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ESTIMATES OF DEVIATIONS FROM EXACT SOLUTIONS FOR BOUNDARY-VALUE PROBLEMS WITH INCOMPRESSIBILITY CONDITION

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Dedicated to the memory of O. A. Ladyzhenskaya

Abstract. Methods of estimating the difference between exact and approximate solutions are considered for boundary-value problems in spaces of solenoidal functions. The estimates obtained apply to any functions in the energy space of the respective problem, and their computation requires solving only finite-dimensional problems. In the paper, two different methods are considered: one involves variational formulations and duality theory, and in the other, estimates are obtained from the integral identities that define generalized solutions of the problems in question. It is shown that estimates of deviations from an exact solution must include an additional penalty term with a factor determined by the constant in the Ladyzhenskaya–Babuška–Brezzi condition.

§1. Introduction

This paper is devoted to estimates of the difference between weak solutions of boundary-value problems and arbitrary functions in the respective function classes that contain these solutions. Such estimates are always required if we need to have reliable information on the accuracy of an approximate computed solution. The classical convergence theory for numerical approximations of partial differential equations makes it possible to construct asymptotic estimates of approximation errors. The ultimate aim in that theory is to prove that the difference between an exact solution u and an approximate solution u_n found in a finite-dimensional subspace of dimension n tends to zero as $n \to \infty$. The proof of asymptotic convergence can be regarded as a justification of the mathematical consistency of the approximation method in question. More precise estimates show the convergence rate, i.e., establish inequalities of the form

$$\|u - u_n\| \leq C(1/n)^k, \tag{1.1}$$

where C and k are some positive numbers. The constant C depends on the solution u and the type of approximations involved. Often, estimates of the form (1.1) are called *a priori* error estimates. Asymptotic *a priori* estimates show the convergence rate for the entire class of approximations in question. However, with the help of them we cannot reliably evaluate the error bound for a specific approximate solution. Furthermore, estimates of type (1.1) are usually based on some additional regularity of exact solutions, which may be absent in many practically important cases. The above reasons stimulated developing

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new error estimation methods able to correctly characterize the accuracy of approximate solutions and applicable both to problems with sufficiently regular solutions and to those without additional regularity. For partial differential equations these questions started receiving serious attention in the 1980s. Nowadays, the investigations devoted to *a posteriori* error control form a new important line in numerical analysis; the purpose is to obtain computable estimates of various norms of the difference between approximate and exact solutions, and also to construct error indicators that show distributions of local errors.

The main approaches to deriving such estimates can be demonstrated by the example of the following simple problem. Let Ω be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$, and let f be a given function in $L_2(\Omega)$. Here and in what follows the norm in the space of square integrable functions in Ω is denoted by $\|\cdot\|$. The weak solution of the first boundary-value problem for the Laplace operator is defined as a function $u \in V_0$ that satisfies the integral identity

$$\int_\Omega \nabla u \cdot \nabla w \, dx = \int_\Omega f w \, dx, \quad w \in V_0, \tag{1.2}$$

where $\cdot$ denotes the scalar product of vectors, and V_0 is the subspace of $W_2^1(\Omega)$ that consists of functions with zero traces on the boundary. Let $v \in V_0$ be an approximate solution of this problem. Then

$$\int_\Omega \nabla(u-v) \cdot \nabla w \, dx = \mathcal{F}_v(w), \quad w \in V_0, \tag{1.3}$$

where $\mathcal{F}_v(w) := \int_\Omega (fw - \nabla v \cdot \nabla w)\, dx$ is a linear functional on V_0 (here the symbol $:=$ means "equals by definition"). It is easily seen that this functional is equal to zero if v coincides with u. In all other cases, the norm of this functional, defined by the relation

$$[\![\mathcal{F}_v]\!] := \sup_{w \in V_0, w \neq 0} \frac{|\mathcal{F}_v(w)|}{\|\nabla w\|}, \tag{1.4}$$

is positive. Therefore, it is natural to call $\mathcal{F}_v$ the *error functional.* Let us show that this norm is indeed a measure of the deviation of v from u. By (1.3) and (1.4), we have

$$\int_\Omega |\nabla(u-v)|^2 \, dx = \mathcal{F}_v(u-v) \leq [\![\mathcal{F}_v]\!] \, \|\nabla(u-v)\|,$$

whence

$$\|\nabla(u-v)\| \leq [\![\mathcal{F}_v]\!]. \tag{1.5}$$

However, we have $|\mathcal{F}_v(w)| \leq \|\nabla(u-v)\| \|\nabla w\|$, so that (1.4) implies the inequality opposite to (1.5). Thus, the norm of the deviation from the exact solution coincides with the norm of $\mathcal{F}_v$. The problem is in the computation of such a norm in practice for a specific v. Clearly, a straightforward computation of the norm with the help of (1.4) is hardly possible. A more promising way is to find some computable upper bounds of $[\![\mathcal{F}_v]\!]$.

In the 1980s and 1990s, in numerical analysis, the so-called *residual method* was used quite often to estimate errors of finite element approximations. This method was suggested in the paper [17] of Babuška and Rheinboldt and in some other publications of these authors.

Further development of this method was exposed in numerous papers of various authors and summarized in the books by Ainsworth and Oden [14], Babuška and Strouboulis [18], and Verfürth [58].

Basically, this method suggests the following way of finding an upper bound for the error functional. First, we assume that v is a Galerkin approximation on a finite-dimensional space $V_h \in V_0$, i.e., $v = u_h$, where

$$\int_\Omega \nabla u_h \cdot \nabla w_h \, dx = \int_\Omega f w_h \, dx, \quad w_h \in V_h.$$

In this case,

$$\begin{aligned} \mathcal{F}_{u_h}(w) &= \int_\Omega \nabla(u - u_h) \cdot \nabla w \, dx \\ &= \int_\Omega \nabla(u - u_h) \cdot \nabla(w - \pi_h w) \, dx, \end{aligned}$$

where π_h is a continuous operator mapping V_0 to V_h (e.g., the Clement interpolation operator [29]). Suppose Ω is split into a collection of subdomains Ω_k, $k = 1, 2, \ldots, M$, and u_h is a smooth function in each subdomain. Then

$$\begin{aligned} \mathcal{F}_{u_h}(w) := &\sum_{k=1}^{M} \int_{\Omega_k} \Delta(u_h - u)(w - \pi_h w) \, dx \\ &+ \sum_{k,l=1}^{M} \int_{\partial\Omega_{kl}} \left[\frac{\partial(u - u_h)}{\partial \nu_{kl}} \right] (w - \pi_h w) \, dx, \end{aligned}$$

where $\partial\Omega_{kl}$ is a common part of the boundaries of Ω_k and Ω_l, ν_{kl} is the unit normal to this boundary, and $[\ ]$ denotes the jump that arises at the boundary $\partial\Omega_{kl}$. If Ω_k are simplexes, then for π_h we have the following estimates (see [29]):

$$\begin{aligned} \|v - \pi_h v\|_{\Omega_k} &\leq \gamma_{1k} \operatorname{diam}(\Omega_k) \|v\|_{1,2,\omega_{1k}(\Omega_k)}, \\ \|v - \pi_h v\|_{\partial\Omega_{kl}} &\leq \gamma_{2k} |\partial\Omega_{kl}|^{1/2} \|v\|_{1,2,\omega_{2k}(\Omega_k)}, \end{aligned}$$

where the right-hand sides involve norms in the space W_2^1, ω_{1k} and ω_{2k} are certain domains containing Ω_k, and the constants γ_{1k} and γ_{2k} depend not only on Ω_k, but also on the form of neighboring subdomains (simplexes). These estimates allow us to present the norm of the error functional as the sum of the local quantities

$$\eta_k^2 := \operatorname{diam}(\Omega_k)^2 \|\Delta_h + f\|_{\Omega_k}^2 + \frac{1}{2} \sum_{l=1}^{M} |\partial\Omega_{kl}| \left\| \left[\frac{\partial u_h}{\partial \nu_{kl}} \right] \right\|_{\partial\Omega_{kl}}^2 \tag{1.6}$$

that are related to the residual on Ω_k and to the jumps of the normal component of the gradient on the boundary. Accordingly, the overall error can be estimated as follows:

$$| \mathcal{F}_{u_h} | \leq \left(\sum_{k=1}^{M} c_k \eta_k^2 \right)^{1/2}, \tag{1.7}$$

where the constants c_k depend on all γ_{1k} and γ_{2k} associated with the simplexes in question. The principal difficulty of this approach to the estimation of $| \mathcal{F}_{u_h} |$ is clearly seen even in this simple example. This difficulty consists in the necessity to find a large number of local constants c_k, which depend on how the domain was discretized and, consequently, change if one sampling is replaced by another. In [26], Carstensen and Funken showed that the exact evaluation of these constants may be a fairy hard task, and using approximate values of γ_{1k} and γ_{2k} may lead to a highly excessive estimate of the total error. In spite of these difficulties, the residual method is widely used in numerical analysis because the quantities η_k are often used as error indicators that show the distribution of local errors in Ω rather than a guaranteed upper bound of the total error.

Naturally, error control problems have attracted much attention of specialists working in computational hydrodynamics. A significant part of the difficulties arising in the process of solving such problems is related to the *incompressibility condition*. Typically, this condition is taken into account by projecting a discrete solution to the set of solenoidal fields or by introducing appropriate penalty terms (see, e.g., Chorin [28], Weinan E and Liu [31], Girault and Raviart [34], Heywood and Rannacher [35], Rannacher [45, 46], Shen [56], Temam [57]). Stationary problems are often solved by passing to a minimax formulation and by using the so-called mixed approximations for the velocity and pressure fields (see, e.g., Brezzi and Douglas [23], Brezzi and Fortin [24]).

For approximations of the Stokes problem, *a posteriori* error estimates constructed by various types of finite element methods were obtained in numerous papers, mainly in the framework of certain modifications of the residual method (see, e.g., Ainsworth and Oden [13], Bank and Welfert [19], Dari, Durán, and Padra [30], Carstensen and Funken [27], Heywood and Rannacher [36], Johnson and Rannacher [37], Johnson, Rannacher and Boman [38], Oden, Wu, and Ainsworth [43], Padra [44], Verfürth [59, 60]). In these estimates, the right-hand side is given by the sum of local quantities η_k that unlike (1.7) involve additional terms that take into account violations of the incompressibility condition.

It should be noted that, for finite element approximations, *a posteriori* estimates were also constructed by special type averagings of approximate solutions and by using the so-called superconvergence phenomenon (see, e.g., Oganesjan and Ruhovec [6], Zlámal [62], Wahlbin [61]). Such a phenomenon may arise if solutions have a higher regularity, which results in certain limitations for applicability. Starting with the paper [63] by Zienkiewicz and Zhu, various averagings of approximate solutions were widely used to obtain indicators of the accuracy of approximate solutions (see, e.g., Carstensen and Bartels [25]). Note that these methods as well as the residual method are justified mathematically only for the Galerkin approximations. Moreover, such estimates give only error indicators and usually provide no guaranteed error bounds.

