Алгебра и анализ
Том. 16 (2004), вып. 5

St. Petersburg Math. J.
Vol. 16 (2005), No. 5, Pages 863–922
S 1061-0022(05)00883-6
Article electronically published on September 23, 2005

HOMOGENIZATION OF A STATIONARY PERIODIC MAXWELL SYSTEM

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Abstract. The homogenization problem is considered for a stationary periodic Maxwell system in $\mathbb{R}^3$ in the small period limit. The behavior of four fields is studied, namely, of the strength of the electric field, the strength of the magnetic field, the electric displacement vector, and the magnetic displacement vector. Each field is represented as a sum of two terms. For some terms uniform approximations in the $L_2(\mathbb{R}^3)$-norm are obtained, together with a precise order estimate for the remainder term.

§0. Introduction

0.1. In the present paper, we consider the homogenization problem for a stationary periodic Maxwell system in the small period limit. This problem was studied intensively; in particular, it was discussed in the books [BaPa, BeLP, ZhKO, Sa]. However, the known results give only the *weak* convergence of solutions to the solution of the "homogenized" system with constant coefficients.

We rely on the abstract approach developed in [BSu1, BSu2]. This approach makes it possible to establish the convergence of resolvents in the operator L_2-norm to the resolvent of the homogenized problem, and simultaneously gives a remainder estimate of precise order. At the same time, the Maxwell operator can be included in the class of differential operators studied in [BSu1, BSu2] only in the case where one of two periodic characteristics of the medium is constant. In [BSu2, Chapter 7], the homogenization problem for the Maxwell operator was considered in the case where $\mu = 1$. Here we study the much more difficult general case, which requires essential modification of the technique. The detailed comparison of the methods and results of the present paper and those of [BSu2] is given below in Subsection 0.8.

0.2. Setting of the problem. We denote $\mathfrak{G} = L_2(\mathbb{R}^3; \mathbb{C}^3)$ and $J = \{\mathbf{f} \in \mathfrak{G} : \operatorname{div} \mathbf{f} = 0\}$. Let Γ be a lattice of periods in $\mathbb{R}^3$, and let $\Omega \subset \mathbb{R}^3$ be the elementary cell of Γ. Assume that the dielectric permittivity $\eta(\mathbf{x})$ and the magnetic permeability $\mu(\mathbf{x})$ are Γ-periodic matrix-valued functions and that η and μ are bounded and uniformly positive. We denote by $\mathbf{u}$ the strength of the electric field and by $\mathbf{v}$ the strength of the magnetic field; $\mathbf{w} = \eta\mathbf{u}$ is the electric displacement vector, and $\mathbf{z} = \mu\mathbf{v}$ is the magnetic displacement vector. We write the Maxwell operator $\mathcal{M}$ in terms of the displacement vectors, assuming that $\mathbf{w}$ and $\mathbf{z}$ are solenoidal. Then $\mathcal{M} = \mathcal{M}(\eta, \mu)$ acts in the space $J \oplus J$ and is given by the formula

$$\mathcal{M}(\eta, \mu) = \begin{pmatrix} 0 & i \operatorname{curl} \mu^{-1} \\ -i \operatorname{curl} \eta^{-1} & 0 \end{pmatrix}$$

on the natural domain. The point $\lambda = i$ is a regular point for $\mathcal{M}$.

2000 *Mathematics Subject Classification.* Primary 35P20, 35Q60.
Key words and phrases. Periodic Maxwell operator, homogenization, effective medium.
This work was supported by RFBR (grant no. 02-01-00798).

Let $\varepsilon > 0$ be a parameter. Consider the family of operators $\mathcal{M}_\varepsilon = \mathcal{M}(\eta^\varepsilon, \mu^\varepsilon)$ with rapidly oscillating coefficients $\eta^\varepsilon(\mathbf{x}) := \eta(\varepsilon^{-1}\mathbf{x})$, $\mu^\varepsilon(\mathbf{x}) := \mu(\varepsilon^{-1}\mathbf{x})$. *Our goal is to study the behavior of the resolvent* $(\mathcal{M}_\varepsilon - iI)^{-1}$ *as* $\varepsilon \to 0$. In other words, we are interested in the behavior of the solution of the equation

$$(0.1)\qquad (\mathcal{M}_\varepsilon - iI)\begin{pmatrix}\mathbf{w}_\varepsilon\\ \mathbf{z}_\varepsilon\end{pmatrix} = \begin{pmatrix}\mathbf{q}\\ \mathbf{r}\end{pmatrix}, \quad \mathbf{q}, \mathbf{r} \in J.$$

The corresponding strengths are given by $\mathbf{u}_\varepsilon = (\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon$ and $\mathbf{v}_\varepsilon = (\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon$. In detail, (0.1) can be written as

$$(0.2)\qquad \begin{cases} i\ \operatorname{curl}(\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon - i\mathbf{w}_\varepsilon = \mathbf{q},\\ -i\ \operatorname{curl}(\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon - i\mathbf{z}_\varepsilon = \mathbf{r},\\ \operatorname{div}\mathbf{w}_\varepsilon = 0, \quad \operatorname{div}\mathbf{z}_\varepsilon = 0.\end{cases}$$

Note that the resolvent of $\mathcal{M}_\varepsilon$ could be considered not only at the point $\lambda = i$, but also at any other nonreal point.

0.3. The main results. We represent each field as a sum of two terms. Namely, $\mathbf{w}_\varepsilon = \mathbf{w}_\varepsilon^{(q)} + \mathbf{w}_\varepsilon^{(r)}$, $\mathbf{z}_\varepsilon = \mathbf{z}_\varepsilon^{(q)} + \mathbf{z}_\varepsilon^{(r)}$, where the pair of vectors $\mathbf{w}_\varepsilon^{(q)}$, $\mathbf{z}_\varepsilon^{(q)}$ is the solution of system (0.2) with $\mathbf{r} = 0$, and the pair of vectors $\mathbf{w}_\varepsilon^{(r)}$, $\mathbf{z}_\varepsilon^{(r)}$ is the solution of system (0.2) with $\mathbf{q} = 0$. Similarly, the fields $\mathbf{u}_\varepsilon$, $\mathbf{v}_\varepsilon$ are also represented as sums.

For some terms, namely, for $\mathbf{u}_\varepsilon^{(q)}$, $\mathbf{w}_\varepsilon^{(q)}$ and $\mathbf{v}_\varepsilon^{(r)}$, $\mathbf{z}_\varepsilon^{(r)}$, *we obtain uniform approximations in the* $\mathfrak{G}$*-norm with a precise order remainder estimate. Such approximations are the main results of the paper.* For the remaining fields we still have only weak convergence in $\mathfrak{G}$ to the corresponding fields in the homogeneous "effective" medium with characteristics η^0, μ^0. Here η^0, μ^0 are the "effective" matrices for the elliptic operators $-\operatorname{div}\eta(\mathbf{x})\nabla$, $-\operatorname{div}\mu(\mathbf{x})\nabla$, respectively. We recall the definition of the matrix μ^0. Let $\mathbf{C} \in \mathbb{C}^3$, and let $\Phi_{\mathbf{C}}(\mathbf{x})$ be a periodic solution of the equation

$$(0.3)\qquad \operatorname{div}\mu(\mathbf{x})(\nabla\Phi_{\mathbf{C}} + \mathbf{C}) = 0, \quad \mathbf{C} \in \mathbb{C}^3.$$

Then

$$\mu^0\mathbf{C} = |\Omega|^{-1}\int_\Omega \mu(\mathbf{x})(\nabla\Phi_{\mathbf{C}} + \mathbf{C})\,d\mathbf{x}, \quad \mathbf{C} \in \mathbb{C}^3.$$

The matrix η^0 is defined similarly. The "elliptic" rule of finding the effective coefficients for the Maxwell operator is well known (see, e.g., [BeLP, ZhKO, Sa]). Let $\mathcal{M}^0 = \mathcal{M}(\eta^0, \mu^0)$ denote the "effective" Maxwell operator with the constant coefficients η^0, μ^0.

Now we describe the character of approximations. For definiteness, we dwell on the case where $\mathbf{q} = 0$. Let $\mathbf{w}_0^{(r)}$, $\mathbf{z}_0^{(r)}$ be the solution of the "homogenized" system

$$(0.4)\qquad (\mathcal{M}^0 - iI)\begin{pmatrix}\mathbf{w}_0^{(r)}\\ \mathbf{z}_0^{(r)}\end{pmatrix} = \begin{pmatrix}0\\ \mathbf{r}\end{pmatrix}.$$

We put $\mathbf{u}_0^{(r)} = (\eta^0)^{-1}\mathbf{w}_0^{(r)}$, $\mathbf{v}_0^{(r)} = (\mu^0)^{-1}\mathbf{z}_0^{(r)}$. Besides the "homogenized" system (0.4), we consider the "correction" system

$$(0.5)\qquad (\mathcal{M}^0 - iI)\begin{pmatrix}\widehat{\mathbf{w}}_\varepsilon^{(r)}\\ \widehat{\mathbf{z}}_\varepsilon^{(r)}\end{pmatrix} = \begin{pmatrix}0\\ \mathbf{r}_\varepsilon\end{pmatrix},$$

where the right-hand side $\mathbf{r}_\varepsilon$ depends on $\mathbf{r}$ and contains some (explicitly described) rapidly oscillating (as $\varepsilon \to 0$) factor with zero mean value. Then the solution of the "correction" system (0.5) weakly converges to zero in $\mathfrak{G}$. We put $\widehat{\mathbf{v}}_\varepsilon^{(r)} = (\mu^0)^{-1}\widehat{\mathbf{z}}_\varepsilon^{(r)}$. *Approximations for the fields* $\mathbf{v}_\varepsilon^{(r)}$, $\mathbf{z}_\varepsilon^{(r)}$ *are chosen in the form*

$$\widetilde{\mathbf{v}}_\varepsilon^{(r)} = (\mathbf{1} + Y^\varepsilon)(\mathbf{v}_0^{(r)} + \widehat{\mathbf{v}}_\varepsilon^{(r)}), \quad \widetilde{\mathbf{z}}_\varepsilon^{(r)} = (\mathbf{1} + G^\varepsilon)(\mathbf{z}_0^{(r)} + \widehat{\mathbf{z}}_\varepsilon^{(r)}), \tag{0.6}$$

where Y^ε, G^ε are appropriate rapidly oscillating periodic matrix-valued functions with zero mean. *We have*

$$\|\mathbf{v}_\varepsilon^{(r)} - \widetilde{\mathbf{v}}_\varepsilon^{(r)}\|_{\mathfrak{G}} \le C\varepsilon\|\mathbf{r}\|_{\mathfrak{G}}, \quad \|\mathbf{z}_\varepsilon^{(r)} - \widetilde{\mathbf{z}}_\varepsilon^{(r)}\|_{\mathfrak{G}} \le C\varepsilon\|\mathbf{r}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{0.7}$$

However, for $\mathbf{u}_\varepsilon^{(r)}$ and $\mathbf{w}_\varepsilon^{(r)}$ we still have only weak convergence to $\mathbf{u}_0^{(r)}$, $\mathbf{w}_0^{(r)}$.

In approximations (0.6) (after opening the parentheses), we distinguish the summands $\mathbf{v}_0^{(r)}$, $\mathbf{z}_0^{(r)}$ that do not depend on ε and are equal to the weak limits of $\mathbf{v}_\varepsilon^{(r)}$, $\mathbf{z}_\varepsilon^{(r)}$; the other summands weakly tend to zero in $\mathfrak{G}$. The rapidly oscillating factors in (0.6) and in $\mathbf{r}_\varepsilon$ are described in terms of the periodic solutions $\Phi_{\mathbf{C}}$ of equations (0.3). Clearly, the precise order estimates (0.7) are much more informative than the weak convergence of the solutions (which follows from (0.7)). For the fields $\mathbf{u}_\varepsilon^{(q)}$, $\mathbf{w}_\varepsilon^{(q)}$, we obtain approximations similar to (0.6), (0.7).

0.4. Reduction to an auxiliary second-order operator. Our method of investigation employs reduction to the homogenization problem for some vector elliptic second-order operator. We explain this, again in the case of $\mathbf{q} = 0$. In this case, (0.2) implies that $\mathbf{z}_\varepsilon^{(r)}$ is the solution of the problem

$$\operatorname{curl}(\eta^\varepsilon)^{-1}\operatorname{curl}(\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon^{(r)} + \mathbf{z}_\varepsilon^{(r)} = i\mathbf{r}, \;\; \operatorname{div}\mathbf{z}_\varepsilon^{(r)} = 0, \quad \mathbf{r} \in J. \tag{0.8}$$

Then $\mathbf{g}_\varepsilon = (\mu^\varepsilon)^{-1/2}\mathbf{z}_\varepsilon^{(r)}$ is the solution of the system

$$(\mu^\varepsilon)^{-1/2}\operatorname{curl}(\eta^\varepsilon)^{-1}\operatorname{curl}(\mu^\varepsilon)^{-1/2}\mathbf{g}_\varepsilon + \mathbf{g}_\varepsilon = i(\mu^\varepsilon)^{-1/2}\mathbf{r}, \quad \operatorname{div}(\mu^\varepsilon)^{1/2}\mathbf{g}_\varepsilon = 0. \tag{0.9}$$

It is convenient to pass from (0.8) to (0.9), because the operator in (0.9) is selfadjoint with respect to the standard inner product in $\mathfrak{G}$, while the operator in (0.8) is selfadjoint with respect to the inner product with the weight $(\mu^\varepsilon)^{-1}$ (which depends on ε).

Next, we extend system (0.9) in order to "remove" the divergence-free condition (depending on the parameter ε). This leads us to consider the operator

$$\mathcal{L} = \mathcal{L}(\mu, \eta, \nu) = \mu^{-1/2}\operatorname{curl}\eta^{-1}\operatorname{curl}\mu^{-1/2} - \mu^{1/2}\nabla\nu\operatorname{div}\mu^{1/2}, \tag{0.10}$$

which is selfadjoint in $\mathfrak{G}$. Here $\nu(\mathbf{x})$ is some bounded and uniformly positive Γ-periodic function. Clearly, for application to the Maxwell operator it would suffice to assume that $\nu(\mathbf{x}) = 1$. The operator $\mathcal{L}$ splits into the orthogonal sum $\mathfrak{G} = G(\mu) \oplus J(\mu)$, where the subspace $G(\mu)$ consists of vector-valued functions of the form $\mu^{1/2}\nabla\varphi$, and $J(\mu)$ is formed by the vector-valued functions $\mathbf{f}$ satisfying the divergence-free condition $\operatorname{div}\mu^{1/2}\mathbf{f} = 0$. Mainly, we are interested in the part of $\mathcal{L}$ acting in $J(\mu)$. Let $\mathcal{P}(\mu)$ denote the orthogonal projection in $\mathfrak{G}$ onto $J(\mu)$.

Let $\mathcal{L}_\varepsilon := \mathcal{L}(\mu^\varepsilon, \eta^\varepsilon, \nu^\varepsilon)$ be an operator of the form (0.10) with rapidly oscillating coefficients. Then the solution of system (0.9) can be written as

$$\mathbf{g}_\varepsilon = (\mathcal{L}_\varepsilon + I)^{-1}(i(\mu^\varepsilon)^{-1/2}\mathbf{r}) = (\mathcal{L}_\varepsilon + I)^{-1}\mathcal{P}(\mu^\varepsilon)(i(\mu^\varepsilon)^{-1/2}\mathbf{r}). \tag{0.11}$$

Thus, our question reduces to the study of the behavior as $\varepsilon \to 0$ of the resolvent $(\mathcal{L}_\varepsilon + I)^{-1}$ (more precisely, of its "solenoidal" part $(\mathcal{L}_\varepsilon + I)^{-1}\mathcal{P}(\mu^\varepsilon)$). The technical part of the paper consists in the study of these objects.

0.5. Inclusion in the framework of the abstract method of [BSu1, BSu2]**.** It is essential that the operator (0.10) admits a factorization $\mathcal{L} = \mathcal{X}^*\mathcal{X}$, where $\mathcal{X} : \mathfrak{G} \to \mathfrak{G}_* = L_2(\mathbb{R}^3; \mathbb{C}^4)$ is the homogeneous first-order differential operator of the form

$$\mathcal{X} = \begin{pmatrix} -i\eta^{-1/2} \operatorname{curl} \mu^{-1/2} \\ -i\nu^{1/2} \operatorname{div} \mu^{1/2} \end{pmatrix}.$$

We decompose the periodic operator $\mathcal{L}$ in the direct integral of operators $\mathcal{L}(\mathbf{k})$ that act in the space $\mathfrak{H} = L_2(\Omega; \mathbb{C}^3)$ and depend on a parameter $\mathbf{k} \in \mathbb{R}^3$. The operator $\mathcal{L}(\mathbf{k})$ corresponds to the differential expression

$$\mu^{-1/2} \operatorname{curl}_{\mathbf{k}} \eta^{-1} \operatorname{curl}_{\mathbf{k}} \mu^{-1/2} - \mu^{1/2} \nabla_{\mathbf{k}} \nu \operatorname{div}_{\mathbf{k}} \mu^{1/2}$$

with periodic boundary conditions. Here $\nabla_{\mathbf{k}} := \nabla + i\mathbf{k}$, $\operatorname{div}_{\mathbf{k}} := \operatorname{div} + i\mathbf{k}\,\cdot$, $\operatorname{curl}_{\mathbf{k}} := \operatorname{curl} + i\mathbf{k}\,\times$. The operator $\mathcal{L}(\mathbf{k})$ also admits factorization: $\mathcal{L}(\mathbf{k}) = \mathcal{X}(\mathbf{k})^*\mathcal{X}(\mathbf{k})$, where $\mathcal{X}(\mathbf{k})$ is defined via $\mathcal{X}$ in the same way as $\mathcal{L}(\mathbf{k})$ is defined via $\mathcal{L}$. The operator $\mathcal{L}(\mathbf{k})$ splits into the orthogonal sum $\mathfrak{H} = G(\mu; \mathbf{k}) \oplus J(\mu; \mathbf{k})$, where $G(\mu; \mathbf{k})$ is the subspace of the periodic vector-valued functions of the form $\mu^{1/2}\nabla_{\mathbf{k}}\varphi$, and $J(\mu; \mathbf{k})$ consists of periodic vector-valued functions $\mathbf{f}$ satisfying $\operatorname{div}_{\mathbf{k}} \mu^{1/2}\mathbf{f} = 0$. In the direct integral decomposition for $\mathcal{L}$, the part $\mathcal{L}_{J(\mu)}$ corresponds to the part $(\mathcal{L}(\mathbf{k}))_{J(\mu;\mathbf{k})}$.

We can apply the abstract method suggested in [BSu1, BSu2] to the operator family $\mathcal{L}(\mathbf{k})$. In [BSu1, BSu2], the family of selfadjoint operators of the form $A(t) = X(t)^*X(t)$ with $t \in \mathbb{R}$, $X(t) = X_0 + tX_1$, was studied. Now the parameter $\mathbf{k}$ is three-dimensional. To avoid this difficulty, we put (cf. [BSu2, Chapter 2]) $\mathbf{k} = t\boldsymbol{\theta}$, $t = |\mathbf{k}|$, $\boldsymbol{\theta} \in \mathbb{S}^2$. Then all the objects will depend on the additional parameter $\boldsymbol{\theta}$, and we need to trace this dependence. We put $\mathcal{L}(\mathbf{k}) =: L(t; \boldsymbol{\theta})$, $\mathcal{X}(\mathbf{k}) =: X(t; \boldsymbol{\theta})$. Then $X(t; \boldsymbol{\theta}) = X_0 + tX_1(\boldsymbol{\theta})$ and $L(t; \boldsymbol{\theta}) = X(t; \boldsymbol{\theta})^*X(t; \boldsymbol{\theta})$.

The main notion of the abstract method of [BSu1, BSu2] is that of the *spectral germ S of the family $A(t)$ at $t = 0$*. In our case, the germ $S(\boldsymbol{\theta})$ of the operator family $L(t; \boldsymbol{\theta})$ depends on $\boldsymbol{\theta}$; $S(\boldsymbol{\theta})$ is a selfadjoint operator acting in the three-dimensional kernel $\mathfrak{N} = \operatorname{Ker} \mathcal{L}(0) = \operatorname{Ker} X_0$. (The precise definition of the germ is given in Subsection 4.3.) Let P denote the orthogonal projection in $\mathfrak{H}$ onto the kernel $\mathfrak{N}$. Calculations show that $S(\boldsymbol{\theta})$ is unitarily equivalent to the germ $S^0(\boldsymbol{\theta})$ corresponding to the "effective" operator $\mathcal{L}^0 = \mathcal{L}(\mu^0, \eta^0, \underline{\nu})$ with the constant effective coefficients. Here $\underline{\nu}^{-1} = |\Omega|^{-1} \int_\Omega \nu(\mathbf{x})^{-1}\, d\mathbf{x}$. We have $S(\boldsymbol{\theta}) = \mathcal{U}^* S^0(\boldsymbol{\theta})\mathcal{U}$, where $\mathcal{U}$ is the unitary operator of "identification of kernels", which maps $\mathfrak{N}$ onto $\mathfrak{N}^0$. Here $\mathfrak{N}^0$ is the similar kernel corresponding to the operator $\mathcal{L}^0$. The kernel $\mathfrak{N}^0$ consists of constant vector-valued functions and can be identified with $\mathbb{C}^3$. The operator $\mathcal{U}$ admits a natural description in terms of solutions of equations (0.3); see Subsection 4.2.

0.6. Approximation of the resolvent near the bottom of the spectrum. The lower edge of the spectrum of the operator $\mathcal{L}$ is the point $\lambda = 0$. The possibility of homogenization is a "threshold effect" related to the behavior of the resolvent near the bottom of the spectrum. This point of view on homogenization problems for periodic operators was successively developed in [BSu1, BSu2]. Accordingly, we need to study the behavior of the resolvent $(\mathcal{L} + \varepsilon^2 I)^{-1}$ for small ε; in its turn, this reduces to the study of the behavior of the resolvent $(L(t; \boldsymbol{\theta}) + \varepsilon^2 I)^{-1}$. Also, we must separate the "solenoidal" parts of these resolvents.

To the operators $L(t; \boldsymbol{\theta})$, we apply the abstract Theorem 1.5.5 of [BSu2] on the approximation of the resolvent in terms of the spectral germ. By that theorem, for small ε the resolvent $(L(t; \boldsymbol{\theta}) + \varepsilon^2 I)^{-1}$ can be approximated (in the operator norm in $\mathfrak{H}$) by the operator $(t^2 S(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}})^{-1}P$. Then, using the relationship between the germs $S(\boldsymbol{\theta})$ and $S^0(\boldsymbol{\theta})$, we can approximate the resolvent $(L(t; \boldsymbol{\theta}) + \varepsilon^2 I)^{-1}$ by the "sandwiched" resolvent

$(\mathcal{U}P)^*(L^0(t;\boldsymbol{\theta})+\varepsilon^2 I)^{-1}(\mathcal{U}P)$ of the operator $L^0(t;\boldsymbol{\theta})$, which corresponds to the effective operator $\mathcal{L}^0$.

Let $\mathcal{P}(\mu;\mathbf{k})$ be the orthogonal projection in $\mathfrak{H}$ onto $J(\mu;\mathbf{k})$. In order to approximate the "solenoidal" part of the resolvent, i.e., the operator $(L(t;\boldsymbol{\theta})+\varepsilon^2 I)^{-1}\mathcal{P}(\mu;t\boldsymbol{\theta})$, we need to study beforehand the behavior of the operator $\mathcal{P}(\mu;t\boldsymbol{\theta})P$ for small t. It turns out that the limit of this operator as $t\to 0$ depends on $\boldsymbol{\theta}$ and is equal to the projection onto some two-dimensional subspace $J_{\boldsymbol{\theta}}$ of the kernel $\mathfrak{N}$. The subspace $J_{\boldsymbol{\theta}}$ is invariant for the germ $S(\boldsymbol{\theta})$. As a result, we approximate the operator $(L(t;\boldsymbol{\theta})+\varepsilon^2 I)^{-1}\mathcal{P}(\mu;t\boldsymbol{\theta})$ by the "sandwiched" solenoidal part of the resolvent of $L^0(t,\boldsymbol{\theta})$, that is, by $(\mathcal{U}P)^*(L^0(t;\boldsymbol{\theta})+\varepsilon^2 I)^{-1}\mathcal{P}(\mu^0;t\boldsymbol{\theta})(\mathcal{U}P)$.

Next, using decomposition of the initial operator $\mathcal{L}$ in the direct integral of the operators $\mathcal{L}(\mathbf{k})=L(t;\boldsymbol{\theta})$, we obtain approximations (in the operator norm in $\mathfrak{G}$) for the resolvent $(\mathcal{L}+\varepsilon^2 I)^{-1}$ by the "sandwiched" resolvent $\mathcal{W}^*(\mathcal{L}^0+\varepsilon^2 I)^{-1}\mathcal{W}$ of $\mathcal{L}^0$. The "solenoidal" part $(\mathcal{L}+\varepsilon^2 I)^{-1}\mathcal{P}(\mu)$ of the resolvent is approximated by the "sandwiched" solenoidal part of the resolvent of the effective operator, i.e., by the operator $\mathcal{W}^*(\mathcal{L}^0+\varepsilon^2 I)^{-1}\mathcal{P}(\mu^0)\mathcal{W}$. The bordering operator $\mathcal{W}$ is described in Subsection 7.2.

Further analysis of the bordering operators $\mathcal{W}$, $\mathcal{W}^*$ allows us to replace them in approximations by the simpler operators of multiplication by some periodic matrix-valued functions $W(\mathbf{x})$, $W(\mathbf{x})^*$. These matrices are expressed explicitly in terms of solutions of equations (0.3).

0.7. The results on homogenization for the operator $\mathcal{L}$. Finally, we apply the scale transformation to obtain approximations in the operator norm in $\mathfrak{G}$ for the resolvent of the operator $\mathcal{L}_\varepsilon$ and for its solenoidal part:

$$\|(\mathcal{L}_\varepsilon+I)^{-1}-(W^\varepsilon)^*(\mathcal{L}^0+I)^{-1}W^\varepsilon\|_{\mathfrak{G}\to\mathfrak{G}}\le C\varepsilon,\quad 0<\varepsilon\le 1, \tag{0.12}$$

$$\|(\mathcal{L}_\varepsilon+I)^{-1}\mathcal{P}(\mu^\varepsilon)-(W^\varepsilon)^*(\mathcal{L}^0+I)^{-1}\mathcal{P}(\mu^0)W^\varepsilon\|_{\mathfrak{G}\to\mathfrak{G}}\le C\varepsilon,\quad 0<\varepsilon\le 1. \tag{0.13}$$

Estimates (0.12), (0.13) *represent the main results of the paper concerning the homogenization problem for the operator* $\mathcal{L}$. The constants in these estimates are controlled explicitly.

Now, the results concerning approximations in the homogenization problem for the Maxwell operator follow from estimate (0.13) and the reduction (described above) of the question to the behavior of the vector-valued function (0.11).

0.8. Comparison with [BSu2]**.** In the case where the magnetic permeability $\mu(\mathbf{x})$ is constant, $\mu(\mathbf{x})=\mu^0$, the operator $\mathcal{L}$ admits a factorization of the form

$$\mathcal{L}=b(\mathbf{D})^*g(\mathbf{x})b(\mathbf{D}), \tag{0.14}$$

where the periodic (4×4)-matrix $g(\mathbf{x})$ and the homogeneous first-order differential operator $b(\mathbf{D}):\mathfrak{G}\to\mathfrak{G}_*$ with constant coefficients are given by

$$g(\mathbf{x})=\begin{pmatrix}\eta(\mathbf{x})^{-1} & 0\\ 0 & \nu(\mathbf{x})\end{pmatrix},\quad b(\mathbf{D})=\begin{pmatrix}-i\,\mathrm{curl}(\mu^0)^{-1/2}\\ -i\,\mathrm{div}(\mu^0)^{1/2}\end{pmatrix}.$$

The class of periodic matrix differential operators admitting factorization as in (0.14) was distinguished and studied in detail in [BSu1, BSu2]. The study of the homogenization problem for the Maxwell operator with $\mu(\mathbf{x})=1$ in [BSu2, Chapter 7] was based on the representation (0.14). In the present paper, the following steps are borrowed from [BSu2, Chapter 7]: representation of each field as a sum of two terms, reduction of the problem to a second-order equation, and further extension of the operator. However, the case of $\mu=\mu^0$ is significantly simpler than the general case: it turns out that if $\mu=\mu^0$, then $\mathfrak{N}=\mathfrak{N}^0$ and $S(\boldsymbol{\theta})=S^0(\boldsymbol{\theta})$, i.e., the kernel $\mathfrak{N}$ and the germ $S(\boldsymbol{\theta})$ for the operator $L(t;\boldsymbol{\theta})$ coincide with those for the effective operator $L^0(t;\boldsymbol{\theta})$. In accordance with the terminology of [BSu2], the operators $L(t;\boldsymbol{\theta})$ and $L^0(t;\boldsymbol{\theta})$ are *threshold equivalent*. In this case, it is possible to approximate the resolvent $(\mathcal{L}_\varepsilon + I)^{-1}$ directly by the resolvent $(\mathcal{L}^0 + I)^{-1}$ of the effective operator (without any bordering). As applied to the Maxwell operator, this leads to the fact that the approximations (0.6) simply coincide with the "homogenized" fields: $\widetilde{\mathbf{v}}_\varepsilon^{(r)} = \mathbf{v}_0^{(r)}$, $\widetilde{\mathbf{z}}_\varepsilon^{(r)} = \mathbf{z}_0^{(r)}$. In the general case, the families $L(t;\boldsymbol{\theta})$ and $L^0(t;\boldsymbol{\theta})$ are not threshold equivalent. Approximations for the resolvent of $\mathcal{L}_\varepsilon$ involve "bordering" operators, and approximations (0.6) for $\mathbf{v}_\varepsilon^{(r)}$, $\mathbf{z}_\varepsilon^{(r)}$ contain rapidly oscillating factors. Besides, it is much more difficult (as compared to the case of $\mu = \mu^0$) to separate the solenoidal parts of the operators in approximations. All this requires essential modification and complication of the technique, as compared with [BSu2]. It should be mentioned that, for $\mu = 1$, our result on approximations for $\mathbf{v}_\varepsilon^{(r)}$, $\mathbf{z}_\varepsilon^{(r)}$ repeats the result of [BSu2, Chapter 7]. At the same time, our analogs of approximations (0.6) for $\mathbf{u}_\varepsilon^{(q)}$, $\mathbf{w}_\varepsilon^{(q)}$ refine the results of [BSu2, Chapter 7], where only weak convergence in $\mathfrak{G}$ to $\mathbf{u}_0^{(q)}$, $\mathbf{w}_0^{(q)}$ was proved for these fields.

0.9. The structure of the paper. §1 contains preliminaries. §2 is devoted to some classes of vector-valued functions; here we analyze the required versions of the Weyl decomposition in $\mathbb{R}^3$ and in Ω. (The material of §2 is based on the results and techniques of [BS1].) In §3, we define the operator $\mathcal{L}$ and the operators $\mathcal{L}(\mathbf{k})$ arising in the direct integral decomposition for $\mathcal{L}$; also, we study the properties of the band functions (the eigenvalues of $\mathcal{L}(\mathbf{k})$). In §4, the general method of [BSu2, Chapter 1] is applied to the study of the operator family $\mathcal{L}(\mathbf{k}) = L(t;\boldsymbol{\theta})$, the spectral germ $S(\boldsymbol{\theta})$ is calculated, and the analytic (in t) branches of eigenvalues and eigenvectors of $L(t;\boldsymbol{\theta})$ are investigated. §5 is devoted to approximation of the operator $\mathcal{P}(\mu;\mathbf{k})P$ for small $|\mathbf{k}|$. Here we prove the crucial technical result of the paper (Theorem 5.1). In §6, we obtain approximations (for small ε) in the operator norm in $\mathfrak{H}$ for the resolvent $(L(t;\boldsymbol{\theta}) + \varepsilon^2 I)^{-1}$ and its solenoidal part. In §7, we find approximations in the operator norm in $\mathfrak{G}$ for the resolvent $(\mathcal{L}+\varepsilon^2 I)^{-1}$ and its solenoidal part. The "bordering" operators $\mathcal{W}$, $\mathcal{W}^*$ are also calculated in §7. In §8 it is shown that, in approximations, the "bordering" operators can be replaced by the simpler operators of multiplication by some periodic matrix-valued functions $W(\mathbf{x})$, $W(\mathbf{x})^*$. Such a replacement only affects the constant in the remainder estimate. In §9, we consider the homogenization problem for the operator $\mathcal{L}_\varepsilon$ and establish estimates of the form (0.12), (0.13). In §10, the results of §9 are adapted for application to the Maxwell operator: the solutions of system (0.8) are approximated uniformly in the $\mathfrak{G}$-norm. In §11, for completeness of exposition, we consider a comparatively elementary question about weak convergence of the solutions and "flows" for system (0.8). (Results of this type are quite traditional in homogenization theory; close results can be found, e.g., in [BeLP].) Finally, §12 contains the main results of the paper on homogenization for the stationary periodic Maxwell system. They follow from the results of §10 by direct recalculation.

0.10. Acknowledgements. The author is deeply grateful to M. Sh. Birman for numerous fruitful discussions and support. The author thanks V. V. Zhikov and A. L. Pyatnitskiĭ for their interest to the present work and useful discussions. The main results of the paper were obtained during the author's visit to the Chalmers University of Technology (Göteborg, Sweden) in December 2003–January 2004. The author thanks Chalmers University and especially Prof. G. Rozenblum for their hospitality. The text of the present paper was written during a visit to the University of Toronto (Canada) in April–May 2004; the author thanks Prof. V. Ivrii for his hospitality.

§1. Preliminaries

1.1. Notation. Let $\mathcal{G}$, $\mathcal{G}_*$ be separable Hilbert spaces. The symbols $(\cdot,\cdot)_{\mathcal{G}}$ and $\|\cdot\|_{\mathcal{G}}$ stand for the inner product and the norm in $\mathcal{G}$, and the symbol $\|\cdot\|_{\mathcal{G}\to\mathcal{G}_*}$ stands for the norm of a linear continuous operator from $\mathcal{G}$ to $\mathcal{G}_*$. By $I = I_{\mathcal{G}}$ we denote the identity operator in $\mathcal{G}$, and by $\mathbf{0}_{\mathcal{G}}$ the null operator. The symbol $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{C}^n$, and $|\cdot|$ is the norm of a vector in $\mathbb{C}^n$. For $z \in \mathbb{C}$, z^+ denotes the complex conjugate number. For vectors $\mathbf{a} = (a^1, a^2, a^3), \mathbf{b} = (b^1, b^2, b^3) \in \mathbb{C}^3$, we use the notation $\mathbf{a}\cdot\mathbf{b} = a^1b^1 + a^2b^2 + a^3b^3$; by $\mathbf{a}\times\mathbf{b}$ we denote the vector product. If s is a (3×3)-matrix, $|s|$ stands for the norm of s viewed as an operator in $\mathbb{C}^3$. The unit (3×3)-matrix is denoted by $\mathbf{1}$. Next, we write $\mathbf{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$, $\partial_j = \partial/\partial x^j$, $D_j = -i\partial_j$, $j = 1,2,3$; $\nabla = (\partial_1, \partial_2, \partial_3)$, $\mathbf{D} = -i\nabla = (D_1, D_2, D_3)$. Notation for the vector differential operations div, curl is standard. The classes L_p of $\mathbb{C}^n$-valued functions in a domain $\mathcal{D} \subset \mathbb{R}^d$ are denoted by $L_p(\mathcal{D};\mathbb{C}^n)$, $1 \le p \le \infty$. By $H^l(\mathcal{D};\mathbb{C}^n)$, $l \in \mathbb{R}$, we denote the Sobolev classes of $\mathbb{C}^n$-valued functions. For $n = 1$, we write simply $L_p(\mathcal{D})$, $H^l(\mathcal{D})$.

We introduce the class

$$\mathcal{H}^1(\mathbb{R}^3) := \Big\{\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla\varphi|^2 + |\mathbf{x}|^{-2}|\varphi|^2)\, d\mathbf{x} < \infty\Big\}$$

with the norm $\|\varphi\|_{\mathcal{H}^1} = \|\nabla\varphi\|_{L_2}$. *Then $\mathcal{H}^1(\mathbb{R}^3)$ is a complete Hilbert space* (see, e.g., [BS2]). The set $C_0^\infty(\mathbb{R}^3)$ is dense in $\mathcal{H}^1(\mathbb{R}^3)$.

The Fourier transformation in $L_2(\mathbb{R}^3;\mathbb{C}^n)$ is denoted by $\mathfrak{F}$.

1.2. Lattices. Fourier series. Let Γ be a lattice in $\mathbb{R}^3$, and let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^3$ be a basis in $\mathbb{R}^3$ generating the lattice Γ. Then

$$\Gamma = \Big\{\mathbf{a} \in \mathbb{R}^3 : \mathbf{a} = \sum_{j=1}^3 \alpha^j \mathbf{a}_j,\ \alpha^j \in \mathbb{Z}\Big\}.$$

Let Ω be the elementary cell of the lattice Γ:

$$\Omega = \Big\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \sum_{j=1}^3 \tau^j \mathbf{a}_j,\ 0 < \tau^j < 1\Big\}.$$

The basis $\{\mathbf{b}^1, \mathbf{b}^2, \mathbf{b}^3\}$ in $\mathbb{R}^3$ dual to $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is defined by the relations $\langle \mathbf{b}^l, \mathbf{a}_j\rangle = 2\pi\delta^l_j$. This basis generates the *lattice $\widetilde{\Gamma}$ dual to the lattice* Γ:

$$\widetilde{\Gamma} = \Big\{\mathbf{b} \in \mathbb{R}^3 : \mathbf{b} = \sum_{j=1}^3 \beta_j \mathbf{b}^j,\ \beta_j \in \mathbb{Z}\Big\}.$$

We denote by $\widetilde{\Omega}$ the *Brillouin zone*

$$\widetilde{\Omega} = \{\mathbf{k} \in \mathbb{R}^3 : |\mathbf{k}| < |\mathbf{k} - \mathbf{b}|,\ 0 \ne \mathbf{b} \in \widetilde{\Gamma}\};$$

$\widetilde{\Omega}$ is a fundamental domain for $\widetilde{\Gamma}$. This domain is a convex centrally symmetric polyhedron. We use the notation $|\Omega| = \operatorname{mes}\Omega$, $|\widetilde{\Omega}| = \operatorname{mes}\widetilde{\Omega}$. Note that $|\Omega|\,|\widetilde{\Omega}| = (2\pi)^3$. Let r_0 be the radius of the ball inscribed into $\operatorname{clos}\widetilde{\Omega}$. Then

$$2r_0 = \min_{0\ne\mathbf{b}\in\widetilde{\Gamma}} |\mathbf{b}|. \tag{1.1}$$

We denote $\mathcal{B}(r) = \{\mathbf{k} \in \mathbb{R}^3 : |\mathbf{k}| \le r\}$, $r > 0$.

In what follows, $\widetilde{H}^l(\Omega;\mathbb{C}^n)$, $l > 0$, stands for the subspace of all functions in $H^l(\Omega;\mathbb{C}^n)$ the Γ-periodic extension of which to $\mathbb{R}^3$ belongs to $H^l_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{C}^3)$. By $\widetilde{C}^\infty(\Omega)$ we denote the class of functions in Ω the Γ-periodic extension of which to $\mathbb{R}^3$ belongs to $C^\infty(\mathbb{R}^3)$. Below, if $\mathbf{u} \in L_2(\Omega;\mathbb{C}^n)$, then $\check{\mathbf{u}}$ denotes the Γ-periodic extension of $\mathbf{u}$ to $\mathbb{R}^3$.

Let $l_2(\widetilde{\Gamma};\mathbb{C}^n)$ be the l_2-space of $\mathbb{C}^n$-valued sequences $\{\mathbf{v_b}\}$, $\mathbf{b} \in \widetilde{\Gamma}$, with the norm $(\sum_{\mathbf{b}\in\widetilde{\Gamma}} |\mathbf{v_b}|^2)^{1/2}$. The discrete Fourier transformation $\{\hat{\mathbf{u}}_\mathbf{b}\} \mapsto \mathbf{u}$,

$$\mathbf{u}(\mathbf{x}) = |\Omega|^{-1/2} \sum_{\mathbf{b}\in\widetilde{\Gamma}} \hat{\mathbf{u}}_\mathbf{b} \exp\left(i\langle \mathbf{b}, \mathbf{x}\rangle\right), \quad \mathbf{x} \in \Omega, \tag{1.2}$$

is associated with the lattice Γ. This transformation is a unitary mapping of $l_2(\widetilde{\Gamma};\mathbb{C}^n)$ onto $L_2(\Omega;\mathbb{C}^n)$:

$$\int_\Omega |\mathbf{u}(\mathbf{x})|^2\, d\mathbf{x} = \sum_{\mathbf{b}\in\widetilde{\Gamma}} |\hat{\mathbf{u}}_\mathbf{b}|^2. \tag{1.3}$$

Let $h^l(\widetilde{\Gamma};\mathbb{C}^n)$, $l > 0$, be the class of $\mathbb{C}^n$-valued sequences $\{\mathbf{v_b}\}$ such that

$$\sum_{\mathbf{b}\in\widetilde{\Gamma}} (|\mathbf{b}|^2 + 1)^l |\mathbf{v_b}|^2 < \infty;$$

then $\mathbf{u} \in \widetilde{H}^l(\Omega;\mathbb{C}^n)$ if and only if $\{\hat{\mathbf{u}}_\mathbf{b}\} \in h^l(\widetilde{\Gamma};\mathbb{C}^n)$.

We have

$$\int_\Omega |(\mathbf{D}+\mathbf{k})\mathbf{u}(\mathbf{x})|^2\, d\mathbf{x} = \sum_{\mathbf{b}\in\widetilde{\Gamma}} |\mathbf{b}+\mathbf{k}|^2 |\hat{\mathbf{u}}_\mathbf{b}|^2, \quad \mathbf{u} \in \widetilde{H}^1(\Omega;\mathbb{C}^n),\ \ \mathbf{k} \in \mathbb{R}^3. \tag{1.4}$$

The convergence of the series in (1.4) is equivalent to the fact that $\mathbf{u} \in \widetilde{H}^1(\Omega;\mathbb{C}^n)$.

1.3. The Gelfand transformation. Initially, the Gelfand transformation $\mathcal{V}$ is defined on the functions belonging to the Schwartz class $\mathcal{S}$ by the formula

$$\begin{aligned}(\mathcal{V}\mathbf{f})(\mathbf{x},\mathbf{k}) = \mathbf{f}_*(\mathbf{x},\mathbf{k}) = |\widetilde{\Omega}|^{-1/2} \sum_{\mathbf{a}\in\Gamma} \exp\left(-i\langle \mathbf{k}, \mathbf{x}+\mathbf{a}\rangle\right)\mathbf{f}(\mathbf{x}+\mathbf{a}),\\ \mathbf{f} \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^n), \quad \mathbf{x} \in \Omega, \quad \mathbf{k} \in \widetilde{\Omega},\end{aligned} \tag{1.5}$$

and $\mathcal{V}$ extends by continuity up to a unitary mapping

$$\mathcal{V} : L_2(\mathbb{R}^3;\mathbb{C}^n) \to \int_{\widetilde{\Omega}} \oplus L_2(\Omega;\mathbb{C}^n)\, d\mathbf{k} =: \mathcal{K}. \tag{1.6}$$

Usually, the parameter $\mathbf{k}$ is called the *quasimomentum*. The relation $\mathbf{f} \in H^1(\mathbb{R}^3;\mathbb{C}^n)$ is equivalent to the fact that $\mathbf{f}_*(\cdot,\mathbf{k}) \in \widetilde{H}^1(\Omega;\mathbb{C}^n)$ for almost every $\mathbf{k} \in \widetilde{\Omega}$ and

$$\int_{\widetilde{\Omega}} \int_\Omega \left(|(\mathbf{D}+\mathbf{k})\mathbf{f}_*(\mathbf{x},\mathbf{k})|^2 + |\mathbf{f}_*(\mathbf{x},\mathbf{k})|^2\right) d\mathbf{x}\, d\mathbf{k} < \infty. \tag{1.7}$$

The integral in (1.7) is equal to the square of the norm of $\mathbf{f}$ in $H^1(\mathbb{R}^3;\mathbb{C}^n)$.

For φ in the Schwartz class $\mathcal{S}(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} |\nabla\varphi(\mathbf{x})|^2\, d\mathbf{x} = \int_{\widetilde{\Omega}} \int_\Omega |(\mathbf{D}+\mathbf{k})\varphi_*(\mathbf{x},\mathbf{k})|^2\, d\mathbf{x}\, d\mathbf{k}. \tag{1.8}$$

This allows us to extend the transformation $\mathcal{V}$ by continuity to $\mathcal{H}^1(\mathbb{R}^3)$. Then the relation $\varphi \in \mathcal{H}^1(\mathbb{R}^3)$ is equivalent to the fact that $\varphi_*(\cdot,\mathbf{k}) \in \widetilde{H}^1(\Omega)$ for almost every $\mathbf{k} \in \widetilde{\Omega}$ and the integral on the right-hand side of (1.8) is finite.

Under the transformation $\mathcal{V}$, the operator in $L_2(\mathbb{R}^3;\mathbb{C}^n)$ of multiplication by a periodic matrix-valued function turns into multiplication by the same matrix-valued function on the fibers of the direct integral $\mathcal{K}$. A linear differential operator $b(\mathbf{D})$ (corresponding to some symbol $b(\boldsymbol{\xi})$) applied to a function $\mathbf{f}$ in $\mathbb{R}^3$ turns into the operator $b(\mathbf{D}+\mathbf{k})$ applied to $\mathbf{f}_*(\cdot,\mathbf{k})$ with periodic boundary conditions.

