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SOME GEOMETRIC PROPERTIES OF CLOSED SPACE CURVES AND CONVEX BODIES

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ABSTRACT. The main results of the paper are as follows.

1. On each smooth closed oriented curve in $\mathbb{R}^n$, there exist two points the oriented tangents at which form an angle greater than $\pi/2 + \sin^{-1}\frac{1}{n-1}$.

2. If n is odd, then an $(n+1)$-gon with equal sides and lying in a hyperplane can be inscribed in each smooth closed Jordan curve in $\mathbb{R}^n$. In particular, a rhombus can be inscribed in each closed curve in $\mathbb{R}^3$.

3. A right prism with rhombic base and an arbitrary ratio of the base edge to the lateral edge can be inscribed in each smooth strictly convex body $K \subset \mathbb{R}^3$.

In the sequel, by a *convex body* $K \subset \mathbb{R}^n$ we mean a compact convex body with nonempty interior. A convex body K is said to be *strictly convex* if it has a unique intersection point with each of its support planes.

As usual, for $A \subset \mathbb{R}^n$, we denote by $\operatorname{conv}(A)$ and by $\operatorname{int}(A)$ the convex hull and the interior of A, respectively.

As usual, by $G_2^+(\mathbb{R}^3)$ (respectively, by $G_2(\mathbb{R}^3)$) we denote the Grassmann manifold of oriented (respectively, nonoriented) planes passing through the origin in $\mathbb{R}^3$, and by $E_2^{(+)}(\mathbb{R}^3) \to G_2^{(+)}(\mathbb{R}^3)$ the tautological vector bundle over $G_2^{(+)}(\mathbb{R}^3)$ in which the fiber over a plane $g \in G_2^{(+)}(R^3)$ is the plane itself regarded as a vector subspace of $\mathbb{R}^3$.

§1. ORIENTED ANGLES BETWEEN ORIENTED TANGENTS TO A SPACE CURVE

Throughout the paper, $\gamma(t)\colon [0,1] \looparrowright \mathbb{R}^n$ is a regularly parametrized curve. We orient the tangents to γ in accordance with the orientation of γ.

We begin with the following obvious statement.

Lemma. *Let $\mathbf{a} \in \mathbb{R}^n$ be a nonzero vector. Then either γ has oriented tangents that form acute as well as obtuse angles with $\mathbf{a}$, or γ lies in the hyperplane orthogonal to $\mathbf{a}$.*

Indeed, multiplying the equation $\int_0^1 \gamma'\,dt = \mathbf{0}$ by $\mathbf{a}$, we obtain $\int_0^1 \gamma'\cdot\mathbf{a}\,dt = 0$; therefore, the function under the integral either takes both positive and negative values, or is zero identically. $\square$

Theorem 1. 1. *For $n \geq 3$, each curve $\gamma\colon [0,1] \looparrowright \mathbb{R}^n$ has two oriented tangents that form an angle greater than $\pi/2 + \sin^{-1}\frac{1}{n-1}$.*

2. *Any two immersed curves $\gamma_1, \gamma_2\colon [0,1] \looparrowright \mathbb{R}^3$ have oriented tangents that form acute and obtuse angles, respectively.*

Both estimates are best possible.

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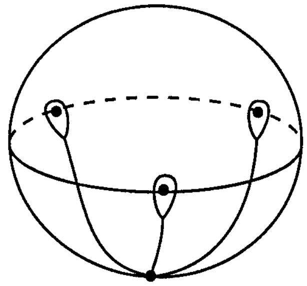

FIGURE 1.

Proof. Translating the unit velocity vectors of γ to the origin $O \in \mathbb{R}^n$, we obtain a curve γ_1, which is the spherical indicatrix of the tangents to γ. We must prove that its angular diameter on the unit sphere exceeds $\pi/2 + \sin^{-1}\frac{1}{n-1}$.

It is well known that $O \in \operatorname{conv}(\gamma_1)$, because otherwise O is separated from γ_1 by a hyperplane P, and the projections of all velocity vectors of γ to the oriented normal to P are codirectional, and, therefore, are all zero because the curve γ is closed.

Consequently, by the Carathéodory principle, O is a convex linear combination of $n+1$ points $A_1, \dots, A_{n+1} \in \gamma_1$ (repetition of points is possible).