Computable error estimates of a different type were obtained in [8]–[11], [47]–[55], and some other papers. These estimates apply to any approximations of the energy function class and contain no constants depending on discretization and the solution method. Therefore, they can be called *functional type a posteriori* estimates, or *estimates of deviations from exact solutions*. For variational problems, such estimates were obtained with the help of the duality theory of the calculus of variations (this method was analyzed in detail in [8, 11, 48]). For example, the corresponding *a posteriori* estimate for (1.2) can be presented in the following form:

$$\|\nabla(u-v)\| \leq \|\nabla v - y\| + C_F\|\operatorname{div} y + f\|. \tag{1.8}$$

Here C_F is a constant in the Friederichs inequality $\|w\| \leq C_F\|\nabla w\|$ for $w \in V_0$, and $y(x)$ is an arbitrary function in the space $H(\Omega, \operatorname{div})$ of all functions in $L_2(\Omega, \mathbb{R}^d)$ that have square integrable divergence. Estimate (1.8) is valid for any $v \in V_0$ and involves only one constant C_F depending on the domain Ω. It is easy to observe that the right-hand side of (1.8) is nonnegative and vanishes if and only if $v = u$ and $y = \nabla u$. Moreover, this estimate is sharp in the sense that the function y can always be taken in such a way that the right-hand side of (1.8) is equal to the left-hand side. In terms of the variational approach, the function y has a clear meaning: y is the basic variable of the dual variational problem. In [11], it was shown that such estimates can also be deduced from variational identities. In essence, this way can be viewed as another method of finding upper estimates of the norm (1.4). Indeed,

$$|\mathcal{F}_v(w)| \leq \left| \int_\Omega (fw - y \cdot \nabla w)\,dx \right| + \left| \int_\Omega (y - \nabla v) \cdot \nabla w\,dx \right|.$$

If $y \in H(\Omega, \operatorname{div})$, then the first and the second integrals are estimated by the quantities $C_F \|\operatorname{div} y + f\| \|\nabla w\|$ and $\|\nabla v - y\| \|\nabla w\|$, respectively, which leads to (1.8).

Estimate (1.8) and other similar estimates can be regarded as generalizations of the classical energy estimates that involve the function v, together with the data of the problem (see, e.g., Ladyzhenskaya [1, 2], Ladyzhenskaya and Ural′tseva [5])).

Note that the right-hand side of (1.8) is a functional depending on v and y. Its exact lower bound is zero and is attained if and only if $v = u$ and $y = \nabla u$. By choosing suitable subspaces for v and y, we can minimize this functional with respect to these variables. Then, the corresponding value of the functional controls the difference between the exact solution and the approximation obtained.

For the Stokes problem, estimates of the type (1.8) were derived for the first time in [51], where the variational approach was used. In this paper, we consider both methods (variational and nonvariational) and show how such estimates can be obtained for other models of viscous incompressible fluids. The outline of the paper is as follows. In §2, we obtain estimates of the deviation from the exact solution for the stationary Stokes problem and some other linear problems. In §3, such estimates are obtained in the general form for problems whose solution lies in a certain subspace. The method used in §§2 and 3 is based upon transformations of the integral identities that define weak solutions of the problems under study. In §4 we use the variational method to study stationary problems in the theory of nonlinear viscous fluids. Here, we consider models with dissipative potentials representable as the sum of a quadratic and a convex functional. Such models belong to the class of the so-called generalized Newtonian fluids introduced by O. A. Ladyzhenskaya (see [3, 40]). Estimates are obtained both for approximations satisfying the incompressibility condition and for those that violate it. In the latter case, the error majorant involves an additional term that presents a penalty for possible violation of the incompressibility condition. The coefficient of this term depends on the constant in the well-known *Ladyzhenskaya–Babuška–Brezzi* condition. It is shown that for the Stokes problem these estimates coincide with those obtained in §2 by the nonvariational method.

§2. Stokes problem

2.1. Ladyzhenskaya–Babuška–Brezzi condition. It is natural to start the consideration of methods of obtaining deviation estimates for boundary-value problems that involve the incompressibility condition with the stationary Stokes problem. Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a bounded domain. We need to find a vector-valued function u (velocity), a tensor-valued function σ (stress), and a scalar function p (pressure) that satisfy the system

$$-\operatorname{div} \sigma + \nabla p = f \quad \text{in } \Omega, \tag{2.1}$$

$$\sigma = \nu \varepsilon(u) \quad \text{in } \Omega, \tag{2.2}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega, \tag{2.3}$$

$$u = u_0 \quad \text{on } \partial\Omega. \tag{2.4}$$

This system describes a slow motion of an incompressible fluid whose viscosity is characterized by a positive constant ν.

In what follows, we denote by $\dot{J}^\infty(\Omega)$ the set of smooth solenoidal functions with compact support in Ω. The closure of $\dot{J}^\infty(\Omega)$ with respect to the norm $\|\nabla v\|$ gives the space $\mathring{J}_2^1(\Omega)$. Let $V := W_2^1(\Omega, \mathbb{R}^d)$, and let $\Sigma := L_2(\Omega, \mathbb{M}^{d\times d})$, where $\mathbb{M}^{d\times d}$ denotes the space of symmetric $(d \times d)$-matrices (tensors); the scalar product of such tensors is denoted by two dots (i.e., a colon). V_0 is the subspace of V that consists of functions with zero traces on $\partial\Omega$. The affine set $V_0 + u_0$ is the subset of V formed by all functions $w+u_0$, where $w \in V_0$, and u_0 is a given solenoidal vector-valued function. Similarly, $\mathring{J}_2^1(\Omega) + u_0$ consists of the functions $w + u_0$, where $w \in \mathring{J}_2^1(\Omega)$. The tensor of small deformations $\varepsilon(v) := \frac{1}{2}(\nabla v + (\nabla v)^T)$ is viewed as an operator from V to Σ. Also, we introduce the Hilbert space $\Sigma_{\mathrm{div}}(\Omega)$, that is, the subspace of Σ formed by the tensor-valued functions τ such that $\operatorname{div}\tau \in L_2$. The scalar product in this space is defined by the relation

$$(\tau, \eta) := \int_\Omega (\tau : \eta + \operatorname{div}\tau \cdot \operatorname{div}\eta)\, dx.$$

The space of square integrable functions with zero mean is denoted by $\widetilde{L}_2(\Omega)$.

Assume that

$$f \in L_2(\Omega, \mathbb{R}^d), \quad u_0 \in W_2^1(\Omega, \mathbb{R}^d), \ \operatorname{div} u_0 = 0.$$

The weak solution of the Stokes problem is defined as the function $u \in \mathring{J}_2^1(\Omega) + u_0$ that satisfies the integral identity

$$\nu \int_\Omega \varepsilon(u) : \varepsilon(v)\, dx = \int_\Omega f \cdot v\, dx, \quad v \in \mathring{J}_2^1(\Omega). \tag{2.5}$$

A solution of this problem exists, is unique, and is a minimizer of the functional

$$I(v) = \int_\Omega \left(\frac{\nu}{2}|\varepsilon(v)|^2 - f \cdot v\right) dx$$

on the set $\mathring{J}_2^1(\Omega) + u_0$. One of the major difficulties of the problem under consideration is that the admissible fields must be subject to the condition $\operatorname{div} u = 0$. Therefore, this condition is often taken into account only approximately, with the help of the penalty method or by introducing Lagrangian multipliers. Then, the question arises how to estimate the deviation from the exact solution for functions that do not belong to the above subspace. To answer this question, we use results of Ladyzhenskaya [1] and Ladyzhenskaya and Solonnikov [4] obtained in the proof of solvability for the Stokes problem in domains with nonsmooth boundary. The first result concerns the possibility of extending a solenoidal field inside a domain so that the resulting function norm will be dominated by the trace norm on the boundary.

Lemma 2.1. *Let Ω be a bounded domain with Lipschitz boundary. Then there exists a positive constant c_Ω depending only on Ω and such that for any vector-valued function $a \in W_2^{1/2}(\partial\Omega)$ satisfying $\int_{\partial\Omega} a \cdot \nu\, dx = 0$ there is a function $\bar{u} \in V_0$ with $\operatorname{div}\bar{u} = 0$ and*

$$\|\nabla \bar{u}\| \le c_\Omega \|a\|_{1/2,\partial\Omega}. \tag{2.6}$$

This lemma implies the following statements.

Lemma 2.2. *Let Ω be a bounded domain with Lipschitz boundary. Then there exists a positive constant $\mathbf{C}_\Omega$ depending on Ω and such that for any $f \in \widetilde{L}_2(\Omega)$ there is a function $\bar{u} \in V_0$ satisfying the condition $\operatorname{div}\bar{u} = f$ and the inequality*

$$\|\nabla \bar{u}\| \le \mathbf{C}_\Omega \|f\|. \tag{2.7}$$

The above lemmas, their consequences, and methods of constructing functions arising in (2.6) and (2.7) were investigated in the paper [4].

Lemma 2.2 has an important corollary. Let $f = \operatorname{div}\widehat{v}$, where $\widehat{v}$ is a function in V_0. Then these exists $u_f \in V_0$ such that

$$\operatorname{div}(\widehat{v} - u_f) = 0, \quad \|\nabla u_f\| \le \mathbf{C}_\Omega \|\operatorname{div}\widehat{v}\|.$$

This means that for the solenoidal field $w_0 = (\widehat{v} - u_f) \in V_0$ we have

$$\|\nabla(\widehat{v} - w_0)\| \le \mathbf{C}_\Omega \| \operatorname{div} \widehat{v}\|. \tag{2.8}$$

Also, Lemma 2.2 implies the following condition, known in the literature as the *Ladyzhenskaya–Babuška–Brezzi* (or LBB) *condition*: there exists a positive constant C_{LBB} such that

$$\inf_{\phi \in \widetilde{L}_2(\Omega), \phi \neq 0} \sup_{w \in V_0, w \neq 0} \frac{\int_\Omega \phi \operatorname{div} w \, dx}{\|\phi\| \, \|\nabla w\|} \ge C_{LBB}. \tag{2.9}$$

Indeed, by Lemma 2.2, for any $\phi \in \widetilde{L}_2(\Omega)$ there is a function $v_\phi \in V_0$ satisfying

$$\operatorname{div} v_\phi = \phi, \quad \|\nabla v_\phi\| \le \mathbf{C}_\Omega \|\phi\|. \tag{2.10}$$

Then

$$\sup_{v \in \mathring{W}_2^1(\Omega), w \neq 0} \frac{\int_\Omega \phi \operatorname{div} v \, dx}{\|\nabla v\| \, \|\phi\|} \ge \frac{\int_\Omega \phi \operatorname{div} v_\phi \, dx}{\|\nabla v_\phi\| \, \|\phi\|} \frac{\|\phi\|}{\|\nabla v_\phi\|} \ge \frac{1}{\mathbf{C}_\Omega},$$

whence we see that (2.9) is true with $C_{LBB} = (\mathbf{C}_\Omega)^{-1}$. Relation (2.9) and its discrete analogs are often used for proving the convergence of approximations in various problems related to the theory of viscous incompressible fluids. In [16, 22] Babuška and Brezzi used this condition to justify the convergence of what is called *mixed* methods, in which a boundary-value problem is reduced to a saddle-point problem for a certain Lagrangian. It should be noted that (2.9) can also be deduced from the Nečas inequality; a simple proof of the latter for domains with Lipschitz boundary can be found in the paper [21] by Bramble. Also, in [21], it was shown that the well-known Korn inequality follows from (2.9).

The constant in the LBB-condition plays an important role in problems involving the incompressibility condition. This constant arises in the projection type error estimates for the velocity and pressure fields and in the corresponding asymptotic rate convergence estimates for mixed methods (this issue was considered in the book [24] by Brezzi and Fortin). Also, the same constant is crucial in the analysis of the rate of convergence for numerical methods based upon the minimax formulation of the problem (see, e.g., Kobelkov and Olshanskiĭ [39]). Below, we shall see that the constant C_{LBB} also arises in *a posteriori* functional type estimates if the exact solution is compared with an approximate solution that does not satisfy the incompressibility condition. Therefore, finding two-sided estimates of C_{LBB} for various domains is an important problem. We note that C_{LBB} can be estimated in terms of C_F and the constant C_P in the Poincaré inequality. Indeed,

$$C_{LBB} = \inf_{q \in \widetilde{L}_2(\Omega), q \neq 0} \mathcal{E}(q),$$

$$\mathcal{E}(q) = \sup_{w \in V_0, w \neq 0} \frac{\int_\Omega q \operatorname{div} w \, dx}{\|q\| \, \|\nabla w\|}.$$

For $q \in \widetilde{W}(\Omega) := \widetilde{L}_2(\Omega) \cap W_2^1(\Omega)$ we have

$$\begin{aligned}\mathcal{E}(q) &= \sup_{w \in V_0, w \neq 0} \frac{\int_\Omega \nabla q \cdot w \, dx}{\|q\| \, \|\nabla w\|} \le \frac{\|\nabla q\|}{\|q\|} \sup_{w \in V_0, w \neq 0} \frac{\|w\|}{\|\nabla w\|} \\ &\le C_F \frac{\|\nabla q\|}{\|q\|}.\end{aligned}$$

Let C_P be the smallest constant in the inequality

$$\|q\| \le C_P \|\nabla q\|, \quad q \in \widetilde{W}(\Omega),$$

i.e.,

$$\inf_{q\in\widetilde{W}(\Omega),q\neq 0}\frac{\|\nabla q\|}{\|q\|}=\frac{1}{C_P}.$$

Then

$$C_{LBB}=\inf_{q\in\widetilde{L}_2(\Omega),q\neq 0}\mathcal{E}(q)\leq\inf_{q\in\widetilde{W}(\Omega),q\neq 0}\mathcal{E}(q)\leq\frac{C_F}{C_P}.$$

This estimate gives some information about the value of C_{LBB}. However, mainly we are interested in the lower estimate. Two-sided estimates for rectangular domains were obtained in [7] by Ol′shanskiĭ and Chizhonkov.

