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PIECEWISE-SMOOTH REFINABLE FUNCTIONS

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Abstract. Univariate piecewise-smooth refinable functions (i.e., compactly supported solutions of the equation $\varphi(\frac{x}{2}) = \sum_{k=0}^{N} c_k \varphi(x-k)$) are classified completely. Characterization of the structure of refinable splines leads to a simple convergence criterion for the subdivision schemes corresponding to such splines, and to explicit computation of the rate of convergence. This makes it possible to prove a factorization theorem about decomposition of any smooth refinable function (not necessarily stable or corresponding to a convergent subdivision scheme) into a convolution of a continuous refinable function and a refinable spline of the corresponding order. These results are applied to a problem of combinatorial number theory (the asymptotics of Euler's partition function). The results of the paper generalize several previously known statements about refinement equations and help to solve two open problems.

§1. Introduction and the statement of the problem

Refinement equations of the type

$$\varphi\Big(\frac{x}{2}\Big) = \sum_{k=0}^{N} c_k \varphi(x-k) \tag{1}$$

(univariate two-scale difference equations with compactly supported mask) have found many applications in wavelets and subdivision algorithms in approximation theory, as well as in the design of curves and surfaces, in probability theory, combinatorial number theory, mathematical physics, and so on (see [1]–[3] and the references therein). The sequence of complex coefficients $c_0, \dots, c_N$ is called the *mask* of the equation ($c_0 c_N \neq 0$, $N \geq 1$); the characteristic polynomial $m(z) = \frac{1}{2}\sum_{k=0}^{N} c_k z^k$ is the *symbol* of the equation. The compactly supported solutions of such equations (refinable functions) are the main object in the construction of wavelets with compact support [4] and in exploring the subdivision algorithms [5, 6]. It is well known that if a refinement equation possesses a compactly supported distributional solution, then $\sum_{k=0}^{N} c_K = 2^r$, where r is an integer. Up to normalization, under this condition the equation always possesses a unique nontrivial compactly supported solution φ in the space of distributions, and this solution is supported on the segment $[0, N]$ (see [1]). The main problem is to compute or estimate the regularity of the solution φ in various function spaces. This solution cannot be infinitely smooth on $\mathbb{R}$; its smoothness is limited by the length N of the support. For any $N \geq 1$ there is no refinement equation possessing a $\mathcal{C}^{N-1}(\mathbb{R})$-solution; the only refinable function belonging to the space W_1^{N-1} (i.e., with summable $(N-1)$st derivative) is the cardinal B-spline $B_{N-1}(x) = \chi_{[0,1)} * \cdots * \chi_{[0,1)}$, which is the convolution of N characteristic functions of the half-interval $[0,1)$ (see [1]). Here we take the half-interval

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$[0,1)$ instead of the segment (otherwise, in the case where $N=1$ the cardinal B-spline B_0 would satisfy the corresponding refinement equation $\varphi(x/2)=\varphi(x)+\varphi(x-1)$ not for all x, but for almost all). The corresponding refinement equation has the symbol $m(z)=\left(\frac{z+1}{2}\right)^N$. All other equations of degree N have solutions of lower smoothness and, as a rule, of a very complicated structure (see [4, 6, 7] for a general discussion of this aspect and for many examples from applications).

Although they are not infinitely smooth, the cardinal B-splines are piecewise infinitely smooth. They are polynomials on each unit segment $[k,k+1]$, $k\in\mathbb{Z}$. This makes them much more convenient to deal with as compared to other refinable functions. A natural question arises in this context: Do there exist refinable splines different from B-splines, and if they do exist, might they be useful in applications, in particular, in construction of compactly supported wavelets and subdivision schemes? The first result in this direction was obtained by Cavaretta, Dahmen, and Micchelli in [6, Chapter 8], where they classified all refinable splines with integral nodes. A complete answer was given by Lawton, Lee, and Shen in [8], where it was shown that all refinable splines have integral nodes and, therefore, are covered by the classification given in [6]. Also we mention the paper [2], where similar results were obtained independently in a weaker form. Before giving the precise statements, we recall some definitions. A function f is called a *spline of order* $l\geq 1$ if there exist points $x_0<\cdots<x_q$ (the nodes of the spline) such that on each segment $(-\infty,x_0],[x_0,x_1],\ldots,[x_{q-1},x_q],[x_q,+\infty)$ this function is a polynomial of degree at most l, and f belongs to $\mathcal{C}^{l-1}(\mathbb{R})$, i.e., the functions $f,f',\ldots,f^{(l-1)}$ are continuous at the nodes. A spline of order $l=0$ is a piecewise constant function with nodes $x_0,\ldots,x_q$. Everywhere below we set $x_0=0$ and $x_q=N$, and we assume that f is identically zero on the rays $(-\infty,x_0)$ and $(x_q,+\infty)$. We say that a node x_k is *proper* if f is not a polynomial on the interval (x_{k-1},x_{k+1}) (here we set $x_{-1}=-\infty, x_{N+1}=+\infty$).

By $\mathcal{Z}=\{z_1,\ldots,z_n\}$ we denote a finite subset of the unit circle $\{z\in\mathbb{C},|z|=1\}$. Some elements of $\mathcal{Z}$ may coincide, in which case we count them with multiplicities. Denote $\mathcal{Z}_s=\mathcal{Z}\cup\{1,\ldots,1\}$ (s units, i.e., the element 1 is repeated s times). Also, let $\mathcal{Z}^2=\{z^2,z\in\mathcal{Z}\}$. Now we formulate the classification of the refinable splines in its strongest version, as given in [8].

Theorem 1 (Lawton, Lee, and Shen, 1995, [8])**.** *Up to normalization, for any integer $N\geq 1$ and any $l\in[0,N-1]$ there exist finitely many splines of order l that solve the refinement equations of degree N. Every such spline $S(x)$ has integral proper nodes and is given by the formula*

$$S(x)=\sum_{k=0}^{r}p_kB_l(x-k), \tag{2}$$

where B_l is the cardinal B-spline of order l, and the p_k are coefficients of the polynomial

$$P(z)=\sum_{k=0}^{r}p_kz^k=p_r\prod_{z_j\in\mathcal{Z}}(z-z_j),$$

p_r is a constant, and $\mathcal{Z}$ is a finite subset of the unit circle such that $\mathcal{Z}_{l+1}^2\subset\mathcal{Z}_{l+1}$.

Remark 1. Recall that we are interested in refinable functions up to their normalization. That is why we set $p_r=1$ in the definition of the polynomial P everywhere below, unless otherwise stipulated.

If $l=0$, the function $S(x)=\sum_{k=0}^{r}p_k\chi_{(0,1]}(x-k)$ is also a spline solution of the same refinement equation; however, it does not coincide with (2) at the integers. To avoid this uncertainty, for $l=0$ we consider solutions continuous from the right.

The reader will have no difficulty in showing that for any N there are finitely many sets $\mathcal{Z}_{l+1}$ of cardinality at most N (counting with multiplicities) and such that $\mathcal{Z}_{l+1}^2 \subset \mathcal{Z}_{l+1}$. Thus, Theorem 1 classifies all refinable splines of a given degree N. For $l = N-1$ we have only the B-spline B_{N-1}, for $l = N-2$ there are two refinable splines $S(x) = B_{N-1}(x) - B_{N-1}(x-1)$ and $S(x) = B_{N-2}(x) + B_{N-2}(x-1)$, and so on. For $l = 0$ we get several refinable step functions with integral nodes. For any refinable spline (2), the corresponding refinement equation has the symbol $m(z) = \left(\frac{z+1}{2}\right)^{l+1} P(z^2)/P(z)$. Therefore, for any N there are only finitely many equations of degree N having spline solutions. In [8] it was shown that all of them, except for the B-splines, possess linearly dependent translates by integers, so that they do not generate wavelets and can hardly be applied in subdivision schemes. This was the conclusion about possible applications of refinable splines.