It is well known that, since γ_1 is arcwise connected, the number of points can be reduced to n. Indeed, moving A_1 towards A_2 along γ, we can find a moment when O lies on the boundary of the polyhedron $\operatorname{conv}\{A_1, \dots, A_{n+1}\}$ (e.g., when $A_1 = A_2$), which allows us to reduce the number of points.

Thus, $O \in \operatorname{conv}\{A_1, \dots, A_n\}$, i.e., $\sum_{i=1}^n x_i \overline{OA_i} = \mathbf{0}$ for some $x_i \geq 0$ with $\sum_{i=1}^n x_i = 1$. For definiteness, let $x_1 = \max_{i \leq n} x_i$. We have

$$\overline{OA_1} = -\sum_{i=2}^n \frac{x_i}{x_1}\overline{OA_i},$$

whence

$$1 = \overline{OA_1^2} = -\sum_{i=2}^n \frac{x_i}{x_1}\overline{OA_i} \cdot \overline{OA_1} = -\sum_{i=1}^n \frac{x_i}{x_1}\cos\angle A_iOA_1.$$

Thus,

$$-1 \geq \sum_i \frac{x_i}{x_1}\cos\angle A_iOA_1,$$

where the sum is taken over all i such that $\cos\angle A_iOA_1 < 0$. Consequently, we have $-1 \geq \sum_i \cos\angle A_iOA_1$, so that the greatest angle $\angle A_kOA_1$ satisfies the inequality $-1/(n-1) \geq \cos\angle A_kOA_1$.

Equality is attained only if $x_1 = x_2 = \cdots = x_n$ and the corresponding points $A_1, \dots, A_n$ are the vertices of a regular simplex. If $O \in \operatorname{int}(\operatorname{conv}(\gamma_1))$, then, by small perturbations of the points A_i, we can arrange that the A_i are not the vertices of a regular simplex, and the above inequality becomes strict. If the indicatrix γ_1 (and, therefore, the curve γ) lies in a hyperplane, then the number of points A_i can be reduced, and, consequently, the inequality can be sharpened.

To prove that the inequality obtained is best possible, we consider the following curve γ_1 (see Figure 1). Let $A_1, \dots, A_n$ be the vertices of the regular simplex inscribed in the equatorial sphere of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. We assume that γ_1 goes around each point A_i in a small neighborhood of A_i and then moves along great circles to the south pole of the sphere S^{n-1}. Let $O \in \operatorname{int}(\operatorname{conv}(\gamma_1))$. Obviously, the angular diameter

FIGURE 2.

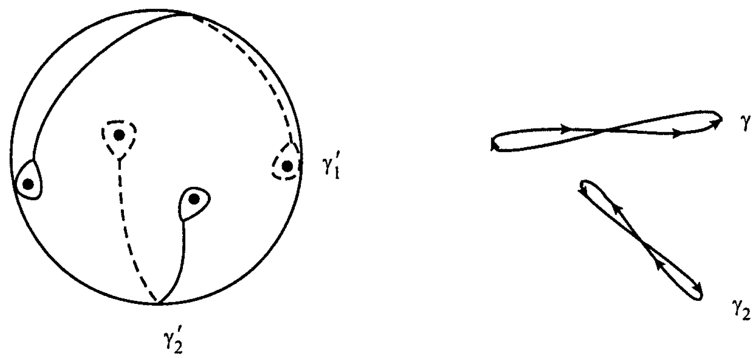

FIGURE 3.

of γ_1 can be made arbitrarily close to the diameter of the simplex $A_1, \dots, A_n$, i.e., to $\pi/2 + \sin^{-1} \frac{1}{n-1}$.

It remains to construct a curve γ such that the indicatrix of tangents of γ is γ_1. Suppose that the parameter t along γ_1 ranges in $[0,1]$. By Gromov's lemma, the condition $O \in \operatorname{int}(\operatorname{conv}(\gamma_1))$ implies the existence of a positive continuous function f on $[0,1]$ such that $\mathbf{0} = \int_0^1 f(t)\gamma_1(t)\,dt$; consequently, γ_1 is the indicatrix of tangents of the closed regular curve $\gamma(s) = \int_0^s f(t)\gamma_1(t)\,dt$.