2.2. Estimates for the velocity field. We pass to estimates of deviations from the field of velocities u corresponding to the exact solution of the Stokes problem. Let $v\in\mathring{J}^1_2(\Omega)+u_0$. Then (2.5) implies the identity

$$\nu\int_\Omega\varepsilon(u-v):\varepsilon(w)\,dx=\int_\Omega(f\cdot w-\nu\varepsilon(v):\varepsilon(w))\,dx,\qquad w\in\mathring{J}^1_2(\Omega). \tag{2.11}$$

For any tensor-valued function $\tau\in\Sigma$, the functional

$$\mathcal{F}_\tau(w):=\int_\Omega(f\cdot w-\tau:\varepsilon(w))\,dx$$

is linear and continuous on $\mathring{J}^1_2(\Omega)$, and as its norm we can take the number

$$[]\mathcal{F}_\tau[]:=\sup_{w\in\mathring{J}^1_2(\Omega),w\neq 0}\frac{\int_\Omega(f\cdot w-\tau:\varepsilon(w))\,dx}{\|\varepsilon(w)\|}.$$

We write (2.11) as

$$\nu\int_\Omega\varepsilon(u-v):\varepsilon(w)\,dx=\mathcal{F}_\tau(w)+\int_\Omega(\tau-\nu\varepsilon(v)):\varepsilon(w)\,dx,\qquad w\in\mathring{J}^1_2(\Omega).$$

Setting $w=u-v$, we arrive at the estimate

$$\nu\|\varepsilon(u-v)\|\leq[]\mathcal{F}_\tau[]+\|\nu\varepsilon(v)-\tau\|. \tag{2.12}$$

Observe that

$$\begin{aligned}[]\mathcal{F}_\tau[]&=\sup_{w\in\mathring{J}^1_2(\Omega),w\neq 0}\frac{\int_\Omega(f\cdot w+q\operatorname{div}w-\tau:\varepsilon(w))\,dx}{\|\varepsilon(w)\|}\\&\leq\sup_{w\in V_0,w\neq 0}\frac{\int_\Omega(f\cdot w+q\operatorname{div}w-\tau:\varepsilon(w))\,dx}{\|\varepsilon(w)\|},\end{aligned}$$

where q is an arbitrary function in $\widetilde{L}_2(\Omega)$. Here the left-hand side is the norm of a linear functional on V_0. Denoting this norm by $|\!|\!|\operatorname{div}\tau+f-\nabla q|\!|\!|$, we obtain

$$\nu\|\varepsilon(u-v)\|\leq\|\nu\varepsilon(v)-\tau\|+|\!|\!|\operatorname{div}\tau+f-\nabla q|\!|\!|. \tag{2.13}$$

It is convenient to rewrite this estimate in the form

$$\begin{aligned}&\nu^2\|\varepsilon(u-v)\|^2\\&\quad\leq(1+\beta)\|\nu\varepsilon(v)-\tau\|^2+\Big(1+\frac{1}{\beta}\Big)|\!|\!|\operatorname{div}\tau+f-\nabla q|\!|\!|^2.\end{aligned} \tag{2.14}$$

Estimates (2.13) and (2.14) are valid for any $\tau \in \Sigma$, $q \in \widetilde{L}_2(\Omega)$, and $\beta > 0$. If $\tau \in \Sigma_{\mathrm{div}}$ and $q \in \widetilde{W}(\Omega)$, then

$$\begin{aligned} |\!|\!| \operatorname{div} \tau + f - \nabla q |\!|\!| &= \sup_{w \in V_0, w \neq 0} \frac{\int_\Omega (f - \nabla q + \operatorname{div} \tau) \cdot w \, dx}{\|\varepsilon(w)\|} \\ &\leq c_F \| \operatorname{div} \tau + f - \nabla q \|, \end{aligned}$$

where c_F is the constant in the inequality $\|w\| \leq c_F \|\varepsilon(w)\|$ for $w \in V_0$. As a result, the right-hand side of the estimate in question is represented as a sum of integrals, namely,

$$\nu^2 \|\varepsilon(u - v)\|^2 \leq (1 + \beta) \|\nu \varepsilon(v) - \tau\|^2 + \frac{(1+\beta) c_F^2}{\beta} \| \operatorname{div} \tau + f - \nabla q \|^2. \tag{2.15}$$

If the comparison function $\widehat{v} \in V_0 + u_0$ does not satisfy the incompressibility condition, then the estimate of its deviation from u can be obtained as follows. By Lemma 2.2, for the function $\widehat{v}_0 := \widehat{v} - u_0$ we can find $w_0 \in \mathring{J}_2^1(\Omega)$ such that $\|\varepsilon(\widehat{v}_0 - \widehat{w}_0\| \leq \mathbf{C}_\Omega \| \operatorname{div} \widehat{v}_0 \|$. Then

$$\nu \|\varepsilon(u - \widehat{v})\| \leq \nu \|\varepsilon(u - \widehat{w}_0 - u_0)\| + \nu \|\varepsilon(\widehat{v}_0 - \widehat{w}_0)\|. \tag{2.16}$$

We use (2.13) to estimate the first norm on the right in this inequality. This yields

$$\begin{aligned} \nu \|\varepsilon(u - \widehat{v})\| &\leq \|\nu \varepsilon(\widehat{w}_0 + u_0) - \tau\| + |\!|\!| \operatorname{div} \tau + f - \nabla q |\!|\!| + \nu \|\varepsilon(\widehat{v}_0 - \widehat{w}_0)\| \\ &\leq \|\nu \varepsilon(\widehat{v}) - \tau\| + |\!|\!| \operatorname{div} \tau + f - \nabla q |\!|\!| + 2\nu \|\varepsilon(\widehat{v}_0 - \widehat{w}_0)\|. \end{aligned}$$

Since $\operatorname{div} u_0 = 0$, we obtain

$$\nu \|\varepsilon(u - \widehat{v})\| \leq \|\nu \varepsilon(\widehat{v}) - \tau\| + |\!|\!| \operatorname{div} \tau + f - \nabla q |\!|\!| + \frac{2\nu}{C_{LBB}} \| \operatorname{div} \widehat{v} \|. \tag{2.17}$$

If $\tau \in \Sigma_{\mathrm{div}}(\Omega)$ and $q \in \widetilde{W}(\Omega)$, then (2.17) implies the estimate

$$\nu \|\varepsilon(u - \widehat{v})\| \leq \|\nu \varepsilon(\widehat{v}) - \tau\| + c_F \| \operatorname{div} \tau + f - \nabla q \| + \frac{2\nu}{C_{LBB}} \| \operatorname{div} \widehat{v} \|. \tag{2.18}$$

Set $\tau = \eta + q\mathbb{I}$, where $\mathbb{I}$ is the unit tensor and $\eta \in \Sigma_{\mathrm{div}}(\Omega)$. Then (2.18) takes the form

$$\nu \|\varepsilon(u - \widehat{v})\| \leq \|\nu \varepsilon(\widehat{v}) - \eta - q\mathbb{I}\| + c_F \| \operatorname{div} \eta + f \| + \frac{2\nu}{C_{LBB}} \| \operatorname{div} \widehat{v} \|.$$

Thus, if the constants c_F and C_{LBB} are known (or we know suitable upper bounds for them), then (2.18) provides a way for evaluating the deviation of $\widehat{v}$ from u. For this, we should select certain finite-dimensional subspaces Σ_k and Q_k for the functions τ (or η) and q, respectively. Minimization of the right-hand side of (2.18) with respect to τ and q gives an estimate for the deviation, and this estimate will be sharper if we increase the dimension of the subspaces involved.

2.3. Estimates for the pressure field. Let $q \in \widetilde{L}_2(\Omega)$ be an approximant of the pressure field p. Then $(p - q) \in \widetilde{L}_2(\Omega)$, and condition (2.9) implies that

$$\sup_{w \in V_0, w \neq 0} \frac{\int_\Omega (p - q) \operatorname{div} w \, dx}{\|p - q\| \, \nabla w} \geq C_{LBB}.$$

Thus, for any small positive ϵ there exists a nonzero function $w_{pq}^\epsilon \in V_0$ such that

$$\int_\Omega (p - q) \operatorname{div} w_{pq}^\epsilon \, dx \geq (C_{LBB} - \epsilon) \|p - q\| \|\nabla w_{pq}^\epsilon\|. \tag{2.19}$$

Since

$$\nu \int_\Omega \varepsilon(u) : \varepsilon(w_{pq}^\epsilon) \, dx = \int_\Omega (f \cdot w_{pq}^\epsilon + p \operatorname{div} w_{pq}^\epsilon) \, dx,$$

we have

$$\int_\Omega (p-q)\operatorname{div} w^\epsilon_{pq}\,dx$$

$$= \nu \int_\Omega \varepsilon(u-\widehat{v}) : \varepsilon(w^\epsilon_{pq})\,dx + \int_\Omega (\nu\varepsilon(\widehat{v})-\tau) : \varepsilon(w^\epsilon_{pq})\,dx$$
$$+ \int_\Omega \left(\tau : \varepsilon(w^\epsilon_{pq}) + \nabla q \cdot w^\epsilon_{pq} - f\cdot w^\epsilon_{pq}\right)\,dx, \tag{2.20}$$

where as $\widehat{v}$ we can take an arbitrary function in V_0+u_0, and as τ an arbitrary tensor-valued function in Σ.

Relations (2.19) and (2.20) lead to the estimates

$$\|p-q\| \le \frac{1}{(C_{LBB}-\epsilon)\|\nabla w^\epsilon_{pq}\|}$$
$$\times\left[\nu\int_\Omega \left(\varepsilon(u-\widehat{v}) : \varepsilon(w^\epsilon_{pq}) + (\nu\varepsilon(\widehat{v})-\tau):\varepsilon(w^\epsilon_{pq})\right)\,dx\right.$$
$$\left.+\int_\Omega\left(-w^\epsilon_{pq}\cdot\operatorname{div}\tau + \nabla q\cdot w^\epsilon_{pq} - f\cdot w^\epsilon_{pq}\right)\,dx\right]$$
$$\le \frac{1}{(C_{LBB}-\epsilon)}\left[\nu\|\varepsilon(u-\widehat{v})\| + \|\nu\varepsilon(\widehat{v})-\tau\| + [\![\operatorname{div}\tau + f - \nabla q]\!]\right].$$

The first term on the right-hand side of this inequality is estimated by (2.17). Since ϵ may be taken arbitrarily small, we obtain the following estimate for the deviation from the exact pressure field:

$$\frac{1}{2}\|p-q\| \le \frac{\nu}{C^2_{LBB}}\|\operatorname{div}\widehat{v}\| + \frac{1}{C_{LBB}}\|\nu\varepsilon(\widehat{v})-\tau\| + \frac{1}{C_{LBB}}[\![\operatorname{div}\tau + f - \nabla q]\!]. \tag{2.21}$$

It is easily seen that the right-hand side of (2.21) consists of the same terms as the right-hand side of (2.17) and vanishes if and only if $\widehat{v}=u$, $\tau=\sigma$, and $p=q$. However, in this case, the dependence of the penalty factors from the constant C_{LBB} is stronger. If $\tau\in\Sigma_{\mathrm{div}}$ and $q\in\widetilde{W}$, then the last term on the right in (2.21) is estimated by the quantity $\|\operatorname{div}\tau + f - \nabla q\|$, which is easy to compute.