However, the next natural question arises: Do there exist piecewise-polynomial refinable functions (not necessarily splines), or piecewise-analytic refinable functions? More precisely:

1) Do there exist refinable functions that are analytic (or, at least, infinitely smooth) on each interval (x_k, x_{k+1}) of some partition $0 = x_0 < x_1 < \cdots < x_{q-1} < x_q = N$ and are different from the splines (2)?

2) More generally, do there exist refinable functions and the corresponding finite partitions such that the smoothness on each interval (x_k, x_{k+1}) exceeds the smoothness on the entire real line, and the refinable functions in question are different from the splines (2)?

We emphasize that in both problems we do not impose any assumptions on the values of the functions at the nodes x_k. A function can be discontinuous at some nodes, or unbounded (and even nonintegrable) on some intervals (x_k, x_{k+1}). A positive answer to any of the above two questions would have led to "locally good" refinable functions that could have been convenient in applications. The first attempt to attack these problems was made in the same monograph [6, Chapter 6]. We shall discuss this in more detail in Remark 2. In the next section we give a negative answer to both questions posed above. Roughly, any "locally good" refinable function must coincide with one of the splines (2). This will follow from Theorem 2, which gives a complete classification of all piecewise-smooth refinable functions.

Nevertheless, this negative result admits an unexpected application to combinatorial number theory (namely, to the problem of asymptotics for Euler's partition function) and helps solving a problem stated by Reznick in 1989 (see [9]). This is the subject of §5 of this paper. In §3 we study the structure of the manifold of all refinable splines and show that these splines can be employed in subdivision algorithms, in spite of their instability (linear dependence of their translates by integers). We classify the splines for which the corresponding subdivision schemes converge, and for all of them we compute the rate of convergence explicitly. In §4 we prove the following factorization theorem: any smooth refinable function (of class $\mathcal{C}^{l+1}$, $l \geq 0$) is the convolution of a continuous refinable function and a refinable spline of order l. This extends a series of earlier results in this direction, obtained in [1, 2, 10] and, in the case of one variable, answers a question stated by Caveretta, Dahmen, and Micchelli in [6].

§2. A classification of piecewise-smooth refinable functions

As usual, we denote by $\mathcal{C}^r(\mathbf{R})$, $r \geq 0$, the space of r times continuously differentiable functions on $\mathbb{R}$; $\mathcal{C}^0(\mathbf{R}) = \mathcal{C}(\mathbf{R})$ is the space of continuous functions. Given $a < b$, we denote by $\mathcal{C}^r(a, b)$, $r \geq 0$, the space of r times continuously differentiable functions on the interval (a, b). A function f belongs to $\mathcal{C}^r(a, b)$ if it belongs to $\mathcal{C}^r[\alpha, \beta]$ for any segment

$[\alpha,\beta] \subset (a,b)$. Note that functions in $\mathcal{C}^r(a,b)$ may have no limits at the endpoints a and b, and may be unbounded and even nonintegrable on (a,b). Let $\omega(h), h \ge 0$, be a monotone nondecreasing continuous function such that $\omega(0)=0$; we introduce the space $\mathcal{C}^\omega(\mathbb{R}) = \{f \in \mathcal{C}^r(\mathbb{R}), \omega(f^{(r)},h) \le Ch^{-r}\omega(h)\}$, where the constant C depends only on f. Here $\omega(g,h) = \sup_{|t|\le h} \|g(\cdot+h)-g(\cdot)\|_{\mathcal{C}}$ is the modulus of continuity, and $r = r(\omega)$ is the largest integer such that $h^{-r}\omega(h) \to 0$, as $h \to 0$ (if $h^{-r}\omega(h) \to 0$ for all r, then $\mathcal{C}^\omega = \mathcal{C}^\infty$). The corresponding spaces on segments and intervals are defined in a similar way. If $\omega(h) = h^\alpha$, $\alpha > 0$, then we get the Lipschitz spaces Lip_α. The Hölder exponent α_f is the supremum of all α such that $f \in \mathrm{Lip}_\alpha$. We recall that if α is an integer, then $\mathcal{C}^\alpha$ is contained in Lip_α, but does not coincide with the latter space.

Suppose that a function f is supported on $[0,N]$ and does not belong to $\mathcal{C}^\omega(\mathbb{R})$, but for some partition $0 = x_0 < x_1 < \cdots < x_q = N$ we have $f|_{(x_k,x_{k+1})} \in \mathcal{C}^\omega(x_k,x_{k+1})$ for any k. In this case we say that a node x_k is *proper* if $f|_{(x_{k-1},x_{k+1})} \notin \mathcal{C}^\omega(x_{k-1},x_{k+1})$ (here we set $x_{-1} = -\infty$ and $x_{N+1} = +\infty$).

Now we consider the refinement equation (1) with the only restriction $c_0c_N \ne 0$. We are interested in the compactly supported solutions of (1), i.e., in the solutions supported on the segment $[0,N]$; these solutions may be nonintegrable.

Theorem 2. a) *If for some $r \ge 0$ a compactly supported solution φ of equation* (1) *does not belong to $\mathcal{C}^r(\mathbb{R})$, but there is a partition $0 = x_0 < x_1 < \cdots < x_q = N$ such that $\varphi|_{(x_k,x_{k+1})} \in \mathcal{C}^r(x_k,x_{k+1})$ for all k, then all proper nodes of this partition are integers, and φ is a spline of the form* (2) *for some $l \le r$.*

b) *If for some $\omega(h)$ a compactly supported solution φ of equation* (1) *does not belong to $\mathcal{C}^\omega(\mathbb{R})$, but there is a partition $0 = x_0 < x_1 < \cdots < x_q = N$ such that $\varphi|_{(x_k,x_{k+1})} \in \mathcal{C}^\omega(x_k,x_{k+1})$ for all k, then all proper nodes are integers, and φ is a spline of the form* (2).

Corollary 1. *Any discontinuous, but piecewise-continuous refinable function is a step function with integral nodes obtained by formula* (2) *with $l = 0$ and $B_l = \chi_{[0,1)}$.*

Moreover, Theorem 2 implies that we can consider piecewise-continuous functions that may fail to have one-sided limits at the nodes and may be nonintegrable on the corresponding intervals. Corollary 1 is still true for such functions.

Corollary 2. *If for a refinable function φ there is a partition such that the Hölder exponent of φ on each interval exceeds the Hölder exponent of φ on $\mathbb{R}$, then φ is a refinable spline.*

Corollary 3. *If for a refinable function φ there is a partition such that φ is $N-1$ times continuously differentiable on each interval, then φ is a refinable spline.*

Subsequently (see Corollary 4), we improve this result. It turns out that even if φ is $N-1$ times continuously differentiable in a neighborhood of an endpoint of the support (i.e., of the point 0 or N), then φ is a refinable spline.

Remark 2. A special case of Corollary 3 was established in [6, Proposition 8.4], where it was shown that if $\varphi \in \mathcal{C}^{N-1}[k,k+1]$, $k = 0,\dots,N-1$, then φ is a spline.

Thus, all piecewise infinitely differentiable (and, therefore, all piecewise-analytic) refinable functions are among the splines classified in Theorem 1. In particular, this shows that the splines (2) are the only refinable functions that can be defined by explicit formulas; all other refinable functions are obtained via passage to a limit and have fractal-like properties.

The proofs of both parts of Theorem 2 are very similar. We give the proof of part a) and then show how to modify that proof to obtain b). The proof starts with two lemmas.