§2. Function classes. The Weyl decomposition

We need information concerning some classes of vector-valued functions in $\mathbb{R}^3$ and in Ω related to the Maxwell operator. In [BS1], such classes were studied for arbitrary domains under the boundary conditions of ideal conductivity. Our case differs from the situation of [BS1]. First, in [BS1] the case of weighted L_2-spaces was considered, while in the present paper we work in the ordinary space L_2. Second, we need classes of functions in Ω depending on a parameter $\mathbf{k} \in \mathbb{R}^3$, and with periodic boundary conditions. Below, in Subsection 2.1, we give the necessary facts (without proof) pertaining to some classes of functions in $\mathbb{R}^3$; these facts can be deduced from the results of [BS1, §1] by direct recalculation. In Subsection 2.2, we consider classes of functions in Ω; here a full exposition is presented. However, in many respects this exposition is parallel to the arguments of [BS1].

2.1. Classes of functions in $\mathbb{R}^3$. Let $s(\mathbf{x})$ be a measurable (3×3)-matrix-valued function in $\mathbb{R}^3$ with real entries and such that

$$c_0 \mathbf{1} \leq s(\mathbf{x}) \leq c_0' \mathbf{1}, \quad \mathbf{x} \in \mathbb{R}^3, \ 0 < c_0 \leq c_0' < \infty. \tag{2.1}$$

In the space $\mathfrak{G} := L_2(\mathbb{R}^3; \mathbb{C}^3)$, we distinguish the "gradient" subspace

$$G(s) = G(\mathbb{R}^3; s) = \{\mathbf{u} = s^{1/2} \nabla \varphi : \varphi \in \mathcal{H}^1(\mathbb{R}^3)\}.$$

The "solenoidal" subspace $J(s) = J(\mathbb{R}^3; s)$ is defined via the orthogonal decomposition

$$\mathfrak{G} = G(\mathbb{R}^3; s) \oplus J(\mathbb{R}^3; s) \tag{2.2}$$

(the Weyl decomposition). Clearly, the subspace $J(s)$ consists of vector-valued functions $\mathbf{u} \in \mathfrak{G}$ such that $\operatorname{div}(s^{1/2}\mathbf{u}) = 0$ (in the sense of distributions). Let $\mathcal{Q}(s)$ be the orthogonal projection in $\mathfrak{G}$ onto $G(s)$, and let $\mathcal{P}(s)$ be the orthogonal projection in $\mathfrak{G}$ onto $J(s)$. The projections $\mathcal{Q}(s)$ and $\mathcal{P}(s)$ act as follows. Let $\mathbf{u} \in \mathfrak{G}$, and let $\varphi \in \mathcal{H}^1(\mathbb{R}^3)$ be a (weak) solution of the equation

$$\operatorname{div} s(\mathbf{x}) \nabla \varphi = \operatorname{div} (s(\mathbf{x}))^{1/2} \mathbf{u}. \tag{2.3}$$

In other words, φ satisfies the identity

$$\int_{\mathbb{R}^3} \langle s(\mathbf{x}) \nabla \varphi, \nabla \psi \rangle \, d\mathbf{x} = \int_{\mathbb{R}^3} \langle (s(\mathbf{x}))^{1/2} \mathbf{u}, \nabla \psi \rangle \, d\mathbf{x}, \quad \psi \in \mathcal{H}^1(\mathbb{R}^3). \tag{2.4}$$

It suffices to check (2.4) for $\psi \in C_0^\infty(\mathbb{R}^3)$. The left-hand side of (2.4) determines an inner product in $\mathcal{H}^1(\mathbb{R}^3)$ (the corresponding norm is equivalent to the standard one), and the right-hand side is an antilinear continuous functional over $\psi \in \mathcal{H}^1(\mathbb{R}^3)$. Hence, by the Riesz theorem, there exists a solution φ; this solution is unique and satisfies $\|s^{1/2} \nabla \varphi\|_{\mathfrak{G}} \leq \|\mathbf{u}\|_{\mathfrak{G}}$. Then

$$\mathcal{Q}(s)\mathbf{u} = s^{1/2} \nabla \varphi, \quad \mathcal{P}(s)\mathbf{u} = \mathbf{u} - s^{1/2} \nabla \varphi. \tag{2.5}$$

Let $\widetilde{s}(\mathbf{x})$ be yet another matrix-valued function satisfying conditions of the same type as $s(\mathbf{x})$.

Lemma 2.1. a) *The operator* $\mathcal{P}(s) s^{1/2} \widetilde{s}^{-1/2}$ *maps* $J(\widetilde{s})$ *bijectively onto* $J(s)$. *The inverse mapping is given by the operator* $\mathcal{P}(\widetilde{s}) \widetilde{s}^{1/2} s^{-1/2}$. b) *The operator* $\mathcal{Q}(s) s^{-1/2} \widetilde{s}^{1/2}$ *maps* $G(\widetilde{s})$ *bijectively onto* $G(s)$. *The inverse mapping is given by the operator* $\mathcal{Q}(\widetilde{s}) \widetilde{s}^{-1/2} s^{1/2}$.

We introduce the following class of vector-valued functions:

$$F(s) = F(\mathbb{R}^3; s) = \{\mathbf{u} \in \mathfrak{G} : \operatorname{div} s^{1/2} \mathbf{u} \in L_2(\mathbb{R}^3), \ \operatorname{curl} s^{-1/2} \mathbf{u} \in \mathfrak{G}\}. \tag{2.6}$$

The class (2.6) is a complete Hilbert space relative to the norm

$$\|\mathbf{u}\|_{F(s)} := \left(\|\mathbf{u}\|_{\mathfrak{G}}^2 + a(s)[\mathbf{u}, \mathbf{u}] \right)^{1/2}, \tag{2.7}$$

where

$$a(s)[\mathbf{u},\mathbf{u}] := \| \operatorname{div} s^{1/2}\mathbf{u}\|^2_{L_2(\mathbb{R}^3)} + \| \operatorname{curl} s^{-1/2}\mathbf{u}\|^2_{\mathfrak{G}}.$$

Note that, for $s = \mathbf{1}$, we have $F(\mathbf{1}) = F(\mathbb{R}^3;\mathbf{1}) = H^1(\mathbb{R}^3;\mathbb{C}^3)$, and $\|\mathbf{u}\|_{F(\mathbf{1})} = \|\mathbf{u}\|_{H^1(\mathbb{R}^3;\mathbb{C}^3)}$.

Lemma 2.2. *The projections $\mathcal{P}(s)$, $\mathcal{Q}(s)$ take the class $F(s)$ into itself.*

We introduce

$$G^1(s) = F(s) \cap G(s), \quad J^1(s) = F(s) \cap J(s).$$

Lemma 2.3. *The sets $G^1(s)$, $J^1(s)$ are closed with respect to the norm* (2.7). *The space $F(s)$ admits the decomposition*

$$F(s) = G^1(s) \oplus J^1(s), \tag{2.8}$$

which is orthogonal relative both to the inner product in $\mathfrak{G}$ and to the inner product in $F(s)$.

Lemma 2.4. a) *The operator $\mathcal{P}(s)s^{1/2}\widetilde{s}^{-1/2}$ maps $J^1(\widetilde{s})$ continuously onto $J^1(s)$. We have*

$$a(s)[\mathbf{u},\mathbf{u}] = a(\widetilde{s})[\mathbf{v},\mathbf{v}], \quad \|\mathbf{u}\|^2_{\mathfrak{G}} \le \|s\|_{L_\infty}\|\widetilde{s}^{-1}\|_{L_\infty}\|\mathbf{v}\|^2_{\mathfrak{G}},$$
$$\mathbf{v} \in J^1(\widetilde{s}), \quad \mathbf{u} = \mathcal{P}(s)s^{1/2}\widetilde{s}^{-1/2}\mathbf{v}.$$

b) *The operator $\mathcal{Q}(s)s^{-1/2}\widetilde{s}^{1/2}$ maps $G^1(\widetilde{s})$ continuously onto $G^1(s)$. We have*

$$a(s)[\mathbf{f},\mathbf{f}] = a(\widetilde{s})[\mathbf{g},\mathbf{g}], \quad \|\mathbf{f}\|^2_{\mathfrak{G}} \le \|s^{-1}\|_{L_\infty}\|\widetilde{s}\|_{L_\infty}\|\mathbf{g}\|^2_{\mathfrak{G}},$$
$$\mathbf{g} \in G^1(\widetilde{s}), \quad \mathbf{f} = \mathcal{Q}(s)s^{-1/2}\widetilde{s}^{1/2}\mathbf{g}.$$

Lemma 2.5. a) *The set $F(s)$ is dense in $\mathfrak{G}$.* b) *The set $J^1(s)$ is dense in $J(s)$, and the set $G^1(s)$ is dense in $G(s)$.*

Lemma 2.6. *The operator $B(s) = \mathcal{P}(\mathbf{1})s^{-1/2}\mathcal{P}(s) + \mathcal{Q}(\mathbf{1})s^{1/2}\mathcal{Q}(s)$ maps $F(s)$ continuously onto $F(\mathbf{1})$. The inverse mapping is given by the operator $\mathcal{P}(s)s^{1/2}\mathcal{P}(\mathbf{1}) + \mathcal{Q}(s)s^{-1/2}\mathcal{Q}(\mathbf{1})$. We have*

$$a(s)[\mathbf{u},\mathbf{u}] = a(\mathbf{1})[\mathbf{v},\mathbf{v}], \quad (C_0(s))^{-1}\|\mathbf{v}\|^2_{\mathfrak{G}} \le \|\mathbf{u}\|^2_{\mathfrak{G}} \le C_0(s)\|\mathbf{v}\|^2_{\mathfrak{G}},$$
$$\mathbf{u} \in F(s), \quad \mathbf{v} = B(s)\mathbf{u},$$

$$C_0(s) := \max\{\|s\|_{L_\infty}, \|s^{-1}\|_{L_\infty}\}. \tag{2.9}$$

2.2. Classes of functions in Ω. Let $\mathbf{k} \in \mathbb{R}^3$ be a parameter. We introduce the differential operations $\nabla_{\mathbf{k}}$, $\operatorname{div}_{\mathbf{k}}$, $\operatorname{curl}_{\mathbf{k}}$ by the following relations:

$$\nabla_{\mathbf{k}}\varphi := \nabla\varphi + i\mathbf{k}\varphi, \quad \operatorname{div}_{\mathbf{k}}\mathbf{f} := \operatorname{div}\mathbf{f} + i\mathbf{k}\cdot\mathbf{f}, \ \ \operatorname{curl}_{\mathbf{k}}\mathbf{f} := \operatorname{curl}\mathbf{f} + i\mathbf{k}\times\mathbf{f}.$$

Suppose that a measurable matrix-valued function $s(\mathbf{x})$ in $\mathbb{R}^3$ has real entries, satisfies conditions (2.1), and *is Γ-periodic*. In the space $\mathfrak{H} := L_2(\Omega;\mathbb{C}^3)$, we distinguish the "gradient" subspace

$$G(s;\mathbf{k}) = G(\Omega;s;\mathbf{k}) = \{\mathbf{u} = s^{1/2}\nabla_{\mathbf{k}}\phi : \phi \in \widetilde{H}^1(\Omega)\}$$

(depending on the parameter $\mathbf{k}$). By definition, the "solenoidal" subspace $J(s;\mathbf{k}) = J(\Omega;s;\mathbf{k})$ is the orthogonal complement of $G(s;\mathbf{k})$:

$$\mathfrak{H} = G(\Omega;s;\mathbf{k}) \oplus J(\Omega;s;\mathbf{k}). \tag{2.10}$$

Clearly, $J(s;\mathbf{k})$ consists of vector-valued functions $\mathbf{u} \in \mathfrak{H}$ such that $\operatorname{div}_{\mathbf{k}}(s^{\frac{1}{2}}\check{\mathbf{u}}) = 0$ (in the sense of distributions). Let $\mathcal{Q}(s;\mathbf{k})$ be the orthogonal projection in $\mathfrak{H}$ onto the subspace $G(s;\mathbf{k})$, and let $\mathcal{P}(s;\mathbf{k})$ be the orthogonal projection in $\mathfrak{H}$ onto $J(s;\mathbf{k})$. The projections

$\mathcal{P}(s;\mathbf{k})$ and $\mathcal{Q}(s;\mathbf{k})$ act as follows. Let $\mathbf{f}\in\mathfrak{H}$, and let $\phi=\phi(\cdot;\mathbf{k})\in\widetilde{H}^1(\Omega)$ be a (weak) periodic solution of the equation

$$\operatorname{div}_{\mathbf{k}} s(\mathbf{x})\nabla_{\mathbf{k}}\phi=\operatorname{div}_{\mathbf{k}}(s(\mathbf{x}))^{1/2}\mathbf{f}. \tag{2.11}$$

In other words, we have

$$\int_\Omega\langle s(\mathbf{x})\nabla_{\mathbf{k}}\phi,\nabla_{\mathbf{k}}\zeta\rangle\,d\mathbf{x}=\int_\Omega\langle(s(\mathbf{x}))^{1/2}\mathbf{f},\nabla_{\mathbf{k}}\zeta\rangle\,d\mathbf{x},\quad \zeta\in\widetilde{H}^1(\Omega). \tag{2.12}$$

For $\mathbf{k}\neq 0 \pmod{\widetilde{\Gamma}}$, the left-hand side of (2.12) determines an inner product in $\widetilde{H}^1(\Omega)$ (the corresponding norm is equivalent to the standard one), and the right-hand side is an antilinear continuous functional over $\zeta\in\widetilde{H}^1(\Omega)$. By the Riesz theorem, there exists a solution ϕ; this solution is unique and satisfies the estimate $\|s^{1/2}\nabla_{\mathbf{k}}\phi\|_{\mathfrak{H}}\le\|\mathbf{f}\|_{\mathfrak{H}}$. For $\mathbf{k}=0$, we look for a solution ϕ in the class $\{\phi\in\widetilde{H}^1(\Omega):\int_\Omega\phi\,d\mathbf{x}=0\}$. Then the Riesz theorem still applies, yielding a unique solution ϕ satisfying the estimate $\|s^{1/2}\nabla\phi\|_{\mathfrak{H}}\le\|\mathbf{f}\|_{\mathfrak{H}}$. We have

$$\mathcal{Q}(s;\mathbf{k})\mathbf{f}=s^{1/2}\nabla_{\mathbf{k}}\phi,\quad \mathcal{P}(s;\mathbf{k})\mathbf{f}=\mathbf{f}-s^{1/2}\nabla_{\mathbf{k}}\phi. \tag{2.13}$$

Let $\widetilde{s}(\mathbf{x})$ be yet another matrix satisfying conditions of the same type as $s(\mathbf{x})$.

Lemma 2.7. a) *The operator $\mathcal{P}(s;\mathbf{k})s^{\frac12}\widetilde{s}^{-\frac12}$ maps $J(\widetilde{s};\mathbf{k})$ bijectively onto $J(s;\mathbf{k})$. The inverse mapping is given by the operator $\mathcal{P}(\widetilde{s};\mathbf{k})\widetilde{s}^{\frac12}s^{-\frac12}$.* b) *The operator $\mathcal{Q}(s;\mathbf{k})s^{-\frac12}\widetilde{s}^{\frac12}$ maps $G(\widetilde{s};\mathbf{k})$ bijectively onto $G(s;\mathbf{k})$. The inverse mapping is given by the operator $\mathcal{Q}(\widetilde{s};\mathbf{k})\widetilde{s}^{-\frac12}s^{\frac12}$.*

Proof. Statement a) follows from the relations

$$\mathcal{P}(s;\mathbf{k})s^{1/2}\widetilde{s}^{-1/2}\mathcal{P}(\widetilde{s};\mathbf{k})=\mathcal{P}(s;\mathbf{k})s^{1/2}\widetilde{s}^{-1/2}, \tag{2.14}$$

$$\mathcal{P}(\widetilde{s};\mathbf{k})\widetilde{s}^{1/2}s^{-1/2}\mathcal{P}(s;\mathbf{k})=\mathcal{P}(\widetilde{s};\mathbf{k})\widetilde{s}^{1/2}s^{-1/2}. \tag{2.15}$$

We prove (2.14); interchanging the roles of s and $\widetilde{s}$ leads to (2.15).

Let $\mathbf{u}\in\mathfrak{H}$,

$$\mathbf{u}=\widetilde{s}^{1/2}\nabla_{\mathbf{k}}\widetilde{\varphi}+\widetilde{\mathbf{u}}, \tag{2.16}$$

where $\widetilde{\mathbf{u}}=\mathcal{P}(\widetilde{s};\mathbf{k})\mathbf{u}$, $\widetilde{s}^{1/2}\nabla_{\mathbf{k}}\widetilde{\varphi}=\mathcal{Q}(\widetilde{s};\mathbf{k})\mathbf{u}$, $\widetilde{\varphi}\in\widetilde{H}^1(\Omega)$. Next, we have

$$s^{1/2}\widetilde{s}^{-1/2}\widetilde{\mathbf{u}}=s^{1/2}\nabla_{\mathbf{k}}\varphi+\mathbf{u}_0, \tag{2.17}$$

where

$$\mathbf{u}_0=\mathcal{P}(s;\mathbf{k})s^{1/2}\widetilde{s}^{-1/2}\widetilde{\mathbf{u}}=\mathcal{P}(s;\mathbf{k})s^{1/2}\widetilde{s}^{-1/2}\mathcal{P}(\widetilde{s};\mathbf{k})\mathbf{u}$$

and

$$s^{1/2}\nabla_{\mathbf{k}}\varphi=\mathcal{Q}(s;\mathbf{k})s^{1/2}\widetilde{s}^{-1/2}\widetilde{\mathbf{u}},\quad \varphi\in\widetilde{H}^1(\Omega).$$

As a result, (2.16) and (2.17) imply that

$$s^{1/2}\widetilde{s}^{-1/2}\mathbf{u}=s^{1/2}\nabla_{\mathbf{k}}(\widetilde{\varphi}+\varphi)+\mathbf{u}_0.$$

Here the first term on the right belongs to $G(s;\mathbf{k})$, and the second belongs to $J(s;\mathbf{k})$. Consequently, $\mathbf{u}_0=\mathcal{P}(s;\mathbf{k})s^{1/2}\widetilde{s}^{-1/2}\mathbf{u}$. This proves (2.14).

Statement b) follows from the relations

$$\mathcal{Q}(s;\mathbf{k})s^{-1/2}\widetilde{s}^{1/2}\mathcal{Q}(\widetilde{s};\mathbf{k})=\mathcal{Q}(s;\mathbf{k})s^{-1/2}\widetilde{s}^{1/2}, \tag{2.18}$$

$$\mathcal{Q}(\widetilde{s};\mathbf{k})\widetilde{s}^{-1/2}s^{1/2}\mathcal{Q}(s;\mathbf{k})=\mathcal{Q}(\widetilde{s};\mathbf{k})\widetilde{s}^{-1/2}s^{1/2}.$$

It suffices to check (2.18), which is equivalent to the relation $\mathcal{Q}(s;\mathbf{k})s^{-1/2}\widetilde{s}^{1/2}\mathcal{P}(\widetilde{s};\mathbf{k})=0$ or

$$\mathcal{P}(s;\mathbf{k})s^{-1/2}\widetilde{s}^{1/2}\mathcal{P}(\widetilde{s};\mathbf{k})=s^{-1/2}\widetilde{s}^{1/2}\mathcal{P}(\widetilde{s};\mathbf{k}). \tag{2.19}$$

Relation (2.19) follows from (2.15) by passage to the adjoint operators in $\mathfrak{H}$. $\square$

We introduce the class of vector-valued functions

$$F(s;\mathbf{k}) = F(\Omega;s;\mathbf{k}) = \{\mathbf{u}\in\mathfrak{H} : \operatorname{div}_{\mathbf{k}} s^{1/2}\check{\mathbf{u}} \in L_{2,\mathrm{loc}}(\mathbb{R}^3),\ \operatorname{curl}_{\mathbf{k}} s^{-1/2}\check{\mathbf{u}} \in \mathfrak{G}_{\mathrm{loc}}\}, \tag{2.20}$$

where $\mathfrak{G}_{\mathrm{loc}} := L_{2,\mathrm{loc}}(\mathbb{R}^3;\mathbb{C}^3)$. The class (2.20) is a complete Hilbert space relative to the norm

$$\|\mathbf{u}\|_{F(s;\mathbf{k})} := \left(\|\mathbf{u}\|^2_{\mathfrak{H}} + a(s;\mathbf{k})[\mathbf{u},\mathbf{u}]\right)^{1/2}, \tag{2.21}$$

where

$$a(s;\mathbf{k})[\mathbf{u},\mathbf{u}] = \|\operatorname{div}_{\mathbf{k}} s^{1/2}\mathbf{u}\|^2_{L_2(\Omega)} + \|\operatorname{curl}_{\mathbf{k}} s^{-1/2}\mathbf{u}\|^2_{\mathfrak{H}}. \tag{2.22}$$

The set (2.20) itself does not depend on $\mathbf{k}$ and may be defined by the formula

$$F(s;\mathbf{k}) = \{\mathbf{u}\in\mathfrak{H} : \operatorname{div} s^{1/2}\check{\mathbf{u}} \in L_{2,\mathrm{loc}}(\mathbb{R}^3),\ \operatorname{curl} s^{-1/2}\check{\mathbf{u}} \in \mathfrak{G}_{\mathrm{loc}}\}. \tag{2.23}$$

However, the norm (2.21) depends on $\mathbf{k}$.

Note that, for $s=\mathbf{1}$, we have

$$F(\Omega;\mathbf{1};\mathbf{k}) = \widetilde{H}^1(\Omega;\mathbb{C}^3) \tag{2.24}$$

and

$$\|\mathbf{u}\|^2_{F(\mathbf{1};\mathbf{k})} = \int_\Omega (|(\mathbf{D}+\mathbf{k})\mathbf{u}|^2 + |\mathbf{u}|^2)\,d\mathbf{x}. \tag{2.25}$$

Indeed, using the Fourier series (1.2) for $\mathbf{u}$, relations (1.3), (1.4), and the definition of the norm (2.21) with $s=\mathbf{1}$, we obtain

$$\begin{aligned}\|\mathbf{u}\|^2_{F(\mathbf{1};\mathbf{k})} &= \sum_{\mathbf{b}\in\widetilde{\Gamma}} (|\hat{\mathbf{u}}_{\mathbf{b}}|^2 + |(\mathbf{b}+\mathbf{k})\cdot\hat{\mathbf{u}}_{\mathbf{b}}|^2 + |(\mathbf{b}+\mathbf{k})\times\hat{\mathbf{u}}_{\mathbf{b}}|^2)\\ &= \sum_{\mathbf{b}\in\widetilde{\Gamma}} (|\hat{\mathbf{u}}_{\mathbf{b}}|^2 + |\mathbf{b}+\mathbf{k}|^2|\hat{\mathbf{u}}_{\mathbf{b}}|^2) = \int_\Omega (|(\mathbf{D}+\mathbf{k})\mathbf{u}|^2 + |\mathbf{u}|^2)\,d\mathbf{x}.\end{aligned}$$

By (2.24), (2.25), the *embedding of the space* $F(\mathbf{1};\mathbf{k})$ *in* $\mathfrak{H}$ *is compact.*

Lemma 2.8. *The projections* $\mathcal{P}(s;\mathbf{k})$, $\mathcal{Q}(s;\mathbf{k})$ *take the class* $F(s;\mathbf{k})$ *into itself.*

Proof. Let $\mathbf{u}\in F(s;\mathbf{k})$,

$$\mathbf{u} = s^{1/2}\nabla_{\mathbf{k}}\varphi + \mathbf{u}_0, \tag{2.26}$$

where $\mathbf{u}_0 = \mathcal{P}(s;\mathbf{k})\mathbf{u}$, $s^{1/2}\nabla_{\mathbf{k}}\varphi = \mathcal{Q}(s;\mathbf{k})\mathbf{u}$. Then $\check{\varphi}(\mathbf{x})$ is a (weak) periodic solution of the equation

$$\operatorname{div}_{\mathbf{k}} s\nabla_{\mathbf{k}}\check{\varphi} = \operatorname{div}_{\mathbf{k}} s^{1/2}\check{\mathbf{u}}. \tag{2.27}$$

The right-hand side of (2.27) belongs to $L_{2,\mathrm{loc}}(\mathbb{R}^3)$ because $\mathbf{u}\in F(s;\mathbf{k})$. Therefore, $\operatorname{div}_{\mathbf{k}} s^{\frac12}(s^{\frac12}\nabla_{\mathbf{k}}\check{\varphi}) \in L_{2,\mathrm{loc}}(\mathbb{R}^3)$. Obviously, $\operatorname{curl}_{\mathbf{k}} s^{-\frac12}(s^{\frac12}\nabla_{\mathbf{k}}\check{\varphi}) = 0$, so that $s^{\frac12}\nabla_{\mathbf{k}}\varphi \in F(s;\mathbf{k})$. Then (2.26) implies that $\mathbf{u}_0\in F(s;\mathbf{k})$. $\square$

We put

$$G^1(s;\mathbf{k}) = G(s;\mathbf{k})\cap F(s;\mathbf{k}),\quad J^1(s;\mathbf{k}) = J(s;\mathbf{k})\cap F(s;\mathbf{k}). \tag{2.28}$$

As a consequence of (2.28) and Lemma 2.8, we obtain the following statement.

Lemma 2.9. *The sets $G^1(s;\mathbf{k})$, $J^1(s;\mathbf{k})$ are closed with respect to the norm* (2.21). *The space $F(s;\mathbf{k})$ admits the decomposition*

$$F(s;\mathbf{k}) = G^1(s;\mathbf{k}) \oplus J^1(s;\mathbf{k}), \tag{2.29}$$

which is orthogonal relative both to the inner product in $\mathfrak{H}$ and to the inner product in $F(s;\mathbf{k})$.

Now we refine Lemma 2.7.

Lemma 2.10. a) *The operator $\mathcal{P}(s;\mathbf{k})s^{\frac12}\widetilde{s}^{-\frac12}$ maps $J^1(\widetilde{s};\mathbf{k})$ continuously onto $J^1(s;\mathbf{k})$. We have*

$$a(s;\mathbf{k})[\mathbf{u},\mathbf{u}] = a(\widetilde{s};\mathbf{k})[\mathbf{v},\mathbf{v}], \tag{2.30}$$

$$\|\mathbf{u}\|^2_{\mathfrak{H}} \le \|s\|_{L_\infty}\|\widetilde{s}^{-1}\|_{L_\infty}\|\mathbf{v}\|^2_{\mathfrak{H}}, \tag{2.31}$$

where $\mathbf{v}\in J^1(\widetilde{s};\mathbf{k})$, $\mathbf{u} = \mathcal{P}(s;\mathbf{k})s^{1/2}\widetilde{s}^{-1/2}\mathbf{v}$.

b) *The operator $\mathcal{Q}(s;\mathbf{k})s^{-1/2}\widetilde{s}^{1/2}$ maps $G^1(\widetilde{s};\mathbf{k})$ continuously onto $G^1(s;\mathbf{k})$. We have*

$$a(s;\mathbf{k})[\mathbf{f},\mathbf{f}] = a(\widetilde{s};\mathbf{k})[\mathbf{g},\mathbf{g}], \tag{2.32}$$

$$\|\mathbf{f}\|^2_{\mathfrak{H}} \le \|s^{-1}\|_{L_\infty}\|\widetilde{s}\|_{L_\infty}\|\mathbf{g}\|^2_{\mathfrak{H}}, \tag{2.33}$$

where $\mathbf{g}\in G^1(\widetilde{s};\mathbf{k})$, $\mathbf{f} = \mathcal{Q}(s;\mathbf{k})s^{-1/2}\widetilde{s}^{1/2}\mathbf{g}$.

Proof. For definiteness, we prove a). Statement b) can be proved by analogy. Suppose $\mathbf{v}\in J^1(\widetilde{s};\mathbf{k})$. This means that $\mathbf{v}\in\mathfrak{H}$, $\operatorname{div}_{\mathbf{k}}\widetilde{s}^{1/2}\check{\mathbf{v}} = 0$, $\operatorname{curl}_{\mathbf{k}}\widetilde{s}^{-1/2}\check{\mathbf{v}}\in\mathfrak{G}_{\mathrm{loc}}$. Let

$$\mathbf{u} = \mathcal{P}(s;\mathbf{k})s^{1/2}\widetilde{s}^{-1/2}\mathbf{v}, \tag{2.34}$$

and let

$$s^{1/2}\widetilde{s}^{-1/2}\check{\mathbf{v}} = s^{1/2}\nabla_{\mathbf{k}}\check{\varphi} + \check{\mathbf{u}}, \tag{2.35}$$

where $\operatorname{div}_{\mathbf{k}} s^{1/2}\check{\mathbf{u}} = 0$. From (2.35) it is seen that $\operatorname{curl}_{\mathbf{k}} s^{-1/2}\check{\mathbf{u}} = \operatorname{curl}_{\mathbf{k}}\widetilde{s}^{-1/2}\check{\mathbf{v}}\in\mathfrak{G}_{\mathrm{loc}}$. Thus, $\mathbf{u}\in J^1(s;\mathbf{k})$, and (2.30) is fulfilled. From (2.34) it follows that

$$\|\mathbf{u}\|^2_{\mathfrak{H}} \le \|s^{1/2}\widetilde{s}^{-1/2}\mathbf{v}\|^2_{\mathfrak{H}} \le \|s\|_{L_\infty}\|\widetilde{s}^{-1}\|_{L_\infty}\|\mathbf{v}\|^2_{\mathfrak{H}},$$

which proves (2.31). Relations (2.30) and (2.31) imply that the mapping $\mathcal{P}(s;\mathbf{k})s^{\frac12}\widetilde{s}^{-\frac12}: J^1(\widetilde{s};\mathbf{k})\to J^1(s;\mathbf{k})$ is continuous, and Lemma 2.7(a) shows that this mapping is bijective. □

Lemma 2.11. a) *The set $F(s;\mathbf{k})$ is dense in $\mathfrak{H}$.* b) *The set $J^1(s;\mathbf{k})$ is dense in $J(s;\mathbf{k})$, and the set $G^1(s;\mathbf{k})$ is dense in $G(s;\mathbf{k})$.*

Proof. By (2.10) and (2.29), statements a) and b) are equivalent. For $s=\mathbf{1}$, statement a) follows from (2.24) and the fact that $\widetilde{H}^1(\Omega;\mathbb{C}^3)$ is dense in $\mathfrak{H}$. Thus, statement b) is also true for $s=\mathbf{1}$. Lemmas 2.7 and 2.10 describe isomorphisms via which b) (and, hence, also a)) is carried over to the case of an arbitrary matrix $s(\mathbf{x})$. □

Lemmas 2.7 and 2.10 readily imply the following statement.

Lemma 2.12. *The operator*

$$B(s;\mathbf{k}) = \mathcal{P}(\mathbf{1};\mathbf{k})s^{-1/2}\mathcal{P}(s;\mathbf{k}) + \mathcal{Q}(\mathbf{1};\mathbf{k})s^{1/2}\mathcal{Q}(s;\mathbf{k})$$

maps $F(s;\mathbf{k})$ continuously onto $F(\mathbf{1};\mathbf{k})$. The inverse mapping is given by the operator

$$\mathcal{P}(s;\mathbf{k})s^{1/2}\mathcal{P}(\mathbf{1};\mathbf{k}) + \mathcal{Q}(s;\mathbf{k})s^{-1/2}\mathcal{Q}(\mathbf{1};\mathbf{k}).$$

We have

$$(C_0(s))^{-1}\|\mathbf{v}\|^2_{\mathfrak{H}} \le \|\mathbf{u}\|^2_{\mathfrak{H}} \le C_0(s)\|\mathbf{v}\|^2_{\mathfrak{H}}, \tag{2.36}$$

$$a(s;\mathbf{k})[\mathbf{u},\mathbf{u}] = a(\mathbf{1};\mathbf{k})[\mathbf{v},\mathbf{v}], \tag{2.37}$$

where $\mathbf{u} \in F(s;\mathbf{k})$*,* $\mathbf{v} = B(s;\mathbf{k})\mathbf{u}$*, and the constant* $C_0(s)$ *is defined by* (2.9)*.*

Lemma 2.12 and the compactness of the embedding of $F(\mathbf{1};\mathbf{k})$ in $\mathfrak{H}$ show that the *embedding of the space* $F(s;\mathbf{k})$ *in* $\mathfrak{H}$ *is compact.*

2.3. The direct integral decompositions for the projections $\mathcal{P}(s)$ and $\mathcal{Q}(s)$. Using the Gelfand transformation $\mathcal{V}$ (see (1.5), (1.6)), we can decompose the projections $\mathcal{P}(s)$ and $\mathcal{Q}(s)$ into direct integrals.

Lemma 2.13. *Let* $\mathcal{P}(s)$ *and* $\mathcal{Q}(s)$ *be the orthogonal projections in* $\mathfrak{G}$ *onto the subspaces* $J(s)$ *and* $G(s)$, *respectively, and let* $\mathcal{P}(s;\mathbf{k})$ *and* $\mathcal{Q}(s;\mathbf{k})$ *be the orthogonal projections in* $\mathfrak{H}$ *onto the subspaces* $J(s;\mathbf{k})$ *and* $G(s;\mathbf{k})$, *respectively. If* $\mathcal{V}$ *is the Gelfand transformation defined by* (1.5) *and* (1.6), *then*

$$\mathcal{V}\mathcal{P}(s)\mathcal{V}^{-1} = \int_{\widetilde{\Omega}} \oplus \mathcal{P}(s;\mathbf{k})\,d\mathbf{k}, \tag{2.38}$$

$$\mathcal{V}\mathcal{Q}(s)\mathcal{V}^{-1} = \int_{\widetilde{\Omega}} \oplus \mathcal{Q}(s;\mathbf{k})\,d\mathbf{k}. \tag{2.39}$$

Proof. Obviously, it suffices to check (2.38), i.e., to prove that

$$(\mathcal{P}(s)\mathbf{u})_*(\cdot,\mathbf{k}) = \mathcal{P}(s;\mathbf{k})\mathbf{u}_*(\cdot,\mathbf{k}), \ \ \mathbf{u} \in \mathfrak{G}, \quad \text{for a.e. } \mathbf{k} \in \widetilde{\Omega},$$

where $\mathbf{u}_* = \mathcal{V}\mathbf{u}$, $(\mathcal{P}(s)\mathbf{u})_* = \mathcal{V}(\mathcal{P}(s)\mathbf{u})$. By (2.5), $\mathcal{P}(s)\mathbf{u} = \mathbf{u} - s^{1/2}\nabla\varphi$, where $\varphi \in \mathcal{H}^1(\mathbb{R}^3)$ satisfies (2.4). Then

$$(\mathcal{P}(s)\mathbf{u})_*(\mathbf{x},\mathbf{k}) = \mathbf{u}_*(\mathbf{x},\mathbf{k}) - (s(\mathbf{x}))^{1/2}\nabla_{\mathbf{k}}\varphi_*(\mathbf{x},\mathbf{k}), \quad \mathbf{x} \in \Omega, \quad \mathbf{k} \in \widetilde{\Omega}. \tag{2.40}$$

Thus, (2.4) means that

$$\begin{aligned} &\int_{\widetilde{\Omega}} d\mathbf{k} \int_{\Omega} \langle s(\mathbf{x})\nabla_{\mathbf{k}}\varphi_*(\mathbf{x},\mathbf{k}), \nabla_{\mathbf{k}}\psi_*(\mathbf{x},\mathbf{k})\rangle\,d\mathbf{x} \\ &\qquad = \int_{\widetilde{\Omega}} d\mathbf{k} \int_{\Omega} \langle (s(\mathbf{x}))^{1/2}\mathbf{u}_*(\mathbf{x},\mathbf{k}), \nabla_{\mathbf{k}}\psi_*(\mathbf{x},\mathbf{k})\rangle\,d\mathbf{x}, \quad \psi \in \mathcal{H}^1(\mathbb{R}^3), \end{aligned} \tag{2.41}$$

and we may assume that $\psi \in \mathcal{S}(\mathbb{R}^3)$. As is well known, the Gelfand transformation $\mathcal{V}$ maps the Schwartz class $\mathcal{S}(\mathbb{R}^3)$ *onto* the set of all infinitely smooth functions $\psi_*(\mathbf{x},\mathbf{k})$ that are Γ-periodic with respect to $\mathbf{x}$ and $\widetilde{\Gamma}$-quasiperiodic with respect to $\mathbf{k}$ (the latter means that the functions $e^{i\langle \mathbf{x},\mathbf{k}\rangle}\psi_*(\mathbf{x},\mathbf{k})$ are $\widetilde{\Gamma}$-periodic in $\mathbf{k}$). We put $\psi_*(\mathbf{x},\mathbf{k}) = \vartheta(\mathbf{x})\zeta(\mathbf{k})$, where $\vartheta \in \widetilde{C}^\infty(\Omega)$, $\zeta \in C_0^\infty(\widetilde{\Omega})$. Then (2.41) implies that, for almost every $\mathbf{k} \in \widetilde{\Omega}$, we have

$$\int_{\Omega} \langle s(\mathbf{x})\nabla_{\mathbf{k}}\varphi_*(\mathbf{x},\mathbf{k}), \nabla_{\mathbf{k}}\vartheta(\mathbf{x})\rangle\,d\mathbf{x} = \int_{\Omega} \langle (s(\mathbf{x}))^{1/2}\mathbf{u}_*(\mathbf{x},\mathbf{k}), \nabla_{\mathbf{k}}\vartheta(\mathbf{x})\rangle\,d\mathbf{x}, \quad \vartheta \in \widetilde{C}^\infty(\Omega).$$

This means that $\varphi_*(\cdot,\mathbf{k}) \in \widetilde{H}^1(\Omega)$ is the weak periodic solution of the equation

$$\operatorname{div}_{\mathbf{k}} s(\mathbf{x})\nabla_{\mathbf{k}}\varphi_*(\mathbf{x},\mathbf{k}) = \operatorname{div}_{\mathbf{k}}(s(\mathbf{x}))^{1/2}\mathbf{u}_*(\mathbf{x},\mathbf{k}) \quad \text{for a.e. } \mathbf{k} \in \widetilde{\Omega}.$$

Then (cf. (2.11), (2.13)) the right-hand side of (2.40) coincides with $\mathcal{P}(s;\mathbf{k})\mathbf{u}_*(\mathbf{x},\mathbf{k})$. $\square$

§3. The operator $\mathcal{L}$

3.1. Definition of the operator $\mathcal{L}$. Suppose that $h(\mathbf{x})$ and $s(\mathbf{x})$ are Γ-periodic measurable (3×3)-matrix-valued functions in $\mathbb{R}^3$ with real entries and such that

$$c_0\mathbf{1} \le h(\mathbf{x}) \le c_0'\mathbf{1}, \quad c_0\mathbf{1} \le s(\mathbf{x}) \le c_0'\mathbf{1}, \qquad \mathbf{x} \in \mathbb{R}^3, \quad 0 < c_0 \le c_0' < \infty. \tag{3.1}$$

Suppose $\nu(\mathbf{x})$ is a real-valued measurable Γ-periodic function in $\mathbb{R}^3$ such that

$$0 < \nu_0 \le \nu(\mathbf{x}) \le \nu_1 < \infty, \quad \mathbf{x} \in \mathbb{R}^3. \tag{3.2}$$

In $\mathfrak{G} = L_2(\mathbb{R}^3; \mathbb{C}^3)$, we consider the operator $\mathcal{L} = \mathcal{L}(s, h, \nu)$ given formally by the differential expression

$$\mathcal{L} = \mathcal{L}(s, h, \nu) = (s(\mathbf{x}))^{-1/2} \operatorname{curl}(h(\mathbf{x}))^{-1} \operatorname{curl}(s(\mathbf{x}))^{-1/2} - (s(\mathbf{x}))^{1/2} \nabla \nu(\mathbf{x}) \operatorname{div}(s(\mathbf{x}))^{1/2}. \tag{3.3}$$

The precise definition of $\mathcal{L}$ is given in terms of the quadratic form

$$\mathfrak{l}[\mathbf{u}, \mathbf{u}] = \int_{\mathbb{R}^3} \Big(\langle (h(\mathbf{x}))^{-1} \operatorname{curl}(s(\mathbf{x}))^{-1/2}\mathbf{u}, \operatorname{curl}(s(\mathbf{x}))^{-1/2}\mathbf{u} \rangle + \nu(\mathbf{x}) |\operatorname{div}(s(\mathbf{x}))^{1/2}\mathbf{u}|^2 \Big)\, d\mathbf{x}, \tag{3.4}$$

$$\operatorname{Dom} \mathfrak{l} = F(\mathbb{R}^3; s), \tag{3.5}$$

where the class $F(\mathbb{R}^3; s)$ is defined by (2.6). Obviously, the form (3.4) is positive. The results of Subsection 2.1 show that under conditions (3.1) and (3.2) the domain (3.5) is dense in $\mathfrak{G}$, and the form (3.4) is closed. Moreover, in the space $F(\mathbb{R}^3; s)$ the form $\mathfrak{l}[\mathbf{u}, \mathbf{u}] + \|\mathbf{u}\|^2_{\mathfrak{G}}$ determines a norm equivalent to the standard one. By definition, the operator $\mathcal{L}$ is the selfadjoint operator in $\mathfrak{G}$ generated by the form (3.4).

Now, we define an operator $\mathcal{X} : \mathfrak{G} \to \mathfrak{G}_* := L_2(\mathbb{R}^3; \mathbb{C}^4)$ by the relation

$$\mathcal{X} = \begin{pmatrix} -ih^{-1/2} \operatorname{curl} s^{-1/2} \\ -i\nu^{1/2} \operatorname{div} s^{1/2} \end{pmatrix}, \quad \operatorname{Dom} \mathcal{X} = F(\mathbb{R}^3; s).$$

Then the form (3.4) can be written as

$$\mathfrak{l}[\mathbf{u}, \mathbf{u}] = \|\mathcal{X}\mathbf{u}\|^2_{\mathfrak{G}_*}, \quad \mathbf{u} \in F(\mathbb{R}^3; s). \tag{3.6}$$

The operator $\mathcal{X}$ is closed together with the form $\mathfrak{l}$. Relation (3.6) means that the operator $\mathcal{L}$ admits a factorization: $\mathcal{L} = \mathcal{X}^*\mathcal{X}$.

Obviously, the decomposition (2.2) reduces $\mathcal{L}$. Formally, the part $\mathcal{L}_{J(s)}$ of $\mathcal{L}$ acting in the "solenoidal" subspace $J(s)$ is given by the differential expression $s^{-\frac{1}{2}} \operatorname{curl} h^{-1} \operatorname{curl} s^{-\frac{1}{2}}$, and the part $\mathcal{L}_{G(s)}$ of $\mathcal{L}$ acting in the "gradient" subspace $G(s)$ corresponds to the expression $-s^{\frac{1}{2}} \nabla \nu \operatorname{div} s^{\frac{1}{2}}$. Mainly, we are interested in the operator $\mathcal{L}_{J(s)}$.

3.2. The operators $\mathcal{L}(\mathbf{k})$. We put

$$\mathfrak{H} = L_2(\Omega; \mathbb{C}^3), \quad \mathfrak{H}_* = L_2(\Omega; \mathbb{C}^4) \tag{3.7}$$

and consider the operator $\mathcal{X}(\mathbf{k}) : \mathfrak{H} \to \mathfrak{H}_*$, $\mathbf{k} \in \mathbb{R}^3$, given by the formula

$$\mathcal{X}(\mathbf{k}) = \begin{pmatrix} -ih^{-1/2} \operatorname{curl}_{\mathbf{k}} s^{-1/2} \\ -i\nu^{1/2} \operatorname{div}_{\mathbf{k}} s^{1/2} \end{pmatrix} \tag{3.8}$$

on the domain

$$\operatorname{Dom} \mathcal{X}(\mathbf{k}) = F(\Omega; s; \mathbf{k}) =: \mathfrak{d}. \tag{3.9}$$

Here $F(\Omega; s; \mathbf{k}) = F(s; \mathbf{k})$ is the class introduced in (2.20), (2.23). The results of Subsection 2.2 show that the domain (3.9) is dense in $\mathfrak{H}$ (and independent of $\mathbf{k}$), and the operator (3.8) is closed. The selfadjoint operator

$$\mathcal{L}(\mathbf{k}) := \mathcal{X}(\mathbf{k})^* \mathcal{X}(\mathbf{k}) : \mathfrak{H} \to \mathfrak{H}, \quad \mathbf{k} \in \mathbb{R}^3, \tag{3.10}$$

is generated by the closed quadratic form

$$\mathfrak{l}(\mathbf{k})[\mathbf{u}, \mathbf{u}] := \|\mathcal{X}(\mathbf{k})\mathbf{u}\|^2_{\mathfrak{H}_*}, \quad \mathbf{u} \in F(s; \mathbf{k}), \ \ \mathbf{k} \in \mathbb{R}^3. \tag{3.11}$$

In the space $F(s; \mathbf{k})$, the form $\mathfrak{l}(\mathbf{k})[\mathbf{u}, \mathbf{u}] + \|\mathbf{u}\|^2_{\mathfrak{H}}$ determines a norm equivalent to the standard one. The spectrum of $\mathcal{L}(\mathbf{k})$ is discrete, because the embedding of $F(s; \mathbf{k})$ in $\mathfrak{H}$ is compact (see Subsection 2.2). The resolvent of the operator $\mathcal{L}(\mathbf{k})$ is compact and depends on $\mathbf{k} \in \mathbb{R}^3$ continuously (with respect to the operator norm).

The decomposition (2.10) reduces $\mathcal{L}(\mathbf{k})$. The part $(\mathcal{L}(\mathbf{k}))_{J(s;\mathbf{k})}$ of $\mathcal{L}(\mathbf{k})$ acting in the subspace $J(s; \mathbf{k})$ corresponds to the differential expression $s^{-1/2} \operatorname{curl}_{\mathbf{k}} h^{-1} \operatorname{curl}_{\mathbf{k}} s^{-1/2}$, and the part $(\mathcal{L}(\mathbf{k}))_{G(s;\mathbf{k})}$ of $\mathcal{L}(\mathbf{k})$ acting in the subspace $G(s; \mathbf{k})$ corresponds to the expression $-s^{1/2} \nabla_{\mathbf{k}} \nu \operatorname{div}_{\mathbf{k}} s^{1/2}$. (The boundary conditions are periodic.)