Estimates (2.17), (2.18), and (2.21) have a clear meaning. They show that the norms of the deviations of $\widehat{v}\in V_0+u_0$ and $q\in\widetilde{L}_2(\Omega)$ from the exact solutions u and p of the Stokes problem are dominated by the sum of the residuals that arise when we substitute in (2.1)–(2.3) the tensor-valued function τ, which can be regarded as an approximant for the stress field σ. In these estimates, the penalty multipliers depend only on the constants c_F, C_{LBB}, and ν.

Remark 2.1. It may happen to be convenient to search an approximate solution v of the stationary Stokes problem in the class W^1_2 without requiring that the boundary condition (2.4) be satisfied exactly (such a situation may arise, e.g., when Ω has a very complicated boundary). In this case, the respective error majorant will involve yet another term, which can be viewed as a penalty for the violation of the boundary condition (for problem (1.2) this question was analyzed in the paper [54]). The errors arising due to discrepancies in boundary conditions of other types can be estimated by the method developed by Repin, Sauter and Smolianski in [55]. For the nonstationary Stokes problem, *a posteriori* error estimates can be obtained by the method exposed in [53]. They have the same structure as for the stationary Stokes problem, with the only difference that integration should be done additionally over the time variable in the limits of the time interval considered.

Remark 2.2. Estimates of the energy norm of the deviation from an exact solution make it possible to obtain exact upper bounds of the error in local norms. For example, consider problem (1.2) and the corresponding estimate (1.8). Let ω be some subdomain of Ω. We are interested in the quantity $\|\nabla(u-v)\|_\omega := \left(\int_\omega |\nabla(u-v)|^2\,dx\right)^{1/2}$. Consider the set

$$V_{0\omega} := \{w \in V_0 \mid w(x) = \alpha, x \in \omega, \alpha \in \mathbb{R}\}.$$

For any $w \in V_{0\omega}$ and any number α, we have

$$\|\nabla(u-v)\|_\omega \le \|\nabla(v+w-u)\|.$$

Therefore,

$$\|\nabla(u-v)\|_\omega \le \|\nabla(v+w)-y\| + C_F\|\operatorname{div} y + f\|.$$

To obtain the sharpest upper estimate for the local norm, we must take the infimum on the right–hand side with respect to $y \in H(\operatorname{div},\Omega)$ and $w \in V_{0\omega}$. Similar local estimates for the Stokes problem follow from (2.15) and (2.18).

2.4. Estimates for other stationary problems.

2.4.1. *Problems with the condition* $\operatorname{div} u = \phi$. In many cases, (2.3) is replaced by the condition

$$\operatorname{div} u = \phi \quad \text{in } \Omega, \tag{2.22}$$

where ϕ is a given function in $\widetilde{L}_2(\Omega)$. Suppose $u_\phi \in V_0$ satisfies the above condition. Then, setting $u = \bar{u} + u_\phi$ and $\bar{u}_0 = u_0 - u_\phi$, we represent the corresponding boundary-value problem in the following form: find $\bar{u} \in \overset{\circ}{J}{}^1_2(\Omega) + \bar{u}_0$ such that

$$-\operatorname{div}\bar{\sigma} + \nabla p = \bar{f} \qquad \text{in } \Omega, \tag{2.23}$$

$$\bar{\sigma} = \nu\varepsilon(\bar{u}) \quad \text{in } \Omega, \tag{2.24}$$

where $\bar{f} = f + \nu \operatorname{div}\varepsilon(u_\phi) \in H^{-1}$. Assume that u is approximated by some $v \in V_0 + u_0$. We write v in the form $v = \bar{v} + u_\phi$. Applying (2.17), we obtain

$$\begin{aligned}\|\nabla(u-v)\| &= \|\nabla(\bar{u}-\bar{v})\| \\ &\le \|\nu\varepsilon(\bar{v}) - \tau\| + |\!|\!| \operatorname{div}\tau + \bar{f} - \nabla q |\!|\!| + \frac{2\nu}{C_{LBB}}\|\operatorname{div}\bar{v}\|.\end{aligned}$$

Here we set $\tau = -\nu\varepsilon(u_\phi) + \eta$, where $\eta \in \Sigma$. Then $\operatorname{div}\tau + \bar{f} = \operatorname{div}\eta + f$ and $\nu\varepsilon(\bar{v}) - \tau = \nu\varepsilon(v) - \eta$. Therefore,

$$\|\nabla(u-v)\| \le \|\nu\varepsilon(v) - \eta\| + |\!|\!| \operatorname{div}\eta + f - \nabla q |\!|\!| + \frac{2\nu}{C_{LBB}}\|\operatorname{div} v - \phi\|. \tag{2.25}$$

2.4.2. *Problems for almost incompressible fluids.* Models of almost incompressible fluids are often used for constructing sequences of functions converging to a solution of the Stokes problem. In this case, the incompressibility condition is replaced by a term that contains the divergence with a large coefficient. Namely, we must find $u_\delta \in V$ satisfying the integral identity

$$\int_\Omega \Big(\nu\varepsilon(u_\delta) : \varepsilon(w) + \frac{1}{\delta}\operatorname{div} u_\delta \operatorname{div} w\Big)\,dx = \int_\Omega f \cdot w\,dx, \quad w \in V_0, \tag{2.26}$$

and the boundary condition

$$u_\delta = u_0 \quad \text{on } \partial\Omega. \tag{2.27}$$

It is not difficult to show (see, e.g., [57]) that u_δ tends to the solution u of the Stokes problem in the W^1_2 norm, and $p_\delta = -\frac{1}{\delta}\operatorname{div} u_\delta \in \widetilde{L}_2(\Omega)$ converges to the corresponding pressure function p in L_2 as $\delta \to 0$.

By (2.17), we obtain an estimate for the difference between u and u_δ. In (2.17), we set $\tau = \tau_\delta := \nu\varepsilon(u_\delta)$ and $q = p_\delta$. Then $\|\nu\varepsilon(u_\delta) - \tau_\delta\| = 0$, and

$$\begin{aligned}\big| \operatorname{div}\tau_\delta + f - \nabla p_\delta \big| &= \sup_{w\in V_0} \frac{\int_\Omega \left(-\nu\varepsilon(u_\delta) : \varepsilon(w) + f\cdot w + p_\delta \operatorname{div} w\right)\,dx}{\|\nabla w\|} \\ &= 0.\end{aligned}$$

Thus, we conclude that

$$\frac{1}{2}\|\varepsilon(u-u_\delta)\| \le \frac{1}{C_{LBB}}\|\operatorname{div} u_\delta\|. \tag{2.28}$$

We observe that the deviation from the exact solution of the Stokes problem is controlled by the norm of the divergence of the problem (2.26)–(2.27). A similar estimate can be obtained for the approximations constructed with the help of the Uzawa algorithm.

2.4.3. *Convection–diffusion equation.* The method described above can be used to derive estimates of deviations from exact solutions for models of processes with convection and diffusion. As an example, we consider the problem

$$-\operatorname{div} A\nabla u + a\cdot\nabla u = f \quad \text{in } \Omega \in \mathbb{R}^2, \tag{2.29}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{2.30}$$

Here A is a symmetric matrix with bounded measurable coefficients and such that

$$\nu_1|y|^2 \le Ay\cdot y \le \nu_2|y|^2, \tag{2.31}$$

and a is a given vector-valued function satisfying the conditions

$$a \in L_\infty(\Omega,\mathbb{R}^d), \quad \operatorname{div} a \in L_\infty, \quad \operatorname{div} a \le 0. \tag{2.32}$$

The solution $u \in \mathring{W}^1_2(\Omega)$ is the function that satisfies the integral identity

$$\int_\Omega \left(A\nabla u\cdot\nabla w + (a\cdot\nabla u)w\right)\,dx = \int_\Omega f w\,dx, \quad w \in \mathring{W}^1_2(\Omega); \tag{2.33}$$

this relation can also be presented in the form

$$\begin{aligned}&\int_\Omega \left(A\nabla(u-v)\cdot\nabla w + (a\cdot\nabla(u-v))w\right)\,dx \\ &\qquad = \int_\Omega (fw - A\nabla v\cdot\nabla w - (a\cdot\nabla v)w)\,dx.\end{aligned} \tag{2.34}$$

Observe that

$$\begin{aligned}\frac{1}{2}\int_\Omega (\operatorname{div} a)(u-v)^2\,dx &= -\frac{1}{2}\int_\Omega a\cdot\nabla((u-v)^2)\,dx \\ &= -\int_\Omega (u-v)a\cdot\nabla(u-v)\,dx.\end{aligned}$$

Setting $w = u - v$, we rewrite (2.34) as follows:

$$\begin{aligned}&|\!|\!|\nabla(u-v)|\!|\!|^2 - \frac{1}{2}\int_\Omega (\operatorname{div} a)(u-v)^2\,dx \\ &\qquad = \int_\Omega \left((f - a\cdot\nabla v + \operatorname{div} y)(u-v) + (y - A\nabla v)\cdot\nabla(u-v)\right)dx,\end{aligned} \tag{2.35}$$

where $y(x)$ is an arbitrary function in $H(\Omega,\operatorname{div})$, and $|\!|\!|z|\!|\!|^2 := \int_\Omega Az\cdot z\,dx$. This yields the estimate

$$\nu_1|\!|\!|\nabla(u-v)|\!|\!| \le \|y - A\nabla v\| + C_F\|f - a\cdot\nabla v + \operatorname{div} y\|. \tag{2.36}$$

If $\operatorname{div} a(x) \leq -2\delta^2$, then (2.35) implies that

$$\begin{aligned} &|\!|\!|\nabla(u-v)|\!|\!|^2 + \delta^2\|u-v\|^2 \\ &\leq \int_\Omega (f - a\cdot\nabla v + \operatorname{div} y)(u-v) + (y - A\nabla v)\cdot\nabla(u-v)\,dx \\ &\leq \sqrt{\frac{1}{\delta^2}\|f - a\cdot\nabla v + \operatorname{div} y\|^2 + \frac{1}{\nu_1^2}\|y - A\nabla v\|^2}\sqrt{|\!|\!|\nabla(u-v)|\!|\!|^2 + \delta^2\|u-v\|^2}, \end{aligned} \tag{2.37}$$

which leads to the estimate

$$\begin{aligned} &|\!|\!|\nabla(u-v)|\!|\!|^2 + \delta^2\|u-v\|^2 \\ &\qquad\leq \frac{1}{\delta^2}\|f - a\cdot\nabla v + \operatorname{div} y\|^2 + \frac{1}{\nu_1^2}\|y - A\nabla v\|^2. \end{aligned} \tag{2.38}$$

2.4.4. *Oseen equation.* The classical formulation of the Oseen boundary-value problem with Dirichlet boundary conditions looks like this:

$$-\nu\Delta u + \operatorname{div}(a\otimes u) = f - \nabla p \quad \text{in } \Omega, \tag{2.39}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega, \tag{2.40}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{2.41}$$