Observe that, in this case, the curve γ itself is C^0-close to the polygonal line formed by the vectors $\overline{OA}_1, \dots, \overline{OA}_n$ going to the vertices of the regular simplex from its center. For $n = 3$, such a curve is depicted in Figure 2. The first statement of the theorem is proved.

The second statement follows from the lemma proved above. The fact that the estimate is best possible is proved by the example of the curves γ_1 and γ_2 in $\mathbb{R}^3$ that are shown in Figure 3 together with their indicatrices γ_1' and γ_2'. The angles between the oriented tangents to γ_1 and γ_2 take values in the interval $[\pi/2 - \varepsilon, \pi]$. If we reverse the orientation on γ_2, then the angles between the oriented tangents to γ_1 and γ_2 will be replaced by the supplementary angles and will take values in the interval $[0, \pi/2 + \varepsilon]$. $\square$

Remarks. 1. Thus, for immersed oriented curves $\gamma_1, \gamma_2 \colon [0,1] \looparrowright \mathbb{R}^3$, we can only state that there exist orthogonal oriented tangents. If $\alpha \neq \pi/2$, then a pair of tangents forming an angle of size α may fail to exist.

2. The author does not know what values can be taken by the angles between oriented tangents for three and more circles immersed in $\mathbb{R}^n$.

3. Probably, it is of some interest to estimate the angle between oriented tangent planes for immersed spheres $S^k \looparrowright \mathbb{R}^n$.

§2. ON RHOMBUSES INSCRIBED IN A SPACE CURVE

In general, a given quadrangle cannot be inscribed via a similarity transformation in a typical circle γ smoothly embedded in $\mathbb{R}^3$. Indeed, the similarity group of $\mathbb{R}^3$ is seven-dimensional, and the property of being inscribed in γ imposes eight restrictions on a quadrangle.

However, there may exist one-parameter families of such types of quadrangles with the property that a similarity image of one of the quadrangles can be inscribed in every smoothly embedded circle $\gamma \hookrightarrow \mathbb{R}^3$.

In [1], two families of such one-parameter nonplanar quadrangles were presented. The first example, found by Stromkvist, consists of the quadrangles $ABCD$ with

$$|AB| = |BC| = |CD| = |DA| \quad \text{and} \quad |AC| = |BD|.$$

In [1] Griffiths states that the family of twisted rectangles with one side turned around the median perpendicular to it has the required property.

Below, we prove that, for each circle smoothly embedded in $\mathbb{R}^3$, a rhombus can be inscribed in it.

Theorem 2. *Let n be odd, let $f\colon S^1 \to \mathbb{R}^n$ be a continuous mapping, and let ρ be a metric on S^1 continuous with respect to the standard metric and such that the limit $\lim_{y\to x} \rho(x,y)/|x-y|$ exists uniformly with respect to x and is a continuous positive function. Then there exist $n+1$ points $A_1, \dots, A_{n+1} \in S^1$ such that*

$$\rho(A_1, A_2) = \rho(A_2, A_3) = \dots = \rho(A_n, A_{n+1}) = \rho(A_{n+1}, A_1)$$

and the points $f(A_1), \dots, f(A_{n+1})$ lie in a hyperplane in $\mathbb{R}^n$.

Proof. The metric ρ on the circle can be approximated as close as we wish by a smooth function on $S^1 \times S^1$ positive outside the diagonal (possibly, with loss of the triangle inequality), with preservation of the local rectification condition for ρ given in the theorem (with the same limit function). Therefore, in the sequel it is assumed that $\rho(x,y)$ is a smooth function of its arguments for $x \neq y$. In the general case the proof is obtained by passing to the limit.

We prove that, for a typical "smooth" function ρ described above and a typical smooth mapping $f: S^1 \to \mathbb{R}^n$, there is an odd number of required sets of points $(A_1, \dots, A_{n+1})$ arranged clockwise in a given order on the circle and considered up to a cyclic permutation.

Let $U \subset (S^1)^{n+1}$ be the open set of ordered collections of points $(A_1, \dots, A_{n+1})$ arranged on the oriented circle S^1 in the given order.