3.3. The band functions. By

$$(0 \le)\ E_1(\mathbf{k}) \le E_2(\mathbf{k}) \le \cdots \le E_j(\mathbf{k}) \le \cdots, \quad \mathbf{k} \in \mathbb{R}^3, \tag{3.12}$$

we denote the consecutive eigenvalues of the operator $\mathcal{L}(\mathbf{k})$. *The band functions $E_j(\mathbf{k})$ are continuous and $\widetilde{\Gamma}$-periodic.* They coincide with the consecutive minima of the ratio

$$\frac{\mathfrak{l}(\mathbf{k})[\mathbf{u}, \mathbf{u}]}{\|\mathbf{u}\|^2_{\mathfrak{H}}}, \quad \mathbf{u} \in F(s; \mathbf{k}). \tag{3.13}$$

Another natural way of enumerating the eigenvalues of $\mathcal{L}(\mathbf{k})$ is related to the splitting of $\mathcal{L}(\mathbf{k})$ as in the orthogonal decomposition (2.10). Let $E_{j,J}(\mathbf{k})$, $j \in \mathbb{N}$, be the consecutive eigenvalues of the operator $(\mathcal{L}(\mathbf{k}))_{J(s;\mathbf{k})}$, and let $E_{j,G}(\mathbf{k})$, $j \in \mathbb{N}$, be the consecutive eigenvalues of the operator $(\mathcal{L}(\mathbf{k}))_{G(s;\mathbf{k})}$. Then the numbers $E_{j,J}(\mathbf{k})$ coincide with the consecutive minima of the ratio (3.13) with $\mathbf{u} \in J^1(s; \mathbf{k})$, and the $E_{j,G}(\mathbf{k})$ coincide with the consecutive minima of the ratio (3.13) with $\mathbf{u} \in G^1(s; \mathbf{k})$. (Recall that the classes $J^1(s; \mathbf{k})$ and $G^1(s; \mathbf{k})$ are introduced in (2.28).) For $\mathbf{k} \in \operatorname{clos} \widetilde{\Omega} \setminus \{0\}$, the functions $E_{j,J}(\mathbf{k})$ and $E_{j,G}(\mathbf{k})$ are continuous. As we shall see below, *some of these functions may fail to be continuous at the point* $\mathbf{k} = 0$ (see Remark 3.1 below).

Now, we estimate the ratio (3.13). From (2.22), (3.8), (3.11), and conditions (3.1), (3.2) it is clear that

$$C_1^{-1} a(s; \mathbf{k})[\mathbf{u}, \mathbf{u}] \le \mathfrak{l}(\mathbf{k})[\mathbf{u}, \mathbf{u}] \le C_2 a(s; \mathbf{k})[\mathbf{u}, \mathbf{u}], \quad \mathbf{u} \in F(s; \mathbf{k}), \tag{3.14}$$

$$C_1 := \max\{\|h\|_{L_\infty}, \|\nu^{-1}\|_{L_\infty}\}, \tag{3.15}$$

$$C_2 := \max\{\|h^{-1}\|_{L_\infty}, \|\nu\|_{L_\infty}\}. \tag{3.16}$$

Next, let $\mathbf{v} = B(s; \mathbf{k})\mathbf{u}$, where the operator $B(s; \mathbf{k}) : F(s; \mathbf{k}) \to F(\mathbf{1}; \mathbf{k})$ is defined in Lemma 2.12. By (2.36), (2.37), and (3.14), we have

$$(C_1 C_0(s))^{-1} \frac{a(\mathbf{1}; \mathbf{k})[\mathbf{v}, \mathbf{v}]}{\|\mathbf{v}\|^2_{\mathfrak{H}}} \le \frac{\mathfrak{l}(\mathbf{k})[\mathbf{u}, \mathbf{u}]}{\|\mathbf{u}\|^2_{\mathfrak{H}}} \le C_2 C_0(s) \frac{a(\mathbf{1}; \mathbf{k})[\mathbf{v}, \mathbf{v}]}{\|\mathbf{v}\|^2_{\mathfrak{H}}}, \quad \mathbf{u} \in F(s; \mathbf{k}). \tag{3.17}$$

Note that, for $\mathbf{u} \in J^1(s; \mathbf{k})$ or $\mathbf{u} \in G^1(s; \mathbf{k})$, the constants in estimates (3.14), (3.17) may be refined with the help of Lemma 2.10. We shall not dwell on this.

By Lemmas 2.10 and 2.12, if $\mathbf{u}$ runs over $J^1(s; \mathbf{k})$, $G^1(s; \mathbf{k})$, or $F(s; \mathbf{k})$, then $\mathbf{v} = B(s; \mathbf{k})\mathbf{u}$ runs over $J^1(\mathbf{1}; \mathbf{k})$, $G^1(\mathbf{1}; \mathbf{k})$, or $F(\mathbf{1}; \mathbf{k}) = \widetilde{H}^1(\Omega; \mathbb{C}^3)$, respectively. We have (see (2.22))

$$a(\mathbf{1};\mathbf{k})[\mathbf{v},\mathbf{v}] = \int_\Omega |(\mathbf{D}+\mathbf{k})\mathbf{v}|^2\, d\mathbf{x}, \quad \mathbf{v} \in \widetilde{H}^1(\Omega;\mathbb{C}^3). \tag{3.18}$$

The closed positive form (3.18) generates a selfadjoint operator $\mathcal{A}_0(\mathbf{k})$ in $\mathfrak{H}$, which corresponds to the differential expression $(\mathbf{D}+\mathbf{k})^2$ and the periodic boundary conditions. Let $E_j^0(\mathbf{k})$, $j \in \mathbb{N}$, be the consecutive eigenvalues of the operator $\mathcal{A}_0(\mathbf{k})$. They coincide with the consecutive minima of the ratio

$$\frac{a(\mathbf{1};\mathbf{k})[\mathbf{v},\mathbf{v}]}{\|\mathbf{v}\|^2_{\mathfrak{H}}}, \quad \mathbf{v} \in \widetilde{H}^1(\Omega;\mathbb{C}^3). \tag{3.19}$$

On the other hand, the operator $\mathcal{A}_0(\mathbf{k})$ splits in accordance with the orthogonal decomposition $\mathfrak{H} = G(\mathbf{1};\mathbf{k}) \oplus J(\mathbf{1};\mathbf{k})$. Let $E^0_{j,J}(\mathbf{k})$, $j \in \mathbb{N}$, be the consecutive eigenvalues of the operator $(\mathcal{A}_0(\mathbf{k}))_{J(\mathbf{1};\mathbf{k})}$, and let $E^0_{j,G}(\mathbf{k})$, $j \in \mathbb{N}$, be the consecutive eigenvalues of the operator $(\mathcal{A}_0(\mathbf{k}))_{G(\mathbf{1};\mathbf{k})}$. The numbers $E^0_{j,J}(\mathbf{k})$ coincide with the consecutive minima of the ratio (3.19) with $\mathbf{v} \in J^1(\mathbf{1};\mathbf{k})$, and the $E^0_{j,G}(\mathbf{k})$ coincide with the consecutive minima of (3.19) with $\mathbf{v} \in G^1(\mathbf{1};\mathbf{k})$.

From (3.17) it follows that

$$(C_1C_0(s))^{-1}E_j^0(\mathbf{k}) \le E_j(\mathbf{k}) \le C_2C_0(s)E_j^0(\mathbf{k}), \quad j \in \mathbb{N}, \tag{3.20}$$

$$(C_1C_0(s))^{-1}E^0_{j,J}(\mathbf{k}) \le E_{j,J}(\mathbf{k}) \le C_2C_0(s)E^0_{j,J}(\mathbf{k}), \quad j \in \mathbb{N}, \tag{3.21}$$

$$(C_1C_0(s))^{-1}E^0_{j,G}(\mathbf{k}) \le E_{j,G}(\mathbf{k}) \le C_2C_0(s)E^0_{j,G}(\mathbf{k}), \quad j \in \mathbb{N}. \tag{3.22}$$

By (1.3) and (1.4), relation (3.19) can be rewritten as

$$\frac{\sum_{\mathbf{b}\in\widetilde{\Gamma}} |\mathbf{b}+\mathbf{k}|^2|\hat{\mathbf{v}}_\mathbf{b}|^2}{\sum_{\mathbf{b}\in\widetilde{\Gamma}} |\hat{\mathbf{v}}_\mathbf{b}|^2}, \quad \{\hat{\mathbf{v}}_\mathbf{b}\} \in h^1(\widetilde{\Gamma};\mathbb{C}^3). \tag{3.23}$$

Then the $E_j^0(\mathbf{k})$ are the consecutive minima of the ratio (3.23); the $E^0_{j,J}(\mathbf{k})$ are the consecutive minima of (3.23) under the additional condition $\hat{\mathbf{v}}_\mathbf{b} \perp (\mathbf{b}+\mathbf{k})$, $\mathbf{b} \in \widetilde{\Gamma}$; and the $E^0_{j,G}(\mathbf{k})$ are the consecutive minima of (3.23) under the additional condition $\hat{\mathbf{v}}_\mathbf{b} \parallel (\mathbf{b}+\mathbf{k})$, $\mathbf{b} \in \widetilde{\Gamma}$. It follows that the numbers $\{E_j^0(\mathbf{k})\}$, $\{E^0_{j,J}(\mathbf{k})\}$, $\{E^0_{j,G}(\mathbf{k})\}$, $j \in \mathbb{N}$, reduce to the numbers $\{|\mathbf{b}+\mathbf{k}|^2\}$, $\mathbf{b} \in \widetilde{\Gamma}$. Herewith, for $E_j^0(\mathbf{k})$, each eigenvalue of the form $|\mathbf{b}+\mathbf{k}|^2$ has multiplicity 3. If $\mathbf{k} \in \operatorname{clos}\widetilde{\Omega}$ and $\mathbf{b}+\mathbf{k} \ne 0$, then, for $E^0_{j,J}(\mathbf{k})$, each eigenvalue of the form $|\mathbf{b}+\mathbf{k}|^2$ has multiplicity 2, and for $E^0_{j,G}(\mathbf{k})$ every such eigenvalue is simple. The case where $\mathbf{b} = \mathbf{k} = 0$ is an exception. The eigenvalue $\lambda = 0$ of the operator $\mathcal{A}_0(0)$ corresponds to the eigenspace $\operatorname{Ker}\mathcal{A}_0(0) = \{\mathbf{v} = \mathbf{C} \in \mathbb{C}^3\}$, which lies entirely in the solenoidal subspace $J(\mathbf{1};0)$. Thus, strictly speaking, if $\mathbf{k} = 0$, we have $E^0_{j,J}(0) = 0$ for $j = 1,2,3$, and $E^0_{1,G}(0) > 0$.

Remark 3.1. From what has been said it follows that the functions $E^0_{3,J}(\mathbf{k})$ and $E^0_{1,G}(\mathbf{k})$ have a (removable) discontinuity at the point $\mathbf{k} = 0$. By (3.21) and (3.22), the same is true for the functions $E_{3,J}(\mathbf{k})$ and $E_{1,G}(\mathbf{k})$. In order to avoid stipulations, *we redefine the functions* $E^0_{3,J}(\mathbf{k})$, $E^0_{1,G}(\mathbf{k})$, $E_{3,J}(\mathbf{k})$, *and* $E_{1,G}(\mathbf{k})$ *at the point* $\mathbf{k} = 0$ *by continuity.*

After such redefinition, for the band functions of $\mathcal{A}_0(\mathbf{k})$ we have the following properties (cf. [BSu2, §2.2]). Clearly,

$$\begin{cases} E_1^0(\mathbf{k}) = E_2^0(\mathbf{k}) = E_3^0(\mathbf{k}) = |\mathbf{k}|^2, & \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}; \\ E^0_{1,J}(\mathbf{k}) = E^0_{2,J}(\mathbf{k}) = |\mathbf{k}|^2, & \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}; \\ E^0_{1,G}(\mathbf{k}) = |\mathbf{k}|^2, & \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}. \end{cases} \tag{3.24}$$

Recall that r_0 is the radius of the ball inscribed in $\operatorname{clos}\widetilde{\Omega}$, and $\mathcal{B}(r) = \{\mathbf{k} \in \mathbb{R}^3 : |\mathbf{k}| \le r\}$. Using (1.1), we obtain:

$$E_1^0(\mathbf{k}) = E_{1,J}^0(\mathbf{k}) = E_{1,G}^0(\mathbf{k}) \ge r^2, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega} \setminus \mathcal{B}(r), \;\; 0 < r \le r_0; \tag{3.25}$$

$$E_4^0(\mathbf{k}) = E_{3,J}^0(\mathbf{k}) = E_{2,G}^0(\mathbf{k}) \ge r_0^2, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}; \tag{3.26}$$

$$E_4^0(0) = E_{3,J}^0(0) = E_{2,G}^0(0) = \min_{0 \ne \mathbf{b} \in \widetilde{\Gamma}} |\mathbf{b}|^2 = 4r_0^2. \tag{3.27}$$

We put

$$c_*^{-1} = C_1 C_0(s) = \max\{\|h\|_{L_\infty}, \|\nu^{-1}\|_{L_\infty}\} \max\{\|s\|_{L_\infty}, \|s^{-1}\|_{L_\infty}\} \tag{3.28}$$

(see (2.9) and (3.15)). By estimates (3.20)–(3.22) and properties (3.24)–(3.27), the band functions of $\mathcal{L}(\mathbf{k})$ satisfy the following relations:

$$\begin{gathered} c_*|\mathbf{k}|^2 \le E_j(\mathbf{k}) \le C_2 C_0(s)|\mathbf{k}|^2, \quad j = 1,2,3, \;\; \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}; \\ c_*|\mathbf{k}|^2 \le E_{j,J}(\mathbf{k}) \le C_2 C_0(s)|\mathbf{k}|^2, \quad j = 1,2, \;\; \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}; \\ c_*|\mathbf{k}|^2 \le E_{1,G}(\mathbf{k}) \le C_2 C_0(s)|\mathbf{k}|^2, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}; \end{gathered} \tag{3.29}$$

$$\begin{gathered} E_1(\mathbf{k}) \ge c_* r^2, \quad E_{1,J}(\mathbf{k}) \ge c_* r^2, \quad E_{1,G}(\mathbf{k}) \ge c_* r^2, \\ \mathbf{k} \in \operatorname{clos}\widetilde{\Omega} \setminus \mathcal{B}(r), \quad 0 < r \le r_0; \end{gathered} \tag{3.30}$$

$$E_4(\mathbf{k}) \ge c_* r_0^2, \;\; E_{3,J}(\mathbf{k}) \ge c_* r_0^2, \;\; E_{2,G}(\mathbf{k}) \ge c_* r_0^2, \quad \mathbf{k} \in \operatorname{clos}\widetilde{\Omega}; \tag{3.31}$$

$$E_4(0) \ge 4c_* r_0^2, \quad E_{3,J}(0) \ge 4c_* r_0^2, \quad E_{2,G}(0) \ge 4c_* r_0^2. \tag{3.32}$$

These properties imply that if $C_2 C_0(s)|\mathbf{k}|^2 < c_* r_0^2$, i.e.,

$$|\mathbf{k}| < c_*^{1/2}(C_2 C_0(s))^{-1/2} r_0 = (C_1 C_2)^{-1/2}(C_0(s))^{-1} r_0, \tag{3.33}$$

then the first three band functions $E_j(\mathbf{k})$, $j = 1,2,3$, coincide piecewise with $E_{1,J}(\mathbf{k})$, $E_{2,J}(\mathbf{k})$, $E_{1,G}(\mathbf{k})$. Under condition (3.33), the operator $\mathcal{L}(\mathbf{k})$ has exactly three eigenvalues lying in the interval $[0, c_* r_0^2)$. For $\mathbf{k} \ne 0$, two of them correspond to the part $(\mathcal{L}(\mathbf{k}))_{J(s;\mathbf{k})}$ of $\mathcal{L}(\mathbf{k})$ acting in the subspace $J(s;\mathbf{k})$, and one of them corresponds to the part $(\mathcal{L}(\mathbf{k}))_{G(s;\mathbf{k})}$ of $\mathcal{L}(\mathbf{k})$ acting in the subspace $G(s;\mathbf{k})$. If $\mathbf{k} = 0$, then the eigenvalue $\lambda = 0$ of multiplicity 3 corresponds to the "solenoidal" part $(\mathcal{L}(0))_{J(s;0)}$ of $\mathcal{L}(0)$ (see Remark 3.1).

3.4. The direct integral for the operator $\mathcal{L}$. In order to represent $\mathcal{L}$ as a direct integral, we apply the Gelfand transformation $\mathcal{V}$ (see Subsection 1.3).

The operators $\mathcal{L}(\mathbf{k})$ allow us to partially diagonalize $\mathcal{L}$ in the direct integral $\mathcal{K}$ (see (1.6) with $n = 3$). If $\mathbf{u} \in \operatorname{Dom}\mathfrak{l} = F(\mathbb{R}^3; s)$, then for $\mathcal{V}\mathbf{u} = \mathbf{u}_*$ we have

$$\mathbf{u}_*(\cdot, \mathbf{k}) \in \operatorname{Dom}\mathfrak{l}(\mathbf{k}) \quad \text{for a.e. } \mathbf{k} \in \widetilde{\Omega}, \tag{3.34}$$

$$\mathfrak{l}[\mathbf{u}, \mathbf{u}] = \int_{\widetilde{\Omega}} \mathfrak{l}(\mathbf{k})[\mathbf{u}_*(\cdot, \mathbf{k}), \mathbf{u}_*(\cdot, \mathbf{k})]\, d\mathbf{k}. \tag{3.35}$$

Conversely, if (3.34) is satisfied for $\mathbf{u}_* \in \mathcal{K}$ and the integral in (3.35) is finite, then $\mathbf{u} = \mathcal{V}^{-1}\mathbf{u}_* \in \operatorname{Dom}\mathfrak{l}$ and (3.35) is true. This means that, in the direct integral $\mathcal{K}$ the operator $\mathcal{L}$ turns into the layerwise multiplication by the operator-valued function $\mathcal{L}(\mathbf{k})$, $\mathbf{k} \in \widetilde{\Omega}$. All this can be expressed by the formula

$$\mathcal{V}\mathcal{L}\mathcal{V}^{-1} = \int_{\widetilde{\Omega}} \oplus \mathcal{L}(\mathbf{k})\, d\mathbf{k}. \tag{3.36}$$

From (3.36) it follows that the *spectrum of* $\mathcal{L}$ *is a union of segments* (*spectral bands*) *that are the ranges of the band functions* (3.12). By (3.29),

$$\min_{\mathbf{k}\in \operatorname{clos}\widetilde{\Omega}} E_j(\mathbf{k}) = E_j(0) = 0, \quad j = 1, 2, 3.$$

Consequently, the lower edge of the spectrum of $\mathcal{L}$ coincides with the point $\lambda = 0$ and is realized as the lower edge of the first three bands. Moreover, (3.31) shows that the lower edge of the fourth band is separated away from the point $\lambda = 0$.

We trace the splitting of operators in the direct integral (3.36). By (2.38), (2.39), and (3.36), the operators $\mathcal{L}\mathcal{P}(s) = \mathcal{L}_{J(s)} \oplus \mathbf{0}_{G(s)}$ and $\mathcal{L}\mathcal{Q}(s) = \mathbf{0}_{J(s)} \oplus \mathcal{L}_{G(s)}$ can be decomposed into the direct integrals of the operators $\mathcal{L}(\mathbf{k})\mathcal{P}(s;\mathbf{k}) = (\mathcal{L}(\mathbf{k}))_{J(s;\mathbf{k})} \oplus \mathbf{0}_{G(s;\mathbf{k})}$ and $\mathcal{L}(\mathbf{k})\mathcal{Q}(s;\mathbf{k}) = \mathbf{0}_{J(s;\mathbf{k})} \oplus (\mathcal{L}(\mathbf{k}))_{G(s;\mathbf{k})}$, respectively:

$$\mathcal{V}\mathcal{L}\mathcal{P}(s)\mathcal{V}^{-1} = \int_{\widetilde{\Omega}} \oplus \mathcal{L}(\mathbf{k})\mathcal{P}(s;\mathbf{k})\,d\mathbf{k}, \tag{3.37}$$

$$\mathcal{V}\mathcal{L}\mathcal{Q}(s)\mathcal{V}^{-1} = \int_{\widetilde{\Omega}} \oplus \mathcal{L}(\mathbf{k})\mathcal{Q}(s;\mathbf{k})\,d\mathbf{k}.$$

§4. Application of the general method to the operators $\mathcal{L}(\mathbf{k})$

4.1. We apply the general method of [BSu2, Chapter 1] to the study of the operator family $\mathcal{L}(\mathbf{k})$ defined in Subsection 3.2. In [BSu2, Chapter 1], the operator pencil of the form $A(t) = X(t)^*X(t)$ with $X(t) = X_0 + tX_1$, $t \in \mathbb{R}$, was studied.

Now the parameter $\mathbf{k}$ is 3-dimensional. We put

$$\mathbf{k} = t\boldsymbol{\theta}, \quad t = |\mathbf{k}|, \quad \boldsymbol{\theta} = |\mathbf{k}|^{-1}\mathbf{k} \in \mathbb{S}^2$$

(cf. [BSu2, Chapter 2]) and view t as the main parameter. Then all objects will depend on the additional parameter $\boldsymbol{\theta}$. We introduce the notation

$$\mathcal{L}(\mathbf{k}) =: L(t;\boldsymbol{\theta}), \quad \mathcal{X}(\mathbf{k}) =: X(t;\boldsymbol{\theta}),$$

where $\mathcal{X}(\mathbf{k}) : \mathfrak{H} \to \mathfrak{H}_*$ is the operator defined in (3.7)–(3.9). By (3.10), the selfadjoint operator $L(t;\boldsymbol{\theta})$ in $\mathfrak{H}$ admits a factorization of the type

$$L(t;\boldsymbol{\theta}) = X(t;\boldsymbol{\theta})^* X(t;\boldsymbol{\theta}) \tag{4.1}$$

(required in the general scheme). By (3.8) and (3.9),

$$X(t;\boldsymbol{\theta}) = X_0 + tX_1(\boldsymbol{\theta}),$$

where $X_0 = \mathcal{X}(0) : \mathfrak{H} \to \mathfrak{H}_*$ is a closed operator given by the expression

$$X_0 = \begin{pmatrix} -ih^{-1/2}\operatorname{curl} s^{-1/2} \\ -i\nu^{1/2}\operatorname{div} s^{1/2} \end{pmatrix}, \quad \operatorname{Dom} X_0 = \mathfrak{d}, \tag{4.2}$$

and $X_1(\boldsymbol{\theta}) : \mathfrak{H} \to \mathfrak{H}_*$ is a bounded operator of the form

$$X_1(\boldsymbol{\theta})\mathbf{u} = \begin{pmatrix} h^{-1/2} & 0 \\ 0 & \nu^{1/2} \end{pmatrix} \begin{pmatrix} \boldsymbol{\theta} \times (s^{-1/2}\mathbf{u}) \\ \boldsymbol{\theta} \cdot (s^{1/2}\mathbf{u}) \end{pmatrix}, \quad \mathbf{u} \in \mathfrak{H}. \tag{4.3}$$

We check that, in our case, the conditions of the general method of [BSu2, Chapter 1] are satisfied. We denote

$$\mathfrak{N} := \operatorname{Ker} \mathcal{L}(0) = \operatorname{Ker} X_0.$$

Let P denote the orthogonal projection in $\mathfrak{H}$ onto the kernel $\mathfrak{N}$. From (4.2) it follows that the kernel $\mathfrak{N}$ is defined by the relations

$$\mathfrak{N} = \{\mathbf{f} \in \mathfrak{H} : \operatorname{curl} s^{-1/2}\check{\mathbf{f}} = 0,\ \operatorname{div} s^{1/2}\check{\mathbf{f}} = 0\}. \tag{4.4}$$

The first relation in (4.4) implies that

$$s(\mathbf{x})^{-1/2}\mathbf{f}(\mathbf{x}) = \mathbf{C} + \nabla\Phi(\mathbf{x}), \quad \mathbf{C} \in \mathbb{C}^3,\ \Phi \in \widetilde{H}^1(\Omega),$$

and the second leads to the following equation for $\Phi = \Phi_{\mathbf{C}} \in \widetilde{H}^1(\Omega)$:

$$\operatorname{div} s(\mathbf{x})(\nabla\Phi_{\mathbf{C}}(\mathbf{x}) + \mathbf{C}) = 0 \tag{4.5}$$

(this equation is understood in the weak sense). A periodic solution $\Phi_{\mathbf{C}}$ of equation (4.5) exists and is defined up to a constant summand; hence, the gradient $\nabla\Phi_{\mathbf{C}}$ is defined uniquely. Thus,

$$\mathfrak{N} = \{\mathbf{f} = s^{1/2}(\mathbf{C} + \nabla\Phi_{\mathbf{C}}) : \mathbf{C} \in \mathbb{C}^3\}. \tag{4.6}$$

Note that $\mathfrak{N} \subset J(s;0)$ (this was already mentioned at the end of Subsection 3.3, but the arguments were different). The functions $\mathbf{f}$ in the kernel $\mathfrak{N}$ are naturally parametrized by vectors $\mathbf{C} \in \mathbb{C}^3$. Observe that for $\mathbf{f} = s^{1/2}(\mathbf{C} + \nabla\Phi_{\mathbf{C}})$ we have $\int_\Omega s^{-1/2}\mathbf{f}\,d\mathbf{x} = |\Omega|\mathbf{C}$. It follows that

$$\dim \mathfrak{N} = 3. \tag{4.7}$$

(This agrees with the results of Subsection 3.3, where (4.7) was proved by variational estimates for the eigenvalues.) Thus, we *have checked that Condition* 1.1 *in* [BSu2, Chapter 1] *is satisfied*: the point $\lambda = 0$ is an isolated point in the spectrum of $\mathcal{L}(0)$, and this point is an eigenvalue of finite multiplicity. Let d^0 be the distance from the point $\lambda = 0$ to the rest of the spectrum of $\mathcal{L}(0)$. By (3.32), we have

$$d^0 \geq 4c_* r_0^2. \tag{4.8}$$

As in [BSu2, §1.1], we fix a number $\delta \in (0, d^0/8)$; it is convenient to assume (cf. (4.8)) that

$$\delta < c_* r_0^2/2. \tag{4.9}$$

Next, we fix a number t^0 such that

$$t^0 \leq \delta^{1/2} \min_{\boldsymbol{\theta}\in\mathbb{S}^2} \|X_1(\boldsymbol{\theta})\|^{-1}.$$

Combining (4.3) with (2.9) and (3.16), we obtain

$$\|X_1(\boldsymbol{\theta})\|^2 \leq \|h^{-1}\|_{L_\infty}\|s^{-1}\|_{L_\infty} + \|\nu\|_{L_\infty}\|s\|_{L_\infty} \leq 2C_2C_0(s).$$

Correspondingly, we put

$$t^0 = \delta^{1/2}(2C_2C_0(s))^{-1/2}. \tag{4.10}$$

By using (4.9) and also the expressions (2.9), (3.16), and (3.28) for the constants, it is easy to check that

$$t^0 \leq (C_1C_2)^{-1/2}(2C_0(s))^{-1} r_0 \leq r_0/2. \tag{4.11}$$

Thus, $\mathcal{B}(t^0) \subset \mathcal{B}(r_0/2) \subset \widetilde{\Omega}$.

We denote by $\mathcal{F}(t;\boldsymbol{\theta};\kappa)$ the spectral projection of the operator $L(t;\boldsymbol{\theta})$ for the closed interval $[0,\kappa]$. As was shown in [BSu2, Chapter 1, Proposition 1.1], for $t \in [0, t^0]$ we have

$$\dim \mathcal{F}(t;\boldsymbol{\theta};\delta)\mathfrak{H} = \dim \mathcal{F}(t;\boldsymbol{\theta};3\delta)\mathfrak{H} = \dim \mathfrak{N} = 3.$$

Thus, the operator $L(t;\boldsymbol{\theta})$ has exactly three eigenvalues (counted with multiplicities) in the interval $[0,\delta]$, and the interval $(\delta, 3\delta)$ is free of the spectrum of $L(t,\boldsymbol{\theta})$. This agrees with what was obtained at the end of Subsection 3.3 on the basis of variational estimates (cf. (3.33) and (4.11)).

By (4.2), for the kernel $\mathfrak{N}_* = \operatorname{Ker} X_0^*$ we have

$$\begin{aligned}\mathfrak{N}_* &= \{(\mathbf{v},\varphi) \in \mathfrak{H}_* : \operatorname{curl}(h^{-1/2}\check{\mathbf{v}}) = 0,\ \nabla(\nu^{1/2}\check{\varphi}) = 0\}\\ &= \{(\mathbf{v},\varphi) : \mathbf{v} = h^{1/2}(\mathbf{C} + \nabla\psi),\ \varphi = \nu^{-1/2}c,\ \mathbf{C} \in \mathbb{C}^3,\ c \in \mathbb{C},\ \psi \in \widetilde{H}^1(\Omega)\}.\end{aligned}$$

Obviously, $\mathfrak{N}_*$ is infinite-dimensional. This *verifies the second assumption of the general method* (the inequality $\dim \mathfrak{N}_* \geq \dim \mathfrak{N}$).

4.2. The operator of identification of kernels. Let s^0 be the constant (3×3)-matrix defined by the relation

$$s^0\mathbf{C} = |\Omega|^{-1} \int_\Omega s(\mathbf{x})(\mathbf{C} + \nabla\Phi_{\mathbf{C}})\, d\mathbf{x}, \quad \mathbf{C} \in \mathbb{C}^3, \tag{4.12}$$

where $\Phi_{\mathbf{C}} \in \widetilde{H}^1(\Omega)$ is the solution of equation (4.5). The matrix s^0 has real entries and is positive definite. This matrix arises in the homogenization problem for the elliptic operator $-\operatorname{div} s(\mathbf{x})\nabla$ and is called the *effective matrix* for $s(\mathbf{x})$ (see, e.g., [ZhKO, BeLP, BSu2]).

If the coefficient $s(\mathbf{x})$ in $\mathcal{L}$ is a constant matrix, then the kernel (4.6) consists of constant vector-valued functions and can be identified with $\mathbb{C}^3$. However, even in this case, it is convenient to preserve parametrization for the elements of the kernel in terms of the vector $\mathbf{C}$, as in (4.6).

In the case where $s(\mathbf{x})$ is the effective matrix s^0, we denote the kernel (4.6) by $\mathfrak{N}^0$. We have

$$\mathfrak{N}^0 = \{\mathbf{f}^0 = (s^0)^{1/2}\mathbf{C} : \mathbf{C} \in \mathbb{C}^3\}. \tag{4.13}$$

Let P_0 denote the orthogonal projection in $\mathfrak{H}$ onto $\mathfrak{N}^0$.

Now, we introduce the "identification of kernels" operator $\mathcal{U} : \mathfrak{N} \to \mathfrak{N}^0$,

$$\mathcal{U} : \quad \mathbf{f} = s^{1/2}(\mathbf{C} + \nabla\Phi_{\mathbf{C}}) \mapsto \mathbf{f}^0 = (s^0)^{1/2}\mathbf{C}, \;\; \mathbf{C} \in \mathbb{C}^3. \tag{4.14}$$

Lemma 4.1. *Let s^0 be the effective matrix for $s(\mathbf{x})$, defined by* (4.12)*. Let $\mathfrak{N}$ and $\mathfrak{N}^0$ be defined by* (4.6) *and* (4.13)*, and let $\mathcal{U}$ be defined by* (4.14)*. Then the operator $\mathcal{U} : \mathfrak{N} \to \mathfrak{N}^0$ is unitary.*

Proof. Let $\mathbf{f} = s^{1/2}(\mathbf{C} + \nabla\Phi_{\mathbf{C}})$, and let $\mathbf{f}^0 = (s^0)^{1/2}\mathbf{C}$, $\mathbf{C} \in \mathbb{C}$. Then, by (4.5), we have

$$\|\mathbf{f}\|^2_{\mathfrak{H}} = \int_\Omega \langle s(\mathbf{x})(\mathbf{C} + \nabla\Phi_{\mathbf{C}}), (\mathbf{C} + \nabla\Phi_{\mathbf{C}})\rangle\, d\mathbf{x} = \left\langle \int_\Omega s(\mathbf{x})(\mathbf{C} + \nabla\Phi_{\mathbf{C}})\, d\mathbf{x}, \mathbf{C} \right\rangle.$$

By (4.12), the right-hand side is equal to $|\Omega|\langle s^0\mathbf{C}, \mathbf{C}\rangle = \|\mathbf{f}^0\|^2_{\mathfrak{H}}$. $\square$

4.3. The spectral germ. The main notion of the general method of [BSu2, Chapter 1] is the *spectral germ* of an operator family at $t = 0$. *The germ $S(\boldsymbol{\theta})$ of the operator family $L(t;\boldsymbol{\theta})$ at $t = 0$* is a selfadjoint operator acting in the 3-dimensional kernel $\mathfrak{N}$. Among two equivalent definitions of the germ, now we choose the *spectral* definition (see [BSu2, Chapter 1, Subsection 1.6]). The operator family $L(t;\boldsymbol{\theta})$ depends on t analytically, and the point $\lambda = 0$ is an isolated eigenvalue of multiplicity 3 for the operator $L(t;\boldsymbol{\theta})$ with $t = 0$. Let t^0 be the number defined by (4.10). By analytic perturbation theory, for $t \in [0, t^0]$ there exist real-analytic (with respect to t) functions $\lambda_l(t;\boldsymbol{\theta})$ (branches of eigenvalues; $l = 1, 2, 3$) and real-analytic $\mathfrak{H}$-valued functions $\mathbf{u}_l(t;\boldsymbol{\theta})$ (branches of eigenfunctions; $l = 1, 2, 3$) such that

$$L(t;\boldsymbol{\theta})\mathbf{u}_l(t;\boldsymbol{\theta}) = \lambda_l(t;\boldsymbol{\theta})\mathbf{u}_l(t;\boldsymbol{\theta}), \quad l = 1, 2, 3, \;\; t \in [0, t^0].$$

The functions $\mathbf{u}_l(t;\boldsymbol{\theta})$, $l = 1, 2, 3$, form an orthonormal basis in the subspace $\mathcal{F}(t;\boldsymbol{\theta};\delta)\mathfrak{H}$. For sufficiently small $t_*(\leq t^0)$, we have the convergent power series expansions

$$\lambda_l(t;\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})t^2 + \cdots, \quad \gamma_l(\boldsymbol{\theta}) \geq 0, \;\; l = 1, 2, 3, \;\; t \in [0, t_*], \tag{4.15}$$

$$\mathbf{u}_l(t;\boldsymbol{\theta}) = \widehat{\mathbf{f}}_l(\boldsymbol{\theta}) + \cdots, \quad l = 1, 2, 3, \;\; t \in [0, t_*]. \tag{4.16}$$

The vectors $\widehat{\mathbf{f}}_l(\boldsymbol{\theta})$, $l = 1, 2, 3$, form an orthonormal basis in the kernel $\mathfrak{N}$.

Now, we give the spectral definition of the germ $S(\boldsymbol{\theta})$: *$S(\boldsymbol{\theta})$ is a selfadjoint operator acting in $\mathfrak{N}$ and such that the numbers $\gamma_l(\boldsymbol{\theta})$ and the vectors $\widehat{\mathbf{f}}_l(\boldsymbol{\theta})$, $l = 1, 2, 3$, are its eigenvalues and eigenvectors*:

$$S(\boldsymbol{\theta})\widehat{\mathbf{f}}_l(\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})\widehat{\mathbf{f}}_l(\boldsymbol{\theta}), \quad l = 1, 2, 3.$$

Clearly, for $t \in [0, t^0]$, the analytic branches of the eigenvalues $\lambda_l(t; \boldsymbol{\theta})$, $l = 1, 2, 3$, *partially coincide* with the band functions $E_j(t\boldsymbol{\theta})$, $j = 1, 2, 3$. (The band functions are enumerated in nondecreasing order, which may violate analyticity.) Then (3.29) implies that

$$\lambda_l(t; \boldsymbol{\theta}) \geq c_* t^2, \quad l = 1, 2, 3, \ \ t \in [0, t^0]. \tag{4.17}$$

It is important that in (4.17) the constant c_* (see (3.28)) and the number t^0 (see (4.10)) do not depend on $\boldsymbol{\theta}$. From (4.15) and (4.17) it follows that

$$\gamma_l(\boldsymbol{\theta}) \geq c_* > 0, \quad l = 1, 2, 3. \tag{4.18}$$

This means that the *germ $S(\boldsymbol{\theta})$ is nondegenerate uniformly with respect to $\boldsymbol{\theta}$*. Inequalities (4.17) show that, in our case, *the last assumption of the general method is satisfied* (see [BSu2, Chapter 1, (5.2)]).

4.4. Calculation of the germ $S(\boldsymbol{\theta})$. We apply the method described in [BSu2] to calculate the germ $S(\boldsymbol{\theta})$. For the operator families admitting a factorization of the form (4.1), the germ also admits a factorization of the form $S(\boldsymbol{\theta}) = R(\boldsymbol{\theta})^* R(\boldsymbol{\theta})$, where $R(\boldsymbol{\theta}) : \mathfrak{N} \to \mathfrak{N}_*$ is the operator defined as follows. For each $\mathbf{f} \in \mathfrak{N}$, we consider a solution $\mathbf{v}_{\boldsymbol{\theta}} \in \mathfrak{d}$ of the equation

$$X_0^*(X_0 \mathbf{v}_{\boldsymbol{\theta}} + X_1(\boldsymbol{\theta})\mathbf{f}) = 0 \tag{4.19}$$

(understood in the weak sense). Such a solution exists and is defined up to a summand belonging to the kernel $\mathfrak{N}$. Here, the element $X_0 \mathbf{v}_{\boldsymbol{\theta}}$ is defined uniquely. Then

$$R(\boldsymbol{\theta})\mathbf{f} = X_0 \mathbf{v}_{\boldsymbol{\theta}} + X_1(\boldsymbol{\theta})\mathbf{f}. \tag{4.20}$$

Equation (4.19) is equivalent to the identity

$$\begin{aligned} &\int_\Omega \langle h^{-1}(\operatorname{curl} s^{-1/2}\mathbf{v}_{\boldsymbol{\theta}} + i\boldsymbol{\theta} \times (s^{-1/2}\mathbf{f})), \operatorname{curl} s^{-1/2}\mathbf{w}\rangle \, d\mathbf{x} \\ &\qquad + \int_\Omega \nu(\operatorname{div} s^{1/2}\mathbf{v}_{\boldsymbol{\theta}} + i\boldsymbol{\theta} \cdot (s^{1/2}\mathbf{f}))(\operatorname{div} s^{1/2}\mathbf{w})^+ \, d\mathbf{x} = 0, \quad \mathbf{w} \in \mathfrak{d}. \end{aligned} \tag{4.21}$$

In accordance with the decomposition (2.10) with $\mathbf{k} = 0$, we write $\mathbf{w}$ as $\mathbf{w} = s^{1/2}\nabla\varphi + \mathbf{w}_0$, where $\operatorname{div} s^{1/2}\check{\mathbf{w}}_0 = 0$. Now we substitute $\mathbf{w}_0$ in place of $\mathbf{w}$ in (4.21). Then the second integral vanishes. Using the identity $\operatorname{curl} s^{-1/2}\mathbf{w} = \operatorname{curl} s^{-1/2}\mathbf{w}_0$, we obtain

$$\int_\Omega \langle h^{-1}(\operatorname{curl} s^{-1/2}\mathbf{v}_{\boldsymbol{\theta}} + i\boldsymbol{\theta} \times (s^{-1/2}\mathbf{f})), \operatorname{curl} s^{-1/2}\mathbf{w}\rangle \, d\mathbf{x} = 0, \quad \mathbf{w} \in \mathfrak{d}. \tag{4.22}$$

From (4.22) it follows that

$$-i \ \operatorname{curl} s^{-1/2}\mathbf{v}_{\boldsymbol{\theta}} + \boldsymbol{\theta} \times (s^{-1/2}\mathbf{f}) = h(\widehat{\mathbf{C}} + \nabla\Psi) \tag{4.23}$$

for some $\widehat{\mathbf{C}} \in \mathbb{C}^3$ and $\Psi \in \widetilde{H}^1(\Omega)$.

We multiply both parts of (4.23) by $\nabla\zeta$, where $\zeta \in \widetilde{H}^1(\Omega)$, and integrate over Ω. By (4.6), $\mathbf{f} = s^{1/2}(\mathbf{C} + \nabla\Phi_{\mathbf{C}})$. The integral of the left-hand side vanishes:

$$\int_\Omega \langle -i \ \operatorname{curl} s^{-1/2}\mathbf{v}_{\boldsymbol{\theta}} + \boldsymbol{\theta} \times (\mathbf{C} + \nabla\Phi_{\mathbf{C}}), \nabla\zeta\rangle \, d\mathbf{x} = 0$$

(we have used the identity $\operatorname{div}(\boldsymbol{\theta} \times \nabla\Phi) = 0$). Consequently,

$$\int_\Omega \langle h(\widehat{\mathbf{C}} + \nabla\Psi), \nabla\zeta\rangle \, d\mathbf{x} = 0, \quad \zeta \in \widetilde{H}^1(\Omega).$$

Thus, $\Psi = \Psi_{\widehat{\mathbf{C}}}$ is a weak periodic solution of the equation

$$\operatorname{div} h(\mathbf{x})(\widehat{\mathbf{C}} + \nabla\Psi_{\widehat{\mathbf{C}}}) = 0.$$

As in (4.12), we define the effective matrix h^0 for $h(\mathbf{x})$ by the relation

$$h^0\widehat{\mathbf{C}} = |\Omega|^{-1} \int_\Omega h(\mathbf{x})(\widehat{\mathbf{C}} + \nabla\Psi_{\widehat{\mathbf{C}}}) \, d\mathbf{x}, \quad \widehat{\mathbf{C}} \in \mathbb{C}^3. \tag{4.24}$$

On the other hand, integrating (4.23), we obtain

$$\int_\Omega h(\mathbf{x})(\widehat{\mathbf{C}}+\nabla\Psi_{\widehat{\mathbf{C}}}) \, d\mathbf{x} = \int_\Omega \Big(-i \operatorname{curl} s^{-1/2}\mathbf{v}_{\boldsymbol{\theta}} + \boldsymbol{\theta} \times (\mathbf{C} + \nabla\Phi_{\mathbf{C}})\Big) \, d\mathbf{x} = |\Omega|\boldsymbol{\theta}\times\mathbf{C}. \tag{4.25}$$

Comparing (4.24) and (4.25), we see that

$$\widehat{\mathbf{C}} = (h^0)^{-1}(\boldsymbol{\theta} \times \mathbf{C}). \tag{4.26}$$

Now we return to (4.21). By (4.22), we have

$$\int_\Omega \nu \left(\operatorname{div} s^{1/2}\mathbf{v}_{\boldsymbol{\theta}} + i\boldsymbol{\theta} \cdot (s^{1/2}\mathbf{f})\right) (\operatorname{div} s^{1/2}\mathbf{w})^+ \, d\mathbf{x} = 0, \quad \mathbf{w} \in \mathfrak{d}.$$

Therefore,

$$-i \operatorname{div} s^{1/2}\mathbf{v}_{\boldsymbol{\theta}} + \boldsymbol{\theta} \cdot s(\mathbf{C} + \nabla\Phi_{\mathbf{C}}) = \nu^{-1}\alpha \tag{4.27}$$

for some constant $\alpha \in \mathbb{C}$. Integrating (4.27) and using (4.12), we obtain

$$\alpha \int_\Omega \nu(\mathbf{x})^{-1} \, d\mathbf{x} = \int_\Omega \boldsymbol{\theta} \cdot s(\mathbf{C} + \nabla\Phi_{\mathbf{C}}) \, d\mathbf{x} = |\Omega|\boldsymbol{\theta} \cdot s^0\mathbf{C},$$

whence

$$\alpha = \underline{\nu}\boldsymbol{\theta} \cdot s^0\mathbf{C}. \tag{4.28}$$

Here

$$\underline{\nu}^{-1} := |\Omega|^{-1} \int_\Omega \nu(\mathbf{x})^{-1} \, d\mathbf{x}. \tag{4.29}$$

The number $\underline{\nu}$ is called the *effective coefficient* for $\nu(\mathbf{x})$.