Here a is a vector-valued function satisfying $\operatorname{div} a = 0$, and $\otimes$ denotes the tensor product of vectors. A function $u \in \mathring{J}_2^1(\Omega)$ that satisfies the integral identity

$$\int_\Omega (\nu\nabla u : \nabla w - (a\otimes u) : \nabla w)\;dx = \int_\Omega f\cdot w\,dx, \qquad w \in \mathring{J}_2^1(\Omega), \tag{2.42}$$

is a weak solution of this problem. Let $v \in \mathring{J}_2^1(\Omega)$. We reshape (2.42) to the form

$$\begin{aligned} &\textstyle\int_\Omega (\nu\nabla(u-v) : \nabla w - (a\otimes(u-v)) : \nabla w)\;dx \\ &\qquad = \int_\Omega (f\cdot w - \nu\nabla v : \nabla w + (a\otimes v) : \nabla w)\;dx, \\ &\qquad\qquad w \in \mathring{J}_2^1(\Omega). \end{aligned} \tag{2.43}$$

Setting $w = u - v$, we observe that

$$\begin{aligned} \int_\Omega (a\otimes w) : \nabla w\,dx &= -\int_\Omega \operatorname{div}(a\otimes w)\cdot w\,dx \\ &= -\int_\Omega (a\cdot\nabla w)\cdot w\,dx. \end{aligned}$$

Since $\nabla(w\cdot w) = 2\nabla w\cdot w$, we have

$$\int_\Omega (a\otimes w) : \nabla w\,dx = -\frac{1}{2}\int_\Omega a\cdot\nabla(w\cdot w)\,dx = 0.$$

Consequently, (2.43) leads to the inequality

$$\nu\|\nabla(u-v)\| \leq \|\tau - \nu\nabla v\| + [\![f - \nabla q - \operatorname{div}(a\otimes v) + \operatorname{div}\tau]\!], \tag{2.44}$$

where $\tau(x) \in \Sigma$ and $q \in \widetilde{L}_2(\Omega)$. Assume that τ and q have some higher regularity (namely, $\tau \in H(\Omega, \operatorname{div})$ and $q \in \widetilde{L}_2(\Omega) \cap W_2^1$). Then, (2.44) implies

$$\nu\|\nabla(u-v)\| \leq \|\tau - \nu\nabla v\| + C_F\|f - \nabla q - \operatorname{div}(a\otimes v) + \operatorname{div}\tau\|, \tag{2.45}$$

which is a generalization of the estimates obtained for the Stokes problem.

If u is approximated by a function $\widehat{v} \in V_0$, then the corresponding estimate for the deviation norm can be deduced in the same way as for the Stokes problem. In this case, the majorant includes an additional term which penalizes the possible violations of the incompressibility condition. This estimate has the following form:

$$
\begin{aligned}
&\nu\|\nabla(u-\widehat{v})\| \\
&\quad \leq \|\tau-\nu\nabla\widehat{v}\| + C_F\|f-\nabla q-\operatorname{div}(a\otimes\widehat{v})+\operatorname{div}\tau\| + \frac{2\nu}{C_{LBB}}\|\operatorname{div}\widehat{v}\|.
\end{aligned} \tag{2.46}
$$

§3. Estimates for deviations from the exact solutions to be searched in a subspace

The method described above can be extended to a wide range of problems whose solutions are searched in a certain subspace of the basic energy space. Suppose V is a reflexive Banach space embedded in a Hilbert space U with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let V_0 be a linear subspace of V, let $\boldsymbol{\Lambda}$ denote a bounded linear operator acting from V_0 to U, and let $\boldsymbol{\Lambda}^* : U \to V_0^*$ denote the operator conjugate to $\boldsymbol{\Lambda}$ in the sense that

$$
(y,\boldsymbol{\Lambda}w) = \langle\boldsymbol{\Lambda}^*y, w\rangle, \quad w\in V_0. \tag{3.1}
$$

Here and in what follows $\langle w^*, w\rangle$ denotes the value of the functional $w^*\in V_0^*$ at $w\in V_0$.

Let $\mathcal{A}\in\mathcal{L}(U,U)$ be a selfadjoint operator satisfying the relation

$$
\nu_1\|y\|^2 \leq (\mathcal{A}y,y) \leq \nu_2\|y\|^2, \quad y\in U, \tag{3.2}
$$

with positive constants ν_1 and ν_2 independent of y. This operator determines an equivalent norm $|\!|\!|y|\!|\!| := (\mathcal{A}y,y)^{1/2}$. Let Y denote the space U endowed with this norm. The inverse operator $\mathcal{A}^{-1}$ is also nondegenerate. It determines another equivalent norm $|\!|\!|y|\!|\!|_* := (\mathcal{A}^{-1}y,y)^{1/2}$, and the space U endowed with this norm will be denoted by Y^*.

We define another pair of mutually conjugate linear operators $B : V_0 \to H$ and $B^* : H \to V_0^*$, where H is a Hilbert space with scalar product $(.,.)_H$. It is convenient to present the spaces and operators under consideration by the following diagram:

$$
\begin{array}{ccccccc}
H & \xleftarrow{B} & V_0 & \xrightarrow{\boldsymbol{\Lambda}} & U & (Y,Y^*) \\
 & & \Updownarrow & & & \\
H & \xrightarrow{B^*} & V_0^* & \xleftarrow{\boldsymbol{\Lambda}^*} & U. &
\end{array}
$$

Consider the following problem: find $p\in H$ and $u\in\mathcal{V}_0$ satisfying the relation

$$
(\mathcal{A}\boldsymbol{\Lambda}u,\boldsymbol{\Lambda}w) + \langle f - B^*p, w\rangle = 0, \quad w\in V_0, \tag{3.3}
$$

where

$$
\mathcal{V}_0 := \{v\in V_0 \mid Bv = 0\}.
$$

Let $v\in\mathcal{V}_0$. From (3.3) it follows that

$$
(\mathcal{A}\boldsymbol{\Lambda}(u-v),\boldsymbol{\Lambda}w) = (y-\mathcal{A}\boldsymbol{\Lambda}v,\boldsymbol{\Lambda}w) - \langle f+\boldsymbol{\Lambda}^*y, w\rangle \tag{3.4}
$$

for any $w\in\mathcal{V}_0$ and $y\in U$. Recalling (3.3) and the relation

$$
\langle B^*q, w\rangle = (q, Bw) = 0,
$$

we arrive at the following estimate:

$$
\begin{aligned}
\sup_{w\in\mathcal{V}_0}\frac{|\langle f+\boldsymbol{\Lambda}^*y,w\rangle|}{|\!|\!|\boldsymbol{\Lambda}w|\!|\!|} &= \sup_{w\in\mathcal{V}_0}\frac{|\langle f+\boldsymbol{\Lambda}^*y-B^*q,w\rangle|}{|\!|\!|\boldsymbol{\Lambda}w|\!|\!|} \\
&\leq \sup_{w\in V_0}\frac{|\langle f+\boldsymbol{\Lambda}^*y-B^*q,w\rangle|}{|\!|\!|\boldsymbol{\Lambda}w|\!|\!|} \leq \frac{1}{\sqrt{\nu_1}}\sup_{w\in V_0}\frac{|\langle f+\boldsymbol{\Lambda}^*y-B^*q,w\rangle|}{\|\boldsymbol{\Lambda}w\|}.
\end{aligned}
$$

The supremum on the right determines a norm to be denoted by $\boldsymbol{|} f+\boldsymbol{\Lambda}^*y-B^*q\boldsymbol{|}$. Thus, (3.4) with $w=u-v$ leads to the estimate

$$\|\!|\boldsymbol{\Lambda}(u-v)|\!\| \leq \|\!|y-\mathcal{A}\boldsymbol{\Lambda}v|\!\|_* + \frac{1}{\sqrt{\nu_1}}\boldsymbol{|} f+\boldsymbol{\Lambda}^*y-B^*q\boldsymbol{|}. \tag{3.5}$$

For elements not belonging to the subspace $\mathcal{V}_0$, their deviations from u can be estimated provided that the operator B satisfies conditions similar to those stated in Lemma 2.2. Namely, assume that there is a constant $\mathbf{c}$ with the property that for any g belonging to $\operatorname{Im} B := \{z\in H \mid \exists v\in V_0 : Bv=z\}$ we can find an element $u_g\in V_0$ such that

$$Bu_g = g \quad \text{and} \quad \|u_g\|_V \leq \mathbf{c}\|g\|. \tag{3.6}$$

Set $g=B\widehat{v}$, where $\widehat{v}$ is an element of V_0. Then we can find $v_g\in V_0$ satisfying $B(\widehat{v}-v_g)=0$ and such that

$$\|v_g\|_V \leq \mathbf{c}\|B\widehat{v}\|. \tag{3.7}$$

Therefore, $w_0=(\widehat{v}-v_g)\in\mathcal{V}_0$ satisfies the relation

$$\|\widehat{v}-w_0\|_V \leq \mathbf{c}\|B\widehat{v}\|.$$

Since $\boldsymbol{\Lambda}$ is a bounded operator, we have

$$\|\boldsymbol{\Lambda}(\widehat{v}-w_0)\| \leq \mathbf{C}\|B\widehat{v}\| \tag{3.8}$$

with constant $\mathbf{C}$ independent of $\widehat{v}$.

Suppose u is compared with a function $\widehat{v}\in V_0$. Let $w_0\in\mathcal{V}_0$ be a function satisfying (3.8). Then

$$\begin{aligned}
\|\!|\boldsymbol{\Lambda}(u-\widehat{v})|\!\| &\leq \|\!|\boldsymbol{\Lambda}(u-w_0)|\!\| + \|\!|\boldsymbol{\Lambda}(\widehat{v}-w_0)|\!\| \\
&\leq \|\!|\mathcal{A}\boldsymbol{\Lambda}w_0-y|\!\|_* + \frac{1}{\sqrt{\nu_1}}\boldsymbol{|} f+\boldsymbol{\Lambda}^*y-B^*q\boldsymbol{|} + \|\!|\boldsymbol{\Lambda}(\widehat{v}-w_0)|\!\| \\
&\leq \|\!|\mathcal{A}\boldsymbol{\Lambda}(\widehat{v}-w_0)|\!\|_* + \|\!|\mathcal{A}\boldsymbol{\Lambda}\widehat{v}-y|\!\|_* + \frac{1}{\sqrt{\nu_1}}\boldsymbol{|} f+\boldsymbol{\Lambda}^*y-B^*q\boldsymbol{|} + \|\!|\boldsymbol{\Lambda}(\widehat{v}-w_0)|\!\| \\
&= 2\|\!|\boldsymbol{\Lambda}(\widehat{v}-w_0)|\!\| + \|\!|\mathcal{A}\boldsymbol{\Lambda}w_0-y|\!\|_* + \frac{1}{\sqrt{\nu_1}}\boldsymbol{|} f+\boldsymbol{\Lambda}^*y-B^*q\boldsymbol{|} \\
&\leq 2\sqrt{\nu_2}\|\boldsymbol{\Lambda}(\widehat{v}-w_0)\| + \|\!|\mathcal{A}\boldsymbol{\Lambda}w_0-y|\!\|_* + \frac{1}{\sqrt{\nu_1}}\boldsymbol{|} f+\boldsymbol{\Lambda}^*y-B^*q\boldsymbol{|}.
\end{aligned} \tag{3.9}$$

Thus, we arrive at the estimate

$$\|\!|\boldsymbol{\Lambda}(u-\widehat{v})|\!\| \leq 2\sqrt{\nu_2}\mathbf{C}\|B\widehat{v}\| + \|\!|\mathcal{A}\boldsymbol{\Lambda}\widehat{v}-y|\!\|_* + \frac{1}{\sqrt{\nu_1}}\boldsymbol{|} f+\boldsymbol{\Lambda}^*y-B^*q\boldsymbol{|}. \tag{3.10}$$

It is easily seen that the problem under consideration can be presented as the following system:

$$\begin{cases}
\langle\boldsymbol{\Lambda}^*\sigma+f-B^*p, w\rangle = 0, \quad w\in V_0,\\
\sigma = \mathcal{A}\boldsymbol{\Lambda}u,\\
Bv = 0.
\end{cases}$$

Each term on the right-hand side in (3.10) is a penalty for the possible violation of one of the equations in this system.