Lemma 1. *If for some $r \ge 0$ a refinable function φ does not belong to $\mathcal{C}^r(\mathbb{R})$, but for some partition $\{x_k\}$ we have $\varphi|_{(x_k,x_{k+1})} \in \mathcal{C}^r(x_k,x_{k+1})$ for all k, then all proper nodes of this partition are integers; moreover, the points 0 and N are proper nodes.*

Proof. Assuming that there are nonintegral proper nodes, we let x_j be the smallest among them. Substituting $x = x_j + h$ in equation (1) and resolving it for $\varphi(x_j + h)$, we obtain

$$\varphi(x_j + h) = \frac{1}{c_0}\varphi\Big(\frac{x_j}{2} + \frac{h}{2}\Big) - \sum_{k=1}^{N} \frac{c_k}{c_0}\varphi\big(x_j - k + h\big). \tag{3}$$

Since the numbers $\frac{x_j}{2}$, $x_j - 1, \dots, x_j - N$, are all nonintegers and are smaller than x_j, it follows that none of them is a proper node. Therefore, the right-hand side of (3) is r times continuously differentiable in h on some small interval $h \in (-\varepsilon, \varepsilon)$. Hence, we have $\varphi \in \mathcal{C}^r(x_j - \varepsilon, x_j + \varepsilon)$, so that the point x_j is not a proper node. Therefore, all proper nodes are integers. Now, let x_j be the smallest proper node. Assume $x_j \ne 0$. Since $\frac{x_j}{2} < x_j$, all the points $\frac{x_j}{2}, x_j - 1, \dots, x_j - N$ are not proper nodes. Using (3), again we conclude that x_j is not a proper node. Thus, $x_j = 0$. To show that $x = N$ is also a proper node, we consider the function $\varphi^*(\cdot) = \varphi(N - \cdot)$, which satisfies the adjoint equation $\varphi^*\big(\frac{x}{2}\big) = \sum_{k=0}^{N} c_{N-k}\varphi^*(x - k)$. The same argument shows that 0 is a proper node for φ^*; hence, N is a proper node for φ. $\square$

Lemma 2. *Under the assumptions of Lemma 1, let s denote the largest integer such that $\varphi \in \mathcal{C}^{s-1}(\mathbb{R})$ (if $\varphi \notin \mathcal{C}(\mathbb{R})$, then we set $s = 0$). Then for any node x_k the limits $\varphi^{(s)}(x_k - 0)$ and $\varphi^{(s)}(x_k + 0)$ exist and are finite.*

Proof. We establish the lemma for the left limits $\varphi^{(s)}(x_k - 0)$. Take the smallest node x_j for which this limit either does not exist or is infinite (we assume that such nodes exist; otherwise there is nothing to prove). Obviously, x_j is a proper node and $j \ne 0$, because for $x_0 = 0$ the limit in question exists and is equal to 0. This shows that at each of the points $\frac{x_j}{2}, x_j - 1, \dots, x_j - N$ the limit $\varphi^{(s)}(x - 0)$ exists and is finite. Now we differentiate both sides of (3) s times with respect to h and take the limit as $h \to -0$. We see that the limit $\varphi^{(s)}(x_j - 0)$ also exists and is finite, which contradicts the definition of x_j. To prove the lemma for the right limits $\varphi^{(s)}(x_k + 0)$, we can invoke the adjoint function φ^* once again. $\square$

Proof of Theorem 2. Let s be the number defined in Lemma 2. The function $\varphi^{(s)}$ is not continuous on $\mathbb{R}$, but it is piecewise continuous (Lemma 2) with integral nodes (Lemma 1). By the same argument as above, we show that $\varphi^{(s)}$ is not continuous at the point 0 (otherwise we denote by x_j the smallest point of discontinuity and use (3) to obtain a contradiction). By Lemma 2, the limit $a = \varphi^{(s)}(+0)$ exists and is finite. Since $\varphi^{(s)}$ is not continuous at the point 0, we have $a \ne 0$. Now we differentiate (1) s times, obtaining $\varphi^{(s)}\big(\frac{x}{2}\big) = \sum_{k=0}^{N} 2^s c_k \varphi^{(s)}(x - k)$. This identity holds true at all nonintegral points x, because by Lemma 1 such points are not proper nodes, so that the function $\varphi^{(s)}$ is well defined at them. For $x \in (0,1)$ this implies $\varphi^{(s)}\big(\frac{x}{2}\big) = 2^s c_0 \varphi^{(s)}(x)$. Iterating n times, we get $\varphi^{(s)}(2^{-n}x) = \big(2^s c_0\big)^n \varphi^{(s)}(x)$. Therefore, $\lim_{n\to\infty}\big(2^s c_0\big)^n \varphi^{(s)}(x) = \lim_{n\to\infty} \varphi^{(s)}(2^{-n}x) = a$. Since $a \ne 0$, we have $\varphi^{(s)}(x) \ne 0$, whence $\lim_{x\to+0}\big(2^s c_0\big)^n = a/\varphi^{(s)}(x)$. Since the left-hand side does not depend on x, we see that $\varphi^{(s)}(x)$ is identically constant on the interval $(0,1)$. Now, let j be the largest integer such that $j \le N - 1$ and $\varphi^{(s)}$ is identically constant on each interval $(k, k+1)$, $k = 0, \dots, j$. We have $j \ge 0$. Suppose $j < N - 1$. Then we differentiate (3) s times and put $x_j = j$. Since each term of the sum $\frac{1}{2^s c_0}\varphi^{(s)}\big(\frac{j}{2} + \frac{h}{2}\big) - \sum_{k=1}^{N} \frac{c_k}{c_0}\varphi^{(s)}(j - k + h)$ is constant on the interval $h \in (0,1)$, so is the function $\varphi^{(s)}(j + h)$. This contradicts the definition of j. Therefore, we have

$j = N-1$, which means that $\varphi^{(s)}$ is a step function with integral nodes. Hence, φ is a spline with integral nodes. Now we apply Theorem 1 and conclude the proof of part a).

Part b) is established in the same way. First, we argue as in Lemma 1 to show that if φ does not belong to the space $\mathcal{C}^{\omega}(\mathbb{R})$, but for some partition $\{x_k\}$ we have $\varphi|_{(x_k, x_{k+1})} \in \mathcal{C}^{\omega}(x_k, x_{k+1})$ for all k, then all proper nodes are integers and the points 0 and N are proper nodes. Then, as in Lemma 2, we define s to be the largest integer such that $\varphi \in \mathcal{C}^{s-1}(\mathbb{R})$ (if $\varphi \notin \mathcal{C}(\mathbb{R})$, then $s=0$). If $\varphi|_{(x_k,x_{k+1})} \in \mathcal{C}^s(x_k, x_{k+1})$ for all k, then, as in Lemma 2, we check that for any node k both limits $\varphi^{(s)}(k-0)$ and $\varphi^{(s)}(k+0)$ exist and are finite. Then we repeat the proof of part a) and conclude that either φ is a spline, or $s = r(\omega)$ and the function $\varphi^{(s)}$ is continuous on $\mathbb{R}$. But this implies $\varphi \in \mathcal{C}^s(\mathbb{R})$, which contradicts the choice of s. It remains to consider the case where $\varphi|_{(x_j,x_{j+1})} \notin \mathcal{C}^s(x_j, x_{j+1})$ for some j. Then $s-1 = r(\omega)$, so that $\varphi \in \mathcal{C}^r(\mathbb{R})$, where $r = r(\omega)$. We show that for every node x_j there exists $\varepsilon_-^j > 0$ such that $\varphi_{(x_j - \varepsilon_-^j, x_j]} \in C^{\omega}(x_j - \varepsilon_-^j, x_j]$, which means that $|\varphi^{(r)}(x) - \varphi^{(r)}(y)| \le C|x-y|^{-r}\omega(|x-y|)$ for all $x, y \in (x_j - \varepsilon_-^j, x_j]$, $x \ne y$. Obviously, this is true for $x_0 = 0$. Let x_j be the smallest node for which the above claim fails. Then, as in the proof of Lemma 2, differentiating (3) r times yields a contradiction. Now we can use the adjoint function φ^* to show the existence of $\varepsilon_+^j > 0$ such that $\varphi_{[x_j, x_j+\varepsilon_+^j)} \in C^{\omega}[x_j, x_j + \varepsilon_+^j)$, $j = 0, \dots, N$. Thus, the function φ belongs to $\mathcal{C}^{\omega}$ on every interval $(x_j - \varepsilon_-^j, x_j + \varepsilon_+^j)$. This implies that $\varphi \in \mathcal{C}^{\omega}(\mathbb{R})$. This contradiction completes the proof. □

We conclude this section with the following proposition, which improves Theorem 2.