By (4.2), (4.3), and (4.20), the operator $R(\boldsymbol{\theta})$ takes a vector $\mathbf{f} = s^{1/2}(\mathbf{C} + \nabla\Phi_{\mathbf{C}}) \in \mathfrak{N}$ to the element

$$R(\boldsymbol{\theta})\mathbf{f} = \begin{pmatrix} h^{-1/2}\left(-i \operatorname{curl} s^{-1/2}\mathbf{v}_{\boldsymbol{\theta}} + \boldsymbol{\theta} \times (\mathbf{C} + \nabla\Phi_{\mathbf{C}})\right) \\ \nu^{1/2}\left(-i \operatorname{div} s^{1/2}\mathbf{v}_{\boldsymbol{\theta}} + \boldsymbol{\theta} \cdot s(\mathbf{C} + \nabla\Phi_{\mathbf{C}})\right) \end{pmatrix}.$$

Now, we calculate the quadratic form of the germ $S(\boldsymbol{\theta}) = R(\boldsymbol{\theta})^*R(\boldsymbol{\theta})$. By (4.19) and (4.20), we have

$$(S(\boldsymbol{\theta})\mathbf{f}, \mathbf{f})_{\mathfrak{H}} = \|R(\boldsymbol{\theta})\mathbf{f}\|^2_{\mathfrak{H}_*} = (X_0\mathbf{v}_{\boldsymbol{\theta}} + X_1(\boldsymbol{\theta})\mathbf{f}, X_1(\boldsymbol{\theta})\mathbf{f})_{\mathfrak{H}_*}.$$

Consequently (see (4.2), (4.3), and (4.6)),

$$\begin{aligned}(S(\boldsymbol{\theta})\mathbf{f}, \mathbf{f})_{\mathfrak{H}} = &\int_\Omega \Big\langle h^{-1}\Big(- i \operatorname{curl} s^{-1/2}\mathbf{v}_{\boldsymbol{\theta}} + \boldsymbol{\theta} \times (\mathbf{C} + \nabla\Phi_{\mathbf{C}})\Big), \boldsymbol{\theta} \times (\mathbf{C} + \nabla\Phi_{\mathbf{C}})\Big\rangle \, d\mathbf{x} \\ &+ \int_\Omega \nu \left(-i \operatorname{div} s^{1/2}\mathbf{v}_{\boldsymbol{\theta}} + \boldsymbol{\theta} \cdot s(\mathbf{C} + \nabla\Phi_{\mathbf{C}})\right) \langle \boldsymbol{\theta}, s(\mathbf{C} + \nabla\Phi_{\mathbf{C}})\rangle \, d\mathbf{x}.\end{aligned}$$

Combining this with (4.23) and (4.27), we obtain

$$(S(\boldsymbol{\theta})\mathbf{f}, \mathbf{f})_{\mathfrak{H}} = \int_\Omega \Big(\langle \widehat{\mathbf{C}} + \nabla\Psi_{\widehat{\mathbf{C}}}, \boldsymbol{\theta} \times (\mathbf{C} + \nabla\Phi_{\mathbf{C}})\rangle + \alpha\langle \boldsymbol{\theta}, s(\mathbf{C} + \nabla\Phi_{\mathbf{C}})\rangle\Big) \, d\mathbf{x}. \tag{4.30}$$

Using the periodicity of the functions $\Phi_{\mathbf{C}}$ and $\Psi_{\widehat{\mathbf{C}}}$ and the identity $\operatorname{div}(\boldsymbol{\theta} \times \nabla\Phi) = 0$, we see that the following terms in (4.30) vanish:

$$\int_\Omega \left(\langle \nabla \Psi_{\widehat{\mathbf{C}}}, \boldsymbol{\theta} \times (\mathbf{C} + \nabla\Phi_{\mathbf{C}})\rangle + \langle \widehat{\mathbf{C}}, \boldsymbol{\theta} \times \nabla\Phi_{\mathbf{C}}\rangle \right) d\mathbf{x} = 0.$$

Therefore, by (4.12), relation (4.30) takes the form

$$(S(\boldsymbol{\theta})\mathbf{f}, \mathbf{f})_{\mathfrak{H}} = |\Omega| \left(\langle \widehat{\mathbf{C}}, \boldsymbol{\theta} \times \mathbf{C}\rangle + \alpha\langle \boldsymbol{\theta}, s^0\mathbf{C}\rangle \right).$$

Finally, we use (4.26) and (4.28) to obtain

$$(S(\boldsymbol{\theta})\mathbf{f}, \mathbf{f})_{\mathfrak{H}} = |\Omega| \Big(\langle (h^0)^{-1}(\boldsymbol{\theta} \times \mathbf{C}), (\boldsymbol{\theta} \times \mathbf{C})\rangle + \underline{\nu} |\langle s^0\boldsymbol{\theta}, \mathbf{C}\rangle|^2 \Big), \qquad \mathbf{f} = s^{1/2}(\mathbf{C} + \nabla\Phi_{\mathbf{C}}), \quad \mathbf{C} \in \mathbb{C}^3. \tag{4.31}$$

4.5. Relationship between $S(\boldsymbol{\theta})$ and $S^0(\boldsymbol{\theta})$. We shall write $S(\boldsymbol{\theta}) = S(\boldsymbol{\theta}; s, h, \nu)$, indicating the dependence of the germ on the coefficients of $\mathcal{L}$ (see (3.3)). For the operator $\mathcal{L}^0$ with constant *effective* coefficients s^0, h^0, $\underline{\nu}$, the germ $S^0(\boldsymbol{\theta}) := S(\boldsymbol{\theta}; s^0, h^0, \underline{\nu})$ acts in the space $\mathfrak{N}^0$ (see (4.13)). In this case, applying (4.31) and using the fact that the constant coefficients themselves play the role of the effective coefficients, we obtain

$$(S^0(\boldsymbol{\theta})\mathbf{f}^0, \mathbf{f}^0)_{\mathfrak{H}} = |\Omega| \Big(\langle (h^0)^{-1}(\boldsymbol{\theta} \times \mathbf{C}), (\boldsymbol{\theta} \times \mathbf{C})\rangle + \underline{\nu} |\langle s^0\boldsymbol{\theta}, \mathbf{C}\rangle|^2 \Big), \qquad \mathbf{f}^0 = (s^0)^{1/2}\mathbf{C}, \quad \mathbf{C} \in \mathbb{C}^3. \tag{4.32}$$

Comparing (4.31) and (4.32), we arrive at the formula

$$(S(\boldsymbol{\theta})\mathbf{f}, \mathbf{f})_{\mathfrak{H}} = (S^0(\boldsymbol{\theta})\mathbf{f}^0, \mathbf{f}^0)_{\mathfrak{H}},$$

where $\mathbf{f} \in \mathfrak{N}$ and $\mathbf{f}^0 \in \mathfrak{N}^0$ are related by the unitary transformation (4.14): $\mathbf{f}^0 = \mathcal{U}\mathbf{f}$. In other words,

$$S(\boldsymbol{\theta}) = \mathcal{U}^* S^0(\boldsymbol{\theta})\mathcal{U}. \tag{4.33}$$

Thus, we have proved the following theorem about the germ in question.

Theorem 4.2. *Let s^0, h^0, $\underline{\nu}$ be the constant effective coefficients defined by* (4.12), (4.24), *and* (4.29). *Let $\mathfrak{N}$ and $\mathfrak{N}^0$ be the spaces defined by* (4.6) *and* (4.13). *Suppose $S(\boldsymbol{\theta}) = S(\boldsymbol{\theta}; s, h, \nu) : \mathfrak{N} \to \mathfrak{N}$ is the spectral germ of the operator family $L(t; \boldsymbol{\theta})$ at $t = 0$, where $L(t; \boldsymbol{\theta})$ corresponds to $\mathcal{L} = \mathcal{L}(s, h, \nu)$ (see* (3.3)*). Let $S^0(\boldsymbol{\theta}) = S(\boldsymbol{\theta}; s^0, h^0, \underline{\nu}) : \mathfrak{N}^0 \to \mathfrak{N}^0$ be the spectral germ of the operator family $L^0(t; \boldsymbol{\theta})$ at $t = 0$, where $L^0(t; \boldsymbol{\theta})$ corresponds to the operator $\mathcal{L}^0 = \mathcal{L}(s^0, h^0, \underline{\nu})$ with constant effective coefficients. Let $\mathcal{U} : \mathfrak{N} \to \mathfrak{N}^0$ be the unitary operator defined by* (4.14). *Then the germs $S(\boldsymbol{\theta})$ and $S^0(\boldsymbol{\theta})$ are unitarily equivalent: relation* (4.33) *is fulfilled.*

The operator

$$\mathcal{L}^0 = (s^0)^{-1/2} \operatorname{curl}(h^0)^{-1} \operatorname{curl}(s^0)^{-1/2} - (s^0)^{1/2}\nabla\underline{\nu} \operatorname{div}(s^0)^{1/2} \tag{4.34}$$

is called the *effective operator* for $\mathcal{L}$.

Remark 4.3. Suppose that, from the outset, the coefficient $s(\mathbf{x})$ in the operator $\mathcal{L}$ is a constant matrix, i.e., $s(\mathbf{x}) = s^0$. Then $\mathfrak{N} = \mathfrak{N}^0$ and $S(\boldsymbol{\theta}; s^0, h, \nu) = S(\boldsymbol{\theta}; s^0, h^0, \underline{\nu})$. In this case, in the terminology of [BSu2, Chapter 1], the operator families $L(t; \boldsymbol{\theta})$ and $L^0(t; \boldsymbol{\theta})$ are threshold equivalent. This allows us to apply the general results of [BSu2] to the Maxwell operator in the case where one of two coefficients is constant (cf. [BSu2, Chapter 7]). If the matrix $s(\mathbf{x})$ is not constant, then the corresponding operator families are not threshold equivalent (in the same sense). However, the germs corresponding to the initial operator and to the effective operator are unitarily equivalent.

Remark 4.4. In what follows, we need estimates for the effective coefficients. It is known (see, e.g., [ZhKO, BeLP, BSu2]) that the effective matrix s^0 satisfies $\underline{s} \le s^0 \le \overline{s}$, where

$$\overline{s} := |\Omega|^{-1} \int_\Omega s(\mathbf{x})\, d\mathbf{x}, \quad \underline{s}^{-1} := |\Omega|^{-1} \int_\Omega (s(\mathbf{x}))^{-1}\, d\mathbf{x}.$$

These inequalities directly imply estimates for the matrix norm of the matrices s^0 and $(s^0)^{-1}$:

$$|s^0| \le \|s\|_{L_\infty}, \quad |(s^0)^{-1}| \le \|s^{-1}\|_{L_\infty}. \tag{4.35}$$

Similar estimates are valid for h^0.

4.6. Splitting the germs $S(\boldsymbol{\theta})$ and $S^0(\boldsymbol{\theta})$. Since the germs $S(\boldsymbol{\theta})$ and $S^0(\boldsymbol{\theta})$ are unitarily equivalent, first we analyze the simpler operator $S^0(\boldsymbol{\theta})$. We consider the following orthogonal decomposition of the 3-dimensional space (4.13), depending on the parameter $\boldsymbol{\theta} \in \mathbb{S}^2$:

$$\mathfrak{N}^0 = G^0_{\boldsymbol{\theta}} \oplus J^0_{\boldsymbol{\theta}}, \tag{4.36}$$

where

$$\begin{aligned} G^0_{\boldsymbol{\theta}} &= \{\mathbf{f}^0 = \beta \mathbf{f}^0_{\boldsymbol{\theta}}, \mathbf{f}^0_{\boldsymbol{\theta}} = (s^0)^{1/2}\boldsymbol{\theta}, \beta \in \mathbb{C}\}, \quad \dim G^0_{\boldsymbol{\theta}} = 1, \\ J^0_{\boldsymbol{\theta}} &= \{\mathbf{f}^0_\perp = (s^0)^{1/2}\mathbf{C}_\perp : \mathbf{C}_\perp \in \mathbb{C}^3, \langle s^0 \mathbf{C}_\perp, \boldsymbol{\theta}\rangle = 0\}, \quad \dim J^0_{\boldsymbol{\theta}} = 2. \end{aligned} \tag{4.37}$$

Let $P_{G^0_{\boldsymbol{\theta}}}$ and $P_{J^0_{\boldsymbol{\theta}}}$ denote the orthogonal projections in $\mathfrak{H}$ onto $G^0_{\boldsymbol{\theta}}$ and onto $J^0_{\boldsymbol{\theta}}$, respectively.

Obviously, the *decomposition* (4.36) *reduces the operator* $S^0(\boldsymbol{\theta})$. By (4.32), for $\mathbf{f}^0 = \beta \mathbf{f}^0_{\boldsymbol{\theta}} \in G^0_{\boldsymbol{\theta}}$ we have

$$(S^0(\boldsymbol{\theta})\mathbf{f}^0, \mathbf{f}^0)_{\mathfrak{H}} = |\Omega| \underline{\nu} \langle s^0\boldsymbol{\theta}, \boldsymbol{\theta}\rangle^2 |\beta|^2, \quad \|\mathbf{f}^0\|^2_{\mathfrak{H}} = |\Omega| \langle s^0 \boldsymbol{\theta}, \boldsymbol{\theta}\rangle |\beta|^2.$$

Hence, in the subspace $G^0_{\boldsymbol{\theta}}$ the germ $S^0(\boldsymbol{\theta})$ has one eigenvalue

$$\gamma_3(\boldsymbol{\theta}) = \underline{\nu}\langle s^0 \boldsymbol{\theta}, \boldsymbol{\theta}\rangle. \tag{4.38}$$

The corresponding eigenvector normalized in $\mathfrak{H}$ is equal (up to a constant factor) to

$$\mathbf{f}^0_3(\boldsymbol{\theta}) = |\Omega|^{-1/2} \langle s^0 \boldsymbol{\theta}, \boldsymbol{\theta}\rangle^{-1/2} \mathbf{f}^0_{\boldsymbol{\theta}}. \tag{4.39}$$

For $\mathbf{f}^0_\perp \in J^0_{\boldsymbol{\theta}}$, we have

$$(S^0(\boldsymbol{\theta})\mathbf{f}^0_\perp, \mathbf{f}^0_\perp)_{\mathfrak{H}} = |\Omega| \langle (h^0)^{-1}(\boldsymbol{\theta} \times \mathbf{C}_\perp), (\boldsymbol{\theta} \times \mathbf{C}_\perp)\rangle, \quad \|\mathbf{f}^0_\perp\|^2_{\mathfrak{H}} = |\Omega| \langle s^0 \mathbf{C}_\perp, \mathbf{C}_\perp\rangle,$$
$$\mathbf{C}_\perp \perp s^0 \boldsymbol{\theta}.$$

In the subspace $J^0_{\boldsymbol{\theta}}$, the germ $S^0(\boldsymbol{\theta})$ has two eigenvalues $\gamma_1(\boldsymbol{\theta})$ and $\gamma_2(\boldsymbol{\theta})$, which correspond to the algebraic problem dealing with the spectrum of the ratio

$$\frac{\langle (h^0)^{-1}(\boldsymbol{\theta} \times \mathbf{C}_\perp), (\boldsymbol{\theta} \times \mathbf{C}_\perp)\rangle}{\langle s^0 \mathbf{C}_\perp, \mathbf{C}_\perp\rangle}, \quad \mathbf{C}_\perp \perp s^0 \boldsymbol{\theta}. \tag{4.40}$$

Let $\mathbf{C}^{(l)}_\perp(\boldsymbol{\theta})$, $l = 1, 2$, be the corresponding eigenvectors normalized by the condition $|\Omega| \langle s^0 \mathbf{C}^{(l)}_\perp(\boldsymbol{\theta}), \mathbf{C}^{(l)}_\perp(\boldsymbol{\theta})\rangle = 1$. Then the normalized eigenvectors $\mathbf{f}^0_1(\boldsymbol{\theta})$, $\mathbf{f}^0_2(\boldsymbol{\theta})$ of the operator $S^0(\boldsymbol{\theta})$ corresponding to the eigenvalues $\gamma_1(\boldsymbol{\theta})$, $\gamma_2(\boldsymbol{\theta})$ are given by the formulas

$$\mathbf{f}^0_l(\boldsymbol{\theta}) = (s^0)^{1/2} \mathbf{C}^{(l)}_\perp(\boldsymbol{\theta}), \quad l = 1, 2. \tag{4.41}$$

If $\gamma_1(\boldsymbol{\theta}) \ne \gamma_2(\boldsymbol{\theta})$, then $\mathbf{f}^0_1(\boldsymbol{\theta})$, $\mathbf{f}^0_2(\boldsymbol{\theta})$ are defined by (4.41) up to phase factors. If $\gamma_1(\boldsymbol{\theta}) = \gamma_2(\boldsymbol{\theta})$, then $\mathbf{f}^0_1(\boldsymbol{\theta})$, $\mathbf{f}^0_2(\boldsymbol{\theta})$ form an arbitrary orthonormal basis in $J^0_{\boldsymbol{\theta}}$.

Under the unitary transformation $\mathcal{U}^{-1} : \mathfrak{N}^0 \to \mathfrak{N}$ (see (4.14)), the decomposition (4.36) turns into the following orthogonal decomposition of the space (4.6):

$$\mathfrak{N} = G_{\boldsymbol{\theta}} \oplus J_{\boldsymbol{\theta}}, \tag{4.42}$$

where

$$G_{\boldsymbol\theta} = \{\mathbf f = \beta \mathbf f_{\boldsymbol\theta},\ \mathbf f_{\boldsymbol\theta} := s^{1/2}(\boldsymbol\theta + \nabla\Phi_{\boldsymbol\theta}), \beta \in \mathbb C\}, \quad \dim G_{\boldsymbol\theta} = 1,$$
$$J_{\boldsymbol\theta} = \{\mathbf f_\perp = s^{1/2}(\mathbf C_\perp + \nabla\Phi_{\mathbf C_\perp}) : \mathbf C_\perp \in \mathbb C^3, \langle s^0 \mathbf C_\perp, \boldsymbol\theta\rangle = 0\}, \quad \dim J_{\boldsymbol\theta} = 2. \tag{4.43}$$

Let $P_{G_{\boldsymbol\theta}}$ and $P_{J_{\boldsymbol\theta}}$ denote the orthogonal projections in $\mathfrak H$ onto the subspaces $G_{\boldsymbol\theta}$ and $J_{\boldsymbol\theta}$, respectively. Obviously, the following "intertwining property" is fulfilled:

$$\mathcal U P_{G_{\boldsymbol\theta}} = P_{G^0_{\boldsymbol\theta}} \mathcal U P, \quad \mathcal U P_{J_{\boldsymbol\theta}} = P_{J^0_{\boldsymbol\theta}} \mathcal U P. \tag{4.44}$$

We recall that P denotes the orthogonal projection in $\mathfrak H$ onto $\mathfrak N$.

The decomposition (4.42) *reduces the operator* $S(\boldsymbol\theta)$. In the subspace $G_{\boldsymbol\theta}$, the germ $S(\boldsymbol\theta)$ has one eigenvalue $\gamma_3(\boldsymbol\theta)$ defined by (4.38). The corresponding normalized eigenvector (defined up to a phase factor) is equal to

$$\mathbf f_3(\boldsymbol\theta) = |\Omega|^{-1/2} \langle s^0 \boldsymbol\theta, \boldsymbol\theta\rangle^{-1/2} \mathbf f_{\boldsymbol\theta}. \tag{4.45}$$

In the subspace $J_{\boldsymbol\theta}$, the germ $S(\boldsymbol\theta)$ has two eigenvalues $\gamma_1(\boldsymbol\theta)$, $\gamma_2(\boldsymbol\theta)$, which correspond to the algebraic problem about the spectrum of the ratio (4.40). The corresponding normalized eigenvectors $\mathbf f_1(\boldsymbol\theta)$, $\mathbf f_2(\boldsymbol\theta)$ of the operator $S(\boldsymbol\theta)$ are given by the relations

$$\mathbf f_l(\boldsymbol\theta) = s^{1/2}(\mathbf C_\perp^{(l)}(\boldsymbol\theta) + \nabla\Phi_{\mathbf C_\perp^{(l)}(\boldsymbol\theta)}), \quad l = 1, 2. \tag{4.46}$$

If $\gamma_1(\boldsymbol\theta) \ne \gamma_2(\boldsymbol\theta)$, then $\mathbf f_1(\boldsymbol\theta)$, $\mathbf f_2(\boldsymbol\theta)$ are defined by (4.46) up to phase factors. If $\gamma_1(\boldsymbol\theta) = \gamma_2(\boldsymbol\theta)$, then $\mathbf f_1(\boldsymbol\theta)$, $\mathbf f_2(\boldsymbol\theta)$ form an arbitrary orthonormal basis in $J_{\boldsymbol\theta}$.

Thus, we have found the *coefficients* $\gamma_l(\boldsymbol\theta)$, $l = 1, 2, 3$, *in the decompositions* (4.15) *for the eigenvalues* $\lambda_l(t; \boldsymbol\theta)$ *of* $L(t; \boldsymbol\theta)$. *In the case where all eigenvalues* $\gamma_l(\boldsymbol\theta)$ *of the germ* $S(\boldsymbol\theta)$ *are distinct, formulas* (4.45) *and* (4.46) *define the "embryos"* $\widehat{\mathbf f}_l(\boldsymbol\theta)$ *of the eigenvectors* $\mathbf u_l(t; \boldsymbol\theta)$ *of* $L(t; \boldsymbol\theta)$ (see (4.16)): $\widehat{\mathbf f}_l(\boldsymbol\theta)$ coincides with $\mathbf f_l(\boldsymbol\theta)$ up to a phase factor, $l = 1, 2, 3$.

Remark 4.5. The *analytic* branches of eigenvalues and eigenvectors of the operator $L(t; \boldsymbol\theta)$, $t \in [0, t^0]$, can always be chosen in such a way that one of the eigenvectors (say, $\mathbf u_3(t; \boldsymbol\theta)$) belongs to the "gradient" subspace $G(s; t\boldsymbol\theta)$ for $t \ne 0$, which implies that the two remaining eigenvectors $\mathbf u_1(t; \boldsymbol\theta)$, $\mathbf u_2(t; \boldsymbol\theta)$ belong to the "solenoidal" subspace $J(s; t\boldsymbol\theta)$. Then, the coefficient $\gamma_3(\boldsymbol\theta)$ in decomposition (4.15) for $\lambda_3(t; \boldsymbol\theta)$ is the eigenvalue of the part of the germ $S(\boldsymbol\theta)$ in the subspace $G_{\boldsymbol\theta}$. The "embryo" $\widehat{\mathbf f}_3(\boldsymbol\theta)$ in the decomposition (4.16) for $\mathbf u_3(t; \boldsymbol\theta)$ coincides up to a phase factor with the eigenvector $\mathbf f_3(\boldsymbol\theta)$ of $S(\boldsymbol\theta)$ in the subspace $G_{\boldsymbol\theta}$ (see (4.45)). The coefficients $\gamma_1(\boldsymbol\theta)$, $\gamma_2(\boldsymbol\theta)$ in the decompositions (4.15) for $\lambda_1(t; \boldsymbol\theta)$, $\lambda_2(t; \boldsymbol\theta)$ are the eigenvalues of the part of $S(\boldsymbol\theta)$ in the subspace $J_{\boldsymbol\theta}$ and correspond to the algebraic problem about the spectrum of the ratio (4.40). The "embryos" $\widehat{\mathbf f}_1(\boldsymbol\theta)$, $\widehat{\mathbf f}_2(\boldsymbol\theta)$ in the decompositions (4.16) for $\mathbf u_1(t; \boldsymbol\theta)$, $\mathbf u_2(t; \boldsymbol\theta)$ belong to $J_{\boldsymbol\theta}$. If $\gamma_1(\boldsymbol\theta) \ne \gamma_2(\boldsymbol\theta)$, then $\widehat{\mathbf f}_1(\boldsymbol\theta)$, $\widehat{\mathbf f}_2(\boldsymbol\theta)$ coincide (up to phase factors) with the eigenvectors $\mathbf f_1(\boldsymbol\theta)$, $\mathbf f_2(\boldsymbol\theta)$ (defined by (4.46)) of $S(\boldsymbol\theta)$ in the subspace $J_{\boldsymbol\theta}$. In Remark 3.1 we already mentioned that for $t = 0$ all three eigenvectors belong to the "solenoidal" subspace $J(s; 0)$: $\mathbf u_l(0; \boldsymbol\theta) = \widehat{\mathbf f}_l(\boldsymbol\theta) \in \mathfrak N \subset J(s; 0)$, $l = 1, 2, 3$. Note that if $\gamma_1(\boldsymbol\theta) = \gamma_2(\boldsymbol\theta)$, then, in general, the knowledge of $S(\boldsymbol\theta)$ does not suffice for finding $\widehat{\mathbf f}_1(\boldsymbol\theta)$, $\widehat{\mathbf f}_2(\boldsymbol\theta)$. Moreover, these "embryos" may fail to be unique. The facts listed in this remark will not be used in what follows; therefore, we omit the detailed arguments.

If the analytic branches of eigenvalues are chosen as described in Remark 4.5, then, clearly, $\lambda_1(t; \boldsymbol\theta)$ and $\lambda_2(t; \boldsymbol\theta)$ partially coincide with $E_{1,J}(t\boldsymbol\theta)$, $E_{2,J}(t\boldsymbol\theta)$, and $\lambda_3(t; \boldsymbol\theta)$ coincides with $E_{1,G}(t\boldsymbol\theta)$.

Remark 4.5 gives a reason to call the operator $(S(\boldsymbol\theta))_{G_{\boldsymbol\theta}}$, which acts in the one-dimensional subspace $G_{\boldsymbol\theta}$, the *spectral germ of the family* $(L(t; \boldsymbol\theta))_{G(s; t\boldsymbol\theta)}$ *at* $t = 0$. Similarly, the operator $(S(\boldsymbol{\theta}))_{J_{\boldsymbol{\theta}}}$ acting in the two-dimensional subspace $J_{\boldsymbol{\theta}}$ is called the *spectral germ of the family* $(L(t;\boldsymbol{\theta}))_{J(s;t\boldsymbol{\theta})}$ *at* $t=0$. By (4.33) and (4.44), we have

$$S(\boldsymbol{\theta})P_{J_{\boldsymbol{\theta}}} = \mathcal{U}^* S^0(\boldsymbol{\theta}) P_{J^0_{\boldsymbol{\theta}}} \mathcal{U},$$
$$S(\boldsymbol{\theta})P_{G_{\boldsymbol{\theta}}} = \mathcal{U}^* S^0(\boldsymbol{\theta}) P_{G^0_{\boldsymbol{\theta}}} \mathcal{U}.$$

This means that the *operators* $(S(\boldsymbol{\theta}))_{J_{\boldsymbol{\theta}}}$ *and* $(S^0(\boldsymbol{\theta}))_{J^0_{\boldsymbol{\theta}}}$, *as well as the operators* $(S(\boldsymbol{\theta}))_{G_{\boldsymbol{\theta}}}$ *and* $(S^0_{\boldsymbol{\theta}})_{G^0_{\boldsymbol{\theta}}}$, *are unitarily equivalent.*

§5. Approximation of the projection onto the solenoidal subspace

5.1. In this section, our goal is to prove the following theorem.

Theorem 5.1. *Let* $\mathcal{P}(s;\mathbf{k})$, $\mathbf{k}=t\boldsymbol{\theta}$, *be the orthogonal projection in* $\mathfrak{H}$ *onto the subspace* $J(s;\mathbf{k})$ *defined by* (2.10). *Let* P *be the orthogonal projection in* $\mathfrak{H}$ *onto the* 3-*dimensional subspace* $\mathfrak{N}$ *defined by* (4.6), *and let* $P_{J_{\boldsymbol{\theta}}}$ *be the orthogonal projection in* $\mathfrak{H}$ *onto the* 2-*dimensional subspace* $J_{\boldsymbol{\theta}}$ *defined by* (4.43). *Let* r_0 *be the radius of the ball inscribed in the Brillouin zone* $\widetilde{\Omega}$. *Then*

$$\|\mathcal{P}(s;t\boldsymbol{\theta})P - P_{J_{\boldsymbol{\theta}}}\|_{\mathfrak{H}\to\mathfrak{H}} \le C_3 t, \quad t\in(0,r_0], \ \ \boldsymbol{\theta}\in\mathbb{S}^2. \tag{5.1}$$

The constant C_3 *depends only on* $\|s\|_{L_\infty}$, $\|s^{-1}\|_{L_\infty}$, *and* r_0.

An explicit expression for C_3 is given below in (5.36). Observe that, *a fortiori*, estimate (5.1) fails for $t=0$: since $\mathfrak{N}\subset J(s;0)$, the operator $\mathcal{P}(s;0)P=P$ is the projection onto a 3-dimensional subspace, and $P_{J_{\boldsymbol{\theta}}}$ is the projection onto a 2-dimensional subspace. This means that the operator $\mathcal{P}(s;\mathbf{k})P$ is discontinuous at $\mathbf{k}=0$, but its limit as $|\mathbf{k}|\to 0$ is equal to $P_{J_{\boldsymbol{\theta}}}$ and depends on the direction $\boldsymbol{\theta}$ of the vector $\mathbf{k}=t\boldsymbol{\theta}$.

Remark 5.2. By direct calculation, it is easy to show that, in the case of a constant matrix $s=s^0$, instead of estimate (5.1) we have the precise identity

$$\mathcal{P}(s^0;t\boldsymbol{\theta})P_0 = P_{J^0_{\boldsymbol{\theta}}}, \quad t>0. \tag{5.2}$$

Here P_0 is the orthogonal projection in $\mathfrak{H}$ onto $\mathfrak{N}^0$ (see (4.13)), and $P_{J^0_{\boldsymbol{\theta}}}$ is the orthogonal projection in $\mathfrak{H}$ onto $J^0_{\boldsymbol{\theta}}$ (see (4.37)).

5.2. Now, we proceed to the proof of Theorem 5.1. Since $\mathcal{P}(s;t\boldsymbol{\theta})P\mathbf{w} - P_{J_{\boldsymbol{\theta}}}\mathbf{w} = \mathcal{P}(s;t\boldsymbol{\theta})\mathbf{u} - P_{J_{\boldsymbol{\theta}}}\mathbf{u}$ for all $\mathbf{w}\in\mathfrak{H}$, where $\mathbf{u}=P\mathbf{w}\in\mathfrak{N}$, *for the proof of* (5.1) *it suffices to establish the estimate*

$$\|\mathcal{P}(s;t\boldsymbol{\theta})\mathbf{u} - P_{J_{\boldsymbol{\theta}}}\mathbf{u}\|_{\mathfrak{H}} \le C_3 t\|\mathbf{u}\|_{\mathfrak{H}}, \quad \mathbf{u}\in\mathfrak{N}, \ \ t\in(0,r_0], \ \ \boldsymbol{\theta}\in\mathbb{S}^2. \tag{5.3}$$

We start with two elementary statements.

Lemma 5.3. *Let* $\mathbf{u}=s^{1/2}(\mathbf{C}+\nabla\Phi_{\mathbf{C}})\in\mathfrak{N}$, *where* $\mathbf{C}\in\mathbb{C}^3$ *and* $\Phi_{\mathbf{C}}\in\widetilde{H}^1(\Omega)$ *is a solution of the equation*

$$\operatorname{div} s(\mathbf{C}+\nabla\Phi_{\mathbf{C}}) = 0. \tag{5.4}$$

Then

$$c_1|\mathbf{C}| \le \|\mathbf{u}\|_{\mathfrak{H}} \le c_2|\mathbf{C}|, \quad c_1=\|s^{-1}\|_{L_\infty}^{-1/2}|\Omega|^{1/2}, \ \ c_2=\|s\|_{L_\infty}^{1/2}|\Omega|^{1/2}. \tag{5.5}$$

Proof. Equation (5.4) implies that

$$\int_\Omega \langle s(\mathbf{C}+\nabla\Phi_{\mathbf{C}}), \nabla\Psi\rangle\, d\mathbf{x} = 0, \quad \Psi\in\widetilde{H}^1(\Omega).$$

Substituting $\Psi=\Phi_{\mathbf{C}}$, we obtain

$$\int_\Omega \langle s(\mathbf{C}+\nabla\Phi_{\mathbf{C}}), \mathbf{C}+\nabla\Phi_{\mathbf{C}}\rangle\, d\mathbf{x} = \int_\Omega \langle s(\mathbf{C}+\nabla\Phi_{\mathbf{C}}), \mathbf{C}\rangle\, d\mathbf{x}.$$

Consequently,

$$\|\mathbf{u}\|_{\mathfrak{H}} = \|s^{1/2}(\mathbf{C} + \nabla\Phi_{\mathbf{C}})\|_{\mathfrak{H}} \le \|s^{1/2}\mathbf{C}\|_{\mathfrak{H}} \le \|s\|_{L_\infty}^{1/2}|\Omega|^{1/2}|\mathbf{C}|,$$

which proves the upper estimate in (5.5).

The periodicity of $\Phi_{\mathbf{C}}$ implies that we have $\int_\Omega s^{-1/2}\mathbf{u}\,d\mathbf{x} = \mathbf{C}|\Omega|$, whence

$$|\mathbf{C}|\,|\Omega| \le \int_\Omega |s^{-1/2}\mathbf{u}|\,d\mathbf{x} \le \|s^{-1}\|_{L_\infty}^{1/2}|\Omega|^{1/2}\|\mathbf{u}\|_{\mathfrak{H}}.$$

This proves the lower estimate in (5.5). □

Lemma 5.4. *Suppose $\boldsymbol{\theta} \in \mathbb{S}^2$ and $\Phi_{\boldsymbol{\theta}} \in \widetilde{H}^1(\Omega)$ solve the equation*

$$\operatorname{div} s(\boldsymbol{\theta} + \nabla\Phi_{\boldsymbol{\theta}}) = 0 \tag{5.6}$$

and satisfy the condition $\int_\Omega \Phi_{\boldsymbol{\theta}}\,d\mathbf{x} = 0$. Then

$$\|\Phi_{\boldsymbol{\theta}}\|_{\mathfrak{H}} \le c_3 = (2r_0)^{-1}\left(\|s\|_{L_\infty}\|s^{-1}\|_{L_\infty}|\Omega|\right)^{1/2}. \tag{5.7}$$

Proof. Equation (5.6) implies that

$$\int_\Omega \langle s\nabla\Phi_{\boldsymbol{\theta}}, \nabla\Phi_{\boldsymbol{\theta}}\rangle\,d\mathbf{x} = -\int_\Omega \langle s\boldsymbol{\theta}, \nabla\Phi_{\boldsymbol{\theta}}\rangle\,d\mathbf{x}.$$

Consequently,

$$\|s^{1/2}\nabla\Phi_{\boldsymbol{\theta}}\|_{\mathfrak{H}} \le \|s^{1/2}\boldsymbol{\theta}\|_{\mathfrak{H}} \le \|s\|_{L_\infty}^{1/2}|\Omega|^{1/2},$$

whence

$$\|\nabla\Phi_{\boldsymbol{\theta}}\|_{\mathfrak{H}} \le \|s^{-1}\|_{L_\infty}^{1/2}\|s\|_{L_\infty}^{1/2}|\Omega|^{1/2}. \tag{5.8}$$

We write the Fourier series for $\Phi_{\boldsymbol{\theta}}$; since the Fourier coefficient with zero index vanishes, with the help of (1.1) we obtain

$$\|\nabla\Phi_{\boldsymbol{\theta}}\|_{\mathfrak{H}}^2 = \sum_{\mathbf{b}\in\widetilde{\Gamma}} |\mathbf{b}|^2|(\hat{\Phi}_{\boldsymbol{\theta}})_{\mathbf{b}}|^2 \ge \left(\min_{0\ne\mathbf{b}\in\widetilde{\Gamma}} |\mathbf{b}|^2\right)\sum_{\mathbf{b}\in\widetilde{\Gamma}} |(\hat{\Phi}_{\boldsymbol{\theta}})_{\mathbf{b}}|^2 = 4r_0^2\|\Phi_{\boldsymbol{\theta}}\|_{\mathfrak{H}}^2. \tag{5.9}$$

Relations (5.8) and (5.9) imply (5.7). □

5.3. So, let $\mathbf{u} = s^{1/2}(\mathbf{C} + \nabla\Phi_{\mathbf{C}}) \in \mathfrak{N}$, and let $\Phi_{\boldsymbol{\theta}}$ be the solution of equation (5.6) fixed by the condition $\int_\Omega \Phi_{\boldsymbol{\theta}}\,d\mathbf{x} = 0$, as in Lemma 5.4. The projection $P_{J_{\boldsymbol{\theta}}}$ acts as follows:

$$P_{J_{\boldsymbol{\theta}}}\mathbf{u} = \mathbf{u} - \beta s^{1/2}(\boldsymbol{\theta} + \nabla\Phi_{\boldsymbol{\theta}}), \quad \beta = \beta(\boldsymbol{\theta}, \mathbf{C}) = \frac{\langle s^0\mathbf{C}, \boldsymbol{\theta}\rangle}{\langle s^0\boldsymbol{\theta}, \boldsymbol{\theta}\rangle}. \tag{5.10}$$

Combining the obvious estimate $|\beta| \le |s^0|^{1/2}|(s^0)^{-1}|^{1/2}|\mathbf{C}|$ with (4.35), we see that

$$|\beta| \le \|s\|_{L_\infty}^{1/2}\|s^{-1}\|_{L_\infty}^{1/2}|\mathbf{C}|. \tag{5.11}$$

In accordance with Subsection 2.2 (see (2.11), (2.13)), the projection $\mathcal{P}(s;\mathbf{k})$ acts by the formula

$$(\mathcal{P}(s;\mathbf{k})\mathbf{u})(\mathbf{x}) = \mathbf{u}(\mathbf{x}) - (s(\mathbf{x}))^{1/2}\nabla_{\mathbf{k}}\phi(\mathbf{x};\mathbf{k}), \tag{5.12}$$

where $\phi(\cdot;\mathbf{k}) \in \widetilde{H}^1(\Omega)$ is the weak solution of the equation

$$\operatorname{div}_{\mathbf{k}} s(\mathbf{x})\nabla_{\mathbf{k}}\phi = \operatorname{div}_{\mathbf{k}}(s(\mathbf{x}))^{1/2}\mathbf{u}.$$

Plugging $\mathbf{u} = s^{1/2}(\mathbf{C} + \nabla\Phi_{\mathbf{C}})$ and using (5.4), we conclude that ϕ satisfies the equation

$$\operatorname{div}_{\mathbf{k}} s(\mathbf{x})\nabla_{\mathbf{k}}\phi = i\mathbf{k}\cdot(s(\mathbf{x})(\mathbf{C} + \nabla\Phi_{\mathbf{C}})). \tag{5.13}$$

Then (5.10) and (5.12) imply that

$$\mathcal{P}(s;\mathbf{k})\mathbf{u} - P_{J_{\boldsymbol{\theta}}}\mathbf{u} = \beta s^{1/2}(\boldsymbol{\theta} + \nabla\Phi_{\boldsymbol{\theta}}) - s^{1/2}\nabla_{\mathbf{k}}\phi. \tag{5.14}$$

The solution $\phi(\mathbf{x};\mathbf{k})$ of equation (5.13) will be sought in the form

$$\phi(\mathbf{x};\mathbf{k}) = -i\beta t^{-1} + \beta\Phi_{\boldsymbol{\theta}}(\mathbf{x}) + t\psi(\mathbf{x};\mathbf{k}),$$

where $\psi(\cdot;\mathbf{k}) \in \widetilde{H}^1(\Omega)$ is a new unknown function. Then

$$\nabla_{\mathbf{k}}\phi = \beta\nabla\Phi_{\boldsymbol{\theta}} + t\nabla_{\mathbf{k}}\psi + it\boldsymbol{\theta}\beta(-it^{-1} + \Phi_{\boldsymbol{\theta}}) = \beta(\boldsymbol{\theta} + \nabla\Phi_{\boldsymbol{\theta}}) + it\boldsymbol{\theta}\beta\Phi_{\boldsymbol{\theta}} + t\nabla_{\mathbf{k}}\psi. \tag{5.15}$$

Substituting (5.15) in (5.14), we obtain

$$\mathcal{P}(s;\mathbf{k})\mathbf{u} - P_{J_{\boldsymbol{\theta}}}\mathbf{u} = -ts^{1/2}(i\boldsymbol{\theta}\beta\Phi_{\boldsymbol{\theta}} + \nabla_{\mathbf{k}}\psi). \tag{5.16}$$

Relations (5.5), (5.7), and (5.11) yield the estimate

$$\|s^{1/2}\boldsymbol{\theta}\beta\Phi_{\boldsymbol{\theta}}\|_{\mathfrak{H}} \le c_4\|\mathbf{u}\|_{\mathfrak{H}}, \quad c_4 = (2r_0)^{-1}\|s\|_{L_\infty}^{3/2}\|s^{-1}\|_{L_\infty}^{3/2}. \tag{5.17}$$

Relations (5.16) and (5.17) show that *for the proof of* (5.3) *it suffices to prove the estimate*

$$\|s^{1/2}\nabla_{\mathbf{k}}\psi\|_{\mathfrak{H}} \le C_4\|\mathbf{u}\|_{\mathfrak{H}}. \tag{5.18}$$

Then inequality (5.3) *follows with the constant* $C_3 = c_4 + C_4$.

5.4. In order to deduce an equation for ψ, we substitute (5.15) in (5.13). Then, by (5.6), the left-hand side of (5.13) takes the form

$$\begin{aligned}&\operatorname{div}_{\mathbf{k}} s\left(\beta(\boldsymbol{\theta} + \nabla\Phi_{\boldsymbol{\theta}}) + it\boldsymbol{\theta}\beta\Phi_{\boldsymbol{\theta}} + t\nabla_{\mathbf{k}}\psi\right)\\ &\quad = t\left(i\beta\langle s(\boldsymbol{\theta} + \nabla\Phi_{\boldsymbol{\theta}}), \boldsymbol{\theta}\rangle + i\beta\operatorname{div}_{\mathbf{k}}(s\boldsymbol{\theta}\Phi_{\boldsymbol{\theta}}) + \operatorname{div}_{\mathbf{k}} s\nabla_{\mathbf{k}}\psi\right).\end{aligned}$$

The right-hand side of (5.13) can be written as $ti\langle s(\mathbf{C} + \nabla\Phi_{\mathbf{C}}), \boldsymbol{\theta}\rangle$. Consequently, ψ satisfies the equation

$$\operatorname{div}_{\mathbf{k}} s\nabla_{\mathbf{k}}\psi = i\langle s(\widetilde{\mathbf{C}} + \nabla\Phi_{\widetilde{\mathbf{C}}}), \boldsymbol{\theta}\rangle - i\beta\operatorname{div}_{\mathbf{k}}(s\boldsymbol{\theta}\Phi_{\boldsymbol{\theta}}), \tag{5.19}$$

where

$$\widetilde{\mathbf{C}} := \mathbf{C} - \beta\boldsymbol{\theta}. \tag{5.20}$$

We seek the solution ψ in the form

$$\psi = \psi_0 + t\psi_1, \tag{5.21}$$

where $\psi_1(\cdot;\mathbf{k}) \in \widetilde{H}^1(\Omega)$ is a new unknown function, and $\psi_0(\cdot;\mathbf{k}) \in \widetilde{H}^1(\Omega)$ satisfies the equation

$$\operatorname{div} s\nabla\psi_0 = i\langle s(\widetilde{\mathbf{C}} + \nabla\Phi_{\widetilde{\mathbf{C}}}), \boldsymbol{\theta}\rangle - i\beta\operatorname{div}(s\boldsymbol{\theta}\Phi_{\boldsymbol{\theta}}). \tag{5.22}$$

The solvability condition for (5.22) is that the right-hand side must be orthogonal in $\mathfrak{H}$ to the constants. This condition is satisfied: by (4.12), (5.10), and (5.20), the integral of the right-hand side vanishes:

$$\int_\Omega i\langle s(\widetilde{\mathbf{C}} + \nabla\Phi_{\widetilde{\mathbf{C}}}), \boldsymbol{\theta}\rangle\, d\mathbf{x} = i|\Omega|\langle s^0\widetilde{\mathbf{C}}, \boldsymbol{\theta}\rangle = i|\Omega|(\langle s^0\mathbf{C}, \boldsymbol{\theta}\rangle - \beta\langle s^0\boldsymbol{\theta}, \boldsymbol{\theta}\rangle) = 0.$$

Equation (5.22) is solvable, and its solution ψ_0 is determined uniquely up to a constant summand. We fix ψ_0 by the condition $\int_\Omega \psi_0\, d\mathbf{x} = 0$.

Lemma 5.5. *Suppose* $\psi_0(\cdot;\mathbf{k}) \in \widetilde{H}^1(\Omega)$ *solves equation* (5.22) *and satisfies the condition* $\int_\Omega \psi_0\, d\mathbf{x} = 0$. *Then*

$$\|s^{1/2}\nabla_{\mathbf{k}}\psi_0\|_{\mathfrak{H}} \le c_5\|\mathbf{u}\|_{\mathfrak{H}}, \quad |\mathbf{k}| \le r_0, \tag{5.23}$$

where c_5 *is the constant defined below in* (5.30).

Proof. Equation (5.22) implies the identity

$$\int_\Omega \langle s(\mathbf{x})\nabla\psi_0, \nabla\psi_0\rangle\, d\mathbf{x} = -i\int_\Omega \Big(\langle s(\mathbf{x})(\widetilde{\mathbf{C}} + \nabla\Phi_{\widetilde{\mathbf{C}}}), \boldsymbol{\theta}\rangle(\psi_0)^+ + \beta\Phi_{\boldsymbol{\theta}}\langle s(\mathbf{x})\boldsymbol{\theta}, \nabla\psi_0\rangle\Big)\, d\mathbf{x}.$$

It follows that

$$\|s^{1/2}\nabla\psi_0\|^2_{\mathfrak{H}} \le \|s(\widetilde{\mathbf{C}} + \nabla\Phi_{\widetilde{\mathbf{C}}})\|_{\mathfrak{H}}\|\psi_0\|_{\mathfrak{H}} + |\beta|\|\Phi_{\boldsymbol{\theta}} s^{1/2}\boldsymbol{\theta}\|_{\mathfrak{H}}\|s^{1/2}\nabla\psi_0\|_{\mathfrak{H}}. \tag{5.24}$$

By Lemma 5.3,

$$\|s(\widetilde{\mathbf{C}} + \nabla\Phi_{\widetilde{\mathbf{C}}})\|_{\mathfrak{H}} \le \|s\|^{1/2}_{L_\infty}\|s^{1/2}(\widetilde{\mathbf{C}} + \nabla\Phi_{\widetilde{\mathbf{C}}})\|_{\mathfrak{H}} \le c_2\|s\|^{1/2}_{L_\infty}|\widetilde{\mathbf{C}}|. \tag{5.25}$$

Next, from (5.10) and (5.20) it is seen that $|(s^0)^{1/2}\widetilde{\mathbf{C}}| \le |(s^0)^{1/2}\mathbf{C}|$. Hence, by (4.35) and (5.5) we have

$$|\widetilde{\mathbf{C}}| \le |s^0|^{1/2}|(s^0)^{-1}|^{1/2}|\mathbf{C}| \le c_1^{-1}\|s\|^{1/2}_{L_\infty}\|s^{-1}\|^{1/2}_{L_\infty}\|\mathbf{u}\|_{\mathfrak{H}}. \tag{5.26}$$

By analogy with (5.9),

$$\|\psi_0\|_{\mathfrak{H}} \le (2r_0)^{-1}\|\nabla\psi_0\|_{\mathfrak{H}} \le (2r_0)^{-1}\|s^{-1}\|^{1/2}_{L_\infty}\|s^{1/2}\nabla\psi_0\|_{\mathfrak{H}}. \tag{5.27}$$

As a consequence of (5.17) and (5.24)–(5.27), we obtain

$$\|s^{1/2}\nabla\psi_0\|_{\mathfrak{H}} \le 2c_4\|\mathbf{u}\|_{\mathfrak{H}}. \tag{5.28}$$

Finally, (5.27) and (5.28) yield the estimate

$$\|s^{1/2}\mathbf{k}\psi_0\|_{\mathfrak{H}} \le t\|s\|^{1/2}_{L_\infty}\|\psi_0\|_{\mathfrak{H}} \le c_4\|s\|^{1/2}_{L_\infty}\|s^{-1}\|^{1/2}_{L_\infty}\|\mathbf{u}\|_{\mathfrak{H}}, \quad t \in [0, r_0]. \tag{5.29}$$

Relations (5.28) and (5.29) directly imply (5.23) with the constant

$$c_5 = \Big(2 + \|s\|^{1/2}_{L_\infty}\|s^{-1}\|^{1/2}_{L_\infty}\Big)\, c_4. \qquad \square \tag{5.30}$$

5.5. In order to deduce an equation for ψ_1, we substitute (5.21) in (5.19) and use (5.22). After cancelling out the factor t, we obtain

$$-\operatorname{div}_{\mathbf{k}} s\nabla_{\mathbf{k}}\psi_1 = i\operatorname{div}_{\mathbf{k}}(s\boldsymbol{\theta}\psi_0) + i\langle s\nabla\psi_0, \boldsymbol{\theta}\rangle - \beta\Phi_{\boldsymbol{\theta}}\langle s\boldsymbol{\theta}, \boldsymbol{\theta}\rangle. \tag{5.31}$$

It remains to estimate the term $t\|s^{1/2}\nabla_{\mathbf{k}}\psi_1\|_{\mathfrak{H}}$.