For the Stokes problem we have $\boldsymbol{\Lambda}v=\varepsilon(v)$ and $\mathcal{A}=\nu\mathbb{I}$, where $\mathbb{I}$ denotes the identity operator and $Bv=-\operatorname{div} v$. It is easily seen that in this case we have $\nu_1=\nu_2=\nu$, and

$$\|\!|\mathcal{A}\boldsymbol{\Lambda}\widehat{v}-y|\!\|_* = \frac{1}{\sqrt{\nu}}\|\nu\varepsilon(v)-y\|.$$

Since $\|\!|\boldsymbol{\Lambda}(u-\widehat{v})|\!\| = \sqrt{\nu}\|\boldsymbol{\Lambda}(u-\widehat{v})\|$, we see that estimate (3.10) coincides with (2.17).

§4. Estimates of deviations from exact solutions for some models of nonlinear viscous fluids

4.1. Mathematical models of generalized Newtonian fluids. We demonstrate the variational method for obtaining estimates of deviations from exact solutions by the example of the model describing the stationary flow of a generalized Newtonian fluid. The classical formulation of this problem is as follows: find a vector-valued function u, a scalar-valued function p, and a tensor-valued function σ that satisfy the system (2.1), (2.3), and (2.4), where (2.2) is replaced by the following *constitutive relation* of a nonlinearly viscous fluid:

$$\sigma \in \partial g(\varepsilon(u)) \quad \text{in } \Omega. \tag{4.1}$$

Here, the function g determines the so-called *dissipative potential* and depends on the physical properties of the fluid, and ∂g denotes the subdifferential of g (if g is Gateaux differentiable, then the subdifferential is well defined and coincides with the derivative g').

In what follows, we consider dissipative potentials of the form

$$g(\varepsilon) = \frac{\nu}{2}|\varepsilon|^2 + \psi(\varepsilon), \tag{4.2}$$

where $\psi : \mathbb{M}^{d\times d} \to \mathbb{R}_+$ is a convex function such that

$$\psi(0) = 0, \quad \psi(\varepsilon) \le c_1|\varepsilon|^2 + c_2, \;\; c_1 > 0. \tag{4.3}$$

The cases where $\psi \equiv 0$ and $\psi(\varepsilon) = k_*|\varepsilon|$ with $k_* > 0$ correspond to the models of the Newtonian and the Bingham fluid, respectively.

Models with dissipative potentials of the above type belong to the class of generalized Newtonian fluids; this class was introduced by Ladyzhenskaya in [3, 40]. At present, a large number of models of viscous fluids of this type are known (see, e.g., [12, 41, 42]). For various models of generalized Newtonian fluids with smooth and nonsmooth potentials, the regularity of weak solutions was investigated in the book [33] by Fuchs and Seregin.

A weak solution of the problem under consideration is defined as a function $u \in \mathring{J}^1_2(\Omega) + u_0$ satisfying the integral identity

$$\int_\Omega \nu\varepsilon(u) : \varepsilon(v-u)\,dx + \Psi(\varepsilon(v)) - \Psi(\varepsilon(u)) \ge \int_\Omega f\cdot(v-u)\,dx, \qquad v \in \mathring{J}^1_2(\Omega) + u_0, \tag{4.4}$$

where $\Psi(\varepsilon) := \int_\Omega \psi(\varepsilon)\,dx$. Also, such a solution can be defined as the minimizer of the following variational problem.

Problem $\mathcal{P}$. *Find $u \in \mathring{J}^1_2(\Omega) + u_0$ such that*

$$J(u) = \inf \mathcal{P} := \inf_{v\in \mathring{J}^1_2(\Omega)+u_0} J(v), \quad J(v) = \int_\Omega g(\varepsilon(v))\,dx - \int_\Omega f\cdot v\,dx.$$

The existence and uniqueness of a minimizer of this problem follows from the known results of the calculus of variations for problems with convex lower semicontinuous functionals (see, e.g., [32]*).*

4.2. Variational method of deriving estimates for deviations from exact solutions. For the class of problems described above, it is convenient to derive estimates of deviations from exact solutions by the *variational* method, which is based on estimation of the quantity in question in terms of the difference between the values of the corresponding functionals. In our case, this estimate has the form

$$\frac{\nu}{2}\int_\Omega |\varepsilon(v-u)|^2\,dx \le J(v) - J(u), \tag{4.5}$$

where v is an arbitrary function in $\mathring{J}_2^1(\Omega) + u_0$. Estimate (4.5) follows from (4.4) and the identity

$$J(v) - J(u) = \int_\Omega \Big(\frac{\nu}{2}|\varepsilon(v-u)|^2 + \nu\varepsilon(u) : \varepsilon(v-u)\Big)\, dx + \Psi(\varepsilon(v)) - \Psi(\varepsilon(u)) - \int_\Omega f\cdot(v-u)\, dx.$$

To estimate $J(u)$ from below, we construct a family of variational problems with functionals defined on a function class wider than $\mathring{J}_2^1(\Omega) + u_0$. We refer to these problems as "perturbed". We take $q \in \widetilde{L}_2(\Omega)$ and $\tau_2 \in \Sigma$ and consider the functional

$$\overline{J}(v) := \int_\Omega \Big(\frac{\nu}{2}|\varepsilon(v)|^2 + \tau_2 : \varepsilon(v) - \psi^*(\tau_2) - f\cdot v - q\operatorname{div}(v-u_0)\Big)\, dx,$$

where $\psi^* : \mathbb{M}^{d\times d} \to \mathbb{R}$ is the functional conjugate to ψ in the sense of Young–Fenchel, i.e.,

$$\psi^*(\kappa^*) = \sup_{\kappa\in\mathbb{M}^{d\times d}} \{\kappa^* : \kappa - \psi(\kappa)\}.$$

Now, the following variational problem arises.

Problem $\overline{\mathcal{P}}$. *Find $\overline{u} \in V_0 + u_0$ such that*

$$\overline{J}(\overline{u}) = \inf \overline{\mathcal{P}} := \inf_{v\in V_0+u_0} \overline{J}(v).$$

It is clear that the minimizer $\overline{u}$ depends on q and τ_2, so that we should denote $\overline{u}$ by $\overline{u}_{q,\tau_2}$ and the problem itself by $\overline{\mathcal{P}}_{q,\tau_2}$. However, for simplicity, we shall not do this, assuming that the bar above means that a quantity depends on the above functions. It is easy to verify that Problem $\overline{\mathcal{P}}$ is uniquely solvable and that

$$\inf \overline{\mathcal{P}} \le \inf \mathcal{P}. \tag{4.6}$$

Indeed, the existence and uniqueness of $\overline{u}$ follows from the properties of the convex functional $\overline{J}$ and the closed set $V_0 + u_0$. In accordance with the definition of ψ^*, for any $v \in \mathring{J}_2^1(\Omega) + u_0$ we have the inequality

$$\overline{J}(v) = \int_\Omega \Big(\frac{\nu}{2}|\varepsilon(v)|^2 + \tau_2 : \varepsilon(v) - \psi^*(\tau_2)\Big)\, dx - \int_\Omega f\cdot v\, dx \le J(v).$$

Therefore,

$$\inf_{v\in V_0+u_0} \overline{J}(v) \le \inf_{v\in \mathring{J}_2^1(\Omega)+u_0} \overline{J}(v) \le \inf_{v\in \mathring{J}_2^1(\Omega)+u_0} J(v) = \inf \mathcal{P},$$

which gives (4.6).

Relations (4.5) and (4.6) imply that

$$\frac{\nu}{2}\int_\Omega |\varepsilon(v-u)|^2\, dx \le J(v) - \inf \overline{\mathcal{P}}. \tag{4.7}$$

Inequality (4.7) cannot be used straightforwardly for estimation of deviations from the exact solution because the value of $\inf \overline{\mathcal{P}}$ is unknown. However, this difficulty can be avoided if we invoke the so-called *dual variational problem* (we denote it by $\overline{\mathcal{P}}^*$), which is a maximization problem for a certain functional. If we manage to establish that $\inf \overline{\mathcal{P}} = \sup \overline{\mathcal{P}}^*$, then $\inf \overline{\mathcal{P}}$ in (4.7) can be replaced by a lower estimate of $\sup \overline{\mathcal{P}}^*$. Estimates obtained in this way will depend on τ_2 and q and also on the variables of the dual problem. Note that there are different variational problems that may be viewed as dual to $\overline{\mathcal{P}}$. The problem is to find a proper version in this collection, namely, a version that leads to estimates convenient for practice and having good accuracy. For the class of problems under consideration, the following version is possible.

Define the Lagrangian

$$\overline{L}(v;\tau_1) := \int_\Omega \Big(\varepsilon(v) : (\tau_1+\tau_2) - \frac{1}{2\nu}|\tau_1|^2 - \psi^*(\tau_2)\Big)\,dx$$
$$- \int_\Omega f\cdot v\,dx - \int_\Omega q\cdot \operatorname{div}\cdot(v-u_0)\,dx.$$

Then

$$\overline{J}(v) = \sup_{\tau_1\in\Sigma} \overline{L}(v;\tau_1),$$

so that Problem $\overline{\mathcal{P}}$ is equivalent to the minimax problem $\inf_{v\in V_0+u_0}\sup_{\tau_1\in\Sigma}\overline{L}(v;\tau_1)$. The corresponding dual problem is $\sup_{\tau_1\in\Sigma}\inf_{v\in V_0+u_0}\overline{L}(v;\tau_1)$. Observe that

$$\inf_{v\in V_0+u_0} \overline{L}(v;\tau_1) = \begin{cases} \overline{I}(\tau_1) & \text{in } \tau_1\in\overline{\Sigma}_f(\Omega),\\ -\infty & \text{in } \tau_1\notin\overline{\Sigma}_f(\Omega),\end{cases}$$

where

$$\overline{I}(\tau_1) = \int_\Omega \Big(\varepsilon(u_0) : (\tau_1+\tau_2) - \frac{1}{2\nu}|\tau_1|^2 - \psi^*(\tau_2) - f\cdot u_0\Big)\,dx,$$

and $\overline{\Sigma}_f$ is the affine subset in Σ that consists of the functions τ satisfying the condition $\operatorname{div}(\tau+\tau_2) = \nabla q - f$ (in the sense of distributions), i.e.,

$$\overline{\Sigma}_f(\Omega) := \Big\{\tau\in\Sigma(\Omega)\ \Big|\ \int_\Omega \varepsilon(w) : (\tau+\tau_2)\,dx = \int_\Omega (f\cdot w + q\operatorname{div} w)\,dx, w\in V_0\Big\}.$$

Thus, we arrive at the following formulation of the dual variational problem.