Proposition 1. *Let φ be a refinable function. If there exists $\varepsilon > 0$ such that the smoothness of φ on the interval $(0, \varepsilon)$ (or on the interval $(N-\varepsilon, N)$) exceeds its smoothness on $\mathbb{R}$, then φ is a refinable spline.*

Proof. We consider only the interval $(0,\varepsilon)$ (the case of $(N-\varepsilon, N)$ will then follow by taking the adjoint function φ^*, as above). For $x \in (0,1)$ equation (1) gives $\varphi\left(\frac{x}{2}\right) = c_0\varphi(x)$. Iterating n times, we get $\varphi(2^{-n}x) = c_0^n \varphi(x)$. Therefore, the smoothness of φ on the interval $(0, 2^{-n})$ is equal to that on the interval $(0,1)$. Taking n so large that $\varepsilon > 2^{-n}$, we see that the smoothness of φ on $(0,1)$ is equal to that on $(0,\varepsilon)$. Now we apply the same trick as in the proofs of Lemmas 1 and 2: denote by j the largest integer such that $j \le N-1$ and the smoothness of φ on each interval $(k, k+1)$, $k = 0, \dots, j$, is not less than that on $(0,\varepsilon)$. We know that $j \ge 0$. We assume that $j < N-1$ and apply (3) for $x_j = j$. The smoothness of each term on the right-hand side of (3) on the interval $h \in (0,1)$ is not less than the smoothness of φ on $(0,\varepsilon)$. Therefore, the same is true for the function $\varphi(j+h)$, which contradicts the choice of j. Thus, we have $j = N-1$, which implies that the smoothness of φ on any interval $(k, k+1)$, where k is an integer, exceeds its smoothness on $\mathbb{R}$. Applying Theorem 2, we conclude the proof. □

Corollary 4. *If for some $\varepsilon > 0$ a refinable function is $N-1$ times continuously differentiable on the interval $(0,\varepsilon)$ or on the interval $(N-\varepsilon, N)$, then this function is a spline given by* (2).

Remark 3. It is well known that the smoothness of a refinable function (not a spline) on some intervals $(a,b) \subset [0,N]$ can exceed its global smoothness. Moreover, there are examples of refinable functions that are not splines but coincide with polynomials on some intervals (see [2]). Proposition 1 shows that this situation can never occur at the extremities of the support, i.e., on the intervals $(0,\varepsilon)$ and $(N-\varepsilon, N)$. In an arbitrary neighborhood of the endpoints of the support, smoothness is the same as on the entire support!

§3. Structure of refinable splines and the convergence of the corresponding subdivision schemes

So, there are no piecewise-smooth refinable functions except for the splines classified in Theorem 1. In [8] it was shown that all such splines except for the cardinal B-splines, for which the polynomial P is equal to 1 identically, are unstable, i.e., their translates by integers are linearly dependent over the space of bounded sequences. In particular, for any such spline its integral translates do not form a Riesz basis of their linear span. Therefore, the splines of this sort cannot generate compactly supported wavelets (see [1]). Indeed, since any such spline is a linear combination of integral translates of the corresponding B-spline, it follows that its multiresolution analysis coincides with that of this B-spline. Thus, refinable splines do not give anything new for wavelets. However, they may be of interest for subdivision schemes, because the stability of the subdivision function is not necessary for the convergence of the corresponding scheme. In fact, the subdivision schemes corresponding to refinable splines may converge, in spite of their instability. For the first time, this phenomenon was observed in [6], where an example of a converging subdivision scheme corresponding to a refinable spline (not a B-spline) was treated. In [6] it was also noted that a refinable spline, even very smooth, may generate a divergent scheme (see [6, §6] for the corresponding examples). In this section we classify all refinable splines generating convergent schemes (Theorem 3). It turns out that for refinable splines convergence can be ensured by a very simple criterion. Moreover, we compute the rate of convergence explicitly. For this, first we characterize the manifold of refinable splines (Proposition 2).

We need some notation. A set $\mathbf{b} = \{\lambda_1, \dots, \lambda_n\}$ of distinct and nonzero complex numbers is said to be *cyclic* if $\lambda_{j+1} = \lambda_j^2$, $j = 1, \dots, n$ (we put $\lambda_{n+1} = \lambda_1$). Clearly, any cyclic set lies on the unit circle $\{z \in \mathbb{C}, |z| = 1\}$. The simplest cyclic set is $\{1\}$; then the two-element set $\{e^{2\pi i/3}, e^{4\pi i/3}\}$ appears, and so on. For any n there are finitely many cyclic sets of n elements. The reader will classify them easily. The smallest cyclic set $\{1\}$ is said to be *trivial*, all other cyclic sets are nontrivial.

Given a cyclic set $\mathbf{b}$, we construct the corresponding *cyclic tree* $\mathcal{T}_{\mathbf{b}}$ as follows: the elements $\lambda_1, \dots, \lambda_n$ of the set $\mathbf{b}$ form the *root* of the tree; they are vertices of level zero (we identify a vertex with the corresponding number). For any λ_j, exactly one of the two numbers $\pm\sqrt{\lambda_j}$ does not belong to $\mathbf{b}$. This number (or vertex) is a neighboring vertex to λ_j, and it belongs to the first level. So, there are n vertices on the zeroth level and n vertices on the first level. Further construction is by induction. Every vertex λ of the kth level ($k \geq 1$) has two neighbors $\pm\sqrt{\lambda}$ at the $(k+1)$st level. Thus, there are $n2^{k-1}$ vertices at the kth level. All vertices of the tree (i.e., the corresponding numbers) are different. Let $\mathcal{A}$ be a subset of vertices of the tree; we assume that $\mathcal{A}$ contains no elements of the root. Some elements of $\mathcal{A}$ may coincide, and we count them with their multiplicities. We say that $\mathcal{A}$ is a *cut set of multiplicity* r ($r \geq 1$) if it has no elements of the root and every infinite path along the tree $\alpha_0 \to \alpha_1 \to \cdots$ that starts at the root (α_k is at the kth level, all paths are without backtracking) has exactly r common elements with the set $\mathcal{A}$. For example, the set of elements of the first level is a cut set of multiplicity 1. It is easy to show that any cut set is finite.

For a given cut set $\mathcal{A}$, we denote by $\mathcal{B}$ the set of vertices blocked by $\mathcal{A}$, i.e., $\mathcal{B} \cap \mathcal{A} = \varnothing$, and every infinite path along the tree that starts at any element $b \in \mathcal{B}$ has at least one common element with $\mathcal{A}$. An element $b \in \mathcal{B}$ has multiplicity $s \geq 1$ if every path starting at b has exactly s common elements with $\mathcal{A}$ (counting multiplicities). All elements of blocked sets will also be counted with their multiplicities. Clearly, any blocked set contains the root, and the multiplicity of each element of the root is equal to the multiplicity of $\mathcal{A}$.

Remark 4. It is not difficult to characterize all cut sets of a given multiplicity r. Consider the set $\mathcal{A}_r$ formed by the elements of the first level taken with multiplicity r. This is a cut set of multiplicity r. We take an arbitrary element $\lambda \in \mathcal{A}_r$ and replace it by the pair of its neighbors at the next level: $\lambda \to \pm\sqrt{\lambda}$. We call this operation *transfer to the next level*. The cardinality of the set increases by 1, and the set remains a cut set of the same multiplicity r. It is not difficult to show that all cut sets of multiplicity r are obtained from $\mathcal{A}_r$ by several transfers to the next level.