We denote

$$M_\rho := \{(A_1, \dots, A_{n+1}) \in U \mid \rho(A_1A_2) = \rho(A_2A_3) = \dots = \rho(A_{n+1}A_1)\} \subset U.$$

For a typical metric ρ as above, the set M_ρ is a smooth one-dimensional compact submanifold of U. (The compactness of M_ρ follows from the fact that no two points A_i in the collection in question can come arbitrarily close to each other. Obviously, only all points simultaneously can come close, but this contradicts the "local rectification" condition imposed on ρ.)

We consider the function

$$F\colon M_\rho \to \mathbb{R}, \quad (A_1, \dots, A_{n+1}) \mapsto (\overline{A_1A_2}, \overline{A_2A_3}, \dots, \overline{A_nA_{n+1}}),$$

where $(\mathbf{v}_1, \dots, \mathbf{v}_{n+1})$ denotes the oriented volume of the parallelepiped constructed by the vectors $\mathbf{v}_1, \dots, \mathbf{v}_{n+1}$. For a typical smooth mapping $f\colon S^1 \to \mathbb{R}^{n+1}$, the condition $F(A_1, \dots, A_{n+1}) = 0$ distinguishes a finite subset in M_ρ invariant with respect to the cyclic permutations of the coordinates $(A_1, \dots, A_{n+1})$.

Since all "metrics" ρ in question and the smooth mappings $f: S^1 \to \mathbb{R}^n$ are smoothly homotopic, the standard topological considerations show that the parity of the number of the collections $(A_1, \dots, A_{n+1})$ considered up to a cyclic permutation of coordinates does not depend on the choice of a specific typical pair (ρ, f).

We prove that, in the typical situation, this number is odd.

For this, we consider the metric ρ on S^1 induced by a smooth embedding $f\colon S^1 \to \mathbb{R}^n$ close to the standard one. Then $M_\rho \cong S^1$, and for each point $A_1 \in S^1$ there is a unique collection $(A_1(t), A_2(t), \dots, A_{n+1}(t)) \in M_\rho$ (smoothly depending on A_1). We have

$$F(A_2, A_3, \dots, A_{n+1}, A_1) = -F(A_1, \dots, A_{n+1})$$

because n is odd. Therefore, in the typical situation, the function F vanishes an odd number of times if $A_1(t) \in (A_1, A_2)$, which completes the proof of Theorem 2. $\square$

In the case where $f : S^1 \hookrightarrow \mathbb{R}^3$ is a smooth embedding and the metric ρ is induced by f, i.e., $\rho(x, y) = |f(x)f(y)|$, we obtain the following statement.

Theorem 3. *If n is odd, then every circle smoothly embedded in $\mathbb{R}^n$ admits inscribing an $(n+1)$-gon with equal sides and lying in a hyperplane. In particular, a rhombus can be inscribed in each circle smoothly embedded in $\mathbb{R}^3$.*

Remarks. 1. In the paper [2], Shnirel′man gave two proofs of the fact that a square can be inscribed in every smooth Jordan curve in the plane. The second proof involves a one-parameter family of rhombuses (similar to the family considered above) inscribed in the curve.

2. N. Yu. Netsvetaev called the author's attention to the fact that Theorem 3 implies the existence of an inscribed square for each smooth Jordan curve on a sphere (because the rhombuses inscribed in a sphere are squares), and implies also the Shnirel′man theorem mentioned above (the limit case as the radius of the sphere increases).

§3. On parallelepipeds inscribed in a three-dimensional convex body

In this section, we apply Theorem 3 to the search of parallelepipeds inscribed in a convex body $K \subset \mathbb{R}^3$. There are octagons in $\mathbb{R}^3$ the homothetic images of which cannot be inscribed in a given smooth convex body $K \subset \mathbb{R}^3$, because the similarity group of $\mathbb{R}^3$ is seven-dimensional, while the property of being inscribed for an octagon is given by eight conditions on the vertices.

However, there exist one-parameter families of octagons such that, for every smooth convex body $K \subset \mathbb{R}^3$, a homothetic image of an octagon in this family can be inscribed in K. Griffiths [1] proved that this property is shared by the family of "twisted regular quadrangular prisms" with a given ratio of the height to the base edge (such "prisms" are the convex hulls of the union of two equal squares lying in planes that are perpendicular to the line through the centers of the squares).