Problem $\overline{\mathcal{P}}^*$. *For given $\tau_2\in\Sigma$ and $q\in\widetilde{L}_2(\Omega)$, find a function $\overline{\sigma}_1\in\overline{\Sigma}_f(\Omega)$ such that*

$$\overline{I}(\overline{\sigma}_1) = \sup\overline{\mathcal{P}}^* := \sup_{\tau_1\in\overline{\Sigma}_f} \overline{I}(\tau_1).$$

Theorem 4.1. *Problem $\overline{\mathcal{P}}^*$ has a unique solution $\overline{\sigma}_1$ satisfying the conditions*

$$\inf\overline{\mathcal{P}} = \overline{J}(\overline{u}) = \sup\overline{\mathcal{P}}^* = \overline{I}(\overline{\sigma}_1), \tag{4.8}$$

$$\nu\varepsilon(\overline{u}) = \overline{\sigma}_1. \tag{4.9}$$

Proof. The functional $-\overline{I}$ is strictly convex and $\overline{\Sigma}_f$ is a convex and closed subset of Σ. Therefore, Problem $\overline{\mathcal{P}}^*$ has a unique solution. We verify (4.8) and (4.9). First, note that $\sup\overline{\mathcal{P}}^* \le \inf\overline{\mathcal{P}}$. This fact follows from the inequality $\sup\inf\overline{L} \le \inf\sup\overline{L}$, which is true for all minimax problems. Next, we use the fact that $\overline{u}$ satisfies the integral identity

$$\int_\Omega (\nu\varepsilon(\overline{u}) : \varepsilon(w) + \tau_2 : \varepsilon(w))\,dx = \int_\Omega (f\cdot w + q\operatorname{div} w)\,dx,$$
$$w\in V_0.$$

It follows that

$$\int_\Omega f\cdot\overline{u}\,dx$$
$$= \int_\Omega (\nu|\varepsilon(\overline{u})|^2 - \nu\varepsilon(\overline{u}) : \varepsilon(u_0) + \tau_2 : \varepsilon(\overline{u}-u_0) + f\cdot u_0 - q\operatorname{div}(\overline{u}-u_0))\,dx.$$

Therefore,

$$\begin{aligned}\inf \overline{\mathcal{P}} &= \overline{J}(\overline{u}) \\ &= \int_\Omega \Big(\frac{\nu}{2}|\varepsilon(\overline{u})|^2 + \tau_2 : \varepsilon(\overline{u}) - \psi^*(\tau_2)\Big)\, dx + \int_\Omega (q \operatorname{div}(\overline{u} - u_0) - f \cdot \overline{u})\, dx \\ &= \int_\Omega \Big(\tau_2 : \varepsilon(u_0) - \frac{\nu}{2}|\varepsilon(\overline{u})|^2 - \psi^*(\tau_2) + \nu\varepsilon(u_0) : \varepsilon(\overline{u})\Big)\, dx - \int_\Omega f \cdot u_0\, dx.\end{aligned}$$

Since $\nu\varepsilon(\overline{u}) \in \overline{\Sigma}_f$, we have $\overline{I}(\nu\varepsilon(\overline{u})) \le \sup \overline{\mathcal{P}}^*$, and

$$\begin{aligned}&\overline{I}(\nu\varepsilon(\overline{u})) \\ &= \int_\Omega \Big(\varepsilon(u_0) : (\nu\varepsilon(\overline{u}) + \tau_2) - \frac{\nu}{2}|\varepsilon(\overline{u})|^2 - \psi^*(\tau_2)\Big)\, dx - \int_\Omega f \cdot u_0\, dx \le \sup \overline{\mathcal{P}}^* \\ &\le \inf \overline{\mathcal{P}} \\ &= \int_\Omega \Big(\tau_2 : \varepsilon(u_0) - \frac{\nu}{2}|\varepsilon(\overline{u})|^2 - \psi^*(\tau_2) + \nu\varepsilon(u_0) : \varepsilon(\overline{u})\Big)\, dx - \int_\Omega f \cdot u_0\, dx.\end{aligned}$$

Consequently, $\nu\varepsilon(\overline{u}) = \overline{\sigma}_1$ and $\inf \overline{\mathcal{P}} = \sup \overline{\mathcal{P}}^*$. □

4.3. Estimates of deviations from exact solutions for solenoidal fields. Suppose $v \in \overset{\circ}{J}{}^1_2(\Omega) + u_0$, i.e., the exact solution u is compared with a vector-valued function v satisfying $\operatorname{div} v = 0$.

From (4.5), (4.7), and (4.8) it follows that for any $v \in \overset{\circ}{J}{}^1_2(\Omega) + u_0$ and any $\tau_{1f} \in \overline{\Sigma}_f(\Omega)$ we have the inequality

$$\int_\Omega \frac{\nu}{2}|\varepsilon(v-u)|^2\, dx \le J(v) - \inf \overline{\mathcal{P}} = J(v) - \sup \overline{\mathcal{P}}^* \le J(v) - \overline{I}(\tau_{1f}). \tag{4.10}$$

We estimate the right-hand side of (4.10) as follows:

$$\begin{aligned}J(\overline{v}) - \overline{I}(\tau_{1f}) \le &\int_\Omega \left(\frac{\nu}{2}|\varepsilon(v)|^2 + \frac{1}{2\nu}|\tau_{1f}|^2 - \varepsilon(u_0) : \tau_{1f}\right) dx \\ &+ \int_\Omega (\psi(\varepsilon(v)) + \psi^*(\tau_2) - \varepsilon(u_0) : \tau_2)\, dx + \int_\Omega f \cdot (u_0 - v)\, dx.\end{aligned}$$

Let $q \in \widetilde{L}_2(\Omega)$. Since $\tau_{1f} \in \overline{\Sigma}_f(\Omega)$ and $v \in \overset{\circ}{J}{}^1_2(\Omega) + u_0$, we have

$$\begin{aligned}\int_\Omega f \cdot (u_0 - v)\, dx &= \int_\Omega (f \cdot (u_0 - v) + q \operatorname{div}(u_0 - v))\, dx \\ &= \int_\Omega \varepsilon(u_0 - v) : (\tau_{1f} + \tau_2)\, dx.\end{aligned}$$

As a result, we obtain the estimate

$$\frac{\nu}{2}\|\varepsilon(v-u)\|^2 \le \mathcal{M}_1(v, \tau_{1f}, \tau_2) := \mathrm{D}_1(\varepsilon(v), \tau_{1f}) + \mathrm{D}_2(\varepsilon(v), \tau_2), \tag{4.11}$$

where

$$\begin{aligned}\mathrm{D}_1(\varepsilon(v), \tau_{1f}) &:= \int_\Omega \left(\tfrac{\nu}{2}|\varepsilon(v)|^2 + \tfrac{1}{2\nu}|\tau_{1f}|^2 - \varepsilon(v) : \tau_{1f}\right) dx = \tfrac{1}{2\nu}\|\nu\varepsilon(v) - \tau_{1f}\|^2; \\ \mathrm{D}_2(\varepsilon(v), \tau_2) &:= \int_\Omega (\psi(\varepsilon(v)) + \psi^*(\tau_2) - \varepsilon(v) : \tau_2)\, dx.\end{aligned}$$

Clearly, both functionals D_1 and D_2 are nonnegative. Moreover, $\mathrm{D}_1(\varepsilon(v), \tau_{1f}) = 0$ if and only if $\tau_{1f} = \nu\varepsilon(v)$. By the properties of conjugate functionals (see, e.g., [11, 32]),

$\mathrm{D}_2(\varepsilon(v), \tau_2) = 0$ if and only if $\tau_{1f} \in \partial\psi(\varepsilon(v))$. Now, it is easy to analyze the meaning of estimates (4.11). For this, we represent the main system in the form

$$-\operatorname{div}(\sigma_1 + \sigma_2) = f - \nabla p \quad \text{in } \Omega, \tag{4.12}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega, \tag{4.13}$$

$$\sigma_1 = \nu\varepsilon(u), \quad \sigma_2 \in \partial\psi(\varepsilon(u)) \quad \text{in } \Omega, \tag{4.14}$$

$$u = u_0 \quad \text{on } \partial\Omega. \tag{4.15}$$

If $v \in \mathring{J}^1_2(\Omega) + u_0$ and $\tau_{1f} \in \overline{\Sigma}_f(\Omega)$, then $-\operatorname{div}(\tau_{1f} + \tau_2) = f - \nabla q$ and $\operatorname{div} v = 0$, so that for v, τ_{1f}, τ_2, and q, relations (4.12), (4.13) and (4.15) are satisfied. Estimate (4.11) shows that, in this case, the energy norm of the deviation from the exact solution is controlled by the quantities $\mathrm{D}_1(\varepsilon(v), \tau_{1f})$ and $\mathrm{D}_2(\varepsilon(v), \tau_2)$, which characterize the "degree of inconsistency" in (4.14). Therefore, the majorant $\mathcal{M}_1(v, \tau_{1f}, \tau_2)$ attains its exact lower bound (which is equal to zero) if and only if $v = u$, $\tau_{1f} = \sigma_1$, and $\tau_2 = \sigma_2$.

Certainly, the condition $\tau_{1f} \in \overline{\Sigma}_f(\Omega)$ is inconvenient, and it is desirable to lift it somehow. This can be done as follows. Let $\tau_1 \notin \overline{\Sigma}_f(\Omega)$. Then

$$\mathrm{D}_1(\varepsilon(v), \tau_{1f}) \le (1+\beta)\mathrm{D}_1(\varepsilon(v), \tau_1) + \frac{1}{2\nu}\Big(1 + \frac{1}{\beta}\Big)\|\tau_{1f} - \tau_1\|^2,$$

where β is an arbitrary positive number. Then, by (4.11),

$$\frac{\nu}{2}\|\varepsilon(v-u)\|^2 \le (1+\beta)\mathrm{D}_1(\varepsilon(v), \tau_1) + \mathrm{D}_2(\varepsilon(v), \tau_2) + \Big(1 + \frac{1}{\beta}\Big)\frac{1}{2\nu}\|\tau_{1f} - \tau_1\|^2,$$

and this estimate is valid for any $\tau_{1f} \in \overline{\Sigma}_f$. We use this in order to estimate the last term. Observe that for any g belonging to the space V_0^* conjugate to V_0, there exists $\tau_g \in \Sigma$ such that $\operatorname{div} \tau_g = g$ (in the sense of distributions) and

$$\|\tau_g\| \le [\![g]\!] := \sup_{V_0} \frac{|\langle g, w\rangle|}{\|\varepsilon(w)\|}.$$

This fact follows from the solvability of the following problem: find $u \in V_0$ satisfying the integral identity $\int_\Omega \varepsilon(u) : \varepsilon(w)\, dx = \langle g, w\rangle$ for any $w \in V_0$, and the relation $\|\nabla u\| \le [\![g]\!]$. Since

$$\operatorname{div}(\tau_1 - \tau_{1f}) = \bar{g} := \operatorname{div}(\tau_1 + \tau_2) + f - \nabla q,$$

we conclude that there is a function $\tau_{\bar{g}}$ satisfying the conditions $\operatorname{div} \tau_{\bar{g}} = \bar{g}$ and $\|\tau_{\bar{g}}\| \le [\![\bar{g}]\!]$. It is easily seen that the function $\widehat{\tau} = \tau_1 - \tau_{\bar{g}}$ satisfies $\operatorname{div} \widehat{\tau} = -\operatorname{div} \tau_2 - f + \nabla q$ and, consequently, belongs to $\overline{\Sigma}_f$. Set $\tau_{1f} = \widehat{\tau}$. Then

$$\|\tau_{1f} - \tau_1\| = \|\widehat{\tau} - \tau_1\| = \|\tau_{\bar{g}}\| \le [\![\operatorname{div}(\tau_1 + \tau_2) + f - \nabla q]\!].$$

Thus, we arrive at the desired estimate

$$\begin{aligned} &\frac{\nu}{2}\|\varepsilon(v-u)\|^2 \\ &\quad \le (1+\beta)\mathrm{D}_1(\varepsilon(v), \tau_1) + \mathrm{D}_2(\varepsilon(v), \tau_2) + \frac{1+\beta}{2\nu\beta}[\![\operatorname{div}(\tau_1 + \tau_2) + f - \nabla q]\!]^2, \end{aligned} \tag{4.16}$$

which is true for any $v \in \mathring{J}^1_2(\Omega) + u_0$, any pair $(\tau_1, \tau_2) \in \Sigma \times \Sigma$, and any $\beta > 0$. The right-hand side of (4.16) is a majorant for the norm of the deviation from the exact solution; we denote this majorant by $\mathcal{M}_2(\beta, v, \tau_1, \tau_2, q)$. If the sum $\tau_1 + \tau_2$ has a somewhat higher regularity, so that $(\tau_1 + \tau_2) \in \Sigma_{\mathrm{div}}(\Omega)$ and, moreover, $q \in \widetilde{W}(\Omega)$, then the last term of the above majorant is estimated by an explicitly computable integral, the same way it was done in §2. As a result, we obtain the estimate

$$
\begin{aligned}
&\frac{\nu}{2}\|\varepsilon(v-u)\|^2 \\
&\qquad\le (1+\beta)\mathrm{D}_1(\varepsilon(v),\tau_1)+\mathrm{D}_2(\varepsilon(v),\tau_2)+\frac{1+\beta}{2\nu\beta}C_F^2\|\operatorname{div}(\tau_1+\tau_2)+f-\nabla q\|^2.
\end{aligned}
\tag{4.17}
$$

Assume that the right-hand side of (4.16) is equal to zero. Then

$$
-\operatorname{div}(\tau_1+\tau_2)=f-\nabla q
$$

in the sense of distributions (if $(\tau_1+\tau_2)\in\Sigma_{\mathrm{div}}(\Omega)$ and $q\in\widetilde{W}(\Omega)$, then this identity is fulfilled in the sense of L_2-functions). Moreover, the vanishing of $\mathrm{D}_1(\varepsilon(v),\tau_1)$ and $\mathrm{D}_2(\varepsilon(v),\tau_2)$ means that $\tau_1=\nu\varepsilon(v)$ and $\tau_2\in\partial\psi(\varepsilon(v))$ almost everywhere. Since $v\in\mathring{J}_2^1(\Omega)+u_0$, relations (4.13) and (4.15) are also satisfied. Consequently, in this case v coincides with the exact solution u, $\tau_1=\sigma_1$, and $\tau_2=\sigma_2$.