Lemma 5.6. *Let $\psi_1(\cdot;\mathbf{k}) \in \widetilde{H}^1(\Omega)$ be the solution of* (5.31). *Then*

$$t\|s^{1/2}\nabla_{\mathbf{k}}\psi_1\|_{\mathfrak{H}} \le c_6\|\mathbf{u}\|_{\mathfrak{H}}, \quad |\mathbf{k}| \le r_0, \tag{5.32}$$

where c_6 is the constant defined below in (5.33), (5.35).

Proof. Equation (5.31) implies that

$$\int_\Omega \langle s\nabla_{\mathbf{k}}\psi_1, \nabla_{\mathbf{k}}\psi_1\rangle\, d\mathbf{x} = \int_\Omega \Big(-i\psi_0\langle s\boldsymbol{\theta}, \nabla_{\mathbf{k}}\psi_1\rangle + i\langle s\nabla\psi_0, \boldsymbol{\theta}\rangle(\psi_1)^+ - \beta\Phi_{\boldsymbol{\theta}}\langle s\boldsymbol{\theta}, \boldsymbol{\theta}\rangle(\psi_1)^+\Big)\, d\mathbf{x}.$$

Consequently,

$$\|s^{1/2}\nabla_{\mathbf{k}}\psi_1\|^2_{\mathfrak{H}} \le \|s^{1/2}\nabla_{\mathbf{k}}\psi_1\|_{\mathfrak{H}}\|s\|^{1/2}_{L_\infty}\|\psi_0\|_{\mathfrak{H}} + \Big(\|s^{1/2}\nabla\psi_0\|_{\mathfrak{H}}\|s\|^{1/2}_{L_\infty} + |\beta|\|\Phi_{\boldsymbol{\theta}}\|_{\mathfrak{H}}\|s\|_{L_\infty}\Big)\|\psi_1\|_{\mathfrak{H}}.$$

Applying (5.5), (5.7), (5.11), (5.17), (5.27), and (5.28), we arrive at the estimate

$$\begin{gathered}\|s^{1/2}\nabla_{\mathbf{k}}\psi_1\|_{\mathfrak{H}} \le \left(c_7 + c_8\frac{\|\psi_1\|_{\mathfrak{H}}}{\|s^{1/2}\nabla_{\mathbf{k}}\psi_1\|_{\mathfrak{H}}}\right)\|\mathbf{u}\|_{\mathfrak{H}},\\ c_7 = (2r_0^2)^{-1}\|s\|^2_{L_\infty}\|s^{-1}\|^2_{L_\infty}, \quad c_8 = 3(2r_0)^{-1}\|s\|^2_{L_\infty}\|s^{-1}\|^{3/2}_{L_\infty}.\end{gathered} \tag{5.33}$$

Using the Fourier series for ψ_1 and assuming that $\mathbf{k} \in \operatorname{clos} \widetilde{\Omega}$, we obtain

$$\|\nabla_{\mathbf{k}} \psi_1\|^2_{\mathfrak{H}} = \sum_{\mathbf{b} \in \widetilde{\Gamma}} |\mathbf{b} + \mathbf{k}|^2 |(\hat{\psi}_1)_{\mathbf{b}}|^2 \geq |\mathbf{k}|^2 \sum_{\mathbf{b} \in \widetilde{\Gamma}} |(\hat{\psi}_1)_{\mathbf{b}}|^2 = t^2 \|\psi_1\|^2_{\mathfrak{H}}.$$

Thus,

$$\frac{\|\psi_1\|_{\mathfrak{H}}}{\|s^{1/2} \nabla_{\mathbf{k}} \psi_1\|_{\mathfrak{H}}} \leq t^{-1} \|s^{-1}\|^{1/2}_{L_\infty}. \tag{5.34}$$

Finally, (5.33) and (5.34) imply that

$$\|s^{1/2} \nabla_{\mathbf{k}} \psi_1\|_{\mathfrak{H}} \leq \left(c_7 + t^{-1} c_8 \|s^{-1}\|^{1/2}_{L_\infty} \right) \|\mathbf{u}\|_{\mathfrak{H}},$$

whence (5.32) follows with the constant

$$c_6 = c_7 r_0 + c_8 \|s^{-1}\|^{1/2}_{L_\infty}. \qquad \square \tag{5.35}$$

Relations (5.21), (5.23), and (5.32) yield estimate (5.18) with the constant $C_4 = c_5 + c_6$. *This completes the proof of Theorem* 5.1. We have proved inequality (5.1) with the constant

$$\begin{aligned} C_3 &= c_4 + C_4 = c_4 + c_5 + c_6 \\ &= (2r_0)^{-1} \left(5\|s\|^2_{L_\infty} \|s^{-1}\|^2_{L_\infty} + 3\|s\|^{3/2}_{L_\infty} \|s^{-1}\|^{3/2}_{L_\infty} \right) \end{aligned} \tag{5.36}$$

(we have used expressions (5.17), (5.30), (5.33), and (5.35) for the constants).

§6. Approximations for the resolvent of $\mathcal{L}(\mathbf{k})$ and its solenoidal part

6.1. Approximation for the operator $(L(t;\boldsymbol{\theta}) + \varepsilon^2 I)^{-1}$. We are going to apply [BSu2, Theorem 1.5.5]; this theorem concerns approximation of the resolvent $(L(t;\boldsymbol{\theta}) + \varepsilon^2 I)^{-1}$ for small ε in terms of the germ $S(\boldsymbol{\theta})$. Estimates (4.17) for the eigenvalues $\lambda_l(t;\boldsymbol{\theta})$, $l = 1, 2, 3$, show that the conditions of this theorem are satisfied. In our case, Theorem 1.5.5 of [BSu2] directly implies the following result.

Theorem 6.1. *Let P be the orthogonal projection in $\mathfrak{H}$ onto the subspace $\mathfrak{N}$ defined by* (4.6), *and let $S(\boldsymbol{\theta})$ be the spectral germ of the family $L(t;\boldsymbol{\theta})$ at $t = 0$* (*see* (4.31)). *If t^0 is the number given by* (4.10), *then for the resolvent*

$$R(t;\boldsymbol{\theta};\varepsilon) := (L(t;\boldsymbol{\theta}) + \varepsilon^2 I)^{-1} \tag{6.1}$$

we have

$$\|R(t;\boldsymbol{\theta};\varepsilon) - (t^2 S(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}})^{-1} P\|_{\mathfrak{H} \to \mathfrak{H}} \leq \mathcal{C}_1 \varepsilon^{-1}, \quad 0 < \varepsilon \leq 1, \;\; t \in [0, t^0]. \tag{6.2}$$

The constant $\mathcal{C}_1$ depends on c_*, δ, and t^0 (*see* (3.16), (3.28), (4.9), *and* (4.10)), *i.e., ultimately, it depends on the radius r_0 of the ball inscribed in $\widetilde{\Omega}$ and on the L_∞-norms of the coefficients s, s^{-1}, h, h^{-1}, ν, ν^{-1}.*

Observe that the norm of each of the two terms in (6.2) is of order $O(\varepsilon^{-2})$, while the norm of the difference is of order of ε^{-1}. Therefore, we may treat relation (6.2) as an approximation for the resolvent (6.1). In [BSu2, §1.5], the following explicit expression for the constant $\mathcal{C}_1$ was given:

$$\mathcal{C}_1 = \beta_1 c_*^{-1/2} (t^0)^{-1} + \beta_2 \delta c_*^{-3/2} (t^0)^{-3} + (3\delta)^{-1}, \tag{6.3}$$

where β_1 and β_2 are absolute constants.

Now we consider the effective operator $L^0(t;\boldsymbol{\theta})$ with the coefficients s^0, h^0, $\underline{\nu}$. We note that (cf. (4.35)) the L_∞-norms of the effective coefficients s^0, $(s^0)^{-1}$, h^0, $(h^0)^{-1}$, $\underline{\nu}$, $\underline{\nu}^{-1}$ do not exceed the corresponding norms of the initial coefficients s, s^{-1}, h, h^{-1}, ν, ν^{-1}. It follows that if $L(t;\boldsymbol{\theta})$ is replaced by $L^0(t;\boldsymbol{\theta})$, then the constants c_* and δ (see (3.28), (4.9)) may only become larger. The constants t^0 and $t^0\delta^{-1/2}$ (see (4.10), (3.16), (2.9)) also may only become larger. Therefore, the constant $\mathcal{C}_1$ (see (6.3)) may only become smaller. This allows us to *assume that the constants t^0 and $\mathcal{C}_1$ for $L^0(t;\boldsymbol{\theta})$ are the same as for $L(t;\boldsymbol{\theta})$*. Applying Theorem 6.1 to the operator $L^0(t;\boldsymbol{\theta})$, for the resolvent

$$R^0(t;\boldsymbol{\theta};\varepsilon) := (L^0(t;\boldsymbol{\theta}) + \varepsilon^2 I)^{-1} \tag{6.4}$$

we obtain

$$\|R^0(t;\boldsymbol{\theta};\varepsilon) - (t^2 S^0(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}^0})^{-1} P_0\|_{\mathfrak{H}\to\mathfrak{H}} \leq \mathcal{C}_1 \varepsilon^{-1}, \quad 0 < \varepsilon \leq 1, \ \ t \in [0, t^0]. \tag{6.5}$$

Here P_0 is the orthogonal projection in $\mathfrak{H}$ onto the subspace $\mathfrak{N}^0$ defined by (4.13), and $S^0(\boldsymbol{\theta})$ is the germ of the family $L^0(t;\boldsymbol{\theta})$.

By (4.33),

$$\begin{aligned}(t^2 S(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}})^{-1} P &= (t^2 \mathcal{U}^* S^0(\boldsymbol{\theta}) \mathcal{U} + \varepsilon^2 \mathcal{U}^* \mathcal{U})^{-1} P \\ &= \mathcal{U}^* (t^2 S^0(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}^0})^{-1} \mathcal{U} P = (\mathcal{U}P)^* (t^2 S^0(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}^0})^{-1} P_0 (\mathcal{U}P).\end{aligned} \tag{6.6}$$

Recall that the unitary "identification of kernels" operator $\mathcal{U} : \mathfrak{N} \to \mathfrak{N}^0$ is defined by (4.14). From (6.2) and (6.6) we obtain

$$\begin{aligned}&\|R(t;\boldsymbol{\theta};\varepsilon) - (\mathcal{U}P)^* (t^2 S^0(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}^0})^{-1} P_0 (\mathcal{U}P)\|_{\mathfrak{H}\to\mathfrak{H}} \leq \mathcal{C}_1 \varepsilon^{-1}, \\ &\qquad\qquad 0 < \varepsilon \leq 1, \quad t \in [0, t^0].\end{aligned} \tag{6.7}$$

On the other hand, multiplying the operators under the norm in (6.5) by $\mathcal{U}P$ from the right and by $(\mathcal{U}P)^*$ from the left, and using the fact that the norms of these factors do not exceed unity, we arrive at the inequality

$$\begin{aligned}&\|(\mathcal{U}P)^* R^0(t;\boldsymbol{\theta};\varepsilon)(\mathcal{U}P) - (\mathcal{U}P)^* (t^2 S^0(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}^0})^{-1} P_0 (\mathcal{U}P)\|_{\mathfrak{H}\to\mathfrak{H}} \leq \mathcal{C}_1 \varepsilon^{-1}, \\ &\qquad\qquad 0 < \varepsilon \leq 1, \quad t \in [0, t^0].\end{aligned} \tag{6.8}$$

Comparing (6.7) and (6.8), we obtain the following result.

Theorem 6.2. *Suppose that the conditions of Theorem* 6.1 *are satisfied. Let $L^0(t;\boldsymbol{\theta})$ be the family corresponding to the effective operator $\mathcal{L}^0$ defined by* (4.34), *and that the resolvent $R^0(t;\boldsymbol{\theta};\varepsilon)$ is defined by* (6.4). *Let $\mathcal{U} : \mathfrak{N} \to \mathfrak{N}^0$ be the unitary operator defined by* (4.14). *Then*

$$\begin{aligned}&\|R(t;\boldsymbol{\theta};\varepsilon) - (\mathcal{U}P)^* R^0(t;\boldsymbol{\theta};\varepsilon)(\mathcal{U}P)\|_{\mathfrak{H}\to\mathfrak{H}} \leq 2\mathcal{C}_1 \varepsilon^{-1}, \\ &\qquad\qquad 0 < \varepsilon \leq 1, \quad t \in [0, t^0].\end{aligned} \tag{6.9}$$

6.2. Approximation of the solenoidal part of the resolvent. Our next goal is to approximate the "solenoidal" part of the resolvent (6.1), i.e., the operator

$$R_J(t;\boldsymbol{\theta};\varepsilon) := \mathcal{P}(s;t\boldsymbol{\theta}) R(t;\boldsymbol{\theta};\varepsilon) = \big((L(t;\boldsymbol{\theta}))_{J(s;t\boldsymbol{\theta})} + \varepsilon^2 I_{J(s;t\boldsymbol{\theta})}\big)^{-1} \oplus \mathbf{0}_{G(s;t\boldsymbol{\theta})}. \tag{6.10}$$

We multiply the operators under the norm in (6.2) by the projection $\mathcal{P}(s;t\boldsymbol{\theta})$ from the left. Then

$$\begin{aligned}&\|R_J(t;\boldsymbol{\theta};\varepsilon) - \mathcal{P}(s;t\boldsymbol{\theta}) P (t^2 S(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}})^{-1} P\|_{\mathfrak{H}\to\mathfrak{H}} \leq \mathcal{C}_1 \varepsilon^{-1}, \\ &\qquad\qquad 0 < \varepsilon \leq 1, \quad t \in (0, t^0].\end{aligned} \tag{6.11}$$

Next, using estimates (4.18) for the eigenvalues of the germ $S(\boldsymbol{\theta})$, we obtain

$$\|(t^2 S(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}})^{-1} P\|_{\mathfrak{H}\to\mathfrak{H}} \leq (c_* t^2 + \varepsilon^2)^{-1}, \quad \varepsilon > 0, \ \ t \geq 0. \tag{6.12}$$

From (6.12) and (5.1) it follows that

$$\begin{aligned}&\|(\mathcal{P}(s;t\boldsymbol{\theta}) P - P_{J_{\boldsymbol{\theta}}})(t^2 S(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}})^{-1} P\|_{\mathfrak{H}\to\mathfrak{H}} \\ &\qquad \leq C_3 t (c_* t^2 + \varepsilon^2)^{-1} \leq C_3 2^{-1} c_*^{-1/2} \varepsilon^{-1}, \quad \varepsilon > 0, \ \ 0 < t \leq r_0.\end{aligned} \tag{6.13}$$

Inequality (6.13) allows us to replace $\mathcal{P}(s;t\boldsymbol{\theta})P$ by $P_{J_{\boldsymbol{\theta}}}$ in (6.11):

$$\begin{aligned} &\|R_J(t;\boldsymbol{\theta};\varepsilon) - (t^2 S(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}})^{-1} P_{J_{\boldsymbol{\theta}}}\|_{\mathfrak{H}\to\mathfrak{H}} \le \mathcal{C}_2 \varepsilon^{-1}, \\ &\quad 0 < \varepsilon \le 1, \quad 0 < t \le t^0, \quad \mathcal{C}_2 = \mathcal{C}_1 + C_3 2^{-1} c_*^{-1/2}. \end{aligned} \tag{6.14}$$

By analogy with (6.10), we introduce the notation

$$\begin{aligned} R_J^0(t;\boldsymbol{\theta};\varepsilon) &:= \mathcal{P}(s^0;t\boldsymbol{\theta}) R^0(t;\boldsymbol{\theta};\varepsilon) \\ &= \big((L^0(t;\boldsymbol{\theta}))_{J(s^0;t\boldsymbol{\theta})} + \varepsilon^2 I_{J(s^0;t\boldsymbol{\theta})} \big)^{-1} \oplus \mathbf{0}_{G(s^0;t\boldsymbol{\theta})}. \end{aligned} \tag{6.15}$$

Multiplying the operators under the norm in (6.5) by $\mathcal{P}(s^0;t\boldsymbol{\theta})$ from the left, and recalling (5.2), we obtain

$$\|R_J^0(t;\boldsymbol{\theta};\varepsilon) - (t^2 S^0(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}^0})^{-1} P_{J^0_{\boldsymbol{\theta}}}\|_{\mathfrak{H}\to\mathfrak{H}} \le \mathcal{C}_1 \varepsilon^{-1}, \quad 0 < \varepsilon \le 1, \ 0 < t \le t^0. \tag{6.16}$$

By (4.33) and (4.44),

$$\begin{aligned} &(t^2 S(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}})^{-1} P_{J_{\boldsymbol{\theta}}} \\ &\quad = \mathcal{U}^* (t^2 S^0(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}^0})^{-1} \mathcal{U} P_{J_{\boldsymbol{\theta}}} = (\mathcal{U}P)^* (t^2 S^0(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}^0})^{-1} P_{J^0_{\boldsymbol{\theta}}} (\mathcal{U}P). \end{aligned} \tag{6.17}$$

Now, comparing (6.14), (6.16), and (6.17), we arrive at the following important result.

Theorem 6.3. *Suppose that the conditions of Theorem* 6.2 *are satisfied. Let the operators* $R_J(t;\boldsymbol{\theta};\varepsilon)$ *and* $R_J^0(t;\boldsymbol{\theta};\varepsilon)$ *be defined by* (6.10) *and* (6.15)*. Then*

$$\begin{aligned} &\|R_J(t;\boldsymbol{\theta};\varepsilon) - (\mathcal{U}P)^* R_J^0(t;\boldsymbol{\theta};\varepsilon)(\mathcal{U}P)\|_{\mathfrak{H}\to\mathfrak{H}} \le \mathcal{C}_3 \varepsilon^{-1}, \\ &0 < \varepsilon \le 1, \quad 0 < t \le t^0, \quad \mathcal{C}_3 = \mathcal{C}_1 + \mathcal{C}_2 = 2\mathcal{C}_1 + C_3 2^{-1} c_*^{-1/2}. \end{aligned} \tag{6.18}$$

6.3. Estimates of the resolvents for $t > t^0$**.** Besides (6.9) and (6.18), we need some elementary estimates of the resolvents for $\mathbf{k} = t\boldsymbol{\theta} \in \operatorname{clos}\widetilde{\Omega} \cap \{\mathbf{k} \in \mathbb{R}^3 : |\mathbf{k}| > t^0\}$. By (3.30), for the lowest eigenvalue $E_1(\mathbf{k})$ of the operator $\mathcal{L}(\mathbf{k}) = L(t;\boldsymbol{\theta})$ we have

$$\|R(t;\boldsymbol{\theta};\varepsilon)\|_{\mathfrak{H}\to\mathfrak{H}} \le (c_*(t^0)^2 + \varepsilon^2)^{-1} \le c_*^{-1}(t^0)^{-2}, \quad \mathbf{k} = t\boldsymbol{\theta} \in \operatorname{clos}\widetilde{\Omega} \cap \{\mathbf{k} : |\mathbf{k}| > t^0\}.$$

The same estimate is valid for $R^0(t;\boldsymbol{\theta};\varepsilon)$, and also for the operators (6.10) and (6.15). Combining this with (6.9) and (6.18), we arrive at the inequalities

$$\begin{aligned} &\|R(t;\boldsymbol{\theta};\varepsilon) - (\mathcal{U}P)^* R^0(t;\boldsymbol{\theta};\varepsilon)(\mathcal{U}P)\|_{\mathfrak{H}\to\mathfrak{H}} \le \mathcal{C}_4 \varepsilon^{-1}, \\ &\qquad 0 < \varepsilon \le 1, \quad \mathbf{k} = t\boldsymbol{\theta} \in \operatorname{clos}\widetilde{\Omega}, \end{aligned} \tag{6.19}$$

$$\begin{aligned} &\|R_J(t;\boldsymbol{\theta};\varepsilon) - (\mathcal{U}P)^* R_J^0(t;\boldsymbol{\theta};\varepsilon)(\mathcal{U}P)\|_{\mathfrak{H}\to\mathfrak{H}} \le \mathcal{C}_5 \varepsilon^{-1}, \\ &\qquad 0 < \varepsilon \le 1, \quad \mathbf{k} = t\boldsymbol{\theta} \in \operatorname{clos}\widetilde{\Omega} \setminus \{0\}, \end{aligned} \tag{6.20}$$

where

$$\mathcal{C}_4 = \max\{2\mathcal{C}_1, 2c_*^{-1}(t^0)^{-2}\}, \tag{6.21}$$

$$\mathcal{C}_5 = \max\{\mathcal{C}_3, 2c_*^{-1}(t^0)^{-2}\} = \max\{2\mathcal{C}_1 + C_3 2^{-1} c_*^{-1/2}, 2c_*^{-1}(t^0)^{-2}\}. \tag{6.22}$$

§7. Approximation for the resolvent of $\mathcal{L}$ and its solenoidal part

7.1. Let $\mathcal{L} = \mathcal{L}(s;h;\nu)$ be the operator in $\mathfrak{G} = L_2(\mathbb{R}^3;\mathbb{C}^3)$ defined in Subsection 3.1. Let $\mathcal{P}(s)$ be the orthogonal projection in $\mathfrak{G}$ onto the subspace $J(s)$ (see Subsection 2.1). Let

$\mathcal{L}^0 = \mathcal{L}(s^0, h^0, \underline{\nu})$ be the effective operator (see (4.34)), and let $\mathcal{P}(s^0)$ be the corresponding orthogonal projection in $\mathfrak{G}$ onto $J(s^0)$. We put

$$R(\varepsilon) = (\mathcal{L} + \varepsilon^2 I)^{-1}, \tag{7.1}$$

$$R^0(\varepsilon) = (\mathcal{L}^0 + \varepsilon^2 I)^{-1}, \tag{7.2}$$

$$R_J(\varepsilon) = \mathcal{P}(s)R(\varepsilon) = (\mathcal{L}_{J(s)} + \varepsilon^2 I_{J(s)})^{-1} \oplus \mathbf{0}_{G(s)}, \tag{7.3}$$

$$R_J^0(\varepsilon) = \mathcal{P}(s^0)R^0(\varepsilon) = (\mathcal{L}^0_{J(s^0)} + \varepsilon^2 I_{J(s^0)})^{-1} \oplus \mathbf{0}_{G(s^0)}. \tag{7.4}$$

Recall that $\mathcal{V}$ denotes the (unitary) Gelfand transformation (see Subsection 1.3). The decompositions (3.36), (3.37) and their analogs for $\mathcal{L}^0$ show that the operators (7.1)–(7.4) can be decomposed into the following direct integrals:

$$R(\varepsilon) = \mathcal{V}^* \left(\int_{\widetilde{\Omega}} \oplus R(t;\boldsymbol{\theta};\varepsilon)\, d\mathbf{k} \right) \mathcal{V}, \tag{7.5}$$

$$R^0(\varepsilon) = \mathcal{V}^* \left(\int_{\widetilde{\Omega}} \oplus R^0(t;\boldsymbol{\theta};\varepsilon)\, d\mathbf{k} \right) \mathcal{V}, \tag{7.6}$$

$$R_J(\varepsilon) = \mathcal{V}^* \left(\int_{\widetilde{\Omega}} \oplus R_J(t;\boldsymbol{\theta};\varepsilon)\, d\mathbf{k} \right) \mathcal{V}, \tag{7.7}$$

$$R_J^0(\varepsilon) = \mathcal{V}^* \left(\int_{\widetilde{\Omega}} \oplus R_J^0(t;\boldsymbol{\theta};\varepsilon)\, d\mathbf{k} \right) \mathcal{V} \tag{7.8}$$

(we have used (6.1), (6.4), (6.10), and (6.15); recall that $t = |\mathbf{k}|$, $\mathbf{k} = t\boldsymbol{\theta}$). The direct integrals of the operators in (7.5)–(7.8) act in the direct integral $\mathcal{K}$ of Hilbert spaces (see (1.6)).

Now, we define a bounded operator $\mathcal{W} : \mathfrak{G} \to \mathfrak{G}$ by the relation

$$\mathcal{W} = \mathcal{V}^* \left(\int_{\widetilde{\Omega}} \oplus (\mathcal{U}P)\, d\mathbf{k} \right) \mathcal{V} = \mathcal{V}^* [\mathcal{U}P] \mathcal{V}. \tag{7.9}$$

Here $[\mathcal{U}P]$ is the $\mathbf{k}$-independent operator in $\mathcal{K}$ that acts as multiplication by the operator $\mathcal{U}P$ in the fibers $\mathfrak{H} = L_2(\Omega; \mathbb{C}^3)$ of the direct integral $\mathcal{K}$. Relations (7.6), (7.9) imply that

$$\begin{aligned} &\mathcal{V}^* \left(\int_{\widetilde{\Omega}} \oplus (\mathcal{U}P)^* R^0(t;\boldsymbol{\theta};\varepsilon)(\mathcal{U}P)\, d\mathbf{k} \right) \mathcal{V} \\ &\qquad = \mathcal{W}^* \mathcal{V}^* \left(\int_{\widetilde{\Omega}} \oplus R^0(t;\boldsymbol{\theta};\varepsilon)\, d\mathbf{k} \right) \mathcal{V}\mathcal{W} = \mathcal{W}^* R^0(\varepsilon) \mathcal{W} \end{aligned} \tag{7.10}$$

(recall that $\mathcal{V}$ is unitary). Similarly, (7.8) and (7.9) yield

$$\mathcal{V}^* \left(\int_{\widetilde{\Omega}} \oplus (\mathcal{U}P)^* R_J^0(t;\boldsymbol{\theta};\varepsilon)(\mathcal{U}P)\, d\mathbf{k} \right) \mathcal{V} = \mathcal{W}^* R_J^0(\varepsilon) \mathcal{W}. \tag{7.11}$$

From (7.5), (7.10) and (7.7), (7.11) we see that

$$\|R(\varepsilon) - \mathcal{W}^* R^0(\varepsilon) \mathcal{W}\|_{\mathfrak{G}\to\mathfrak{G}} = \operatorname{ess\,sup}_{\mathbf{k}\in\widetilde{\Omega}} \|R(t;\boldsymbol{\theta};\varepsilon) - (\mathcal{U}P)^* R^0(t;\boldsymbol{\theta};\varepsilon)(\mathcal{U}P)\|_{\mathfrak{H}\to\mathfrak{H}}, \tag{7.12}$$

$$\|R_J(\varepsilon) - \mathcal{W}^* R_J^0(\varepsilon) \mathcal{W}\|_{\mathfrak{G}\to\mathfrak{G}} = \operatorname{ess\,sup}_{\mathbf{k}\in\widetilde{\Omega}} \|R_J(t;\boldsymbol{\theta};\varepsilon) - (\mathcal{U}P)^* R_J^0(t;\boldsymbol{\theta};\varepsilon)(\mathcal{U}P)\|_{\mathfrak{H}\to\mathfrak{H}}. \tag{7.13}$$

Now, combining (7.12) and (7.13) with estimates (6.19) and (6.20), we arrive at the following results.

Theorem 7.1. *In the notation* (7.1), (7.2), *and* (7.9), *we have*

$$\|R(\varepsilon) - \mathcal{W}^* R^0(\varepsilon) \mathcal{W}\|_{\mathfrak{G}\to\mathfrak{G}} \le \mathcal{C}_4 \varepsilon^{-1}, \quad 0 < \varepsilon \le 1, \tag{7.14}$$

where $\mathcal{C}_4$ *is the constant defined by* (6.21).

Theorem 7.2. *In the notation* (7.3), (7.4), *and* (7.9), *we have*

$$\|R_J(\varepsilon) - \mathcal{W}^* R_J^0(\varepsilon)\mathcal{W}\|_{\mathfrak{G}\to\mathfrak{G}} \le \mathcal{C}_5\varepsilon^{-1}, \quad 0 < \varepsilon \le 1, \tag{7.15}$$

where $\mathcal{C}_5$ *is the constant defined by* (6.22).

Remark 7.3. Ultimately, the constants $\mathcal{C}_4, \mathcal{C}_5$ depend on the L_∞-norms of the coefficients $s, s^{-1}, h, h^{-1}, \nu, \nu^{-1}$, and on r_0. Note that the operators $R_J(\varepsilon)$ and $R_J^0(\varepsilon)$ do not depend on the coefficient ν (because the part of $\mathcal{L}(s,h,\nu)$ in the subspace $J(s)$ is one and the same for all ν). Consequently, estimate (7.15) is valid with the constant $\mathcal{C}_5$ corresponding to the operator $\mathcal{L}(s,h,\nu)$ *with arbitrary* $\nu(\mathbf{x})$, in particular, with $\nu(\mathbf{x}) = 1$. In this case, $\mathcal{C}_5$ only depends on the L_∞-norms of the coefficients s, s^{-1}, h, h^{-1}, and on r_0.

7.2. The bordering operators. Now, we study the operators $\mathcal{W}$, $\mathcal{W}^*$ (see (7.9)) that border the resolvents $R^0(\varepsilon)$ or $R_J^0(\varepsilon)$ in the approximations (7.14), (7.15). It suffices to consider the operator

$$\mathcal{W}^* = \mathcal{V}^*[(\mathcal{U}P)^*]\mathcal{V} = \mathcal{V}^*[\mathcal{U}^*P_0]\mathcal{V}. \tag{7.16}$$

Here we have used the obvious identity

$$(\mathcal{U}P)^* = (P_0\mathcal{U}P)^* = P\mathcal{U}^*P_0 = \mathcal{U}^*P_0.$$

Proposition 7.4. *For any function* $\mathbf{u} \in \mathfrak{G}$ *we have*

$$([P_0]\mathcal{V}\mathbf{u})(\mathbf{k}) = |\Omega|^{-1/2}(\mathfrak{F}\mathbf{u})(\mathbf{k}), \quad \mathbf{k} \in \widetilde{\Omega}, \tag{7.17}$$

where $\mathfrak{F}$ *is the Fourier transformation in* $\mathbb{R}^3$.

Proof. It suffices to check (7.17) for the functions $\mathbf{u}$ in the Schwartz class $\mathcal{S}(\mathbb{R}^3;\mathbb{C}^3)$. Then $\mathbf{u}_*(\mathbf{x},\mathbf{k}) = (\mathcal{V}\mathbf{u})(\mathbf{x},\mathbf{k})$ is defined in accordance with (1.5). Now, to the function $\mathbf{u}_*(\cdot,\mathbf{k}) \in \mathfrak{H}$ we apply the projection P_0 onto the subspace $\mathfrak{N}^0 \subset \mathfrak{H}$ of constant vector-valued functions

$$\begin{aligned}(P_0\mathbf{u}_*)(\mathbf{k}) &= |\Omega|^{-1}\int_\Omega \mathbf{u}_*(\mathbf{x},\mathbf{k})\,d\mathbf{x} = |\Omega|^{-1}|\widetilde{\Omega}|^{-1/2}\int_\Omega \sum_{\mathbf{a}\in\Gamma} e^{-i\langle\mathbf{k},\mathbf{x}+\mathbf{a}\rangle}\mathbf{u}(\mathbf{x}+\mathbf{a})\,d\mathbf{x}\\ &= |\Omega|^{-1/2}(2\pi)^{-3/2}\sum_{\mathbf{a}\in\Gamma}\int_{\Omega+\mathbf{a}} e^{-i\langle\mathbf{k},\mathbf{y}\rangle}\mathbf{u}(\mathbf{y})\,d\mathbf{y}\\ &= |\Omega|^{-1/2}(2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-i\langle\mathbf{k},\mathbf{y}\rangle}\mathbf{u}(\mathbf{y})\,d\mathbf{y} = |\Omega|^{-1/2}(\mathfrak{F}\mathbf{u})(\mathbf{k})\end{aligned}$$

(we have used the relation $|\Omega||\widetilde{\Omega}| = (2\pi)^3$). $\square$

By (4.14), the operator $\mathcal{U}^* = \mathcal{U}^{-1} : \mathfrak{N}^0 \to \mathfrak{N}$ takes the vector $\mathbf{f}^0 = (s^0)^{1/2}\mathbf{C} \in \mathfrak{N}^0$ to $\mathbf{f} = s^{1/2}(\mathbf{C} + \nabla\Phi_{\mathbf{C}}) \in \mathfrak{N}$. Let $\mathbf{e}_j$, $j = 1,2,3$, be the standard unit vectors in $\mathbb{C}^3$. We put

$$\mathbf{C}_j := (s^0)^{-1/2}\mathbf{e}_j, \ \ \Phi_j := \Phi_{\mathbf{C}_j}, \quad j = 1,2,3. \tag{7.18}$$

Obviously, the vectors $\mathbf{e}_j = (s^0)^{1/2}\mathbf{C}_j$, $j = 1,2,3$, form an orthogonal basis in $\mathfrak{N}^0$: $(\mathbf{e}_j,\mathbf{e}_l)_{\mathfrak{H}} = |\Omega|\delta_{jl}$. Then, since $\mathcal{U}$ is unitary, the functions

$$\mathbf{g}_j = \mathcal{U}^*\mathbf{e}_j = s^{1/2}(\mathbf{C}_j + \nabla\Phi_j), \quad j = 1,2,3, \tag{7.19}$$

form an orthogonal basis in $\mathfrak{N}$. The operator $\mathcal{U}^*$ takes the constant vector-valued function $\mathbf{f}^0 \in \mathfrak{N}^0$ to the function

$$(\mathcal{U}^*\mathbf{f}^0)(\mathbf{x}) = \sum_{j=1}^{3}\langle\mathbf{f}^0,\mathbf{e}_j\rangle\mathbf{g}_j(\mathbf{x}).$$

In other words, if $\mathbb{C}^3$-valued functions are represented as columns of their coordinates in the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, then the *operator* $\mathcal{U}^*$ *reduces to multiplication by the* (3×3)*-matrix* $W^*(\mathbf{x})$*, the columns of which are the coordinates of the vectors* $\mathbf{g}_j(\mathbf{x})$:

$$(\mathcal{U}^* \mathbf{f}^0)(\mathbf{x}) = W^*(\mathbf{x}) \mathbf{f}^0. \tag{7.20}$$

From (7.17) and (7.20) it follows that

$$([\mathcal{U}^* P_0] \mathcal{V} \mathbf{u})(\mathbf{x}, \mathbf{k}) = |\Omega|^{-1/2} W^*(\mathbf{x}) (\mathfrak{F}\mathbf{u})(\mathbf{k}), \quad \mathbf{x} \in \Omega, \ \mathbf{k} \in \widetilde{\Omega}. \tag{7.21}$$

Finally, in order to calculate the operator $\mathcal{W}^*$ (see (7.16)), we consider its sesquilinear form, assuming that $\mathbf{u}, \mathbf{v} \in \mathcal{S}(\mathbb{R}^3; \mathbb{C}^3)$. Using (7.21), (1.5), and the periodicity of the matrix-valued function $W^*(\mathbf{x})$, we obtain

(7.22)
$$\begin{aligned}
(\mathcal{W}^* \mathbf{u}, \mathbf{v})_{\mathfrak{G}} &= ([\mathcal{U}^* P_0] \mathcal{V} \mathbf{u}, \mathcal{V} \mathbf{v})_{\mathcal{K}} \\
&= |\Omega|^{-1/2} |\widetilde{\Omega}|^{-1/2} \int_\Omega d\mathbf{x} \int_{\widetilde{\Omega}} d\mathbf{k} \sum_{\mathbf{a} \in \Gamma} \exp(i \langle \mathbf{k}, \mathbf{x} + \mathbf{a} \rangle) \langle W^*(\mathbf{x}) (\mathfrak{F}\mathbf{u})(\mathbf{k}), \mathbf{v}(\mathbf{x} + \mathbf{a}) \rangle \\
&= (2\pi)^{-3/2} \sum_{\mathbf{a} \in \Gamma} \int_{\Omega + \mathbf{a}} d\mathbf{y} \int_{\widetilde{\Omega}} d\mathbf{k} \, \exp(i \langle \mathbf{k}, \mathbf{y} \rangle) \langle W^*(\mathbf{y}) (\mathfrak{F}\mathbf{u})(\mathbf{k}), \mathbf{v}(\mathbf{y}) \rangle \\
&= (2\pi)^{-3/2} \int_{\mathbb{R}^3} d\mathbf{y} \int_{\widetilde{\Omega}} d\mathbf{k} \, \exp(i \langle \mathbf{k}, \mathbf{y} \rangle) \langle W^*(\mathbf{y}) (\mathfrak{F}\mathbf{u})(\mathbf{k}), \mathbf{v}(\mathbf{y}) \rangle.
\end{aligned}$$

We introduce the pseudodifferential operator Π that acts in $\mathfrak{G}$ and the symbol of which is the indicator $\chi_{\widetilde{\Omega}}(\mathbf{k})$ of the set $\widetilde{\Omega}$:

$$(\Pi \mathbf{u})(\mathbf{y}) = (2\pi)^{-3/2} \int_{\widetilde{\Omega}} e^{i \langle \mathbf{k}, \mathbf{y} \rangle} (\mathfrak{F}\mathbf{u})(\mathbf{k}) \, d\mathbf{k}. \tag{7.23}$$

Then (7.22) can be rewritten as

$$(\mathcal{W}^* \mathbf{u}, \mathbf{v})_{\mathfrak{G}} = \int_{\mathbb{R}^3} \langle W^*(\mathbf{y}) (\Pi \mathbf{u})(\mathbf{y}), \mathbf{v}(\mathbf{y}) \rangle \, d\mathbf{y}, \quad \mathbf{u}, \mathbf{v} \in \mathcal{S}(\mathbb{R}^3; \mathbb{C}^3).$$

This implies the following statement.

Proposition 7.5. *Let* Π *be the pseudodifferential operator defined by* (7.23)*. Let* $W^*(\mathbf{x})$ *be the matrix-valued function with the columns* $\mathbf{g}_j(\mathbf{x})$, $j = 1, 2, 3$, *defined by* (7.19)*. Then the operator* (7.16) *can be represented as* $\mathcal{W}^* = W^* \Pi$.

Thus, the *operator* $\mathcal{W}^*$ *is the composition of the pseudodifferential operator* Π *and the operator of multiplication by the periodic matrix-valued function* $W^*(\mathbf{x})$. The matrix $W^*(\mathbf{x})$ is expressed in terms of the gradients of the periodic solutions $\Phi_j(\mathbf{x})$ of the elliptic equations $\operatorname{div} s(\mathbf{x})(\mathbf{C}_j + \nabla \Phi_j) = 0$ (see (7.18)). Hence, in general, for an arbitrary matrix $s(\mathbf{x})$ satisfying condition (3.1) only, the matrix $W^*(\mathbf{x})$ is not bounded. Nevertheless, the composition $W^* \Pi$ is a bounded operator in $\mathfrak{G}$, which is clear from (7.16). Also, this fact can easily be verified directly. We shall not dwell on this direct verification. As is seen from (7.9), (7.16), we have

$$\|\mathcal{W}\|_{\mathfrak{G} \to \mathfrak{G}} = \|\mathcal{W}^*\|_{\mathfrak{G} \to \mathfrak{G}} \le 1. \tag{7.24}$$

§8. Other approximations for the operators $R(\varepsilon)$ and $R_J(\varepsilon)$

8.1. The properties of the solutions $\Phi_{\mathbf{C}}$.[1] In Theorems 7.1 and 7.2, we obtained approximations for $R(\varepsilon)$ and $R_J(\varepsilon)$ in terms of the "sandwiched" operators $R^0(\varepsilon)$ and $R^0_J(\varepsilon)$. The role of the bordering operators was played by $\mathcal{W}^*$ and $\mathcal{W}$. In Proposition

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[1]The author thanks A. A. Arkhipova for consultation and help in clarifying the properties of the solutions $\Phi_{\mathbf{C}}$.

7.5, the operator $\mathcal{W}^*$ was represented as $W^*\Pi$. Our goal in this section is to show that these approximations remain valid if the bordering operators $\mathcal{W}^*$ and $\mathcal{W}$ are replaced by the operators of multiplication by the periodic matrix-valued functions $W^*(\mathbf{x})$ and $W(\mathbf{x})$, respectively. In other words, the pseudodifferential operator Π can be replaced by the identity operator I.

For this, we need some properties of the periodic solutions $\Phi_{\mathbf{C}}$ of equation (4.5). The following statement is a consequence of [LU, Chapter III, Theorem 13.1].

Proposition 8.1. *Suppose that a measurable periodic matrix $s(\mathbf{x})$ with real entries satisfies condition* (3.1). *Let $\mathbf{C} \in \mathbb{C}^3$, and let $\Phi_{\mathbf{C}}(\mathbf{x})$ be the* (*weak*) *periodic solution of* (4.5) *satisfying the additional condition $\int_\Omega \Phi_{\mathbf{C}}(\mathbf{x})\,d\mathbf{x} = 0$. Then $\Phi_{\mathbf{C}} \in L_\infty(\mathbb{R}^3)$ and*

$$\|\Phi_{\mathbf{C}}\|_{L_\infty} \le \widehat{C}|\mathbf{C}|, \tag{8.1}$$

where the constant $\widehat{C}$ only depends on $\|s\|_{L_\infty}$, $\|s^{-1}\|_{L_\infty}$, and on Ω.

Now, we prove the following statement.

Proposition 8.2. *Suppose that the conditions of Proposition* 8.1 *are satisfied. Let $\mathbf{f} = s^{1/2}(\mathbf{C}+\nabla\Phi_{\mathbf{C}})$. Then the operator $[\mathbf{f}]$ of multiplication by the column $\mathbf{f}(\mathbf{x})$ is a continuous operator from $H^1(\mathbb{R}^3)$ to $\mathfrak{G} = L_2(\mathbb{R}^3;\mathbb{C}^3)$, and*

$$\|[\mathbf{f}]\|_{H^1(\mathbb{R}^3)\to\mathfrak{G}} \le \widehat{C}_1|\mathbf{C}|, \quad \widehat{C}_1 = (2\|s\|_{L_\infty})^{1/2}\max\{2\widehat{C}, 1\}. \tag{8.2}$$

Proof. For a periodic solution $\Phi_{\mathbf{C}} \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ of equation (4.5), we have

$$\int_{\mathbb{R}^3} \langle s(\mathbf{x})(\mathbf{C}+\nabla\Phi_{\mathbf{C}}), \nabla\Psi\rangle\,d\mathbf{x} = 0 \tag{8.3}$$

for any function $\Psi \in H^1(\mathbb{R}^3)$ with compact support. Let $v \in C_0^\infty(\mathbb{R}^3)$. Then in (8.3) we can put $\Psi(\mathbf{x}) = \Phi_{\mathbf{C}}(\mathbf{x})|v(\mathbf{x})|^2$:

$$\int_{\mathbb{R}^3} \langle s(\mathbf{x})(\mathbf{C}+\nabla\Phi_{\mathbf{C}}), \nabla\Phi_{\mathbf{C}}\rangle|v|^2\,d\mathbf{x} + \int_{\mathbb{R}^3} \langle s(\mathbf{x})(\mathbf{C}+\nabla\Phi_{\mathbf{C}}), \nabla|v|^2\rangle\Phi_{\mathbf{C}}^+\,d\mathbf{x} = 0.$$

Consequently,

$$\begin{aligned}
\int_{\mathbb{R}^3} |\mathbf{f}|^2|v|^2\,d\mathbf{x} &= \int_{\mathbb{R}^3} \langle s(\mathbf{x})(\mathbf{C}+\nabla\Phi_{\mathbf{C}}), \mathbf{C}+\nabla\Phi_{\mathbf{C}}\rangle|v|^2\,d\mathbf{x} \\
&= \int_{\mathbb{R}^3} \langle s(\mathbf{x})(\mathbf{C}+\nabla\Phi_{\mathbf{C}}), \mathbf{C}\rangle|v|^2\,d\mathbf{x} - \int_{\mathbb{R}^3} \langle s(\mathbf{x})(\mathbf{C}+\nabla\Phi_{\mathbf{C}}), \nabla|v|^2\rangle\Phi_{\mathbf{C}}^+\,d\mathbf{x} \\
&\le \|s\|_{L_\infty}^{1/2}|\mathbf{C}|\int_{\mathbb{R}^3} |\mathbf{f}||v|^2\,d\mathbf{x} + 2\|s\|_{L_\infty}^{1/2}\|\Phi_{\mathbf{C}}\|_{L_\infty}\int_{\mathbb{R}^3} |\mathbf{f}||v||\nabla v|\,d\mathbf{x} \\
&\le \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{f}|^2|v|^2\,d\mathbf{x} + \|s\|_{L_\infty}|\mathbf{C}|^2\int_{\mathbb{R}^3} |v|^2\,d\mathbf{x} + 4\|s\|_{L_\infty}\|\Phi_{\mathbf{C}}\|_{L_\infty}^2\int_{\mathbb{R}^3} |\nabla v|^2\,d\mathbf{x}.
\end{aligned}$$

As a result, by (8.1), we obtain

$$\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{f}|^2|v|^2\,d\mathbf{x} \le \|s\|_{L_\infty}|\mathbf{C}|^2\left(\int_{\mathbb{R}^3} |v|^2\,d\mathbf{x} + 4\widehat{C}^2\int_{\mathbb{R}^3} |\nabla v|^2\,d\mathbf{x}\right), \quad v \in C_0^\infty(\mathbb{R}^3),$$

which implies (8.2). $\square$

Proposition 8.2 elementarily implies the following statement, in which we use the notation $\mathfrak{G}^l := H^l(\mathbb{R}^3;\mathbb{C}^3)$, $l \in \mathbb{R}$.