Theorem 4. *Let $K \subset \mathbb{R}^3$ be a smooth strictly convex body, let $x > 0$, and let l be a line. Then:*

1. *There exists a parallelepiped with rhombic base and a lateral edge parallel to l that is inscribed in K and has the ratio of the base edge to the lateral edge equal to x.*

2 *There exists a right prism that is inscribed in K and has rhombic base and the ratio of the base edge to the lateral edge equal to x.*

Proof. 1. Performing the Steiner symmetrization of the body K with respect to the plane P passing through the point $O \in \mathbb{R}^3$ and perpendicular to the given line l, we obtain a smooth strictly convex body $K_P \subset \mathbb{R}^3$.

The lengths of chords of K and K_P parallel to l vary in the interval $(0, h_P]$, where h_P is the distance between the support planes of K_p parallel to P.

The endpoints of the chords of K_P parallel to l and having a fixed length $a \in (0, h_P)$ form two equal smooth plane curves γ_1 and γ_2 parallel to P, and, for the body K, the endpoints of similar chords also form two smooth space curves γ_1' and γ_2' that are diffeomorphically projected onto γ_1 and γ_2 along l.

Inscribing a rhombus in γ_1' by Theorem 3, we see that in K we can inscribe a parallelepiped having rhombic base with side c and such that the lateral edge of it is parallel to l and has an arbitrary length $a \in (0, h_p)$; in the typical situation, the number of such parallelepipeds is odd. It is clear that $\lim_{a\to 0} c/a = +\infty$ and $\lim_{a\to h_p} c/a = 0$.

For a typical body K, i.e., for an open set of such bodies dense in the C^1-topology, and for a fixed plane P, the parallelepipeds in question form a one-dimensional noncompact manifold in the twelve-dimensional manifold of parallelepipeds in $\mathbb{R}^3$. Since, for almost all fixed a, the number of such parallelepipeds is odd, there exists a continuous branch of such parallelepipeds (and, in fact, an odd number of such branches) connecting the parallelepipeds with lateral edges of lengths ε and $h_p - \varepsilon$ for a sufficiently small $\varepsilon > 0$. Since, for sufficiently small $\varepsilon > 0$, the former parallelepipeds have a small ratio of edges $c/a < x$, and the latter ones have a large ratio of edges $c/a > x$, we can find a parallelepiped required in item 1 of Theorem 4. It is easily seen that, typically, the number of such parallelepipeds is odd, namely, there are an odd number of them on each of the continuous branches considered above and, in general, an even number on each of the other components.

2. Let M_x denote the set of parallelepipeds inscribed in K, having a rhombic base, and such that the ratio of the base edge to the lateral edge is equal to x. For a typical strictly convex smooth body K, the set M_x is a compact two-dimensional manifold because, obviously, the parallelepipeds described above cannot degenerate.

We define a continuous mapping $M_x \to E_2^+(\mathbb{R}^3)$, where $E_2^+(\mathbb{R}^3)$ is the space of the tautological vector bundle $E_2^+(\mathbb{R}^3) \to G_2^+(\mathbb{R}^3)$. For a parallelepiped $m \in M_x$, we consider oriented planes P^+ and P^- perpendicular to a lateral edge of the parallelepiped (these planes differ only in orientation). With m, we associate the projection of the outer unit normal to the upper face of m with respect to P^+ (respectively, P^-) to the plane P^+ (respectively, P^-). By construction, it is clear that the projections of m to P^+ and P^- differ in sign.

Therefore, we obtain a continuous mapping ϕ from M_x to the space of the tangent bundle over the projective space $G_2(\mathbb{R}^3)$, after identifying the subspaces P^+ and P^- via central symmetry with respect to the origin.

It follows that the image of M_x in the space of the tangent bundle over the projective space intersects a typical fiber at an odd number of points. Therefore, this image realizes a generator of the two-dimensional homology group with coefficients in $\mathbb{Z}_2$ and has intersection index 1 modulo 2 with the zero section. Consequently, the image of M_x in some plane $P^+(P^-)$ is zero, which proves the second statement of Theorem 4. $\square$

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