The following fact is well known in combinatorics (see, e.g., [6, §8] and the references therein). The reader will have no difficulty in proving it on his own.

Lemma 3. *Any finite set $\mathcal{Z} \subset \mathbb{C}\backslash\{0\}$ such that $\mathcal{Z}^2 \subset \mathcal{Z}$ is a disjoint union of several sets $\mathcal{B}_1, \dots, \mathcal{B}_s$ blocked by the corresponding cut sets $\mathcal{A}_1, \dots, \mathcal{A}_s$ of some cyclic trees $\mathcal{T}_1, \dots, \mathcal{T}_s$.*

This fact enables us to classify all refinable splines given in Theorem 1.

Proposition 2. *For any refinable spline $S(x)$ of order $l \geq 0$ there exists a unique family of different cyclic trees $\mathcal{T}_j, j = 0, \dots, s$, and their cut sets $\mathcal{A}_j$ with multiplicities r_j such that*

1) *the tree T_0 has a trivial root, the other trees have nontrivial roots, and the cut set $\mathcal{A}_0$ has multiplicity $r_0 \geq l + 1$;*

2) *the spline $S(x)$ is obtained by formula* (2) *with the polynomial*

$$P(z) = (z-1)^{-l-1} \prod_{j=0}^{s} \prod_{b \in \mathcal{B}_j} (z - b), \tag{4}$$

where the $\mathcal{B}_j$ are the sets blocked by $\mathcal{A}_j$. The symbol of the corresponding refinement equation is

$$m(z) = 2^{-l-1} \prod_{j=0}^{s} \prod_{a \in \mathcal{A}_j} (z - a). \tag{5}$$

Conversely, for any family of different trees $\mathcal{T}_j$, $j = 0, \dots, s$, and their cut sets $\mathcal{A}_j$ of multiplicities $r_j \geq 1$, where the tree $\mathcal{T}_0$ has a trivial root, and for any $l \in \{0, \dots, r_0 - 1\}$, the solution of the refinement equation with symbol (5) *is a spline of order l. This spline is obtained by formula* (2) *with the polynomial P given by* (4).

This proposition characterizes the structure of the family of refinable splines; also, it enables us to answer the question about the convergence of the corresponding subdivision schemes and cascade algorithms, and, moreover, to find the rate of convergence. First, we recall some notation.

The subdivision operator Γ corresponding to a mask $\{c_0, \dots, c_N\}$ acts on the space of bounded sequences l_∞ by the formula $(\Gamma g)_k = \sum_i c_{k-2i} g_i$, where $g = (g_i)_{i \in \mathbb{Z}} \in l_\infty$ (here and below we set $c_k = 0$ for $k < 0$ and for $k > N$). The subdivision scheme converges for a given $g \in l_\infty$ if there exists a continuous function f such that $\|f(2^{-n}\cdot) - \Gamma^n g\|_\infty \to 0$ as $n \to \infty$, where $\|\cdot\|_\infty$ is the uniform norm of the space l_∞. The subdivision scheme is said to be convergent if it converges for every initial sequence $g \in l_\infty$. It is well known that if a subdivision scheme converges, then the corresponding refinement equation possesses a continuous solution φ and, moreover, for any $g \in l_\infty$ the limit function f is decomposed as follows:

$$f(\cdot) = \sum_k g_k \varphi(\cdot - k), \tag{6}$$

where the solution φ is normalized by the condition $\int \varphi(x)\,dx = 1$ (see [5]). The convergence of a subdivision scheme is equivalent to the convergence of the corresponding cascade algorithm, i.e., to the relation $\|T^n f_0 - \varphi\|_{\mathcal{C}} \to 0$ as $n \to \infty$, where $[Tf](x) = \sum_k c_k f(2x-k)$ is the so-called transition operator, and f_0 is an arbitrary compactly supported continuous function satisfying the partition of unity condition $\sum_k f_0(x-k) \equiv 1$ (this is necessary for convergence). Often, the role of f_0 is played by the function $\chi_{[0,1)}$ (see [1, 2]). The following conditions necessary for convergence are well known: the solution φ of the corresponding refinement equation must be continuous and the symbol m must satisfy $m(1) = 1$ and $m(-1) = 0$. In [6] it was shown that under these conditions the subdivision scheme converges provided the solution φ is stable. However, long ago it was observed that, actually, stability is not necessary. This means that, generally speaking, refinable splines (which are never stable except for the B-splines) are not forbidden to generate convergent subdivision schemes. We shall formulate the main result of this section in the general case, for the convergence in the spaces $\mathcal{C}^l$. We say that a cascade algorithm converges in $\mathcal{C}^l$ ($l \ge 0$) if the sequence $T^n f_0$ converges in this space (i.e., with the first $l-1$ derivatives) to φ for any initial function f_0 that is compactly supported, has bounded lth derivative, and satisfies $\sum_k f_0(x-k) \equiv 1$ and $\sum_k (x-k)^r f_0(x-k) \equiv \text{const}$, $r = 1, \ldots, l$. Often, the role of f_0 is played by B_l. The convergence of subdivision schemes in $\mathcal{C}^l$ is defined similarly by using the derived schemes (see [5, 6]). The rate of convergence of a cascade algorithm/subdivision scheme is $\nu = l + \liminf_{k\to\infty} -k^{-1}\log_2 \|T^k B_l - \varphi\|_{\mathcal{C}^l}$, where $l \ge 0$ is the largest integer for which this scheme converges in $\mathcal{C}^l$.

Theorem 3. *The cascade algorithm/subdivision scheme corresponding to a refinable spline converges in $\mathcal{C}^k$, $k \ge 0$, if and only if the following conditions are fulfilled*:

1) $r_0 = l + 1 \ge k + 2$;
2) $l - r_j \ge k$ *for all* $j = 1, \ldots, s$;
3) *the point* $z = -1$ *is contained in* $\mathcal{A}_0$ *with multiplicity* $r(-1) \ge k + 1$,

where l, $\mathcal{A}_j$, and r_j are as in Proposition 2. *The rate of convergence is given by the formula*

$$\nu = \min\{r(-1), l, r_0 - r_1, \ldots, r_0 - r_s\}. \tag{7}$$

Thus, the problem of computing the rate of convergence, which is very difficult for general subdivision schemes, becomes elementary for refinable splines. It suffices to decompose the roots of the symbol into cut sets $\mathcal{A}_j$ and to find their multiplicities.

Example 1. Consider the refinement equation with the following symbol:

$$m(z) = \Big(\frac{z+1}{2}\Big)^a \Big(\frac{z^2+1}{2}\Big)^b (z^2 - z + 1)^r,$$

where a, b, and r are some nonnegative integers. From Proposition 2 it follows that the corresponding refinable function φ is a spline of order $l = a + b - 1$. The roots of the symbol split into two cut sets $\mathcal{A}_0$ and $\mathcal{A}_1$. The set $\mathcal{A}_0$ of multiplicity $a + b$ is a cut set of the tree $\mathcal{T}_0$ with trivial root and contains $\{-1\}$ with multiplicity a and $\pm i = \pm\sqrt{-1}$ with multiplicity b, and the set $\mathcal{A}_1$ of multiplicity r is a cut set of the tree $\mathcal{T}_1$ with the root $\{e^{2\pi i/3}, e^{4\pi i/3}\}$ and contains two elements $\pm e^{\pi i/3}$ with multiplicity r. By Theorem 3, the corresponding subdivision scheme converges if and only if $a \ge 1$, $a + b \ge 2$, and $a + b \ge r + 1$. The rate of convergence is equal to $\nu = \min\{a, a+b-1, a+b-r\}$. For example, if $a = 0$, $b = 3$, and $r = 1$, then the function φ is a spline of order 2; φ is piecewise quadratic with integral nodes and the support of φ is $[0, 8]$. We have $\alpha_\varphi = 2$, but the scheme diverges, because $a = 0$. If $a = 2$, $b = 1$, and $r = 1$, then the function φ is again a piecewise quadratic spline with the same support and with the same regularity α_φ. In this case the scheme converges even in $\mathcal{C}^1$, and $\nu = 2$. If $a = 2$, $b = 1$, and $r = 2$, then φ is also a spline of order 2 with $\alpha_\varphi = 2$, the corresponding scheme converges in $\mathcal{C}$, but not in $\mathcal{C}^1$, and the rate of convergence is $\nu = 1$.