Corollary 8.3. 1°. *The operator $[\nabla\Phi_{\mathbf{C}}]$ of multiplication by the column $\nabla\Phi_{\mathbf{C}}(\mathbf{x})$ acts continuously from $H^1(\mathbb{R}^3)$ to $\mathfrak{G}$; the adjoint operator $[\nabla\Phi_{\mathbf{C}}]^*$ of multiplication by the row $(\nabla\Phi_{\mathbf{C}}(\mathbf{x}))^*$ acts continuously from $\mathfrak{G}$ to $H^{-1}(\mathbb{R}^3)$. We have*

$$\|[\nabla\Phi_{\mathbf{C}}]\|_{H^1(\mathbb{R}^3)\to\mathfrak{G}} \le \widehat{C}_2|\mathbf{C}|, \quad \widehat{C}_2 = \widehat{C}_1\|s^{-1}\|^{1/2}_{L_\infty} + 1,$$
$$\|[\nabla\Phi_{\mathbf{C}}]^*\|_{\mathfrak{G}\to H^{-1}(\mathbb{R}^3)} \le \widehat{C}_2|\mathbf{C}|.$$

2°. *Let $\mathbf{g}_j$, $j=1,2,3$, be defined by* (7.18), (7.19). *Then*

$$\|[\mathbf{g}_j]\|_{H^1(\mathbb{R}^3)\to\mathfrak{G}} \le \widehat{C}_1|(s^0)^{-1/2}| \le \widehat{C}_1\|s^{-1}\|^{1/2}_{L_\infty}, \quad j=1,2,3.$$

3°. *The operator of multiplication by the matrix $W^*(\mathbf{x})$ acts continuously from $\mathfrak{G}^1$ to $\mathfrak{G}$, and*

$$\|[W^*]\|_{\mathfrak{G}^1\to\mathfrak{G}} \le \widehat{C}_3 := \sqrt{3}\widehat{C}_1\|s^{-1}\|^{1/2}_{L_\infty}. \tag{8.4}$$

4°. *The operator of multiplication by the matrix $W(\mathbf{x})$ acts continuously from $\mathfrak{G}$ to $\mathfrak{G}^{-1}$, and*

$$\|[W]\|_{\mathfrak{G}\to\mathfrak{G}^{-1}} \le \widehat{C}_3.$$

8.2. Analysis of the sandwiched resolvent. We put (see Proposition 7.5)

$$\mathcal{Y}(\varepsilon)^* := \mathcal{W}^*(R^0(\varepsilon))^{1/2} = W^*\Pi(R^0(\varepsilon))^{1/2}. \tag{8.5}$$

Then

$$\mathcal{W}^*R^0(\varepsilon)\mathcal{W} = \mathcal{Y}(\varepsilon)^*\mathcal{Y}(\varepsilon), \tag{8.6}$$
$$\mathcal{W}^*R^0_J(\varepsilon)\mathcal{W} = \mathcal{Y}(\varepsilon)^*\mathcal{P}(s^0)\mathcal{Y}(\varepsilon). \tag{8.7}$$

We denote

$$\widetilde{\mathcal{Y}}(\varepsilon)^* = W^*(I-\Pi)(R^0(\varepsilon))^{1/2}. \tag{8.8}$$

Our goal is to show that the operator (8.8) is bounded in $\mathfrak{G}$, and its norm is estimated by a constant independent of ε. This will allow us to replace the approximating operators (8.6) and (8.7) in the approximations (7.14) and (7.15) by $W^*R^0(\varepsilon)W$ and $W^*R^0_J(\varepsilon)W$.

Proposition 8.4. *The operator $(I-\Pi)(R^0(\varepsilon))^{1/2} : \mathfrak{G}\to\mathfrak{G}^1$ is continuous, and*

$$\|(I-\Pi)(R^0(\varepsilon))^{1/2}\|_{\mathfrak{G}\to\mathfrak{G}^1} \le c_*^{-1/2}(1+r_0^{-2})^{1/2}. \tag{8.9}$$

The constant c_* is defined by (3.28), *and r_0 is the radius of the ball inscribed in $\widetilde{\Omega}$.*

Proof. In the Fourier representation, the operator $\mathcal{L}^0$ (see (4.34)) acts on a function $\mathbf{u}(\mathbf{x})$ as the multiplication of its Fourier image $(\mathfrak{F}\mathbf{u})(\boldsymbol{\xi})$ by the symbol $l^0(\boldsymbol{\xi})$, $\boldsymbol{\xi}\in\mathbb{R}^3$. The symbol $l^0(\boldsymbol{\xi})$ is the (3×3)-matrix corresponding to the quadratic form

$$\langle l^0(\boldsymbol{\xi})\mathbf{z},\mathbf{z}\rangle = \langle (h^0)^{-1}(\boldsymbol{\xi}\times((s^0)^{-1/2}\mathbf{z})), \boldsymbol{\xi}\times((s^0)^{-1/2}\mathbf{z})\rangle + \underline{\nu}|\langle (s^0)^{1/2}\mathbf{z},\boldsymbol{\xi}\rangle|^2, \quad \boldsymbol{\xi}\in\mathbb{R}^3, \quad \mathbf{z}\in\mathbb{C}^3. \tag{8.10}$$

It follows that the matrix-valued function $l^0(\boldsymbol{\xi})$ is a second-order homogeneous function of $\boldsymbol{\xi}$:

$$l^0(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2 l^0(\boldsymbol{\theta}), \quad \boldsymbol{\theta} = |\boldsymbol{\xi}|^{-1}\boldsymbol{\xi}\in\mathbb{S}^2. \tag{8.11}$$

Combining (8.10) and (4.32), we see that

$$\frac{\langle l^0(\boldsymbol{\theta})\mathbf{z},\mathbf{z}\rangle}{|\mathbf{z}|^2} = \frac{(S^0(\boldsymbol{\theta})\mathbf{z},\mathbf{z})_{\mathfrak{H}}}{\|\mathbf{z}\|^2_{\mathfrak{H}}}, \quad \mathbf{z}\in\mathbb{C}^3\setminus\{0\}.$$

Recalling estimates (4.18) for the eigenvalues of the germ $S^0(\boldsymbol{\theta})$, we arrive at a lower estimate for the symbol (8.11):

$$l^0(\boldsymbol{\xi}) \geq c_* |\boldsymbol{\xi}|^2 \mathbf{1}, \quad \boldsymbol{\xi} \in \mathbb{R}^3. \tag{8.12}$$

In the Fourier representation, the resolvent $R^0(\varepsilon)$ (see (7.2)) corresponds to the symbol $(l^0(\boldsymbol{\xi}) + \varepsilon^2 \mathbf{1})^{-1}$, while the operator $(I - \Pi)(R^0(\varepsilon))^{1/2}$ (see (7.23)) corresponds to the symbol $(1 - \chi_{\widetilde{\Omega}}(\boldsymbol{\xi}))(l^0(\boldsymbol{\xi}) + \varepsilon^2 \mathbf{1})^{-1/2}$. Let $\mathbf{u} \in \mathfrak{G}$ and $\mathbf{v} := (I - \Pi)(R^0(\varepsilon))^{1/2}\mathbf{u}$. Then

$$(\mathfrak{F}\mathbf{v})(\boldsymbol{\xi}) = (1 - \chi_{\widetilde{\Omega}}(\boldsymbol{\xi}))(l^0(\boldsymbol{\xi}) + \varepsilon^2 \mathbf{1})^{-1/2}(\mathfrak{F}\mathbf{u})(\boldsymbol{\xi}). \tag{8.13}$$

By (8.12) and (8.13), and by the estimate $|\boldsymbol{\xi}| \geq r_0$ for $\boldsymbol{\xi} \in \mathbb{R}^3 \setminus \widetilde{\Omega}$, we have

$$\begin{aligned} \|\mathbf{v}\|^2_{\mathfrak{G}^1} &= \int_{\mathbb{R}^3} (|\boldsymbol{\xi}|^2 + 1)|(\mathfrak{F}\mathbf{v})(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \leq \int_{\mathbb{R}^3 \setminus \widetilde{\Omega}} (|\boldsymbol{\xi}|^2 + 1)(c_*|\boldsymbol{\xi}|^2 + \varepsilon^2)^{-1} |(\mathfrak{F}\mathbf{u})(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \\ &\leq c_*^{-1}(1 + r_0^{-2}) \int_{\mathbb{R}^3 \setminus \widetilde{\Omega}} |(\mathfrak{F}\mathbf{u})(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \leq c_*^{-1}(1 + r_0^{-2}) \|\mathbf{u}\|^2_{\mathfrak{G}}. \end{aligned}$$

This proves (8.9). □

Now, (8.4) and (8.9) directly imply the following statement.

Proposition 8.5. *The operators* $\widetilde{\mathcal{Y}}(\varepsilon)^*$ (*see* (8.8)) *and* $\widetilde{\mathcal{Y}}(\varepsilon)$ *are bounded in* $\mathfrak{G}$. *We have*

$$\|\widetilde{\mathcal{Y}}(\varepsilon)^*\|_{\mathfrak{G} \to \mathfrak{G}} = \|\widetilde{\mathcal{Y}}(\varepsilon)\|_{\mathfrak{G} \to \mathfrak{G}} \leq \widehat{C}_4 := c_*^{-1/2}(1 + r_0^{-2})^{1/2} \widehat{C}_3. \tag{8.14}$$

Obviously, (7.2) implies that $\|(R^0(\varepsilon))^{1/2}\|_{\mathfrak{G} \to \mathfrak{G}} \leq \varepsilon^{-1}$, $\varepsilon > 0$. Combining this with (7.24), we obtain an estimate for the norm of $\mathcal{Y}(\varepsilon)^*$ (see (8.5)), and then also for the norm of $\mathcal{Y}(\varepsilon)$:

$$\|\mathcal{Y}(\varepsilon)^*\|_{\mathfrak{G} \to \mathfrak{G}} = \|\mathcal{Y}(\varepsilon)\|_{\mathfrak{G} \to \mathfrak{G}} \leq \varepsilon^{-1}, \quad \varepsilon > 0. \tag{8.15}$$

By (8.5), (8.6), and (8.8), we have

$$\begin{aligned} W^* R^0(\varepsilon) W - \mathcal{W}^* R^0(\varepsilon) \mathcal{W} &= (\mathcal{Y}(\varepsilon)^* + \widetilde{\mathcal{Y}}(\varepsilon)^*)(\mathcal{Y}(\varepsilon) + \widetilde{\mathcal{Y}}(\varepsilon)) - \mathcal{Y}(\varepsilon)^* \mathcal{Y}(\varepsilon) \\ &= \widetilde{\mathcal{Y}}(\varepsilon)^* \mathcal{Y}(\varepsilon) + \mathcal{Y}(\varepsilon)^* \widetilde{\mathcal{Y}}(\varepsilon) + \widetilde{\mathcal{Y}}(\varepsilon)^* \widetilde{\mathcal{Y}}(\varepsilon). \end{aligned}$$

Then (8.14) and (8.15) yield the estimate

$$\begin{aligned} &\|W^* R^0(\varepsilon) W - \mathcal{W}^* R^0(\varepsilon) \mathcal{W}\|_{\mathfrak{G} \to \mathfrak{G}} \\ &\quad \leq 2\|\widetilde{\mathcal{Y}}(\varepsilon)\|_{\mathfrak{G} \to \mathfrak{G}} \|\mathcal{Y}(\varepsilon)\|_{\mathfrak{G} \to \mathfrak{G}} + \|\widetilde{\mathcal{Y}}(\varepsilon)\|^2_{\mathfrak{G} \to \mathfrak{G}} \leq 2\widehat{C}_4 \varepsilon^{-1} + \widehat{C}_4^2, \quad \varepsilon > 0. \end{aligned} \tag{8.16}$$

Similarly, we use (8.7) to obtain

$$\|W^* R^0_J(\varepsilon) W - \mathcal{W}^* R^0_J(\varepsilon) \mathcal{W}\|_{\mathfrak{G} \to \mathfrak{G}} \leq 2\widehat{C}_4 \varepsilon^{-1} + \widehat{C}_4^2, \quad \varepsilon > 0. \tag{8.17}$$

8.3. Approximation for $R(\varepsilon)$ and $R_J(\varepsilon)$. Combining (7.14), (7.15) and (8.16), (8.17), we arrive at the following results.

Theorem 8.6. *Let* $R(\varepsilon)$, $R^0(\varepsilon)$ *be the operators defined by* (7.1), (7.2). *Let* $W^*(\mathbf{x})$ *be the matrix with the columns* $\mathbf{g}_j(\mathbf{x})$, $j = 1, 2, 3$, *defined by* (7.18), (7.19). *Then*

$$\|R(\varepsilon) - W^* R^0(\varepsilon) W\|_{\mathfrak{G} \to \mathfrak{G}} \leq \mathcal{C}_6 \varepsilon^{-1}, \quad 0 < \varepsilon \leq 1, \ \mathcal{C}_6 = \mathcal{C}_4 + 2\widehat{C}_4 + \widehat{C}_4^2, \tag{8.18}$$

where $\mathcal{C}_4$ *is defined by* (6.21), *and* $\widehat{C}_4$ *is defined by* (8.14).

Theorem 8.7. *Let* $R_J(\varepsilon)$, $R^0_J(\varepsilon)$ *be the operators defined by* (7.3), (7.4). *Let* $W^*(\mathbf{x})$ *be the same matrix as in Theorem* 8.6. *Then*

$$\|R_J(\varepsilon) - W^* R^0_J(\varepsilon) W\|_{\mathfrak{G} \to \mathfrak{G}} \leq \mathcal{C}_7 \varepsilon^{-1}, \quad 0 < \varepsilon \leq 1, \ \mathcal{C}_7 = \mathcal{C}_5 + 2\widehat{C}_4 + \widehat{C}_4^2, \tag{8.19}$$

where $\mathcal{C}_5$ *is defined by* (6.22), *and* $\widehat{C}_4$ *is defined by* (8.14).

Remark 8.8. Ultimately, the constants $\mathcal{C}_6$, $\mathcal{C}_7$ in estimates (8.18), (8.19) depend on the L_∞-norms of the coefficients s, s^{-1}, h, h^{-1}, ν, ν^{-1} and on the parameters of the lattice Γ. As in Remark 7.3, we may assume that the constant $\mathcal{C}_7$ corresponds to the operator $\mathcal{L}(s, h, 1)$ with $\nu(\mathbf{x}) = 1$; therefore, it only depends on the L_∞-norms of the coefficients s, s^{-1}, h, h^{-1} and on the parameters of the lattice Γ.

Now we look at how the approximating operators $W^* R^0(\varepsilon) W$ and $W^* R^0_J(\varepsilon) W = W^* \mathcal{P}(s^0) R^0(\varepsilon) W$ act. By Corollary 8.3(4°), the operator $[W]$ continuously maps $\mathfrak{G}$ into $\mathfrak{G}^{-1}$. The resolvent $R^0(\varepsilon)$, viewed as a pseudodifferential operator of order -2 with the symbol $(l^0(\boldsymbol{\xi}) + \varepsilon^2 \mathbf{1})^{-1}$, continuously maps $\mathfrak{G}^{-1}$ into $\mathfrak{G}^1$. By Corollary 8.3(3°), $[W^*]$ is a continuous operator from $\mathfrak{G}^1$ to $\mathfrak{G}$. As a result, the operator $W^* R^0(\varepsilon) W$ is continuous in $\mathfrak{G}$. In order to check that the operator $W^* R^0_J(\varepsilon) W$ is also continuous in $\mathfrak{G}$, we must use the fact that the projection $\mathcal{P}(s^0)$ is a continuous operator in $\mathfrak{G}^1$. Indeed, $\mathcal{P}(s^0)$ acts as a zeroth-order pseudodifferential operator. The symbol $p(\boldsymbol{\xi})$ of the operator $\mathcal{P}(s^0)$ is a (3×3)-matrix-valued function homogeneous in $\boldsymbol{\xi} \in \mathbb{R}^3$ of order zero. We have

$$p(\boldsymbol{\xi})\mathbf{z} = \mathbf{z} - \langle s^0 \boldsymbol{\xi}, \boldsymbol{\xi} \rangle^{-1} \langle (s^0)^{1/2} \mathbf{z}, \boldsymbol{\xi} \rangle (s^0)^{1/2} \boldsymbol{\xi}, \quad \boldsymbol{\xi} \in \mathbb{R}^3 \setminus 0, \ \mathbf{z} \in \mathbb{C}^3. \tag{8.20}$$

Therefore, $\mathcal{P}(s^0)$ continuously maps $\mathfrak{G}^1$ into $\mathfrak{G}^1$.

§9. Homogenization problem for the operator $\mathcal{L}$

Below, if $\varphi(\mathbf{x})$ is a measurable Γ-periodic function in $\mathbb{R}^3$, *we systematically use the notation* $\varphi^\varepsilon(\mathbf{x}) := \varphi(\varepsilon^{-1}\mathbf{x})$, $\varepsilon > 0$. For the operator $\mathcal{L} = \mathcal{L}(s, h, \nu)$ defined in Subsection 3.1, we consider the following family of $(\varepsilon\Gamma)$-periodic operators:

$$\mathcal{L}_\varepsilon := \mathcal{L}(s^\varepsilon, h^\varepsilon, \nu^\varepsilon), \quad \varepsilon > 0. \tag{9.1}$$

The coefficients of the operator (9.1) are rapidly oscillating as $\varepsilon \to 0$. The problem is to study the behavior as $\varepsilon \to 0$ of the resolvent $(\mathcal{L}_\varepsilon + I)^{-1}$, and also of its "solenoidal" part $(\mathcal{L}_\varepsilon + I)^{-1} \mathcal{P}(s^\varepsilon)$.

9.1. Approximation of the resolvent $(\mathcal{L}_\varepsilon + I)^{-1}$. The operator (9.1) is the selfadjoint operator in $\mathfrak{G}$ generated by the quadratic form

$$\begin{aligned} \mathfrak{l}_\varepsilon[\mathbf{u}, \mathbf{u}] = &\int_{\mathbb{R}^3} \langle (h^\varepsilon(\mathbf{x}))^{-1} \operatorname{curl}(s^\varepsilon(\mathbf{x}))^{-1/2}\mathbf{u}, \operatorname{curl}(s^\varepsilon(\mathbf{x}))^{-1/2}\mathbf{u} \rangle \, d\mathbf{x} \\ &+ \int_{\mathbb{R}^3} \nu^\varepsilon(\mathbf{x}) |\operatorname{div}(s^\varepsilon(\mathbf{x}))^{1/2}\mathbf{u}|^2 \, d\mathbf{x}, \quad \operatorname{Dom} \mathfrak{l}_\varepsilon = F(\mathbb{R}^3; s^\varepsilon) \end{aligned} \tag{9.2}$$

(the class $F(\mathbb{R}^3; s^\varepsilon)$ is defined by (2.6)). *Let T_ε denote the unitary scale transformation in* $\mathfrak{G}$: $(T_\varepsilon \mathbf{u})(\mathbf{y}) = \varepsilon^{3/2} \mathbf{u}(\varepsilon \mathbf{y})$, $\varepsilon > 0$. Relation (9.2) implies that

$$\mathcal{L}_\varepsilon = \varepsilon^{-2} T_\varepsilon^* \mathcal{L} T_\varepsilon, \quad \varepsilon > 0,$$

whence

$$(\mathcal{L}_\varepsilon + I)^{-1} = \varepsilon^2 T_\varepsilon^* (\mathcal{L} + \varepsilon^2 I)^{-1} T_\varepsilon = \varepsilon^2 T_\varepsilon^* R(\varepsilon) T_\varepsilon. \tag{9.3}$$

Here we have used the notation (7.1). For the operator $\mathcal{L}^0$ (see (4.34)) with constant coefficients, identity (9.3) turns into the even simpler relation

$$(\mathcal{L}^0 + I)^{-1} = \varepsilon^2 T_\varepsilon^* (\mathcal{L}^0 + \varepsilon^2 I)^{-1} T_\varepsilon = \varepsilon^2 T_\varepsilon^* R^0(\varepsilon) T_\varepsilon \tag{9.4}$$

(we have used (7.2)).

Let $W(\mathbf{x})$ be a periodic matrix-valued function occurring in Theorems 8.6 and 8.7. Obviously, $T_\varepsilon^* [W] T_\varepsilon = [W^\varepsilon]$, i.e., under the scale transformation the operator of multiplication by the periodic function $W(\mathbf{x})$ turns into the operator of multiplication by the rapidly oscillating function $W^\varepsilon(\mathbf{x}) = W(\varepsilon^{-1}\mathbf{x})$. Since T_ε is unitary, by (9.4) we have

$$\varepsilon^2 T_\varepsilon^* W^* R^0(\varepsilon) W T_\varepsilon = \varepsilon^2 (W^\varepsilon)^* T_\varepsilon^* R^0(\varepsilon) T_\varepsilon W^\varepsilon = (W^\varepsilon)^* (\mathcal{L}^0 + I)^{-1} W^\varepsilon. \tag{9.5}$$

Subtracting (9.5) from (9.3) and using Theorem 8.6, we arrive at the following theorem.

Theorem 9.1. *Let $\mathcal{L}_\varepsilon$ be the operator defined by* (9.1)*. Let $\mathcal{L}^0$ be the effective operator defined by* (4.34)*. Let $W^*(\mathbf{x})$ be a periodic matrix-valued function with the columns $\mathbf{g}_j(\mathbf{x})$, $j = 1, 2, 3$, defined by* (7.18), (7.19), *and let $W^\varepsilon(\mathbf{x}) = W(\varepsilon^{-1}\mathbf{x})$. Then*

$$\|(\mathcal{L}_\varepsilon + I)^{-1} - (W^\varepsilon)^* (\mathcal{L}^0 + I)^{-1} W^\varepsilon\|_{\mathfrak{G}\to\mathfrak{G}} \le \mathcal{C}_6 \varepsilon, \quad 0 < \varepsilon \le 1. \tag{9.6}$$

The constant $\mathcal{C}_6$ depends on the L_∞-norms of the coefficients s, s^{-1}, h, h^{-1}, ν, ν^{-1} and on the parameters of the lattice Γ.

We emphasize that the approximating operator $(W^\varepsilon)^*(\mathcal{L}^0 + I)^{-1}W^\varepsilon$ in (9.6) involves the resolvent of the effective operator $\mathcal{L}^0$ *with constant coefficients*. At the same time, the approximating operator still depends on ε via the explicitly described rapidly oscillating factors W^ε, $(W^\varepsilon)^*$. This dependence can be eliminated only by passing to the weak operator limit (see Subsection 9.3 below), i.e., by essential deterioration of the quality of convergence.

9.2. Approximation of the solenoidal part of the resolvent of $\mathcal{L}_\varepsilon$. By applying the explicit description of the projection $\mathcal{P}(s)$ (see Subsection 2.1), it is easy to show that

$$T_\varepsilon^* \mathcal{P}(s) T_\varepsilon = \mathcal{P}(s^\varepsilon). \tag{9.7}$$

We multiply both parts of (9.3) by the projection $\mathcal{P}(s^\varepsilon)$ and use (7.3) and (9.7), obtaining

$$\mathcal{P}(s^\varepsilon)(\mathcal{L}_\varepsilon + I)^{-1} = \varepsilon^2 T_\varepsilon^* \mathcal{P}(s) R(\varepsilon) T_\varepsilon = \varepsilon^2 T_\varepsilon^* R_J(\varepsilon) T_\varepsilon. \tag{9.8}$$

Similarly, multiplying (9.4) by $\mathcal{P}(s^0)$ and using (7.4) and (9.7), we get

$$\mathcal{P}(s^0)(\mathcal{L}^0 + I)^{-1} = \varepsilon^2 T_\varepsilon^* R_J^0(\varepsilon) T_\varepsilon. \tag{9.9}$$

Note that the operators on the left-hand sides of (9.8) and (9.9) can also be written as

$$\begin{aligned} \mathcal{P}(s^\varepsilon)(\mathcal{L}_\varepsilon + I)^{-1} &= ((\mathcal{L}_\varepsilon)_{J(s^\varepsilon)} + I_{J(s^\varepsilon)})^{-1} \oplus \mathbf{0}_{G(s^\varepsilon)}, \\ \mathcal{P}(s^0)(\mathcal{L}^0 + I)^{-1} &= ((\mathcal{L}^0)_{J(s^0)} + I_{J(s^0)})^{-1} \oplus \mathbf{0}_{G(s^0)}. \end{aligned}$$

By analogy with (9.5), we can use (9.9) to obtain

$$\varepsilon^2 T_\varepsilon^* W^* R_J^0(\varepsilon) W T_\varepsilon = \varepsilon^2 (W^\varepsilon)^* T_\varepsilon^* R_J^0(\varepsilon) T_\varepsilon W^\varepsilon = (W^\varepsilon)^* \mathcal{P}(s^0)(\mathcal{L}^0 + I)^{-1} W^\varepsilon. \tag{9.10}$$

Subtracting (9.10) from (9.8) and applying Theorem 8.7 (and also Remark 8.8), we arrive at the following result.

Theorem 9.2. *Suppose that the conditions of Theorem* 9.1 *are satisfied. Let $\mathcal{P}(s^\varepsilon)$ and $\mathcal{P}(s^0)$ be the orthogonal projections in $\mathfrak{G}$ onto the subspaces $J(s^\varepsilon)$ and $J(s^0)$* (*see* (2.2))*, respectively. Then*

$$\|\mathcal{P}(s^\varepsilon)(\mathcal{L}_\varepsilon + I)^{-1} - (W^\varepsilon)^* \mathcal{P}(s^0)(\mathcal{L}^0 + I)^{-1} W^\varepsilon\|_{\mathfrak{G}\to\mathfrak{G}} \le \mathcal{C}_7 \varepsilon, \quad 0 < \varepsilon \le 1. \tag{9.11}$$

The constant $\mathcal{C}_7$ depends on the L_∞-norms of the coefficients s, s^{-1}, h, h^{-1} and on the parameters of the lattice Γ.

9.3. On the weak limit of the resolvent. Under the conditions of Theorems 9.1 and 9.2, the *weak operator limits in* $\mathfrak{G}$ can be calculated for the resolvent $(\mathcal{L}_\varepsilon + I)^{-1}$ and for its solenoidal part. For this, we need the following elementary fact known as the "mean value property" (see, e.g., [ZhKO]).

Proposition 9.3. 1°. *Let $\psi(\mathbf{x})$ be a Γ-periodic function of class $L_{2,\mathrm{loc}}(\mathbb{R}^3)$. We put*

$$\overline{\psi} := |\Omega|^{-1} \int_\Omega \psi(\mathbf{x})\, d\mathbf{x}.$$

Then, as $\varepsilon \to 0$, the functions $\psi^\varepsilon(\mathbf{x}) = \psi(\varepsilon^{-1}\mathbf{x})$ weakly tend to $\overline{\psi}$ in $L_{2,\mathrm{loc}}(\mathbb{R}^3)$.

2°. *Let $\varphi \in L_\infty(\mathbb{R}^3)$ be a Γ-periodic function, and let $u \in L_2(\mathbb{R}^3)$. Then, as $\varepsilon \to 0$, the functions $\varphi^\varepsilon(\mathbf{x})u(\mathbf{x})$ weakly tend to $\overline{\varphi}u(\mathbf{x})$ in $L_2(\mathbb{R}^3)$.*

Now, we prove the following statement.

Theorem 9.4. *Under the conditions of Theorem* 9.1, *the resolvent $(\mathcal{L}_\varepsilon + I)^{-1}$ has a weak operator limit*:

$$(w, \mathfrak{G} \to \mathfrak{G})\text{-}\lim_{\varepsilon\to 0}(\mathcal{L}_\varepsilon + I)^{-1} = \overline{W}^*(\mathcal{L}^0 + I)^{-1}\overline{W}. \tag{9.12}$$

Proof. By (9.6), it suffices to find the weak limit of the operator

$$(W^\varepsilon)^*(\mathcal{L}^0 + I)^{-1}W^\varepsilon. \tag{9.13}$$

First, we show that the norm of this operator in $\mathfrak{G}$ is bounded uniformly with respect to ε.

As was shown in Corollary 8.3(3°), the operator $[W^*]$ acts continuously from $\mathfrak{G}^1$ to $\mathfrak{G}$, and estimate (8.4) is valid. Then the same is true for the operator $[(W^\varepsilon)^*]$, and its $(\mathfrak{G}^1 \to \mathfrak{G})$-norm is bounded uniformly in ε:

$$\|[(W^\varepsilon)^*]\|_{\mathfrak{G}^1\to\mathfrak{G}} \le \widehat{C}_3, \quad 0 < \varepsilon \le 1. \tag{9.14}$$

Indeed, estimate (8.4) means that

$$\int_{\mathbb{R}^3} |W^*(\mathbf{x})\mathbf{v}(\mathbf{x})|^2\, d\mathbf{x} \le \widehat{C}_3^2 \int_{\mathbb{R}^3} (|\nabla_{\mathbf{x}}\mathbf{v}(\mathbf{x})|^2 + |\mathbf{v}(\mathbf{x})|^2)\, d\mathbf{x}, \quad \mathbf{v} \in \mathfrak{G}^1. \tag{9.15}$$

Substituting $\mathbf{y} = \varepsilon\mathbf{x}$ and $\mathbf{u}(\mathbf{y}) = \mathbf{v}(\mathbf{x})$, we rewrite (9.15) as

$$\int_{\mathbb{R}^3} |W^*(\varepsilon^{-1}\mathbf{y})\mathbf{u}(\mathbf{y})|^2\, d\mathbf{y} \le \widehat{C}_3^2 \int_{\mathbb{R}^3} (\varepsilon^2|\nabla_{\mathbf{y}}\mathbf{u}(\mathbf{y})|^2 + |\mathbf{u}(\mathbf{y})|^2)\, d\mathbf{y}, \quad \mathbf{u} \in \mathfrak{G}^1.$$

This proves (9.14). By duality, (9.14) implies that the operator $[W^\varepsilon]$ acts continuously from $\mathfrak{G}$ to $\mathfrak{G}^{-1}$, and

$$\|[W^\varepsilon]\|_{\mathfrak{G}\to\mathfrak{G}^{-1}} \le \widehat{C}_3, \quad 0 < \varepsilon \le 1. \tag{9.16}$$

The resolvent $(\mathcal{L}^0 + I)^{-1}$, viewed as a pseudodifferential operator of order -2 with the symbol $(l^0(\boldsymbol{\xi}) + \mathbf{1})^{-1}$, continuously maps $\mathfrak{G}^{-1}$ into $\mathfrak{G}^1$, and we have

$$\|(\mathcal{L}^0 + I)^{-1}\|_{\mathfrak{G}^{-1}\to\mathfrak{G}^1} \le \max\{c_*^{-1}, 1\} \tag{9.17}$$

by (8.12). From (9.14), (9.16), and (9.17) it follows that the norm of the operator (9.13) is uniformly bounded:

$$\|(W^\varepsilon)^*(\mathcal{L}^0 + I)^{-1}W^\varepsilon\|_{\mathfrak{G}\to\mathfrak{G}} \le \widehat{C}_3^2 \max\{c_*^{-1}, 1\}, \quad 0 < \varepsilon \le 1. \tag{9.18}$$

By (9.18), it suffices to find the limit

$$\lim_{\varepsilon\to 0}((W^\varepsilon)^*(\mathcal{L}^0 + I)^{-1}W^\varepsilon\mathbf{F}, \mathbf{G})_{\mathfrak{G}}, \quad \mathbf{F}, \mathbf{G} \in C_0^\infty(\mathbb{R}^3; \mathbb{C}^3). \tag{9.19}$$

By the mean value property (Proposition 9.3), we have

$$(w, \mathfrak{G})\text{-}\lim_{\varepsilon\to 0} W^\varepsilon\mathbf{F} = \overline{W}\mathbf{F}, \quad (w, \mathfrak{G})\text{-}\lim_{\varepsilon\to 0} W^\varepsilon\mathbf{G} = \overline{W}\mathbf{G}.$$

Let $\zeta \in C_0^\infty(\mathbb{R}^3)$ be such that $\zeta(\mathbf{x}) = 1$ for $\mathbf{x} \in \operatorname{supp} \mathbf{F}$. Then the operator $(\mathcal{L}^0 + I)^{-1}$ in (9.19) can be replaced by the *compact* operator $(\mathcal{L}^0 + I)^{-1}[\zeta]$. Hence, $(\mathcal{L}^0 + I)^{-1} W^\varepsilon \mathbf{F}$ strongly converges in $\mathfrak{G}$ to $(\mathcal{L}^0 + I)^{-1}\overline{W}\mathbf{F}$. Now, it is clear that the limit in (9.19) coincides with $((\mathcal{L}^0 + I)^{-1}\overline{W}\mathbf{F}, \overline{W}\mathbf{G})_{\mathfrak{G}}$. This proves (9.12). $\square$

The following theorem can be proved by analogy.

Theorem 9.5. *Under the conditions of Theorem* 9.2, *there exists a weak operator limit for the operator* $\mathcal{P}(s^\varepsilon)(\mathcal{L}_\varepsilon + I)^{-1}$:

$$(w, \mathfrak{G} \to \mathfrak{G})\text{-}\lim_{\varepsilon \to 0} \mathcal{P}(s^\varepsilon)(\mathcal{L}_\varepsilon + I)^{-1} = \overline{W}^* \mathcal{P}(s^0)(\mathcal{L}^0 + I)^{-1}\overline{W}.$$

§10. Adaptation of the results of §9 to applications to the Maxwell operator

10.1. For further application to the homogenization problem for the Maxwell operator, we need to reformulate the results of §9 in terms natural for the following problem:

$$\operatorname{curl}(h^\varepsilon)^{-1}\operatorname{curl}(s^\varepsilon)^{-1}\boldsymbol{\psi}_\varepsilon + \boldsymbol{\psi}_\varepsilon = \mathbf{F}, \quad \operatorname{div}\boldsymbol{\psi}_\varepsilon = 0, \tag{10.1}$$

where $\mathbf{F} \in J(\mathbf{1})$, i.e., $\mathbf{F} \in \mathfrak{G}$ and $\operatorname{div}\mathbf{F} = 0$. We are also interested in the behavior of the function $\boldsymbol{\phi}_\varepsilon = (s^\varepsilon)^{-1}\boldsymbol{\psi}_\varepsilon$, which is the solution of the problem

$$\operatorname{curl}(h^\varepsilon)^{-1}\operatorname{curl}\boldsymbol{\phi}_\varepsilon + s^\varepsilon\boldsymbol{\phi}_\varepsilon = \mathbf{F}, \quad \operatorname{div} s^\varepsilon\boldsymbol{\phi}_\varepsilon = 0. \tag{10.2}$$

Introducing $\mathbf{g}_\varepsilon = (s^\varepsilon)^{-1/2}\boldsymbol{\psi}_\varepsilon$, we see that

$$(s^\varepsilon)^{-1/2}\operatorname{curl}(h^\varepsilon)^{-1}\operatorname{curl}(s^\varepsilon)^{-1/2}\mathbf{g}_\varepsilon + \mathbf{g}_\varepsilon = (s^\varepsilon)^{-1/2}\mathbf{F}, \quad \operatorname{div}(s^\varepsilon)^{1/2}\mathbf{g}_\varepsilon = 0,$$

which can be rewritten as

$$\mathcal{L}_\varepsilon\mathbf{g}_\varepsilon + \mathbf{g}_\varepsilon = (s^\varepsilon)^{-1/2}\mathbf{F}, \quad \mathbf{g}_\varepsilon \in J(s^\varepsilon).$$

For the right-hand side, we automatically have $(s^\varepsilon)^{-1/2}\mathbf{F} \in J(s^\varepsilon)$. Then

$$\mathbf{g}_\varepsilon = \mathcal{P}(s^\varepsilon)(\mathcal{L}_\varepsilon + I)^{-1}(s^\varepsilon)^{-1/2}\mathbf{F}, \tag{10.3}$$

$$\boldsymbol{\psi}_\varepsilon = (s^\varepsilon)^{1/2}\mathbf{g}_\varepsilon, \quad \boldsymbol{\phi}_\varepsilon = (s^\varepsilon)^{-1/2}\mathbf{g}_\varepsilon. \tag{10.4}$$

By (9.11), the following functions play the role of approximations for the functions (10.4):

$$\widetilde{\boldsymbol{\psi}}_\varepsilon = (s^\varepsilon)^{1/2}(W^\varepsilon)^*(\mathcal{L}^0 + I)^{-1}\mathcal{P}(s^0)W^\varepsilon(s^\varepsilon)^{-1/2}\mathbf{F}, \tag{10.5}$$

$$\widetilde{\boldsymbol{\phi}}_\varepsilon = (s^\varepsilon)^{-1}\widetilde{\boldsymbol{\psi}}_\varepsilon. \tag{10.6}$$

Moreover, from (9.11) and (10.3)–(10.6) it follows that

$$\|\boldsymbol{\psi}_\varepsilon - \widetilde{\boldsymbol{\psi}}_\varepsilon\|_{\mathfrak{G}} \le \mathcal{C}_8\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1, \ \mathcal{C}_8 = \mathcal{C}_7\|s\|_{L_\infty}^{1/2}\|s^{-1}\|_{L_\infty}^{1/2}, \tag{10.7}$$

$$\|\boldsymbol{\phi}_\varepsilon - \widetilde{\boldsymbol{\phi}}_\varepsilon\|_{\mathfrak{G}} \le \mathcal{C}_9\varepsilon\|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1, \ \mathcal{C}_9 = \mathcal{C}_7\|s^{-1}\|_{L_\infty}. \tag{10.8}$$

10.2. Now, we transform expressions (10.5) and (10.6). Since (see (7.18) and (7.19)) the matrix $W^*(\mathbf{x})$ has columns $\mathbf{g}_j(\mathbf{x}) = s(\mathbf{x})^{1/2}(\mathbf{C}_j + \nabla\Phi_j(\mathbf{x}))$, $j = 1, 2, 3$, the matrix $\widetilde{W}^*(\mathbf{x}) := s(\mathbf{x})^{-1/2}W^*(\mathbf{x})$ *has columns* $\mathbf{C}_j + \nabla\Phi_j(\mathbf{x})$, $j = 1, 2, 3$. Relations (10.5) and (10.6) can be rewritten as

$$\widetilde{\boldsymbol{\psi}}_\varepsilon = s^\varepsilon(\widetilde{W}^\varepsilon)^*\boldsymbol{\omega}_\varepsilon, \quad \widetilde{\boldsymbol{\phi}}_\varepsilon = (\widetilde{W}^\varepsilon)^*\boldsymbol{\omega}_\varepsilon, \tag{10.9}$$

where $\boldsymbol{\omega}_\varepsilon = (\mathcal{L}^0 + I)^{-1}\mathcal{P}(s^0)\widetilde{W}^\varepsilon\mathbf{F}$. We denote

$$\boldsymbol{\psi}_\varepsilon^0 := (s^0)^{1/2}\boldsymbol{\omega}_\varepsilon, \quad \boldsymbol{\phi}_\varepsilon^0 := (s^0)^{-1}\boldsymbol{\psi}_\varepsilon^0 = (s^0)^{-1/2}\boldsymbol{\omega}_\varepsilon. \tag{10.10}$$

Then $\boldsymbol{\psi}_\varepsilon^0$ is the solution of the problem

$$\operatorname{curl}(h^0)^{-1}\operatorname{curl}(s^0)^{-1}\boldsymbol{\psi}_\varepsilon^0 + \boldsymbol{\psi}_\varepsilon^0 = (s^0)^{1/2}\mathcal{P}(s^0)\widetilde{W}^\varepsilon\mathbf{F}, \quad \operatorname{div}\boldsymbol{\psi}_\varepsilon^0 = 0. \tag{10.11}$$

Let $\mathfrak{G}((s^0)^{-1}) = L_2(\mathbb{R}^3;\mathbb{C}^3;(s^0)^{-1})$ denote the L_2-space with the weight $(s^0)^{-1}$. The inner product in this space is given by $(\mathbf{u},\mathbf{v})_{\mathfrak{G}((s^0)^{-1})} = ((s^0)^{-1}\mathbf{u},\mathbf{v})_{\mathfrak{G}}$. The set $J(\mathbf{1}) = \{\mathbf{u}\in\mathfrak{G} : \operatorname{div}\mathbf{u} = 0\}$ is a closed subspace in $\mathfrak{G}((s^0)^{-1})$. Let $\widetilde{\mathcal{P}}(s^0)$ be the orthogonal projection in $\mathfrak{G}((s^0)^{-1})$ onto $J(\mathbf{1})$. It is easily seen that $(s^0)^{1/2}\mathcal{P}(s^0) = \widetilde{\mathcal{P}}(s^0)(s^0)^{1/2}$. Then the right-hand side in (10.11) coincides with $\widetilde{\mathcal{P}}(s^0)(s^0)^{1/2}\widetilde{W}^\varepsilon\mathbf{F}$. *By* $V^*(\mathbf{x})$ *we denote the matrix with the columns* $\mathbf{e}_j+\nabla\Phi_{\mathbf{e}_j}(\mathbf{x})$, $j=1,2,3$. It is elementary to check that $\widetilde{W}^*(\mathbf{x}) = V^*(\mathbf{x})(s^0)^{-1/2}$, whence $(s^0)^{1/2}\widetilde{W}(\mathbf{x}) = V(\mathbf{x})$. Therefore, the right-hand side in (10.11) coincides with $\widetilde{\mathcal{P}}(s^0)V^\varepsilon\mathbf{F}$. As a result, (10.11) can be rewritten as

$$\operatorname{curl}(h^0)^{-1}\operatorname{curl}(s^0)^{-1}\boldsymbol{\psi}^0_\varepsilon + \boldsymbol{\psi}^0_\varepsilon = \widetilde{\mathcal{P}}(s^0)V^\varepsilon\mathbf{F}, \quad \operatorname{div}\boldsymbol{\psi}^0_\varepsilon = 0. \tag{10.12}$$

Note that the right-hand side in (10.12) belongs to the class $J^{-1}(\mathbf{1}) := \{\mathbf{v}\in\mathfrak{G}^{-1},\ \operatorname{div}\mathbf{v} = 0\}$. Indeed, by Corollary 8.3(1°), the operator $[V]$ acts continuously from $\mathfrak{G}$ to $\mathfrak{G}^{-1}$. The same is true for $[V^\varepsilon]$. Then $V^\varepsilon\mathbf{F}\in\mathfrak{G}^{-1}$. The projection $\widetilde{\mathcal{P}}(s^0)$ acts as a zeroth-order pseudodifferential operator, and, by continuity, extends to a continuous operator in $\mathfrak{G}^{-1}$. The image $\widetilde{\mathcal{P}}(s^0)\mathfrak{G}^{-1}$ coincides with $J^{-1}(\mathbf{1})$. Thus, for the solution of problem (10.12), we have $\boldsymbol{\psi}^0_\varepsilon\in\mathfrak{G}^1$. Also, $\boldsymbol{\phi}^0_\varepsilon\in\mathfrak{G}^1$ by (10.10).

Now, (10.9) and (10.10) imply that

$$\widetilde{\boldsymbol{\psi}}_\varepsilon = s^\varepsilon(V^\varepsilon)^*(s^0)^{-1}\boldsymbol{\psi}^0_\varepsilon, \quad \widetilde{\boldsymbol{\phi}}_\varepsilon = (V^\varepsilon)^*\boldsymbol{\phi}^0_\varepsilon. \tag{10.13}$$

10.3. The matrix $V^*(\mathbf{x})$ can be represented as $V^*(\mathbf{x}) = \mathbf{1}+\widetilde{V}^*(\mathbf{x})$, where $\widetilde{V}^*(\mathbf{x})$ is the *matrix with the columns* $\nabla\Phi_{\mathbf{e}_j}(\mathbf{x})$, $j=1,2,3$. Obviously, $\widetilde{V}^*(\mathbf{x})$ *has zero mean value*: $\int_\Omega\widetilde{V}^*(\mathbf{x})\,d\mathbf{x} = 0$. Then the right-hand side of (10.12) can be written as $\widetilde{\mathcal{P}}(s^0)V^\varepsilon\mathbf{F} = \mathbf{F}+\widetilde{\mathcal{P}}(s^0)\widetilde{V}^\varepsilon\mathbf{F}$. Naturally, the solution $\boldsymbol{\psi}^0_\varepsilon$ of problem (10.12) can be represented as

$$\boldsymbol{\psi}^0_\varepsilon = \boldsymbol{\psi}_0 + \widehat{\boldsymbol{\psi}}_\varepsilon, \tag{10.14}$$

where $\boldsymbol{\psi}_0$ does not depend on ε and satisfies the “homogenized” problem

$$\operatorname{curl}(h^0)^{-1}\operatorname{curl}(s^0)^{-1}\boldsymbol{\psi}_0 + \boldsymbol{\psi}_0 = \mathbf{F}, \quad \operatorname{div}\boldsymbol{\psi}_0 = 0, \tag{10.15}$$

and $\widehat{\boldsymbol{\psi}}_\varepsilon$ is the solution of the “correction” problem

$$\operatorname{curl}(h^0)^{-1}\operatorname{curl}(s^0)^{-1}\widehat{\boldsymbol{\psi}}_\varepsilon + \widehat{\boldsymbol{\psi}}_\varepsilon = \widetilde{\mathcal{P}}(s^0)\widetilde{V}^\varepsilon\mathbf{F}, \quad \operatorname{div}\widehat{\boldsymbol{\psi}}_\varepsilon = 0. \tag{10.16}$$

Since the right-hand side of (10.16) belongs to $J^{-1}(\mathbf{1})$, we have $\widehat{\boldsymbol{\psi}}_\varepsilon\in\mathfrak{G}^1$. Note that $\widehat{\boldsymbol{\psi}}_\varepsilon$ weakly tends to zero in $\mathfrak{G}$. This can easily be checked by analogy with the proof of Theorem 9.4. One should use the presence of the factor $\widetilde{V}^\varepsilon$ on the right-hand side of (10.16) and the fact that $\widetilde{V}^\varepsilon$ weakly tends to zero in $L_{2,\mathrm{loc}}$ (by the mean value property).

