ in Theorem $2^{\prime}$ in the case where $w=a \equiv 1$.