To prove Theorem 3, we need several auxiliary results. For a cut set $\mathcal{A}$ of some tree $\mathcal{T}$, denote by $P_{\mathcal{A}}(z)$ the polynomial $\prod_{a\in\mathcal{A}}(z-a)$ (the elements $a \in \mathcal{A}$ are counted with their multiplicities). Next, given a polynomial $Q(z)$ and an arbitrary finite set $Y \subset \mathbb{C}$, we put $\rho(Q,Y) = \prod_{y\in Y} Q(y)$. Clearly, $\rho(\cdot, Y)$ is a multiplicative function on the ring of polynomials. The reader can easily prove the following lemma by induction, or find the proof in [11].

Lemma 4. *Let $\mathcal{A}$ be a cut set of a cyclic tree $\mathcal{T}$ with root $\mathbf{b}$, and let r be the multiplicity of $\mathcal{A}$. Also, let $\mathbf{b}_1$ be an arbitrary cyclic set of cardinality n. Then*

$$\rho(P_{\mathcal{A}}, \mathbf{b}_1) = \begin{cases} 2^{rn} & \text{if } \mathbf{b}_1 = \mathbf{b}, \\ 1 & \text{if } \mathbf{b}_1 \neq \mathbf{b}. \end{cases} \tag{8}$$

The next statement was proved in [12].

Proposition 3. *A cascade algorithm/subdivision scheme converges in $\mathcal{C}^k$ ($k \geq 0$) if and only if the following three conditions are satisfied*:

a) *the solution φ of the corresponding refinement equation belongs to $\mathcal{C}^k$;*
b) *$m(1) = 1$, and the number $z = -1$ is a root of the symbol m of multiplicity $r(-1) \geq k+1$;*
c) *for any cut set $\mathcal{A}$ corresponding to a tree $\mathcal{T}$ with a nontrivial root $\mathbf{b}$ and satisfying $m(\mathcal{A}) = 0$ (i.e., m is divisible by the polynomial $P_{\mathcal{A}}$) we have $\rho(m, \mathbf{b}) < 2^{-k}$.*

The rate of convergence is given by the formula

$$\nu = \min\Big\{\alpha_\varphi, r(-1), -\frac{1}{n_j}\log_2|\rho(m, \mathbf{b}_j)|, j = 1, \dots, s\Big\}, \tag{9}$$

where α_φ is the Hölder exponent of φ, $\mathcal{A}_j$, $\mathbf{b}_j$, $j = 1, \dots, s$, are all possible cut sets and nontrivial roots as in c), *and n_j is the cardinality of $\mathbf{b}_j$.*

Proof of Theorem 3. Consider a refinable spline given by formula (2). Obviously, it belongs to $\mathcal{C}^{l-1}$ and does not belong to $\mathcal{C}^l$. Item a) of Proposition 3 implies that $l - 1 \geq k$, whence $l + 1 \geq k + 2$. Furthermore, the symbol m of the equation is given by (5), and by Proposition 2 we have $m = 2^{-l-1}\prod_{i=0}^{s} P_{\mathcal{A}_i}$. Applying Lemma 4 and using the multiplicativity of the function $\rho(\cdot, \mathbf{b}_j)$, we obtain $\rho(m, \mathbf{b}_j) = 2^{-(l+1)n_j}\prod_{i=0}^{s}\rho(P_{\mathcal{A}_i}, \mathbf{b}_j) = 2^{n_j(r_j-l-1)}$ for every $j = 0, \dots, s$. For the trivial root $b_0 = \{1\}$ this yields $m(1) = \rho(m, 1) = 2^{r_0-l-1}$. By Proposition 3 we have $m(1) = 1$, so that $r_0 = l + 1$. For the nontrivial cyclic sets $\mathbf{b}_j$, $j \geq 1$, we get $-\frac{1}{n_j}\log_2|\rho(m, \mathbf{b}_j)| = -\frac{1}{n_j}\log_2 2^{n_j(r_j-l-1)} = l + 1 - r_j = r_0 - r_j$. Now it remains to apply item c) of Proposition 3 and formula (9). ☐

Remark 5. Using the same arguments, it is easy to compute the rate of convergence of the subdivision schemes corresponding to splines in other function spaces, e.g., in L^p (see [13] for the definitions and properties of subdivision schemes in L^p).

§4. A factorization theorem for refinable functions

Theorem 2 in §2 restricts the set of piecewise smooth refinable functions. All but finitely many refinement equations of a given degree have “bad” irregular solutions. In this sense the result is negative. As a compensation, however, we can establish the following positive fact.

Theorem 4. *Any $\mathcal{C}^{l+1}$-refinable function φ ($l \geq 0$) can be presented as the convolution $\varphi = S * \widetilde{\varphi}$, where S is a refinable spline of order l and $\widetilde{\varphi}$ is a continuous refinable function. Moreover, $\alpha_\varphi = \alpha_{\widetilde{\varphi}} + l + 1$.*

Thus, any smooth refinable function is the convolution of a refinable spline and a continuous refinable function.

Remark 6. In terms of the smoothness of $\widetilde{\varphi}$, we can express not only that of φ, but also the rate of convergence of the corresponding subdivision scheme. By using Proposition 3 and Theorem 3, it can be shown that

$$\nu = \min\Big\{ r(-1), \widetilde{\nu} + l + 1, -\frac{1}{n_j} \log_2 |\rho(\widetilde{m}, \mathbf{b}_j)| + r_0 - r_j, j = 1, \dots, s \Big\}, \tag{10}$$

provided $m(1) = 1$, where $r(-1)$ is the multiplicity of the root $z = -1$ of the symbol m.

Remark 7. Theorem 4 reduces the study of smooth refinable functions to continuous refinable functions. Results in this direction can be found in many publications on refinement equations and wavelets, starting with the classical paper [6]. For refinable functions with orthogonal translates by integers, the statement of Theorem 4 is well known and was mentioned in [2, 14]. In fact, this result can be extended without essential modifications to all stable refinable functions (see [3]). In both cases, the spline $S(x)$ is a B-spline and the polynomial P is equal to 1 identically. As to general refinement equations, for $l = 0$ an analog of Theorem 4 was established in [2]. In [10] Villemoes proved that any refinable function is a linear combination of integral shifts of a stable refinable function having the same smoothness. In [12] it was shown that any smooth refinable function is the convolution of a continuous (and even stable) refinable function and a spline. However, in contrast to Theorem 4, this spline is not necessarily refinable, i.e., it may fail to satisfy a refinement equation. In this context we note that the statement of Theorem 4 cannot be improved in the sense that the function $\widetilde{\varphi}$ may fail to be stable. In fact, there are equations whose smooth solutions cannot be written as the convolution of a refinable spline and a stable refinable function.

Caveretta, Dahmen, and Micchelli (see [6, §2]) were the first to observe that convolution with a smooth refinable function increases both smoothness and the rate of convergence of subdivision schemes. Therefore, taking the convolution with, say, a refinable spline is a convenient way to produce smooth refinable functions and rapidly convergent schemes. The following problem was stated in [6, Remark 2.6]: Is the converse true? Is taking a convolution the only way to obtain smooth refinable functions? Can any smooth refinable function φ be written as some convolution of simpler refinable functions? For univariate refinement equations, Theorem 4 answers this question in the affirmative. Moreover, we see that one of the functions to be convolved can always be taken as a refinable spline whose order is smaller by 1 than the smoothness of φ. Theorem 4 and formula (10) express smoothness and the rate of convergence in terms of the corresponding parameters of the convolution operands. Here, of course, it should be mentioned that these results concern only univariate refinement equations, while the general question was stated in [6] for the multivariate case.