We put

$$\boldsymbol{\phi}_0 := (s^0)^{-1}\boldsymbol{\psi}_0, \quad \widehat{\boldsymbol{\phi}}_\varepsilon := (s^0)^{-1}\widehat{\boldsymbol{\psi}}_\varepsilon. \tag{10.17}$$

Then, by (10.10) and (10.14),

$$\boldsymbol{\phi}^0_\varepsilon = \boldsymbol{\phi}_0 + \widehat{\boldsymbol{\phi}}_\varepsilon. \tag{10.18}$$

Note that $\widehat{\boldsymbol{\phi}}_\varepsilon\in\mathfrak{G}^1$, $\widehat{\boldsymbol{\phi}}_\varepsilon$ weakly tends to zero in $\mathfrak{G}$, and $\boldsymbol{\phi}_0$ is the solution of the “homogenized” problem

$$\operatorname{curl}(h^0)^{-1}\operatorname{curl}\boldsymbol{\phi}_0 + s^0\boldsymbol{\phi}_0 = \mathbf{F}, \quad \operatorname{div}s^0\boldsymbol{\phi}_0 = 0. \tag{10.19}$$

Next, the mean value of the matrix $s(\mathbf{x})V^*(\mathbf{x})$ with the columns $s(\mathbf{x})(\mathbf{e}_j+\nabla\Phi_{\mathbf{e}_j}(\mathbf{x}))$, $j=1,2,3$, is equal to s^0 (see (4.12)). We put

$$G(\mathbf{x}) := s(\mathbf{x})V^*(\mathbf{x})(s^0)^{-1} - \mathbf{1}. \tag{10.20}$$

Then the matrix $G(\mathbf{x})$ has zero mean value. Now, using (10.14), (10.18), (10.20), we can rewrite (10.13) as

$$\widetilde{\boldsymbol{\psi}}_\varepsilon = (\mathbf{1} + G^\varepsilon)(\boldsymbol{\psi}_0 + \widehat{\boldsymbol{\psi}}_\varepsilon), \tag{10.21}$$

$$\widetilde{\boldsymbol{\phi}}_\varepsilon = (\mathbf{1} + (\widetilde{V}^\varepsilon)^*)(\boldsymbol{\phi}_0 + \widehat{\boldsymbol{\phi}}_\varepsilon). \tag{10.22}$$

Once again, from (10.21) and (10.22) we see directly that $\widetilde{\boldsymbol{\psi}}_\varepsilon, \widetilde{\boldsymbol{\phi}}_\varepsilon \in \mathfrak{G}$. Indeed, the functions $\boldsymbol{\psi}_0 + \widehat{\boldsymbol{\psi}}_\varepsilon$ and $\boldsymbol{\phi}_0 + \widehat{\boldsymbol{\phi}}_\varepsilon$ belong to $\mathfrak{G}^1$, and the operators of multiplication by G^ε and by $(\widetilde{V}^\varepsilon)^*$ act continuously from $\mathfrak{G}^1$ to $\mathfrak{G}$ by Corollary 8.3(1°).

In (10.21) and (10.22), after opening the parentheses, we distinguish the terms $\boldsymbol{\psi}_0$ and $\boldsymbol{\phi}_0$, which are independent of ε and play the role of the weak limits for $\boldsymbol{\psi}_\varepsilon$, $\boldsymbol{\phi}_\varepsilon$:

$$(w, \mathfrak{G})\text{-}\lim_{\varepsilon\to 0} \boldsymbol{\psi}_\varepsilon = \boldsymbol{\psi}_0, \quad (w, \mathfrak{G})\text{-}\lim_{\varepsilon\to 0} \boldsymbol{\phi}_\varepsilon = \boldsymbol{\phi}_0.$$

Indeed, all other summands in (10.21) and (10.22), namely, $\widehat{\boldsymbol{\psi}}_\varepsilon$, $\widehat{\boldsymbol{\phi}}_\varepsilon$, $G^\varepsilon(\boldsymbol{\psi}_0 + \widehat{\boldsymbol{\psi}}_\varepsilon)$, and $(\widetilde{V}^\varepsilon)^*(\boldsymbol{\phi}_0 + \widehat{\boldsymbol{\phi}}_\varepsilon)$ weakly tend to zero in $\mathfrak{G}$. This can be proved by analogy with the proof of Theorem 9.4. One should use the fact that, by the mean value property, the functions $\widetilde{V}^\varepsilon$ and G^ε weakly tend to zero in $L_{2,\mathrm{loc}}$.

10.4. We summarize. Combining estimates (10.7) and (10.8) with the representations (10.21) and (10.22), we deduce the following theorem.

Theorem 10.1. *Let $\boldsymbol{\psi}_\varepsilon$ be the solution of problem* (10.1), *where $\mathbf{F} \in J(\mathbf{1})$, and let $\boldsymbol{\phi}_\varepsilon = (s^\varepsilon)^{-1}\boldsymbol{\psi}_\varepsilon$. Let $\widetilde{V}^*(\mathbf{x})$ be the matrix with the columns $\nabla\Phi_{\mathbf{e}_j}(\mathbf{x})$, $j = 1, 2, 3$, and let $G(\mathbf{x})$ be the matrix defined by* (10.20). *Let $\widetilde{\mathcal{P}}(s^0)$ be the orthogonal projection in the weighted space $\mathfrak{G}((s^0)^{-1})$ onto the solenoidal subspace $J(\mathbf{1})$. If $\boldsymbol{\psi}_0$ is the solution of the "homogenized" problem* (10.15), *$\widehat{\boldsymbol{\psi}}_\varepsilon$ is the solution of the "correction" problem* (10.16), *and $\boldsymbol{\phi}_0$ and $\widehat{\boldsymbol{\phi}}_\varepsilon$ are defined by* (10.17), *then the functions $\widetilde{\boldsymbol{\psi}}_\varepsilon$ and $\widetilde{\boldsymbol{\phi}}_\varepsilon$ defined by* (10.21) *and* (10.22) *play the role of approximations for $\boldsymbol{\psi}_\varepsilon$ and $\boldsymbol{\phi}_\varepsilon$. We have*

$$\|\boldsymbol{\psi}_\varepsilon - \widetilde{\boldsymbol{\psi}}_\varepsilon\|_{\mathfrak{G}} \le \mathcal{C}_8 \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1,$$
$$\|\boldsymbol{\phi}_\varepsilon - \widetilde{\boldsymbol{\phi}}_\varepsilon\|_{\mathfrak{G}} \le \mathcal{C}_9 \varepsilon \|\mathbf{F}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1.$$

The constants $\mathcal{C}_8$ and $\mathcal{C}_9$ depend on the L_∞-norms of the coefficients s, s^{-1}, h, h^{-1} and on the parameters of the lattice Γ. The solutions $\boldsymbol{\psi}_\varepsilon$ and $\boldsymbol{\phi}_\varepsilon$ weakly tend in $\mathfrak{G}$ to $\boldsymbol{\psi}_0$ and $\boldsymbol{\phi}_0$, respectively.

§11. Weak convergence of solutions and flows

11.1. Consider problem (10.2). As was shown in Theorem 10.1, the solutions $\boldsymbol{\phi}_\varepsilon$ weakly tend to the solution $\boldsymbol{\phi}_0$ of the "homogenized" problem (10.19). In this section we show that, moreover, the functions $\operatorname{curl}\boldsymbol{\phi}_\varepsilon$ converge weakly to $\operatorname{curl}\boldsymbol{\phi}_0$, and the "flows" $(h^\varepsilon)^{-1}\operatorname{curl}\boldsymbol{\phi}_\varepsilon$ converge weakly to the "homogenized flow" $(h^0)^{-1}\operatorname{curl}\boldsymbol{\phi}_0$. The results of such type are traditional for homogenization theory. Close results (though there are differences) can be found, e.g., in [BeLP].

Our goal in this section is the proof of the following theorem.

Theorem 11.1. *Let $\boldsymbol{\phi}_\varepsilon$ be the* (*weak*) *solution of the problem*

$$\operatorname{curl}(h^\varepsilon)^{-1}\operatorname{curl}\boldsymbol{\phi}_\varepsilon + s^\varepsilon\boldsymbol{\phi}_\varepsilon = \mathbf{F}, \quad \operatorname{div} s^\varepsilon\boldsymbol{\phi}_\varepsilon = 0, \tag{11.1}$$

where $\mathbf{F} \in J(\mathbf{1})$. Let $\boldsymbol{\phi}_0$ be the solution of the "homogenized" problem

$$\operatorname{curl}(h^0)^{-1}\operatorname{curl}\boldsymbol{\phi}_0 + s^0\boldsymbol{\phi}_0 = \mathbf{F}, \quad \operatorname{div} s^0\boldsymbol{\phi}_0 = 0. \tag{11.2}$$

Then, as $\varepsilon \to 0$, *we have the following*:

1°. *the solutions* $\boldsymbol{\phi}_\varepsilon$ *weakly tend in* $\mathfrak{G}$ *to* $\boldsymbol{\phi}_0$;

2°. *the functions* $\boldsymbol{\psi}_\varepsilon := s^\varepsilon \boldsymbol{\phi}_\varepsilon$ *weakly tend in* $\mathfrak{G}$ *to* $\boldsymbol{\psi}_0 := s^0 \boldsymbol{\phi}_0$;

3°. *the functions* $\operatorname{curl} \boldsymbol{\phi}_\varepsilon$ *weakly tend in* $\mathfrak{G}$ *to* $\operatorname{curl} \boldsymbol{\phi}_0$;

4°. *the flows* $\mathbf{p}_\varepsilon := (h^\varepsilon)^{-1} \operatorname{curl} \boldsymbol{\phi}_\varepsilon$ *weakly tend in* $\mathfrak{G}$ *to* $\mathbf{p}_0 := (h^0)^{-1} \operatorname{curl} \boldsymbol{\phi}_0$.

Though statements 1° and 2° of Theorem 11.1 are contained in Theorem 10.1, here we present a simple proof of these statements, which does not depend on the preceding material. This emphasizes the fact that the last statement of Theorem 10.1 is much simpler than the statements concerning approximations of solutions in the $\mathfrak{G}$-norm.

11.2. We proceed to the proof of Theorem 11.1. Equation (11.1) means that $\boldsymbol{\phi}_\varepsilon \in \mathfrak{G}$, $\operatorname{curl} \boldsymbol{\phi}_\varepsilon \in \mathfrak{G}$, and

$$((h^\varepsilon)^{-1} \operatorname{curl} \boldsymbol{\phi}_\varepsilon, \operatorname{curl} \mathbf{g})_{\mathfrak{G}} + (s^\varepsilon \boldsymbol{\phi}_\varepsilon, \mathbf{g})_{\mathfrak{G}} = (\mathbf{F}, \mathbf{g})_{\mathfrak{G}} \tag{11.3}$$

for any vector-valued function $\mathbf{g} \in \mathfrak{G}$ such that $\operatorname{curl} \mathbf{g} \in \mathfrak{G}$.

We put $\mathbf{g} = \boldsymbol{\phi}_\varepsilon$ in (11.3). Then

$$\|(h^\varepsilon)^{-1/2} \operatorname{curl} \boldsymbol{\phi}_\varepsilon\|^2_{\mathfrak{G}} + \|(s^\varepsilon)^{1/2} \boldsymbol{\phi}_\varepsilon\|^2_{\mathfrak{G}} \le \|\mathbf{F}\|_{\mathfrak{G}} \|\boldsymbol{\phi}_\varepsilon\|_{\mathfrak{G}}. \tag{11.4}$$

From (11.4) it follows that

$$\|\boldsymbol{\phi}_\varepsilon\|_{\mathfrak{G}} \le \|s^{-1}\|_{L_\infty} \|\mathbf{F}\|_{\mathfrak{G}}, \tag{11.5}$$

$$\|\operatorname{curl} \boldsymbol{\phi}_\varepsilon\|_{\mathfrak{G}} \le \|h\|^{1/2}_{L_\infty} \|s^{-1}\|^{1/2}_{L_\infty} \|\mathbf{F}\|_{\mathfrak{G}}, \tag{11.6}$$

whence

$$\|\boldsymbol{\psi}_\varepsilon\|_{\mathfrak{G}} \le \|s\|_{L_\infty} \|s^{-1}\|_{L_\infty} \|\mathbf{F}\|_{\mathfrak{G}}, \tag{11.7}$$

$$\|\mathbf{p}_\varepsilon\|_{\mathfrak{G}} \le \|h^{-1}\|_{L_\infty} \|h\|^{1/2}_{L_\infty} \|s^{-1}\|^{1/2}_{L_\infty} \|\mathbf{F}\|_{\mathfrak{G}}. \tag{11.8}$$

By the uniform boundedness properties (11.5)–(11.8), for some subsequence $\varepsilon_j \to 0$ the following limits exist:

$$(w, \mathfrak{G})\text{-}\lim_{\varepsilon_j \to 0} \boldsymbol{\phi}_{\varepsilon_j} = \boldsymbol{\phi} \in \mathfrak{G}, \tag{11.9}$$

$$(w, \mathfrak{G})\text{-}\lim_{\varepsilon_j \to 0} \boldsymbol{\psi}_{\varepsilon_j} = \boldsymbol{\psi} \in \mathfrak{G}, \tag{11.10}$$

$$(w, \mathfrak{G})\text{-}\lim_{\varepsilon_j \to 0} \operatorname{curl} \boldsymbol{\phi}_{\varepsilon_j} = \operatorname{curl} \boldsymbol{\phi} \in \mathfrak{G}, \tag{11.11}$$

$$(w, \mathfrak{G})\text{-}\lim_{\varepsilon_j \to 0} \mathbf{p}_{\varepsilon_j} = \mathbf{p} \in \mathfrak{G}. \tag{11.12}$$

Our goal is to show that $\boldsymbol{\phi} = \boldsymbol{\phi}_0$, $\boldsymbol{\psi} = \boldsymbol{\psi}_0$, $\mathbf{p} = \mathbf{p}_0$. Then relations (11.9)–(11.12) will imply all the statements of the theorem.

We pass to the limit as $\varepsilon_j \to 0$ in identity (11.3). By (11.10) and (11.12), we have

$$(\mathbf{p}, \operatorname{curl} \mathbf{g})_{\mathfrak{G}} + (\boldsymbol{\psi}, \mathbf{g})_{\mathfrak{G}} = (\mathbf{F}, \mathbf{g})_{\mathfrak{G}}, \quad \mathbf{g} \in \mathfrak{G}, \ \operatorname{curl} \mathbf{g} \in \mathfrak{G}. \tag{11.13}$$

Consequently,

$$\operatorname{curl} \mathbf{p} = \mathbf{F} - \boldsymbol{\psi} \in \mathfrak{G}. \tag{11.14}$$

Next, passing to the limit in the relation $\operatorname{div} \boldsymbol{\psi}_\varepsilon = 0$ (see (11.1)), we obtain

$$\operatorname{div} \boldsymbol{\psi} = 0. \tag{11.15}$$

11.3. We recall the representation (4.24) for the effective matrix h^0:

$$h^0\mathbf{C} = |\Omega|^{-1}\int_\Omega h(\mathbf{x})(\mathbf{C}+\nabla\Psi_{\mathbf{C}}(\mathbf{x}))\,d\mathbf{x}, \quad \mathbf{C}\in\mathbb{C}^3, \tag{11.16}$$

where $\Psi_{\mathbf{C}} \in \widetilde{H}^1(\Omega)$ is a solution of the equation $\operatorname{div} h(\mathbf{x})(\mathbf{C}+\nabla\Psi_{\mathbf{C}}) = 0$. We fix the solution $\Psi_{\mathbf{C}}$ by the condition $\int_\Omega \Psi_{\mathbf{C}}(\mathbf{x})\,d\mathbf{x} = 0$. We put $\boldsymbol{\rho}(\mathbf{x}) = \mathbf{C}+\nabla\Psi_{\mathbf{C}}(\mathbf{x})$, and extend $\boldsymbol{\rho}(\mathbf{x})$ up to a Γ-periodic function $\boldsymbol{\rho} \in \mathfrak{G}_{\mathrm{loc}} = L_{2,\mathrm{loc}}(\mathbb{R}^3;\mathbb{C}^3)$. The mean value property (see Proposition 9.3) and identity (11.16) show that

$$(w,\mathfrak{G}_{\mathrm{loc}})\text{-}\lim_{\varepsilon\to 0} h^\varepsilon\boldsymbol{\rho}^\varepsilon = h^0\mathbf{C}. \tag{11.17}$$

Lemma 11.2. *Let $\zeta \in C_0^\infty(\mathbb{R}^3)$. Then*

$$\lim_{\varepsilon_j\to 0}(\zeta\boldsymbol{\rho}^\varepsilon, \operatorname{curl}\boldsymbol{\phi}_\varepsilon)_{\mathfrak{G}} = (\zeta\mathbf{C}, \operatorname{curl}\boldsymbol{\phi})_{\mathfrak{G}}. \tag{11.18}$$

Proof. The expression under the limit sign can be represented as the sum

$$(\zeta\mathbf{C}, \operatorname{curl}\boldsymbol{\phi}_\varepsilon)_{\mathfrak{G}} + (\zeta(\nabla\Psi_{\mathbf{C}})^\varepsilon, \operatorname{curl}\boldsymbol{\phi}_\varepsilon)_{\mathfrak{G}}. \tag{11.19}$$

By (11.11), the limit of the first term (with respect to the subsequence $\varepsilon_j \to 0$) coincides with the right-hand side of (11.18). Thus, it suffices to show that the second term in (11.19) tends to zero. We have

$$\begin{aligned}\Theta_1(\varepsilon) &:= (\zeta(\nabla\Psi_{\mathbf{C}})^\varepsilon, \operatorname{curl}\boldsymbol{\phi}_\varepsilon)_{\mathfrak{G}} = \varepsilon(\zeta\nabla(\Psi_{\mathbf{C}}^\varepsilon), \operatorname{curl}\boldsymbol{\phi}_\varepsilon)_{\mathfrak{G}}\\ &= \varepsilon(\nabla(\zeta\Psi_{\mathbf{C}}^\varepsilon) - \Psi_{\mathbf{C}}^\varepsilon\nabla\zeta, \operatorname{curl}\boldsymbol{\phi}_\varepsilon)_{\mathfrak{G}} = -\varepsilon(\Psi_{\mathbf{C}}^\varepsilon\nabla\zeta, \operatorname{curl}\boldsymbol{\phi}_\varepsilon)_{\mathfrak{G}},\end{aligned}$$

whence

$$|\Theta_1(\varepsilon)| \le \varepsilon\|\operatorname{curl}\boldsymbol{\phi}_\varepsilon\|_{\mathfrak{G}}\|\Psi_{\mathbf{C}}^\varepsilon\nabla\zeta\|_{\mathfrak{G}}. \tag{11.20}$$

By (11.6), the norms $\|\operatorname{curl}\boldsymbol{\phi}_\varepsilon\|_{\mathfrak{G}}$ are uniformly bounded. Next, by the mean value property, the functions $\Psi_{\mathbf{C}}^\varepsilon$ weakly tend to zero in $L_{2,\mathrm{loc}}(\mathbb{R}^3)$ as $\varepsilon \to 0$. Since $\zeta \in C_0^\infty(\mathbb{R}^3)$, it follows that the functions $\Psi_{\mathbf{C}}^\varepsilon\nabla\zeta$ weakly tend to zero in $\mathfrak{G}$. Hence, the norms $\|\Psi_{\mathbf{C}}^\varepsilon\nabla\zeta\|_{\mathfrak{G}}$ are uniformly bounded. Now, relation (11.20) implies that $\lim_{\varepsilon\to 0}\Theta_1(\varepsilon) = 0$. □

11.4. The obvious identity

$$(\zeta\boldsymbol{\rho}^\varepsilon, \operatorname{curl}\boldsymbol{\phi}_\varepsilon)_{\mathfrak{G}} = (\zeta h^\varepsilon\boldsymbol{\rho}^\varepsilon, (h^\varepsilon)^{-1}\operatorname{curl}\boldsymbol{\phi}_\varepsilon)_{\mathfrak{G}} = (\zeta h^\varepsilon\boldsymbol{\rho}^\varepsilon, \mathbf{p}_\varepsilon)_{\mathfrak{G}}$$

allows us to use a different method to calculate the limit in (11.18). Namely,

$$(\zeta\boldsymbol{\rho}^\varepsilon, \operatorname{curl}\boldsymbol{\phi}_\varepsilon)_{\mathfrak{G}} = (\zeta h^\varepsilon\boldsymbol{\rho}^\varepsilon, \mathbf{p})_{\mathfrak{G}} + \Theta_2(\varepsilon), \tag{11.21}$$

$$\Theta_2(\varepsilon) := (\zeta h^\varepsilon\boldsymbol{\rho}^\varepsilon, \mathbf{p}_\varepsilon - \mathbf{p})_{\mathfrak{G}}. \tag{11.22}$$

By (11.17),

$$\lim_{\varepsilon\to 0}(\zeta h^\varepsilon\boldsymbol{\rho}^\varepsilon, \mathbf{p})_{\mathfrak{G}} = (\zeta h^0\mathbf{C}, \mathbf{p})_{\mathfrak{G}}. \tag{11.23}$$

Lemma 11.3. *For the term $\Theta_2(\varepsilon)$ defined by* (11.22), *we have*

$$\lim_{\varepsilon_j\to 0}\Theta_2(\varepsilon) = 0. \tag{11.24}$$

Proof. Since $\operatorname{div} h(\mathbf{x})\boldsymbol{\rho}(\mathbf{x}) = 0$, where $h(\mathbf{x})\boldsymbol{\rho}(\mathbf{x})$ is a Γ-periodic vector-valued function of class $\mathfrak{G}_{\mathrm{loc}}$ with the mean value $h^0\mathbf{C}$ (see (11.16)), there exists a Γ-periodic vector-valued potential $\mathbf{A}(\mathbf{x})$ such that

$$h(\mathbf{x})\boldsymbol{\rho}(\mathbf{x}) = h^0\mathbf{C} + \operatorname{curl}\mathbf{A}(\mathbf{x}), \quad \operatorname{div}\mathbf{A}(\mathbf{x}) = 0, \quad \int_\Omega \mathbf{A}(\mathbf{x})\,d\mathbf{x} = 0, \tag{11.25}$$

and $\mathbf{A} \in H^1_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{C}^3)$. We have

$$h^\varepsilon\boldsymbol{\rho}^\varepsilon = h^0\mathbf{C} + (\operatorname{curl}\mathbf{A})^\varepsilon = h^0\mathbf{C} + \varepsilon\operatorname{curl}(\mathbf{A}^\varepsilon).$$

Then (11.22) implies that

$$\Theta_2(\varepsilon) = (\zeta h^0 \mathbf{C}, \mathbf{p}_\varepsilon - \mathbf{p})_{\mathfrak{G}} + \varepsilon(\zeta \operatorname{curl}(\mathbf{A}^\varepsilon), \mathbf{p}_\varepsilon - \mathbf{p})_{\mathfrak{G}} =: \Theta_2'(\varepsilon) + \Theta_2''(\varepsilon). \tag{11.26}$$

By (11.12),

$$\lim_{\varepsilon_j \to 0} \Theta_2'(\varepsilon) = 0. \tag{11.27}$$

We transform $\Theta_2''(\varepsilon)$:

$$\Theta_2''(\varepsilon) = \varepsilon(\operatorname{curl}(\zeta \mathbf{A}^\varepsilon), \mathbf{p}_\varepsilon - \mathbf{p})_{\mathfrak{G}} - \varepsilon((\nabla\zeta) \times \mathbf{A}^\varepsilon, \mathbf{p}_\varepsilon - \mathbf{p})_{\mathfrak{G}} =: \widetilde{\Theta}_2''(\varepsilon) + \widehat{\Theta}_2''(\varepsilon). \tag{11.28}$$

For the second term in (11.28), we have

$$|\widehat{\Theta}_2''(\varepsilon)| \le \varepsilon \|(\nabla\zeta) \times \mathbf{A}^\varepsilon\|_{\mathfrak{G}} \|\mathbf{p}_\varepsilon - \mathbf{p}\|_{\mathfrak{G}}.$$

By (11.8), the norms $\|\mathbf{p}_\varepsilon - \mathbf{p}\|_{\mathfrak{G}}$ are uniformly bounded. The norms $\|(\nabla\zeta) \times \mathbf{A}^\varepsilon\|_{\mathfrak{G}}$ are bounded due to the weak convergence of $\mathbf{A}^\varepsilon$ (to zero) in $\mathfrak{G}_{\mathrm{loc}}$. Consequently,

$$\lim_{\varepsilon \to 0} \widehat{\Theta}_2''(\varepsilon) = 0. \tag{11.29}$$

Using identities (11.3) and (11.13) with $\mathbf{g} = \zeta \mathbf{A}^\varepsilon$, we obtain

$$\begin{aligned} \widetilde{\Theta}_2''(\varepsilon) &= \varepsilon(\operatorname{curl}(\zeta \mathbf{A}^\varepsilon), (h^\varepsilon)^{-1} \operatorname{curl} \boldsymbol{\phi}_\varepsilon)_{\mathfrak{G}} - \varepsilon(\operatorname{curl}(\zeta \mathbf{A}^\varepsilon), \mathbf{p})_{\mathfrak{G}} \\ &= \varepsilon(\zeta \mathbf{A}^\varepsilon, \mathbf{F} - \boldsymbol{\psi}_\varepsilon)_{\mathfrak{G}} - \varepsilon(\zeta \mathbf{A}^\varepsilon, \mathbf{F} - \boldsymbol{\psi})_{\mathfrak{G}} = \varepsilon(\zeta \mathbf{A}^\varepsilon, \boldsymbol{\psi} - \boldsymbol{\psi}_\varepsilon)_{\mathfrak{G}}. \end{aligned} \tag{11.30}$$

By (11.7), the norms $\|\boldsymbol{\psi} - \boldsymbol{\psi}_\varepsilon\|_{\mathfrak{G}}$ are uniformly bounded. The norms $\|\zeta \mathbf{A}^\varepsilon\|_{\mathfrak{G}}$ are uniformly bounded due to the weak convergence of $\mathbf{A}^\varepsilon$ in $\mathfrak{G}_{\mathrm{loc}}$. Hence, (11.30) implies that $\lim_{\varepsilon\to 0} \widetilde{\Theta}_2''(\varepsilon) = 0$. Combining this with (11.26)–(11.29), we arrive at (11.24). $\square$

Relations (11.21), (11.23), and (11.24) show that

$$\lim_{\varepsilon_j \to 0} (\zeta \boldsymbol{\rho}^\varepsilon, \operatorname{curl} \boldsymbol{\phi}_\varepsilon)_{\mathfrak{G}} = (\zeta h^0 \mathbf{C}, \mathbf{p})_{\mathfrak{G}}. \tag{11.31}$$

Comparing (11.18) and (11.31), we obtain

$$(\zeta \mathbf{C}, \operatorname{curl} \boldsymbol{\phi})_{\mathfrak{G}} = (\zeta h^0 \mathbf{C}, \mathbf{p})_{\mathfrak{G}}, \quad \mathbf{C} \in \mathbb{C}^3, \ \ \zeta \in C_0^\infty(\mathbb{R}^3).$$

Thus,

$$\mathbf{p} = (h^0)^{-1} \operatorname{curl} \boldsymbol{\phi}. \tag{11.32}$$

11.5. We recall the representation (4.12) for the effective matrix s^0:

$$s^0 \mathbf{C} = |\Omega|^{-1} \int_\Omega s(\mathbf{x})(\mathbf{C} + \nabla \Phi_{\mathbf{C}}(\mathbf{x}))\, d\mathbf{x}, \quad \mathbf{C} \in \mathbb{C}^3, \tag{11.33}$$

where $\Phi_{\mathbf{C}} \in \widetilde{H}^1(\Omega)$ is a solution of the equation $\operatorname{div} s(\mathbf{x})(\mathbf{C} + \nabla\Phi_{\mathbf{C}}) = 0$. We fix the solution $\Phi_{\mathbf{C}}$ by the condition $\int_\Omega \Phi_{\mathbf{C}}(\mathbf{x})\, d\mathbf{x} = 0$. We put $\boldsymbol{\tau}(\mathbf{x}) = \mathbf{C} + \nabla\Phi_{\mathbf{C}}(\mathbf{x})$, and extend $\boldsymbol{\tau}(\mathbf{x})$ up to a Γ-periodic function $\boldsymbol{\tau} \in \mathfrak{G}_{\mathrm{loc}}$. The mean value property and (11.33) show that

$$(w, \mathfrak{G}_{\mathrm{loc}})\text{-}\lim_{\varepsilon \to 0} s^\varepsilon \boldsymbol{\tau}^\varepsilon = s^0 \mathbf{C}. \tag{11.34}$$

Lemma 11.4. *Let $\zeta \in C_0^\infty(\mathbb{R}^3)$. Then*

$$\lim_{\varepsilon_j \to 0} (\zeta \boldsymbol{\tau}^\varepsilon, s^\varepsilon \boldsymbol{\phi}_\varepsilon)_{\mathfrak{G}} = (\zeta \mathbf{C}, \boldsymbol{\psi})_{\mathfrak{G}}. \tag{11.35}$$

Proof. We have

$$(\zeta\boldsymbol{\tau}^\varepsilon, s^\varepsilon\boldsymbol{\phi}_\varepsilon)_{\mathfrak{G}} = (\zeta\mathbf{C}, \boldsymbol{\psi}_\varepsilon)_{\mathfrak{G}} + \Theta_3(\varepsilon), \quad \Theta_3(\varepsilon) := (\zeta(\nabla\Phi_{\mathbf{C}})^\varepsilon, \boldsymbol{\psi}_\varepsilon)_{\mathfrak{G}}. \tag{11.36}$$

By (11.10),

$$\lim_{\varepsilon_j\to 0}(\zeta\mathbf{C}, \boldsymbol{\psi}_\varepsilon)_{\mathfrak{G}} = (\zeta\mathbf{C}, \boldsymbol{\psi})_{\mathfrak{G}}. \tag{11.37}$$

We show that

$$\lim_{\varepsilon\to 0}\Theta_3(\varepsilon) = 0. \tag{11.38}$$

From (11.36) it follows that

$$\Theta_3(\varepsilon) = \varepsilon(\zeta\nabla(\Phi_{\mathbf{C}}^\varepsilon), \boldsymbol{\psi}_\varepsilon)_{\mathfrak{G}} = \varepsilon(\nabla(\zeta\Phi_{\mathbf{C}}^\varepsilon), \boldsymbol{\psi}_\varepsilon)_{\mathfrak{G}} - \varepsilon(\Phi_{\mathbf{C}}^\varepsilon(\nabla\zeta), \boldsymbol{\psi}_\varepsilon)_{\mathfrak{G}}. \tag{11.39}$$

Since $\operatorname{div}\boldsymbol{\psi}_\varepsilon = 0$ (see (11.1)), the first term on the right-hand side of (11.39) is equal to zero. Hence, (11.39) implies the estimate

$$|\Theta_3(\varepsilon)| \le \varepsilon\|\Phi_{\mathbf{C}}^\varepsilon(\nabla\zeta)\|_{\mathfrak{G}}\|\boldsymbol{\psi}_\varepsilon\|_{\mathfrak{G}}. \tag{11.40}$$

By (11.7), the norms $\|\boldsymbol{\psi}_\varepsilon\|_{\mathfrak{G}}$ are uniformly bounded. The norms $\|\Phi_{\mathbf{C}}^\varepsilon(\nabla\zeta)\|_{\mathfrak{G}}$ are uniformly bounded due to the weak convergence of $\Phi_{\mathbf{C}}^\varepsilon$ in $L_{2,\mathrm{loc}}$. Therefore, (11.40) implies (11.38). Comparing (11.36)–(11.38), we arrive at (11.35). □

Now, we calculate the limit in (11.35) in a different way. We have

$$(\zeta\boldsymbol{\tau}^\varepsilon, s^\varepsilon\boldsymbol{\phi}_\varepsilon)_{\mathfrak{G}} = (\zeta s^\varepsilon\boldsymbol{\tau}^\varepsilon, \boldsymbol{\phi})_{\mathfrak{G}} + \Theta_4(\varepsilon), \tag{11.41}$$

$$\Theta_4(\varepsilon) := (\zeta s^\varepsilon\boldsymbol{\tau}^\varepsilon, \boldsymbol{\phi}_\varepsilon - \boldsymbol{\phi})_{\mathfrak{G}}. \tag{11.42}$$

By (11.34), we obtain

$$\lim_{\varepsilon\to 0}(\zeta s^\varepsilon\boldsymbol{\tau}^\varepsilon, \boldsymbol{\phi})_{\mathfrak{G}} = (\zeta s^0\mathbf{C}, \boldsymbol{\phi})_{\mathfrak{G}}. \tag{11.43}$$

Lemma 11.5. *For the term $\Theta_4(\varepsilon)$ defined by* (11.42), *we have*

$$\lim_{\varepsilon_j\to 0}\Theta_4(\varepsilon) = 0. \tag{11.44}$$

Proof. Since $\operatorname{div} s(\mathbf{x})\boldsymbol{\tau}(\mathbf{x}) = 0$, and $s(\mathbf{x})\boldsymbol{\tau}(\mathbf{x})$ is a Γ-periodic vector-valued function of class $\mathfrak{G}_{\mathrm{loc}}$ with the mean value $s^0\mathbf{C}$, there exists a Γ-periodic vector-valued potential $\widetilde{\mathbf{A}}(\mathbf{x})$ such that

$$s(\mathbf{x})\boldsymbol{\tau}(\mathbf{x}) = s^0\mathbf{C} + \operatorname{curl}\widetilde{\mathbf{A}}(\mathbf{x}), \quad \operatorname{div}\widetilde{\mathbf{A}}(\mathbf{x}) = 0, \ \int_\Omega \widetilde{\mathbf{A}}(\mathbf{x})\,d\mathbf{x} = 0.$$

Moreover, $\widetilde{\mathbf{A}} \in H^1_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{C}^3)$. We have $s^\varepsilon\boldsymbol{\tau}^\varepsilon = s^0\mathbf{C} + \varepsilon\operatorname{curl}(\widetilde{\mathbf{A}}^\varepsilon)$. Then (11.42) implies that

$$\Theta_4(\varepsilon) = (\zeta s^0\mathbf{C}, \boldsymbol{\phi}_\varepsilon - \boldsymbol{\phi})_{\mathfrak{G}} + \varepsilon(\zeta\operatorname{curl}(\widetilde{\mathbf{A}}^\varepsilon), \boldsymbol{\phi}_\varepsilon - \boldsymbol{\phi})_{\mathfrak{G}} =: \Theta_4'(\varepsilon) + \Theta_4''(\varepsilon).$$

By (11.9), $\Theta_4'(\varepsilon) \to 0$ as $\varepsilon_j \to 0$. Next,

$$\Theta_4''(\varepsilon) = \varepsilon(\widetilde{\mathbf{A}}^\varepsilon, \operatorname{curl}\zeta(\boldsymbol{\phi}_\varepsilon - \boldsymbol{\phi}))_{\mathfrak{G}} = \varepsilon(\widetilde{\mathbf{A}}^\varepsilon, \zeta\operatorname{curl}(\boldsymbol{\phi}_\varepsilon - \boldsymbol{\phi}))_{\mathfrak{G}} + \varepsilon(\widetilde{\mathbf{A}}^\varepsilon, (\nabla\zeta)\times(\boldsymbol{\phi}_\varepsilon - \boldsymbol{\phi}))_{\mathfrak{G}}.$$

Hence,

$$|\Theta_4''(\varepsilon)| \le \varepsilon\|\zeta\widetilde{\mathbf{A}}^\varepsilon\|_{\mathfrak{G}}\|\operatorname{curl}\boldsymbol{\phi}_\varepsilon - \operatorname{curl}\boldsymbol{\phi}\|_{\mathfrak{G}} + \varepsilon\|\boldsymbol{\phi}_\varepsilon - \boldsymbol{\phi}\|_{\mathfrak{G}}\|\widetilde{\mathbf{A}}^\varepsilon \times \nabla\zeta\|_{\mathfrak{G}}.$$

By (11.5) and (11.6), the norms $\|\boldsymbol{\phi}_\varepsilon - \boldsymbol{\phi}\|_{\mathfrak{G}}$ and $\|\operatorname{curl}(\boldsymbol{\phi}_\varepsilon - \boldsymbol{\phi})\|_{\mathfrak{G}}$ are uniformly bounded, and, by the weak convergence of $\widetilde{\mathbf{A}}^\varepsilon$ in $\mathfrak{G}_{\mathrm{loc}}$, the norms $\|\zeta\widetilde{\mathbf{A}}^\varepsilon\|_{\mathfrak{G}}$ and $\|\widetilde{\mathbf{A}}^\varepsilon \times \nabla\zeta\|_{\mathfrak{G}}$ are also uniformly bounded. Consequently, $\Theta_4''(\varepsilon) \to 0$ as $\varepsilon \to 0$. □

From (11.41), (11.43), and (11.44) it follows that

$$\lim_{\varepsilon_j \to 0} (\zeta \boldsymbol{\tau}^\varepsilon, s^\varepsilon \boldsymbol{\phi}_\varepsilon)_{\mathfrak{G}} = (\zeta s^0 \mathbf{C}, \boldsymbol{\phi})_{\mathfrak{G}}. \tag{11.45}$$

Comparing (11.35) and (11.45), and using the arbitrariness of $\mathbf{C} \in \mathbb{C}^3$ and $\zeta \in C_0^\infty(\mathbb{R}^3)$, we conclude that

$$\boldsymbol{\psi} = s^0 \boldsymbol{\phi}. \tag{11.46}$$

11.6. Now, relations (11.14), (11.32), and (11.46) imply that $\boldsymbol{\phi}$ satisfies the equation

$$\operatorname{curl}(h^0)^{-1} \operatorname{curl} \boldsymbol{\phi} + s^0 \boldsymbol{\phi} = \mathbf{F}.$$

This equation is the same as the first equation in (11.2) for $\boldsymbol{\phi}_0$. Moreover, from (11.15) and (11.46) it is clear that $\operatorname{div} s^0 \boldsymbol{\phi} = 0$. Consequently, $\boldsymbol{\phi}$ is the solution of problem (11.2), whence $\boldsymbol{\phi} = \boldsymbol{\phi}_0$ by uniqueness. Then (11.32) implies that $\mathbf{p} = (h^0)^{-1} \operatorname{curl} \boldsymbol{\phi}_0 = \mathbf{p}_0$, and (11.46) yields $\boldsymbol{\psi} = s^0 \boldsymbol{\phi}_0 = \boldsymbol{\psi}_0$. As has already been mentioned, this implies all the statements of Theorem 11.1.

§12. Homogenization for the periodic Maxwell system

12.1. Setting of the problem. Suppose that the dielectric permittivity $\eta(\mathbf{x})$ and the magnetic permeability $\mu(\mathbf{x})$ are Γ-periodic measurable (3×3)-matrix-valued functions in $\mathbb{R}^3$ with real entries, and

$$\begin{gathered} c_0 \mathbf{1} \le \eta(\mathbf{x}) \le c_0' \mathbf{1}, \quad c_0 \mathbf{1} \le \mu(\mathbf{x}) \le c_0' \mathbf{1}, \\ \mathbf{x} \in \mathbb{R}^3, \quad 0 < c_0 \le c_0' < \infty. \end{gathered} \tag{12.1}$$

Recalling the notation $\mathfrak{G} = L_2(\mathbb{R}^3; \mathbb{C}^3)$, $\mathfrak{G}^l = H^l(\mathbb{R}^3; \mathbb{C}^3)$, $l \in \mathbb{R}$, we put $J^l = J^l(\mathbf{1}) := \{\mathbf{f} \in \mathfrak{G}^l : \operatorname{div} \mathbf{f} = 0\}$, $J = J^0$. Let $\mathfrak{G}(\eta^{-1}) = L_2(\mathbb{R}^3; \mathbb{C}^3; \eta^{-1})$ denote the "weighted" space with inner product $(\mathbf{u}, \mathbf{v})_{\mathfrak{G}(\eta^{-1})} = (\eta^{-1}\mathbf{u}, \mathbf{v})_{\mathfrak{G}}$. The space $\mathfrak{G}(\mu^{-1})$ is defined similarly. The set J is a closed subspace in $\mathfrak{G}$ and in both weighted spaces $\mathfrak{G}(\eta^{-1})$ and $\mathfrak{G}(\mu^{-1})$.

Below $\mathbf{u}, \mathbf{v}$ stand for the *strengths of the electric and magnetic fields*, $\mathbf{w} = \eta \mathbf{u}$ *is the electric displacement vector*, *and* $\mathbf{z} = \mu \mathbf{v}$ *is the magnetic displacement vector.*

As in [BSu2, Chapter 7], *we write the Maxwell operator* $\mathcal{M} = \mathcal{M}(\eta, \mu)$ *in terms of the displacement vectors, assuming that these vectors are solenoidal.* Then the operator $\mathcal{M}$ acts in the space $J \oplus J$ $(\subset \mathfrak{G} \oplus \mathfrak{G})$, which does not depend on the coefficients η and μ; $\mathcal{M}$ is given by the formula

$$\mathcal{M} = \mathcal{M}(\eta, \mu) = \begin{pmatrix} 0 & i \operatorname{curl} \mu^{-1} \\ -i \operatorname{curl} \eta^{-1} & 0 \end{pmatrix} \tag{12.2}$$

on the domain

$$\operatorname{Dom} \mathcal{M}(\eta, \mu) = \{(\mathbf{w}, \mathbf{z}) : \mathbf{w} \in J,\ \mathbf{z} \in J,\ \operatorname{curl} \eta^{-1} \mathbf{w} \in \mathfrak{G},\ \operatorname{curl} \mu^{-1} \mathbf{z} \in \mathfrak{G}\}. \tag{12.3}$$

The operator $\mathcal{M}$ is closed but not selfadjoint in $J \oplus J$ (relative to the standard inner product in $\mathfrak{G} \oplus \mathfrak{G}$). However, the operator $\mathcal{M}$ is selfadjoint in the space $J \oplus J$ viewed as a subspace in $\mathfrak{G}(\eta^{-1}) \oplus \mathfrak{G}(\mu^{-1})$.

Let $\varepsilon > 0$ be a parameter. Recall the definition $\varphi^\varepsilon(\mathbf{x}) := \varphi(\varepsilon^{-1}\mathbf{x})$ for a measurable Γ-periodic function $\varphi(\mathbf{x})$. We introduce the family of operators

$$\mathcal{M}_\varepsilon = \mathcal{M}(\eta^\varepsilon, \mu^\varepsilon), \quad \varepsilon > 0,$$

acting in $J \oplus J$. Here $\operatorname{Dom} \mathcal{M}_\varepsilon$ (see (12.3)) depends on ε.

Our goal is to study the behavior of the resolvent $(\mathcal{M}_\varepsilon - iI)^{-1}$ *as* $\varepsilon \to 0$. Consider the equation

$$(\mathcal{M}_\varepsilon - iI) \begin{pmatrix} \mathbf{w}_\varepsilon \\ \mathbf{z}_\varepsilon \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \mathbf{r} \end{pmatrix}, \quad \mathbf{q}, \mathbf{r} \in J. \tag{12.4}$$

The corresponding strengths are given by the relations $\mathbf{u}_\varepsilon = (\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon$, $\mathbf{v}_\varepsilon = (\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon$. In detail, (12.4) can be written as

$$\begin{cases} i \operatorname{curl}(\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon - i\mathbf{w}_\varepsilon = \mathbf{q}, \\ -i \operatorname{curl}(\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon - i\mathbf{z}_\varepsilon = \mathbf{r}, \\ \operatorname{div}\mathbf{w}_\varepsilon = 0, \quad \operatorname{div}\mathbf{z}_\varepsilon = 0. \end{cases} \tag{12.5}$$

It is useful (cf. [BSu2]) to represent the solutions as sums:

$$\mathbf{w}_\varepsilon = \mathbf{w}_\varepsilon^{(q)} + \mathbf{w}_\varepsilon^{(r)}, \quad \mathbf{z}_\varepsilon = \mathbf{z}_\varepsilon^{(q)} + \mathbf{z}_\varepsilon^{(r)},$$

where the pair of vectors $\mathbf{w}_\varepsilon^{(q)}$, $\mathbf{z}_\varepsilon^{(q)}$ is the solution of system (12.5) with $\mathbf{r} = 0$, and the pair of vectors $\mathbf{w}_\varepsilon^{(r)}$, $\mathbf{z}_\varepsilon^{(r)}$ is the solution of system (12.5) with $\mathbf{q} = 0$. Correspondingly, the fields $\mathbf{u}_\varepsilon$ and $\mathbf{v}_\varepsilon$ are also represented as sums.