We summarize the above in the following statement.

Theorem 4.2. 1. *Suppose $\beta\in\mathbb{R}_+$, $v\in\mathring{J}_2^1(\Omega)+u_0$, $\tau_1\in\Sigma$, $\tau_2\in\Sigma$, and $q\in\widetilde{L}_2(\Omega)$; then the functional $\mathcal{M}_2(\beta,v,\tau_1,\tau_2,q)$ majorizes the quantity $\|\varepsilon(v-u)\|^2$.*

2. *For any $\beta\in\mathbb{R}_+$, the infimum of this functional on the set $(\mathring{J}_2^1(\Omega)+u_0)\times\Sigma\times\Sigma\times\widetilde{L}_2(\Omega)$ is equal to zero, and it is attained if and only if $v=u$, $\tau_1=\sigma_1$, $\tau_2=\sigma_2$, and $q=p$.*

We illustrate this result by two examples.

Example 1. For the Stokes problem we have $\psi(\varepsilon)\equiv 0$. Set $\tau_2=0$. Then $\mathrm{D}_2(\varepsilon(v),\tau_2)\equiv 0$, and (4.16) takes the form (2.14).

Example 2. For the Bingham model we have $\psi(\varepsilon)=k_*|\varepsilon|$, and

$$
\psi^*(\tau(x))=\begin{cases}0 & \text{if } |\tau(x)|\le k_*,\\ +\infty & \text{if } |\tau(x)|>k_*.\end{cases}
$$

Therefore,

$$
\mathrm{D}_2(\varepsilon(v),\tau_2)=\int_\Omega (k_*|\varepsilon(v)|-\varepsilon(v)\cdot\tau_2)\,dx
$$

if τ_2 satisfies the condition $|\tau_2(x)|\le k_*$ almost everywhere. Otherwise, $\mathrm{D}_2(\varepsilon(v),\tau_2)=+\infty$. Then (4.16) takes the form

$$
\begin{aligned}
&\frac{\nu}{2}\|\varepsilon(v-u)\|^2 \\
&\qquad\le \int_\Omega\Big(\frac{(1+\beta)}{2\nu}|\nu\varepsilon(v)-\tau_1|^2+k_*|\varepsilon(v)|-\varepsilon(v):\tau_2\Big)\,dx \\
&\qquad\quad+\frac{1+\beta}{2\beta\nu}\,\|\!|\operatorname{div}(\tau_1+\tau_2)+f-\nabla q\|\!|^2.
\end{aligned}
\tag{4.18}
$$

If τ_1, τ_2, and q are such that $q\in\widetilde{W}$ and $\operatorname{div}(\tau_1+\tau_2)\in L_2(\Omega)$, then the last term in (4.18) is estimated by an integral.

It is well known that a Bingham fluid may have two zones: the *congestion zone* Ω_0 (where $\varepsilon(u)\equiv 0$) and the *flow zone* Ω_1 (where $|\varepsilon(u)|>0$). Assume that the right-hand side of (4.18) vanishes for some functions v, τ_1, τ_2, and q. Then in Ω_0 we have $\varepsilon(v)=0$, and, consequently, $\tau_1 = 0$ and $\operatorname{div}\tau_2 + f - \nabla q = 0$ for some τ_2 satisfying $|\tau_2(x)| \le 1$. At the same time, in the flow zone Ω_1 we have

$$\tau_2 = k_* \frac{\varepsilon(v)}{|\varepsilon(v)|}, \quad \tau_1 = \nu\varepsilon(v), \quad \operatorname{div}(\tau_1 + \tau_2) + f - \nabla q = 0.$$

4.4. Estimates of deviations from exact solutions for velocity fields of general type. Suppose the exact solution is compared with a function $\widehat{v} \in V_0 + u_0$. To obtain estimates of the difference between $\widehat{v}$ and u in the energy norm we can apply the same method as was used in §2 for the Stokes problem. Namely, take $v \in \mathring{J}^1_2(\Omega) + u_0$. For any $\gamma > 0$ we have

$$\|\varepsilon(\widehat{v} - u)\|^2 \le \frac{(1+\gamma)}{\gamma}\|\varepsilon(\widehat{v} - v)\|^2 + (1+\gamma)\|\varepsilon(v-u)\|^2.$$

Since $v \in \mathring{J}^1_2(\Omega) + u_0$, the second term is estimated by (4.16). As a result, we obtain an estimate involving a function v, which may be chosen arbitrarily in the set $\mathring{J}^1_2(\Omega) + u_0$. Therefore, we can pass to the infimum over all such functions. All the details of this procedure can be found in [52]. For this reason, here we omit all technical details and present only the final estimate:

$$\begin{aligned} &\frac{\nu}{2}\|\varepsilon(\widehat{v} - u)\|^2 \\ &\quad \le c_1 \mathrm{D}_1(\varepsilon(\widehat{v}), \tau_1) + c_2\big(\mathrm{D}_2(\varepsilon(\widehat{v}), \tau_2) + \tfrac{\delta}{2\nu}\|\widehat{\eta}_2 - \tau_2\|^2\big) \\ &\qquad + c_3 \big\| \operatorname{div}(\tau_1 + \tau_2) + f - \nabla q \big\|^2 + c_4 \mathrm{R}_{\mathrm{div}}(\widehat{v}), \end{aligned} \tag{4.19}$$

where $\widehat{\eta}_2 \in \partial\psi(\varepsilon(\widehat{v}))$, and the coefficients c_1, c_2, c_3, and c_4 are computed in terms of the positive constants γ, δ, α_1, α_2, and α_3 by the formulas

$$c_1 = (1+\gamma)\Big(1 + \alpha_1 + \frac{1}{\alpha_3}\Big), \quad c_2 = c_4 = (1+\gamma), \quad c_3 = (1 + \alpha_2 + \alpha_3)\,\frac{1+\gamma}{2\nu}.$$

The functional $\mathrm{R}_{\mathrm{div}}$ is a penalty for the possible violation of the incompressibility condition. It is defined by the relation

$$\mathrm{R}_{\mathrm{div}}(\widehat{v}) = \inf_{v \in \mathring{J}^1_2(\Omega) + u_0} \Big\{ \frac{\mu}{2}\|\varepsilon(\widehat{v} - v)\|^2 + \mathcal{R}(v, \widehat{v}) \Big\}.$$

If the potential ψ is differentiable, then $\mathcal{R}(v, \widehat{v})$ is defined by the formula

$$\mathcal{R}(v, \widehat{v}) = \int_\Omega (\psi'(v) - \psi'(\widehat{v})) : \varepsilon(v - \widehat{v})\, dx,$$

where $\mu = \nu\big(1 + \frac{1}{\gamma} + \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\delta}\big)$.

The right-hand side of (4.19) will be denoted by $\mathcal{M}_3(v, \tau_1, \tau_2, q, \delta, \gamma, \alpha_1, \alpha_2, \alpha_3)$; it is a nonnegative functional. It is easily seen that, for any positive parameters, this functional vanishes if and only if the velocity field $\widehat{v} \in V_0 + u_0$ and the tensor-valued functions τ_1 and τ_2 (which correspond to the two parts of the stress field) satisfy the conditions

$$\begin{aligned} \operatorname{div}(\tau_1 + \tau_2) + f - \nabla q = 0 &\quad \text{in } \Omega, \\ \operatorname{div}\widehat{v} = 0 &\quad \text{in } \Omega, \\ \tau_1 = \nu(\varepsilon(\widehat{v})), \ \ \tau_2 = \partial\psi(\varepsilon(\widehat{v})) &\quad \text{in } \Omega. \end{aligned}$$

This means that

$$\widehat{v} = u, \quad \tau_1 = \sigma_1, \quad \tau_2 = \sigma_2, \quad q = p. \tag{4.20}$$

All the above is summarized in the following statement.

Theorem 4.3. 1. *For any* $\tau_1 \in \Sigma$, $\tau_2 \in \Sigma$, $q \in \widetilde{L}_2(\Omega)$ *and any positive* γ, δ, α_1, α_2, *and* α_3, *the functional* $\mathcal{M}_3(\widehat{v}, \tau_1, \tau_2, q)$ *is a majorant of the energy norm of the difference between* $\widehat{v}$ *and the exact solution* u.

2. *For any positive* γ, δ, α_1, α_2, *and* α_3, *the infimum of this functional over all* $\widehat{v} \in V_0 + u_0$, $\tau_1 \in \Sigma$, $\tau_2 \in \Sigma$, *and* $q \in \widetilde{L}_2(\Omega)$ *is equal to zero, and it is attained if these variables coincide with the corresponding components of the true solution, i.e., if* (4.20) *is fulfilled.*

The first two terms of $\mathcal{M}_3$ can be computed directly. The third can be estimated by a computable quantity provided $\tau_1 + \tau_2 \in \Sigma_{\mathrm{div}}(\Omega)$ and $q \in \widetilde{W}(\Omega)$. Thus, it remains to obtain a realistic and computable bound for the term $\mathrm{R}_{\mathrm{div}}(\widehat{v})$. In the paper [52], it was shown how to do this for the Bingham model and for models with power growth dissipative potentials.

Note that estimate (4.19) can be reshaped to a more compact form if we set $\tau_2(\widehat{v}) \in \partial\psi(\varepsilon(\widehat{v}))$. Then

$$\begin{aligned}&\frac{\nu}{2}\|\varepsilon(\widehat{v} - u)\|^2 \\ &\qquad \leq c_1 \mathrm{D}_1(\varepsilon(\widehat{v}), \tau_1) + c_3 \big|\!\big|\!\big| \operatorname{div}(\tau_1 + \tau_2(\widehat{v})) + f - \nabla q \big|\!\big|\!\big|^2 + c_4 \mathrm{R}_{\mathrm{div}}(\widehat{v}).\end{aligned}$$

In particular, if ψ is a differentiable functional, then

$$\begin{aligned}&\frac{\nu}{2}\|\varepsilon(\widehat{v} - u)\|^2 \\ &\qquad \leq c_1 \mathrm{D}_1(\varepsilon(\widehat{v}), \tau_1) + c_3 \big|\!\big|\!\big| \operatorname{div}(\tau_1 + \psi'(\varepsilon(\widehat{v}))) + f - \nabla q \big|\!\big|\!\big|^2 + c_4 \mathrm{R}_{\mathrm{div}}(\widehat{v}).\end{aligned}$$

Finally, we note that the majorants $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ can be viewed as new functionals defined on all possible approximations of the fields u, σ^*, and p. These majorants are weighted sums of the residuals in relations (4.12)–(4.15), and they vanish only if the exact solution is substituted in the majorant. The majorants constructed for many other problems have a similar form (see, e.g., [10, 11, 50, 53]). These observations make it possible to suggest that the majorants for the energy norm of the deviations from exact solutions should consist of terms that may be thought of as penalties for the possible violation of each of the relations, and the corresponding coefficients are determined by the constants in the embedding theorems associated with the mathematical formulation of the problem in question.

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