Proof of Theorem 4. Again we refer to the paper [12], where it was shown that the symbol of a refinement equation admitting a $\mathcal{C}^{l+1}$-solution can be decomposed as $m(z) = \widetilde{m}(z)2^{-l-1}\prod_{j=0}^{s}\prod_{a\in\mathcal{A}_j}(z-a)$, where $\widetilde{m}$ is a polynomial such that $\widetilde{m}(1) = 1$ and the $\mathcal{A}_j$ are cut sets of some trees $\mathcal{T}_j$; the root of the tree $\mathcal{T}_0$ is trivial and $r_0 \geq l + 1$. Moreover, the solution $\widetilde{\varphi}$ of the refinement equation with the symbol $\widetilde{m}$ is continuous, and $\alpha_{\widetilde{\varphi}} = \alpha_{\varphi} - l - 1$. It remains to note that the solution of the refinement equation with the symbol $2^{-l-1}\prod_{j=0}^{s}\prod_{a\in\mathcal{A}_j}(z-a)$ is a refinable spline, which will be denoted by S (Proposition 2). Then the function $S*\widetilde{\varphi}$ satisfies the refinement equation with the symbol m, and this completes the proof. $\square$

Remark 8. The factorization theorem can be improved in several ways. For example, its complete analog is valid in the space W_p^{l+1}. Moreover, in fact, any $\mathcal{C}^{l+1}$-refinable function involves a spline of order $l+1$, and not merely l. More precisely: any $\mathcal{C}^k$-refinable function φ $(k \geq 0)$ is the convolution $\varphi = S * \widetilde{\varphi}$, where S is a refinable spline of order k and $\widetilde{\varphi}$ is a refinable function. In particular, any continuous refinable function involves a refinable step function. This fact is proved in the same way as Theorem 4. However, in contrast to Theorem 4, the function $\widetilde{\varphi}$ is not necessarily continuous and, in general, is a distribution, possibly irregular.

§5. Application to combinatorial number theory

Theorem 2, in which the piecewise-smooth refinable functions are classified, has quite an unexpected application in combinatorial number theory. We use this theorem to solve a problem concerning the asymptotics of the Euler binary partition function.

For an arbitrary integer $d \geq 2$, the binary partition function $b(k) = b(d,k)$ is defined on the set of nonnegative integers k as the total number of different binary expansions $k = \sum_{j=0}^{\infty} d_j 2^j$, where the "digits" d_j take values in the set $0, \dots, d-1$. For $d = \infty$ the value $b(k)$ is the number of such expansions with arbitrary nonnegative integral digits. For the first time, the partition function appeared for $d = \infty$ in the work of Leonhard Euler (see [15]), in connection with some power series. Clearly, for $d = 2$ we have $b(k) \equiv 1$. For $d \geq 3$ such a binary expansion is not necessarily unique, and the following problem arises: to characterize the total number $b(k)$ of these expansions asymptotically as $k \to \infty$. In the context of various problems of number theory and power series, this problem was studied by Mahler [16], de Bruijn [17], Reznick [9], Knuth [18]; see also [19, 20, 21]. Most of those papers were devoted to the case where $d = \infty$. The first results for finite d were obtained by Tanturri in 1918 (see [20] and two references therein). In [9] Reznick showed that if $d = 2^{r+1}$, where $r \geq 0$ is an integer, then the partition function has the following simple asymptotics: $b(k) = C_r k^r + o(k^r)$ as $k \to \infty$. Here C_r is an effective constant and, as usual, $o(x)$ means a value that tends to zero after being divided by x as $x \to \infty$. It was also noted in [9] that this asymptotics can also be derived from the results of Tanturri. For the other even $d = 2n$ we have

$$C_n^1 k^{\log_2 n} \leq b(k) \leq C_n^2 k^{\log_2 n},$$

where C_n^1, C_n^2 are positive constants (see [9]). We denote

$$\mu_1 = \liminf_{k\to\infty} k^{-\log_2 n} b(k); \quad \mu_2 = \limsup_{k\to\infty} k^{-\log_2 n} b(k).$$

Since $C_n^1 \leq \mu_1 \leq \mu_2 \leq C_n^2$, we see that for any n the values μ_1 and μ_2 are both positive and finite. If n is an integral power of 2, then $\mu_1 = \mu_2$. So, in this case the partition function has a very simple asymptotic behavior: $b(k) \sim c k^{\log_2 n}$ as $k \to \infty$, where $c = \mu_1 = \mu_2$. However, for general n this is not always the case. In [9] Reznick showed (referring also to the earlier paper [22] by Carlitz) that if $d = 6$ and $n = 3$, then $\mu_1 \neq \mu_2$. The question for other values of n was formulated as an open problem. Is it true that $\mu_1 = \mu_2$ only for the numbers n that are integral powers of 2? The following theorem gives the answer.

Theorem 5. *If $\mu_1 = \mu_2$, then $n = 2^r$ for some integer $r \geq 0$.*

Remark 9. For odd values of d the asymptotic behavior of $b(k)$ is more complicated (see [9, 21]).

We shall prove this theorem and, moreover, express the values μ_1 and μ_2 in terms of a special refinable function (Proposition 4 and Corollary 5). Also, we indicate a method of computing these values for any n. Before doing this, we make some observations. First, we extend the function $b(k)$ to the negative values of k by zero: $b(k) = 0$ for all integers $k < 0$. It can easily be checked that for every $k \in \mathbb{Z}$ we have the following recurrence relations:

$$b(2k) = \sum_{j=0}^{n-1} b(k-j); \quad b(2k+1) = \sum_{j=0}^{n-1} b(k-j). \tag{11}$$

These formulas look very similar to iterations of a subdivision algorithm. Indeed, if we take the initial sequence $g_0 = 1, g_i = 0$ for $i \neq 0$, then for every $j \geq 0$ we obtain

$$\left(\Gamma^j g\right)_k = n^{-j} b(k) \quad \text{for all } k \leq 2^j - 1,$$

where Γ is the subdivision operator related to the mask $\{c_0, \ldots, c_{2n-1}\} = \{\frac{1}{n}, \ldots, \frac{1}{n}\}$. This is shown by simple induction on j. For $j = 0$, indeed we have $(\Gamma^0 g)_k = g_k = b(k)$ for all $k \leq 2^0 - 1 = 0$. Passage from j to $j+1$ is realized directly by formulas (11).

A subdivision scheme with positive coefficients always converges provided $m(1) = 1$, $m(-1) = 0$ (see [23]). Therefore, so does the scheme with the mask $\{c_0, \ldots, c_{2n-1}\} = \{\frac{1}{n}, \ldots, \frac{1}{n}\}$. From (6) it follows that this scheme converges to a continuous solution φ of the corresponding refinement equation and $\int \varphi(x)\,dx = 1$. By the definition of convergent schemes, we have $\|\Gamma^j g - \varphi(2^{-j}\cdot)\|_\infty \to 0$ as $j \to \infty$. Denoting $\delta_j = \max_{k \leq 2^j - 1} |n^{-j} b(k) - \varphi(2^{-j}k)|$, we see that $\delta_j \to 0$ as $j \to \infty$.

This proves the following statement.

Lemma 5. *For every $j \geq 0$ and $k \leq 2^j - 1$ we have*

$$\left| k^{-\log_2 n} b(k) - \left(2^{-j}k\right)^{-\log_2 n} \varphi\left(2^{-j}k\right) \right| \leq (2^{-j}k)\delta_j,$$

where $\delta_j \to 0$ as $j \to \infty$.

If we take k in the segment $[2^{j-s}, 2^{j-s+1}]$, where $s \geq 1$ is an integer, then $\frac{k}{2^j} \in [2^{-s}, 2^{1-s}]$. Now we denote $M(s,j) = \{\frac{k}{2^j}, 2^{j-s} \leq k \leq 2^{j-s+1}\}$ and $\psi(x) = x^{-\log_2 n}\varphi(x)$. Since φ is continuous, $\psi(x)$ is uniformly continuous on the segment $[2^{-s}, 2^{1-s}]$. Therefore, $\min_{x \in [2^{-s}, 2^{1-s}]} \psi(x) = \lim_{j \to +\infty} \min_{x \in M(s,j)} \psi(x)$. By Lemma 5, the quantity $\min_{x \in M(s,j)} \psi(x)$ is equivalent to $\min_{2^{j-s} \leq k \leq 2^{j-s+1}} k^{-\log_2 n} b(k)$ as $j \to \infty$. Clearly,

$$\lim_{j \to \infty} \min_{2^{j-s} \leq k \leq 2^{j-s+1}} k^{-\log_2 n} b(k) = \liminf_{k \to \infty} k^{-\log_2 n} b(k).$$

Thus,

$$\inf_{x \in [2^{-s}, 2^{1-s}]} \psi(x) = \liminf_{k \to \infty} k^{-\log_2 n} b(k) = \mu_1.$$

The same argument shows that $\sup_{x \in [2^{-s}, 2^{1-s}]} \psi(x) = \mu_2$. Thus, we have found expressions for μ_1 and μ_2.

Proposition 4. *Let $n \geq 1$ be an integer, and let $b(k) = b(2n, k)$ be the corresponding partition function. Then for any integer $s \geq 1$ we have*

$$\mu_1 = \min_{x \in [2^{-s}, 2^{1-s}]} \psi(x); \quad \mu_2 = \max_{x \in [2^{-s}, 2^{1-s}]} \psi(x),$$

where $\psi(x) = x^{-\log_2 n}\varphi(x)$, and φ is the continuous solution of the refinement equation with the symbol $m(z) = \frac{1}{2n}\sum_{k=0}^{2n-1} z^k$.

These formulas make it possible to compute both μ_1 and μ_2 with an arbitrarily prescribed accuracy. For this, we need only to approximate the function φ. This can be done, e.g., with the help of subdivision schemes.

Corollary 5.

$$\mu_1 = \inf_{x \in (0,1)} \psi(x); \quad \mu_2 = \sup_{x \in (0,1)} \psi(x).$$

Proof of Theorem 5. If $\mu_1 = \mu_2$, then $\inf_{x\in(0,1)} x^{-\log_2 n}\varphi(x) = \sup_{x\in(0,1)} x^{-\log_2 n}\varphi(x)$ by Corollary 5. Hence, on the interval $(0,1)$ we have $\varphi(x) \equiv Cx^{\log_2 n}$, where C is a constant. Since this function is infinitely smooth on $(0,1)$, Corollary 4 shows that φ is a refinable spline. Consequently, $\log_2 n$ is an integer, which completes the proof. $\square$

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References

[1] I. Daubechies and J. Lagarias, *Two-scale difference equations.* I. *Existence and global regularity of solutions*, SIAM J. Math. Anal. **22** (1991), 1388–1410. MR1112515 (92d:39001)

[2] L. Berg and G. Plonka, *Some notes on two-scale difference equations*, Functional Equations and Inequalities, Math. Appl., vol. 518, Kluwer Acad. Publ., Dordrecht, 2000, pp. 7–29. MR1792070 (2002f:39049)

[3] V. Protasov, *A complete solution characterizing smooth refinable functions*, SIAM J. Math. Anal. **31** (2000), no. 6, 1332–1350 (electronic). MR1766557 (2001j:42033)

[4] I. Daubechies and J. Lagarias, *Two-scale difference equations.* II. *Local regularity, infinite products of matrices and fractals*, SIAM J. Math. Anal. **23** (1992), 1031–1079. MR1166574 (93g:39001)

[5] N. Dyn, J. A. Gregory, and D. Levin, *Analysis of uniform binary subdivision schemes for curve design*, Constr. Approx. **7** (1991), 127–147. MR1101059 (92d:65027)

[6] A. S. Cavaretta, W. Dahmen, and C. A. Micchelli, *Stationary subdivision*, Mem. Amer. Math. Soc. **93** (1991), no. 453, 186 pp. MR1079033 (92h:65017)

[7] L. Berg and G. Plonka, *Spectral properties of two-slanted matrices*, Results Math. **35** (1999), no. 3–4, 201–215. MR1694902 (2000c:15009)

[8] W. Lawton, S. L. Lee, and Z. Shen, *Characterization of compactly supported refinable splines*, Adv. Comput. Math. **3** (1995), no. 1–2, 137–145. MR1314906 (95m:41020)

[9] B. Reznick, *Some binary partition functions*, Analytic Number Theory (Allerton Park, IL, 1989), Progr. Math., vol. 85, Birkhäuser Boston, Boston, MA, 1990, pp. 451–477. MR1084197 (91k:11092)

[10] L. Villemoes, *Wavelet analysis of refinement equations*, SIAM J. Math. Anal. **25** (1994), no. 5, 1433–1460. MR1289147 (96f:39009)

[11] V. Protasov, *The correlation between the convergence of subdivision processes and solvability of refinement equations*, Algorithms for Approximation IV (Proc. of the 2001 Internat. Sympos., Huddersfield, England, July 15–20, 2001), pp. 394–401.

[12] ———, *The stability of subdivision operator at its fixed point*, SIAM J. Math. Anal. **33** (2001), no. 2, 448–460 (electronic). MR1857979 (2002h:26023)

[13] R. Q. Jia, *Subdivision schemes in L_p spaces*, Adv. Comput. Math. **3** (1995), no. 4, 309–341. MR1339166 (96d:65028)

[14] I. Daubechies, *Ten lectures on wavelets*, CBMS-NSF Regional Conf. Ser. in Appl. Math., vol. 61, SIAM, Philadelphia, PA, 1992, 357 pp. MR1162107 (93e:42045)

[15] L. Euler, *Introductio in analysis infinitorum*, Opera Omnia Ser. Prima Opera Math., vol. 8, Teubner, Leipzig, 1922.

[16] K. Mahler, *On a special functional equation*, J. London Math. Soc. **15** (1940), 115–123. MR0002921 (2:133e)

[17] N. G. de Bruijn, *On Mahler's partition problem*, Indag. Math. (N.S.) **10** (1948), 210–220. MR0025502 (10:16d)

[18] D. E. Knuth, *An almost linear recurrence*, Fibonacci Quart. **4** (1966), 117–128. MR0199168 (33:7317)

[19] R. F. Churchhouse, *Congruence properties of the binary partition function*, Proc. Cambridge Philos. Soc. **66** (1969), 371–376. MR0248102 (40:1356)

[20] A. Tanturri, *Sul numero delle partizioni d'un numero in potenze di* 2, Atti. Accad. Naz. Lincei **27** (1918), 399–403.

[21] V. Yu. Protasov, *Asymptotics of the partition function*, Mat. Sb. **191** (2000), no. 3, 65–98; English transl., Sb. Math. **191** (2000), no. 3–4, 381–414. MR1773255 (2001h:11134)

[22] L. Carlitz, *Generating functions and partition problems*, Proc. Sympos. Pure Math., vol. 8, Amer. Math. Soc., Providence, RI , 1965, pp. 144–169. MR0175796 (31:72)

[23] C. A. Micchelli and H. Prautzsch, *Uniform refinement of curves*, Linear Algebra Appl. **114/115** (1989), 841–870. MR0986909 (90k:65088)

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