12.2. The case where $\mathbf{q} = 0$. If $\mathbf{q} = 0$, system (12.5) takes the form

$$\begin{cases} \mathbf{w}_\varepsilon^{(r)} = \operatorname{curl}(\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon^{(r)}, \\ \operatorname{curl}(\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon^{(r)} + \mathbf{z}_\varepsilon^{(r)} = i\mathbf{r}, \\ \operatorname{div}\mathbf{w}_\varepsilon^{(r)} = 0, \quad \operatorname{div}\mathbf{z}_\varepsilon^{(r)} = 0. \end{cases} \tag{12.6}$$

The corresponding strengths are given by

$$\mathbf{u}_\varepsilon^{(r)} = (\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon^{(r)}, \quad \mathbf{v}_\varepsilon^{(r)} = (\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon^{(r)}. \tag{12.7}$$

From (12.6) it is clear that $\mathbf{z}_\varepsilon^{(r)}$ is the solution of the problem

$$\operatorname{curl}(\eta^\varepsilon)^{-1}\operatorname{curl}(\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon^{(r)} + \mathbf{z}_\varepsilon^{(r)} = i\mathbf{r}, \ \operatorname{div}\mathbf{z}_\varepsilon^{(r)} = 0, \quad \mathbf{r} \in J, \tag{12.8}$$

and $\mathbf{v}_\varepsilon^{(r)}$ is the solution of the problem

$$\operatorname{curl}(\eta^\varepsilon)^{-1}\operatorname{curl}\mathbf{v}_\varepsilon^{(r)} + \mu^\varepsilon\mathbf{v}_\varepsilon^{(r)} = i\mathbf{r}, \ \operatorname{div}\mu^\varepsilon\mathbf{v}_\varepsilon^{(r)} = 0, \quad \mathbf{r} \in J. \tag{12.9}$$

By (12.6) and (12.7), $\mathbf{w}_\varepsilon^{(r)}$ and $\mathbf{u}_\varepsilon^{(r)}$ can be expressed in terms of $\mathbf{v}_\varepsilon^{(r)}$:

$$\mathbf{w}_\varepsilon^{(r)} = \operatorname{curl}\mathbf{v}_\varepsilon^{(r)}, \quad \mathbf{u}_\varepsilon^{(r)} = (\eta^\varepsilon)^{-1}\operatorname{curl}\mathbf{v}_\varepsilon^{(r)}. \tag{12.10}$$

Problem (12.8) is of the form (10.1), and problem (12.9) is of the form (11.1) with the coefficients $h = \eta$, $s = \mu$ and with the right-hand side $\mathbf{F} = i\mathbf{r}$. Applying Theorem 10.1, we can find approximations $\widetilde{\mathbf{v}}_\varepsilon^{(r)}$, $\widetilde{\mathbf{z}}_\varepsilon^{(r)}$ for $\mathbf{v}_\varepsilon^{(r)}$ and $\mathbf{z}_\varepsilon^{(r)}$ (with respect to the norm in $\mathfrak{G}$). Applying Theorem 11.1, we obtain results about the weak convergence of $\mathbf{v}_\varepsilon^{(r)}$, $\mathbf{z}_\varepsilon^{(r)}$, and also of $\mathbf{w}_\varepsilon^{(r)} = \operatorname{curl}\mathbf{v}_\varepsilon^{(r)}$ and $\mathbf{u}_\varepsilon^{(r)}$ (the latter plays the role of the "flow" for $\mathbf{v}_\varepsilon^{(r)}$; see (12.10)).

We formulate the results. Let μ^0 be the effective matrix for $\mu(\mathbf{x})$. Recall the expression for μ^0. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be the standard unit vectors in $\mathbb{C}^3$, and let $\Phi_{\mathbf{e}_j} \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ be the (weak) Γ-periodic solution of the equation

$$\operatorname{div}\mu(\mathbf{x})(\mathbf{e}_j + \nabla\Phi_{\mathbf{e}_j}(\mathbf{x})) = 0, \quad j = 1, 2, 3.$$

If $\widetilde{\mu}(\mathbf{x})$ denotes the matrix with the columns $\mu(\mathbf{x})(\mathbf{e}_j + \nabla\Phi_{\mathbf{e}_j}(\mathbf{x}))$, $j = 1, 2, 3$, then

$$\mu^0 = |\Omega|^{-1}\int_\Omega \widetilde{\mu}(\mathbf{x})\,d\mathbf{x}.$$

The matrix with columns $\nabla\Phi_{\mathbf{e}_j}(\mathbf{x})$ is denoted by $Y(\mathbf{x})$. We put $G(\mathbf{x}) := \widetilde{\mu}(\mathbf{x})(\mu^0)^{-1} - \mathbf{1}$.

The effective matrix η^0 for $\eta(\mathbf{x})$ is defined similarly. Namely, let $\Psi_{\mathbf{e}_j} \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ be the (weak) Γ-periodic solution of the equation

$$\operatorname{div}\eta(\mathbf{x})(\mathbf{e}_j + \nabla\Psi_{\mathbf{e}_j}(\mathbf{x})) = 0, \quad j = 1, 2, 3.$$

We denote by $\widetilde{\eta}(\mathbf{x})$ the matrix with the columns $\eta(\mathbf{x})(\mathbf{e}_j + \nabla\Psi_{\mathbf{e}_j}(\mathbf{x}))$, $j = 1, 2, 3$. Then

$$\eta^0 = |\Omega|^{-1} \int_\Omega \widetilde{\eta}(\mathbf{x})\, d\mathbf{x}.$$

The matrix with columns $\nabla\Psi_{\mathbf{e}_j}(\mathbf{x})$ is denoted by $Z(\mathbf{x})$. We put $K(\mathbf{x}) := \widetilde{\eta}(\mathbf{x})(\eta^0)^{-1} - \mathbf{1}$.

Let $\mathcal{M}^0 = \mathcal{M}(\eta^0, \mu^0)$ be the Maxwell operator (12.2) with $\eta = \eta^0$ and $\mu = \mu^0$. Let $(\mathbf{w}_0^{(r)}, \mathbf{z}_0^{(r)})$ be the solution of the "homogenized" system

$$(\mathcal{M}^0 - iI) \begin{pmatrix} \mathbf{w}_0^{(r)} \\ \mathbf{z}_0^{(r)} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{r} \end{pmatrix},$$

or, in detail,

$$\begin{cases} \mathbf{w}_0^{(r)} = \operatorname{curl}(\mu^0)^{-1}\mathbf{z}_0^{(r)}, \\ \operatorname{curl}(\eta^0)^{-1}\mathbf{w}_0^{(r)} + \mathbf{z}_0^{(r)} = i\mathbf{r}, \\ \operatorname{div} \mathbf{w}_0^{(r)} = 0, \quad \operatorname{div} \mathbf{z}_0^{(r)} = 0. \end{cases} \tag{12.11}$$

Let

$$\mathbf{u}_0^{(r)} = (\eta^0)^{-1}\mathbf{w}_0^{(r)}, \quad \mathbf{v}_0^{(r)} = (\mu^0)^{-1}\mathbf{z}_0^{(r)}. \tag{12.12}$$

Obviously, the fields $\mathbf{w}_0^{(r)}$, $\mathbf{z}_0^{(r)}$, $\mathbf{u}_0^{(r)}$, $\mathbf{v}_0^{(r)}$ belong to the class $\mathfrak{G}^1$. From (12.11) it follows that $\mathbf{z}_0^{(r)}$ is the solution of the problem of the form (10.15), and $\mathbf{v}_0^{(r)}$ is the solution of the problem of the form (11.2) with $h^0 = \eta^0$, $s^0 = \mu^0$, and $\mathbf{F} = i\mathbf{r}$. Herewith, $\mathbf{w}_0^{(r)} = \operatorname{curl} \mathbf{v}_0^{(r)}$ and $\mathbf{u}_0^{(r)} = (\eta^0)^{-1} \operatorname{curl} \mathbf{v}_0^{(r)}$.

Let $\widetilde{\mathcal{P}}(\mu^0)$ denote the orthogonal projection in the weighted space $\mathfrak{G}((\mu^0)^{-1})$ onto the subspace J. The projection $\widetilde{\mathcal{P}}(\eta^0)$ is defined similarly. We put

$$\mathbf{r}_\varepsilon := \widetilde{\mathcal{P}}(\mu^0)(Y^\varepsilon)^*\mathbf{r}. \tag{12.13}$$

Note (see Corollary 8.3(1°)) that the operator of multiplication by the matrix-valued function $(Y^\varepsilon(\mathbf{x}))^*$ continuously maps $\mathfrak{G}$ into $\mathfrak{G}^{-1}$. Next, the projection $\widetilde{\mathcal{P}}(\mu^0)$ acts as a zeroth-order pseudodifferential operator; therefore, this projection is continuous in $\mathfrak{G}^{-1}$. The image $\widetilde{\mathcal{P}}(\mu^0)\mathfrak{G}^{-1}$ coincides with J^{-1}. Thus, $\mathbf{r}_\varepsilon \in J^{-1}$. Let $(\widehat{\mathbf{w}}_\varepsilon^{(r)}, \widehat{\mathbf{z}}_\varepsilon^{(r)})$ be the solution of the "correction" system

$$(\mathcal{M}^0 - iI) \begin{pmatrix} \widehat{\mathbf{w}}_\varepsilon^{(r)} \\ \widehat{\mathbf{z}}_\varepsilon^{(r)} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{r}_\varepsilon \end{pmatrix}. \tag{12.14}$$

Here we keep in mind that the operator $\mathcal{M}^0$ (with constant coefficients) extends up to a continuous operator from $J \oplus J$ to $J^{-1} \oplus J^{-1}$. In detail, (12.14) takes the form

$$\begin{cases} \widehat{\mathbf{w}}_\varepsilon^{(r)} = \operatorname{curl}(\mu^0)^{-1}\widehat{\mathbf{z}}_\varepsilon^{(r)}, \\ \operatorname{curl}(\eta^0)^{-1}\widehat{\mathbf{w}}_\varepsilon^{(r)} + \widehat{\mathbf{z}}_\varepsilon^{(r)} = i\mathbf{r}_\varepsilon, \\ \operatorname{div} \widehat{\mathbf{w}}_\varepsilon^{(r)} = 0, \quad \operatorname{div} \widehat{\mathbf{z}}_\varepsilon^{(r)} = 0. \end{cases} \tag{12.15}$$

We put

$$\widehat{\mathbf{v}}_\varepsilon^{(r)} = (\mu^0)^{-1}\widehat{\mathbf{z}}_\varepsilon^{(r)}. \tag{12.16}$$

From (12.15) it follows that $\widehat{\mathbf{z}}_\varepsilon^{(r)}$ is the solution of the problem

$$\operatorname{curl}(\eta^0)^{-1} \operatorname{curl}(\mu^0)^{-1}\widehat{\mathbf{z}}_\varepsilon^{(r)} + \widehat{\mathbf{z}}_\varepsilon^{(r)} = i\mathbf{r}_\varepsilon, \quad \operatorname{div} \widehat{\mathbf{z}}_\varepsilon^{(r)} = 0, \tag{12.17}$$

which is of the form (10.16). If $\mathbf{r}_\varepsilon \in J^{-1}$, then for the solution of (12.17) we have $\widehat{\mathbf{z}}_\varepsilon^{(r)} \in \mathfrak{G}^1$. Then (see (12.16)) $\widehat{\mathbf{v}}_\varepsilon^{(r)} \in \mathfrak{G}^1$, and $\widehat{\mathbf{w}}_\varepsilon^{(r)} = \operatorname{curl} \widehat{\mathbf{v}}_\varepsilon^{(r)} \in \mathfrak{G}$.

Applying Theorems 10.1 and 11.1 to systems (12.8) and (12.9), we arrive at the following theorem.

Theorem 12.1. *Suppose that the* Γ*-periodic measurable matrix-valued functions* η *and* μ *with real entries satisfy conditions* (12.1). *Let* $(\mathbf{w}_\varepsilon^{(r)}, \mathbf{z}_\varepsilon^{(r)})$ *be the solution of system* (12.6) *with* $\mathbf{r} \in J$, *and let* $\mathbf{u}_\varepsilon^{(r)}$, $\mathbf{v}_\varepsilon^{(r)}$ *be defined by* (12.7). *Let* $(\mathbf{w}_0^{(r)}, \mathbf{z}_0^{(r)})$ *be the solution of the "homogenized" system* (12.11), *and let* $\mathbf{u}_0^{(r)}, \mathbf{v}_0^{(r)}$ *be defined by* (12.12). *Let* $(\widehat{\mathbf{w}}_\varepsilon^{(r)}, \widehat{\mathbf{z}}_\varepsilon^{(r)})$ *be the solution of the "correction" system* (12.15), *and let* $\widehat{\mathbf{v}}_\varepsilon^{(r)}$ *be defined by* (12.16). *Let* $Y^\varepsilon(\mathbf{x})$ *be the matrix with the columns* $(\nabla\Phi_{\mathbf{e}_j})(\varepsilon^{-1}\mathbf{x})$, $j = 1, 2, 3$, *and let* $G^\varepsilon(\mathbf{x}) := \widetilde{\mu}^\varepsilon(\mathbf{x})(\mu^0)^{-1} - \mathbf{1}$. *We put*

$$\widetilde{\mathbf{v}}_\varepsilon^{(r)} = (\mathbf{1} + Y^\varepsilon)(\mathbf{v}_0^{(r)} + \widehat{\mathbf{v}}_\varepsilon^{(r)}), \tag{12.18}$$

$$\widetilde{\mathbf{z}}_\varepsilon^{(r)} = (\mathbf{1} + G^\varepsilon)(\mathbf{z}_0^{(r)} + \widehat{\mathbf{z}}_\varepsilon^{(r)}). \tag{12.19}$$

Then the following is true.

1°. *For the strength of the magnetic field* $\mathbf{v}_\varepsilon^{(r)}$, *we have the following approximation in the* $\mathfrak{G}$*-norm*:

$$\|\mathbf{v}_\varepsilon^{(r)} - \widetilde{\mathbf{v}}_\varepsilon^{(r)}\|_{\mathfrak{G}} \leq C\varepsilon\|\mathbf{r}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1. \tag{12.20}$$

2°. *As* $\varepsilon \to 0$, $\mathbf{v}_\varepsilon^{(r)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{v}_0^{(r)}$, *and* $\operatorname{curl} \mathbf{v}_\varepsilon^{(r)}$ *weakly tends in* $\mathfrak{G}$ *to* $\operatorname{curl} \mathbf{v}_0^{(r)}$.

3°. *For the magnetic displacement vector* $\mathbf{z}_\varepsilon^{(r)}$, *we have the following approximation in the* $\mathfrak{G}$*-norm*:

$$\|\mathbf{z}_\varepsilon^{(r)} - \widetilde{\mathbf{z}}_\varepsilon^{(r)}\|_{\mathfrak{G}} \leq C\varepsilon\|\mathbf{r}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1. \tag{12.21}$$

4°. *As* $\varepsilon \to 0$, $\mathbf{z}_\varepsilon^{(r)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{z}_0^{(r)}$.

5°. *As* $\varepsilon \to 0$, *the strength of the electric field* $\mathbf{u}_\varepsilon^{(r)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{u}_0^{(r)}$. *For* $\operatorname{curl} \mathbf{u}_\varepsilon^{(r)} = i\mathbf{r} - \mathbf{z}_\varepsilon^{(r)}$, *we have the following approximation*:

$$\| \operatorname{curl} \mathbf{u}_\varepsilon^{(r)} - i\mathbf{r} + \widetilde{\mathbf{z}}_\varepsilon^{(r)}\|_{\mathfrak{G}} \leq C\varepsilon\|\mathbf{r}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \leq 1. \tag{12.22}$$

6°. *As* $\varepsilon \to 0$, *the electric displacement vector* $\mathbf{w}_\varepsilon^{(r)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{w}_0^{(r)}$.

The constants in estimates (12.20)–(12.22) *only depend on the* L_∞*-norms of the matrix-valued functions* η, η^{-1}, μ, μ^{-1} *and on the parameters of the lattice* Γ.

Note that, *a fortiori*, the functions (12.18) and (12.19) belong to $\mathfrak{G}$, since $(\mathbf{v}_0^{(r)} + \widehat{\mathbf{v}}_\varepsilon^{(r)}) \in \mathfrak{G}^1$, $(\mathbf{z}_0^{(r)} + \widehat{\mathbf{z}}_\varepsilon^{(r)}) \in \mathfrak{G}^1$, and the operators of multiplication by Y^ε and G^ε continuously map $\mathfrak{G}^1$ into $\mathfrak{G}$.

In the approximations (12.18) and (12.19) (after opening the parentheses), we distinguish the terms $\mathbf{v}_0^{(r)}$, $\mathbf{z}_0^{(r)}$, which do not depend on ε and are equal to the weak limits for $\mathbf{v}_\varepsilon^{(r)}$, $\mathbf{z}_\varepsilon^{(r)}$. The other summands weakly tend to zero in $\mathfrak{G}$.

12.3. The case where $\mathbf{r} = 0$ is treated in a similar way. In this case, system (12.5) takes the form

$$\begin{cases} \operatorname{curl}(\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon^{(q)} - \mathbf{w}_\varepsilon^{(q)} = -i\mathbf{q}, \\ \mathbf{z}_\varepsilon^{(q)} = -\operatorname{curl}(\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon^{(q)}, \\ \operatorname{div} \mathbf{w}_\varepsilon^{(q)} = 0, \quad \operatorname{div} \mathbf{z}_\varepsilon^{(q)} = 0. \end{cases} \tag{12.23}$$

For the strengths of the electric and magnetic fields, we have

$$\mathbf{u}_\varepsilon^{(q)} = (\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon^{(q)}, \quad \mathbf{v}_\varepsilon^{(q)} = (\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon^{(q)}. \tag{12.24}$$

From (12.23) it is clear that $\mathbf{w}_\varepsilon^{(q)}$ is the solution of the problem

$$\operatorname{curl}(\mu^\varepsilon)^{-1}\operatorname{curl}(\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon^{(q)} + \mathbf{w}_\varepsilon^{(q)} = i\mathbf{q}, \ \ \operatorname{div}\mathbf{w}_\varepsilon^{(q)} = 0, \quad \mathbf{q} \in J, \tag{12.25}$$

and $\mathbf{u}_\varepsilon^{(q)}$ is the solution of the problem

$$\operatorname{curl}(\mu^\varepsilon)^{-1}\operatorname{curl}\mathbf{u}_\varepsilon^{(q)} + \eta^\varepsilon\mathbf{u}_\varepsilon^{(q)} = i\mathbf{q}, \ \ \operatorname{div}\eta^\varepsilon\mathbf{u}_\varepsilon^{(q)} = 0, \quad \mathbf{q} \in J. \tag{12.26}$$

The fields $\mathbf{v}_\varepsilon^{(q)}$ and $\mathbf{z}_\varepsilon^{(q)}$ can be expressed in terms of $\mathbf{u}_\varepsilon^{(q)}$:

$$\mathbf{z}_\varepsilon^{(q)} = -\operatorname{curl}\mathbf{u}_\varepsilon^{(q)}, \quad \mathbf{v}_\varepsilon^{(q)} = -(\mu^\varepsilon)^{-1}\operatorname{curl}\mathbf{u}_\varepsilon^{(q)}. \tag{12.27}$$

Problem (12.25) is of the form (10.1), and problem (12.26) is of the form (11.1) with $h = \mu$, $s = \eta$, and $\mathbf{F} = i\mathbf{q}$. Theorems 10.1 and 11.1 are applicable.

We formulate the results. Let $(\mathbf{w}_0^{(q)}, \mathbf{z}_0^{(q)})$ be the solution of the "homogenized" system

$$(\mathcal{M}^0 - iI)\begin{pmatrix}\mathbf{w}_0^{(q)} \\ \mathbf{z}_0^{(q)}\end{pmatrix} = \begin{pmatrix}\mathbf{q} \\ 0\end{pmatrix},$$

or, in detail,

$$\begin{cases} \operatorname{curl}(\mu^0)^{-1}\mathbf{z}_0^{(q)} - \mathbf{w}_0^{(q)} = -i\mathbf{q}, \\ \mathbf{z}_0^{(q)} = -\operatorname{curl}(\eta^0)^{-1}\mathbf{w}_0^{(q)}, \\ \operatorname{div}\mathbf{w}_0^{(q)} = 0, \quad \operatorname{div}\mathbf{z}_0^{(q)} = 0. \end{cases} \tag{12.28}$$

We put

$$\mathbf{u}_0^{(q)} = (\eta^0)^{-1}\mathbf{w}_0^{(q)}, \quad \mathbf{v}_0^{(q)} = (\mu^0)^{-1}\mathbf{z}_0^{(q)}. \tag{12.29}$$

Let $(\widehat{\mathbf{w}}_\varepsilon^{(q)}, \widehat{\mathbf{z}}_\varepsilon^{(q)})$ be the solution of the "correction" system

$$(\mathcal{M}_0 - iI)\begin{pmatrix}\widehat{\mathbf{w}}_\varepsilon^{(q)} \\ \widehat{\mathbf{z}}_\varepsilon^{(q)}\end{pmatrix} = \begin{pmatrix}\mathbf{q}_\varepsilon \\ 0\end{pmatrix}, \tag{12.30}$$

where $\mathbf{q}_\varepsilon := \widetilde{\mathcal{P}}(\eta^0)(Z^\varepsilon)^*\mathbf{q}$. In detail, (12.30) has the form

$$\begin{cases} \operatorname{curl}(\mu^0)^{-1}\widehat{\mathbf{z}}_\varepsilon^{(q)} - \widehat{\mathbf{w}}_\varepsilon^{(q)} = -i\mathbf{q}_\varepsilon, \\ \widehat{\mathbf{z}}_\varepsilon^{(q)} = -\operatorname{curl}(\eta^0)^{-1}\widehat{\mathbf{w}}_\varepsilon^{(q)}, \\ \operatorname{div}\widehat{\mathbf{w}}_\varepsilon^{(q)} = 0, \quad \operatorname{div}\widehat{\mathbf{z}}_\varepsilon^{(q)} = 0. \end{cases} \tag{12.31}$$

We put

$$\widehat{\mathbf{u}}_\varepsilon^{(q)} = (\eta^0)^{-1}\widehat{\mathbf{w}}_\varepsilon^{(q)}. \tag{12.32}$$

Applied to systems (12.25) and (12.26), Theorems 10.1 and 11.1 yield the following theorem.

Theorem 12.2. *Suppose that the Γ-periodic measurable matrix-valued functions η and μ with real entries satisfy conditions* (12.1)*. Let $(\mathbf{w}_\varepsilon^{(q)}, \mathbf{z}_\varepsilon^{(q)})$ be the solution of system* (12.23) *with $\mathbf{q} \in J$, and let $\mathbf{u}_\varepsilon^{(q)}$, $\mathbf{v}_\varepsilon^{(q)}$ be defined by* (12.24)*. Let $(\mathbf{w}_0^{(q)}, \mathbf{z}_0^{(q)})$ be the solution of the "homogenized" system* (12.28)*, and let $\mathbf{u}_0^{(q)}, \mathbf{v}_0^{(q)}$ be defined by* (12.29)*. Let $(\widehat{\mathbf{w}}_\varepsilon^{(q)}, \widehat{\mathbf{z}}_\varepsilon^{(q)})$ be the solution of the "correction" system* (12.31)*, and let $\widehat{\mathbf{u}}_\varepsilon^{(q)}$ be defined by* (12.32)*. Let $Z^\varepsilon(\mathbf{x})$ be the matrix with the columns $(\nabla\Psi_{\mathbf{e}_j})(\varepsilon^{-1}\mathbf{x})$, $j = 1, 2, 3$. We put $K^\varepsilon(\mathbf{x}) := \widetilde{\eta}^\varepsilon(\mathbf{x})(\eta^0)^{-1} - \mathbf{1}$ and*

$$\widetilde{\mathbf{u}}_\varepsilon^{(q)} = (\mathbf{1} + Z^\varepsilon)(\mathbf{u}_0^{(q)} + \widehat{\mathbf{u}}_\varepsilon^{(q)}), \tag{12.33}$$

$$\widetilde{\mathbf{w}}_\varepsilon^{(q)} = (\mathbf{1} + K^\varepsilon)(\mathbf{w}_0^{(q)} + \widehat{\mathbf{w}}_\varepsilon^{(q)}). \tag{12.34}$$

Then the following is true:

1°. *For the strength of the electric field* $\mathbf{u}_\varepsilon^{(q)}$, *we have the following approximation in the* $\mathfrak{G}$*-norm*:

$$\|\mathbf{u}_\varepsilon^{(q)} - \widetilde{\mathbf{u}}_\varepsilon^{(q)}\|_{\mathfrak{G}} \le C\varepsilon\|\mathbf{q}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{12.35}$$

2°. *As* $\varepsilon \to 0$, $\mathbf{u}_\varepsilon^{(q)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{u}_0^{(q)}$, *and* $\operatorname{curl}\mathbf{u}_\varepsilon^{(q)}$ *weakly tends in* $\mathfrak{G}$ *to* $\operatorname{curl}\mathbf{u}_0^{(q)}$.

3°. *For the electric displacement vector* $\mathbf{w}_\varepsilon^{(q)}$, *we have the following approximation in the* $\mathfrak{G}$*-norm*:

$$\|\mathbf{w}_\varepsilon^{(q)} - \widetilde{\mathbf{w}}_\varepsilon^{(q)}\|_{\mathfrak{G}} \le C\varepsilon\|\mathbf{q}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{12.36}$$

4°. *As* $\varepsilon \to 0$, $\mathbf{w}_\varepsilon^{(q)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{w}_0^{(q)}$.

5°. *As* $\varepsilon \to 0$, *the strength of the magnetic field* $\mathbf{v}_\varepsilon^{(q)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{v}_0^{(q)}$. *For* $\operatorname{curl}\mathbf{v}_\varepsilon^{(q)} = \mathbf{w}_\varepsilon^{(q)} - i\mathbf{q}$, *we have the following approximation*:

$$\|\operatorname{curl}\mathbf{v}_\varepsilon^{(q)} + i\mathbf{q} - \widetilde{\mathbf{w}}_\varepsilon^{(q)}\|_{\mathfrak{G}} \le C\varepsilon\|\mathbf{q}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{12.37}$$

6°. *As* $\varepsilon \to 0$, *the magnetic displacement vector* $\mathbf{z}_\varepsilon^{(q)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{z}_0^{(q)}$.

The constants in estimates (12.35)–(12.37) *only depend on the* L_∞*-norms of the matrix-valued functions* η, η^{-1}, μ, μ^{-1} *and on the parameters of the lattice* Γ.

In (12.33) and (12.34) (after opening the parentheses), we distinguish the terms $\mathbf{u}_0^{(q)}$, $\mathbf{w}_0^{(q)}$, which do not depend on ε and are equal to the weak limits for $\mathbf{u}_\varepsilon^{(q)}$, $\mathbf{w}_\varepsilon^{(q)}$. The other summands weakly tend to zero in $\mathfrak{G}$.

12.4. Discussion of the results. Obviously, Theorems 12.1 and 12.2 contain the known results about weak convergence of the (total) electric and magnetic fields to the corresponding fields in a medium with homogenized coefficients η^0, μ^0. The "elliptic" rule of finding the effective characteristics for the Maxwell operator is well known (see, e.g., [BeLP, ZhKO, Sa]). However, the sharp order estimates (12.20), (12.21), (12.35), (12.36), pertaining to the fields $\mathbf{v}_\varepsilon^{(r)}$, $\mathbf{z}_\varepsilon^{(r)}$ and $\mathbf{u}_\varepsilon^{(q)}$, $\mathbf{w}_\varepsilon^{(q)}$, are much more informative than the results about weak convergence (which follow from these estimates). The approximations (12.18), (12.19), and (12.33), (12.34) are expressed in terms of the solutions of the "homogenized" Maxwell systems (12.11) and (12.28) with constant "effective" coefficients η^0, μ^0 and with right-hand sides independent of ε, and also in terms of the solutions of the "correction" systems (12.15) and (12.31), also with the constant coefficients η^0, μ^0, but with right-hand sides depending on ε. The approximations (and the right-hand sides of systems (12.15) and (12.31)) involve the rapidly oscillating factors Y^ε, G^ε, and Z^ε, K^ε. These factors cannot be eliminated without deterioration of the quality of convergence.

For the fields $\mathbf{u}_\varepsilon^{(r)}$, $\mathbf{w}_\varepsilon^{(r)}$ and $\mathbf{v}_\varepsilon^{(q)}$, $\mathbf{z}_\varepsilon^{(q)}$, we have not succeeded in finding appropriate approximations in the $\mathfrak{G}$-norm and obtained only the weak convergence. The reason (in the framework of the technique applied) is that these fields play the role of the flows or the curls (see (12.10) and (12.27)) for the solutions $\mathbf{v}_\varepsilon^{(r)}$ and $\mathbf{u}_\varepsilon^{(q)}$ of the corresponding second-order equations (see (12.9) and (12.26)). While for the solutions $\mathbf{v}_\varepsilon^{(r)}$ and $\mathbf{u}_\varepsilon^{(q)}$ themselves we obtain uniform approximations in the $\mathfrak{G}$-norm, for the flows and curls one can hardly expect more than weak convergence (cf. [BSu2]).

Once again (see the discussion in §0), let us have a look at the crucial steps of our method, which allowed us to obtain the approximations described in Theorems 12.1 and 12.2. They are: taking into account the divergence free conditions; representation of the Maxwell system in terms of the displacement vectors with fixed solenoidal right-hand sides $\mathbf{q}$, $\mathbf{r}$ (see (12.4)); representation of each field as a sum of two terms; passage to the second-order operator; extension of this operator by removing the divergence free conditions. All these steps were also used in [BSu2, Chapter 7] in the study of the case where $\mu = 1$. However, for $\mu = 1$ the situation is principally simpler, because in that case the germ of the initial second-order operator coincides with the germ of the effective operator: $S(\boldsymbol{\theta}) = S^0(\boldsymbol{\theta})$ (see Remark 4.3). In the general case these germs do not coincide, which causes a lot of (new) difficulties.

12.5. The case where $\mu = \mu^0$ or $\eta = \eta^0$. Now, we distinguish the case where one of two characteristics of the medium is constant. We start with the case of $\mu = \mu^0$. Then $\Phi_{\mathbf{e}_j}(\mathbf{x}) = 0$, whence $Y(\mathbf{x}) = 0$, $\widetilde{\mu}(\mathbf{x}) = \mu^0$, $G(\mathbf{x}) = 0$. Next, the right-hand side of the "correction" system (see (12.13)) also vanishes: $\mathbf{r}_\varepsilon = 0$, so that the solutions of system (12.14) are trivial. As a result, the approximations (12.18) and (12.19) coincide with the solutions of the "homogenized" system, $\widetilde{\mathbf{v}}_\varepsilon^{(r)} = \mathbf{v}_0^{(r)}$, $\widetilde{\mathbf{z}}_\varepsilon^{(r)} = \mathbf{z}_0^{(r)}$, and estimates (12.20), (12.21) turn into the simpler relations

$$\begin{aligned} \|\mathbf{v}_\varepsilon^{(r)} - \mathbf{v}_0^{(r)}\|_{\mathfrak{G}} &\le C\varepsilon \|\mathbf{r}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1, \ \ \mu = \mu^0, \\ \|\mathbf{z}_\varepsilon^{(r)} - \mathbf{z}_0^{(r)}\|_{\mathfrak{G}} &\le C\varepsilon \|\mathbf{r}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1, \ \ \mu = \mu^0. \end{aligned} \tag{12.38}$$

In the case where $\mu = 1$, this repeats the result of Theorem 3.1(1°) in [BSu2, Chapter 7]. At the same time, the approximations (12.35), (12.36) for the fields $\mathbf{u}_\varepsilon^{(q)}$, $\mathbf{w}_\varepsilon^{(q)}$ essentially refine the results of [BSu2] for the case of $\mu = 1$. The method used in [BSu2] gave only weak convergence for $\mathbf{u}_\varepsilon^{(q)}$, $\mathbf{w}_\varepsilon^{(q)}$.

If $\mu = \mu^0$ and $\mathbf{r} \in J^{-1}$, then the $\mathfrak{G}^1$-norm of the solution $\mathbf{v}_\varepsilon^{(r)}$ of problem (12.9), as well as the $\mathfrak{G}^1$-norm of the solution $\mathbf{v}_0^{(r)}$ of the similar "homogenized" problem, satisfies the standard estimates in terms of $C\|\mathbf{r}\|_{\mathfrak{G}^{-1}}$ (with constant C independent of ε). Thus,

$$\|\mathbf{v}_\varepsilon^{(r)} - \mathbf{v}_0^{(r)}\|_{\mathfrak{G}^1} \le C \|\mathbf{r}\|_{\mathfrak{G}^{-1}}, \quad \mathbf{r} \in J^{-1}, \ \ \mu = \mu^0. \tag{12.39}$$

Interpolating between (12.38) and (12.39), we obtain

$$\begin{gathered} \|\mathbf{v}_\varepsilon^{(r)} - \mathbf{v}_0^{(r)}\|_{\mathfrak{G}^l} \le C_l \varepsilon^{1-l} \|\mathbf{r}\|_{\mathfrak{G}^{-l}}, \\ \mu = \mu^0, \quad \mathbf{r} \in J^{-l}, \quad 0 < \varepsilon \le 1, \quad 0 \le l < 1. \end{gathered} \tag{12.40}$$

Similarly,

$$\begin{gathered} \|\mathbf{z}_\varepsilon^{(r)} - \mathbf{z}_0^{(r)}\|_{\mathfrak{G}^l} \le C_l \varepsilon^{1-l} \|\mathbf{r}\|_{\mathfrak{G}^{-l}}, \\ \mu = \mu^0, \quad \mathbf{r} \in J^{-l}, \quad 0 < \varepsilon \le 1, \ \ 0 \le l < 1. \end{gathered} \tag{12.41}$$

For $\mu = 1$, estimates (12.40) and (12.41) repeat the result of Theorem 3.1(2°) in [BSu2, Chapter 7].

In the case where $\eta = \eta^0$, we obtain the following simple approximations for the fields $\mathbf{u}_\varepsilon^{(q)}$, $\mathbf{w}_\varepsilon^{(q)}$:

$$\begin{aligned} \|\mathbf{u}_\varepsilon^{(q)} - \mathbf{u}_0^{(q)}\|_{\mathfrak{G}} &\le C\varepsilon \|\mathbf{q}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1, \ \ \eta = \eta^0, \\ \|\mathbf{w}_\varepsilon^{(q)} - \mathbf{w}_0^{(q)}\|_{\mathfrak{G}} &\le C\varepsilon \|\mathbf{q}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1, \ \ \eta = \eta^0, \end{aligned}$$

which can be supplemented by the interpolational estimates

$$\begin{gathered} \|\mathbf{u}_\varepsilon^{(q)} - \mathbf{u}_0^{(q)}\|_{\mathfrak{G}^l} \le C_l \varepsilon^{1-l} \|\mathbf{q}\|_{\mathfrak{G}^{-l}}, \\ \|\mathbf{w}_\varepsilon^{(q)} - \mathbf{w}_0^{(q)}\|_{\mathfrak{G}^l} \le C_l \varepsilon^{1-l} \|\mathbf{q}\|_{\mathfrak{G}^{-l}}, \\ \eta = \eta^0, \quad \mathbf{q} \in J^{-l}, \quad 0 < \varepsilon \le 1, \quad 0 \le l < 1. \end{gathered}$$

Now the fields $\mathbf{v}_\varepsilon^{(r)}$, $\mathbf{z}_\varepsilon^{(r)}$ satisfy relations (12.20) and (12.21).

In the general case of variable periodic coefficients $\eta(\mathbf{x})$ and $\mu(\mathbf{x})$, interpolation is less natural, because, instead of the $\mathfrak{G}^1$-norm of the solutions, we can estimate only the norm in some function class depending on ε.

12.6. The "dual" problem. Let $\mathbf{q} \in J$. We put

$$\mathbf{R}_\varepsilon := \operatorname{curl}(\eta^\varepsilon)^{-1}\mathbf{q} \quad (\in J^{-1}).$$

It is directly seen that if $(\mathbf{w}_\varepsilon^{(q)}, \mathbf{z}_\varepsilon^{(q)})$ is the solution of (12.23), i.e.,

$$(\mathcal{M}_\varepsilon - iI)\begin{pmatrix}\mathbf{w}_\varepsilon^{(q)}\\ \mathbf{z}_\varepsilon^{(q)}\end{pmatrix} = \begin{pmatrix}\mathbf{q}\\ 0\end{pmatrix}, \tag{12.42}$$

then the pair of vectors

$$\mathbf{w}_\varepsilon^{(R)} = i\mathbf{q} - \mathbf{w}_\varepsilon^{(q)}, \quad \mathbf{z}_\varepsilon^{(R)} = -\mathbf{z}_\varepsilon^{(q)} \tag{12.43}$$

is the solution of the problem

$$(\mathcal{M}_\varepsilon - iI)\begin{pmatrix}\mathbf{w}_\varepsilon^{(R)}\\ \mathbf{z}_\varepsilon^{(R)}\end{pmatrix} = \begin{pmatrix}0\\ \mathbf{R}_\varepsilon\end{pmatrix}. \tag{12.44}$$

Problem (12.44) *is called the dual problem with respect to* (12.42). We put

$$\begin{aligned}\mathbf{u}_\varepsilon^{(R)} &= (\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon^{(R)} = i(\eta^\varepsilon)^{-1}\mathbf{q} - \mathbf{u}_\varepsilon^{(q)},\\ \mathbf{v}_\varepsilon^{(R)} &= (\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon^{(R)} = -\mathbf{v}_\varepsilon^{(q)}.\end{aligned} \tag{12.45}$$

Applying Theorem 12.2, we immediately obtain results on the behavior of the solutions of the dual problem (12.44). By (12.35), (12.36), (12.43), and (12.45), $\mathbf{w}_\varepsilon^{(R)}$ and $\mathbf{u}_\varepsilon^{(R)}$ can be approximated by the fields

$$\widetilde{\mathbf{w}}_\varepsilon^{(R)} = i\mathbf{q} - \widetilde{\mathbf{w}}_\varepsilon^{(q)}, \tag{12.46}$$

$$\widetilde{\mathbf{u}}_\varepsilon^{(R)} = i(\eta^\varepsilon)^{-1}\mathbf{q} - \widetilde{\mathbf{u}}_\varepsilon^{(q)}, \tag{12.47}$$

where $\widetilde{\mathbf{u}}_\varepsilon^{(q)}$ and $\widetilde{\mathbf{w}}_\varepsilon^{(q)}$ are defined by (12.33) and (12.34). We have

$$\|\mathbf{w}_\varepsilon^{(R)} - \widetilde{\mathbf{w}}_\varepsilon^{(R)}\|_{\mathfrak{G}} \le C\varepsilon\|\mathbf{q}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1, \tag{12.48}$$

$$\|\mathbf{u}_\varepsilon^{(R)} - \widetilde{\mathbf{u}}_\varepsilon^{(R)}\|_{\mathfrak{G}} \le C\varepsilon\|\mathbf{q}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{12.49}$$

Next, if $(\mathbf{w}_0^{(q)}, \mathbf{z}_0^{(q)})$ is the solution of the "homogenized" system (12.28), then the pair of vectors

$$\mathbf{w}_0^{(R)} = i\mathbf{q} - \mathbf{w}_0^{(q)}, \quad \mathbf{z}_0^{(R)} = -\mathbf{z}_0^{(q)}$$

is the solution of the dual "homogenized" system

$$(\mathcal{M}^0 - iI)\begin{pmatrix}\mathbf{w}_0^{(R)}\\ \mathbf{z}_0^{(R)}\end{pmatrix} = \begin{pmatrix}0\\ \mathbf{R}\end{pmatrix}, \quad \mathbf{R} := \operatorname{curl}(\eta^0)^{-1}\mathbf{q}. \tag{12.50}$$

We put

$$\begin{aligned}\mathbf{u}_0^{(R)} &= (\eta^0)^{-1}\mathbf{w}_0^{(R)} = i(\eta^0)^{-1}\mathbf{q} - \mathbf{u}_0^{(q)},\\ \mathbf{v}_0^{(R)} &= (\mu^0)^{-1}\mathbf{z}_0^{(R)} = -\mathbf{v}_0^{(q)}.\end{aligned} \tag{12.51}$$

Then Theorem 12.2 implies results about *weak convergence* of all four fields $\mathbf{u}_\varepsilon^{(R)}$, $\mathbf{w}_\varepsilon^{(R)}$, $\mathbf{v}_\varepsilon^{(R)}$, $\mathbf{z}_\varepsilon^{(R)}$. Here, the last three fields tend to the corresponding fields $\mathbf{w}_0^{(R)}$, $\mathbf{v}_0^{(R)}$, and $\mathbf{z}_0^{(R)}$, while, in general, the weak limit for $\mathbf{u}_\varepsilon^{(R)}$ does not coincide with $\mathbf{u}_0^{(R)}$. Indeed, by the mean value property (see Proposition 9.3, 2°), $(\eta^\varepsilon)^{-1}\mathbf{q}$ weakly tends in $\mathfrak{G}$ to $\underline{\eta}^{-1}\mathbf{q}$, where

$$\underline{\eta}^{-1} := |\Omega|^{-1}\int_\Omega \eta(\mathbf{x})^{-1}\,d\mathbf{x}.$$

Hence, (12.45) and (12.51) imply that

$$(w,\mathfrak{G})\text{-}\lim_{\varepsilon\to 0}\mathbf{u}_\varepsilon^{(R)} = i\underline{\eta}^{-1}\mathbf{q} - \mathbf{u}_0^{(q)} = i(\underline{\eta}^{-1} - (\eta^0)^{-1})\mathbf{q} + \mathbf{u}_0^{(R)}.$$

As a result, we arrive at the following theorem.

Theorem 12.3. *Let* $\mathbf{q}\in J$, *and let* $\mathbf{R}_\varepsilon = \operatorname{curl}(\eta^\varepsilon)^{-1}\mathbf{q}$. *Let* $(\mathbf{w}_\varepsilon^{(R)},\mathbf{z}_\varepsilon^{(R)})$ *be the solution of problem* (12.44), *and let* $\mathbf{u}_\varepsilon^{(R)} = (\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon^{(R)}$, $\mathbf{v}_\varepsilon^{(R)} = (\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon^{(R)}$. *Let* $(\mathbf{w}_0^{(R)},\mathbf{z}_0^{(R)})$ *be the solution of the "homogenized" problem* (12.50), *and let* $\mathbf{u}_0^{(R)} = (\eta^0)^{-1}\mathbf{w}_0^{(R)}$, $\mathbf{v}_0^{(R)} = (\mu^0)^{-1}\mathbf{z}_0^{(R)}$. *Let* $\widetilde{\mathbf{w}}_\varepsilon^{(R)}$, $\widetilde{\mathbf{u}}_\varepsilon^{(R)}$ *be defined by* (12.46), (12.47). *Then the following is true.*

1°. *The strength of the electric field* $\mathbf{u}_\varepsilon^{(R)}$ *admits an approximation in the* $\mathfrak{G}$*-norm with estimate* (12.49).

2°. *As* $\varepsilon\to 0$, $\mathbf{u}_\varepsilon^{(R)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{u}_0^{(R)} + i(\underline{\eta}^{-1} - (\eta^0)^{-1})\mathbf{q}$.

3°. *The electric displacement vector* $\mathbf{w}_\varepsilon^{(R)}$ *admits an approximation in the* $\mathfrak{G}$*-norm with estimate* (12.48).

4°. *As* $\varepsilon\to 0$, $\mathbf{w}_\varepsilon^{(R)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{w}_0^{(R)}$.

5°. *As* $\varepsilon\to 0$, *the strength of the magnetic field* $\mathbf{v}_\varepsilon^{(R)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{v}_0^{(R)}$. *For* $\operatorname{curl}\mathbf{v}_\varepsilon^{(R)} = \mathbf{w}_\varepsilon^{(R)}$ *we have approximation in the* $\mathfrak{G}$*-norm* (*see* (12.48)).

6°. *As* $\varepsilon\to 0$, *the magnetic displacement vector* $\mathbf{z}_\varepsilon^{(R)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{z}_0^{(R)}$.

The problem dual to (12.6) is considered by analogy. Let

$$\mathbf{r}\in J,\quad \mathbf{Q}_\varepsilon := \operatorname{curl}(\mu^\varepsilon)^{-1}\mathbf{r}.$$

If $(\mathbf{w}_\varepsilon^{(r)},\mathbf{z}_\varepsilon^{(r)})$ is the solution of system (12.6), i.e.,

$$(\mathcal{M}_\varepsilon - iI)\begin{pmatrix}\mathbf{w}_\varepsilon^{(r)}\\ \mathbf{z}_\varepsilon^{(r)}\end{pmatrix} = \begin{pmatrix}0\\ \mathbf{r}\end{pmatrix}, \tag{12.52}$$

then the pair of vectors

$$\mathbf{w}_\varepsilon^{(Q)} = \mathbf{w}_\varepsilon^{(r)},\quad \mathbf{z}_\varepsilon^{(Q)} = -i\mathbf{r} + \mathbf{z}_\varepsilon^{(r)} \tag{12.53}$$

is the solution of the problem

$$(\mathcal{M}_\varepsilon - iI)\begin{pmatrix}\mathbf{w}_\varepsilon^{(Q)}\\ \mathbf{z}_\varepsilon^{(Q)}\end{pmatrix} = \begin{pmatrix}\mathbf{Q}_\varepsilon\\ 0\end{pmatrix}. \tag{12.54}$$

Problem (12.54) *is called the dual problem with respect to* (12.52). We put

$$\mathbf{u}_\varepsilon^{(Q)} = (\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon^{(Q)} = \mathbf{u}_\varepsilon^{(r)}, \tag{12.55}$$

$$\mathbf{v}_\varepsilon^{(Q)} = (\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon^{(Q)} = -i(\mu^\varepsilon)^{-1}\mathbf{r} + \mathbf{v}_\varepsilon^{(r)}. \tag{12.56}$$

Applying Theorem 12.1, we immediately obtain results on the behavior of the solutions of the dual problem (12.54). The fields $\mathbf{v}_\varepsilon^{(Q)}$ and $\mathbf{z}_\varepsilon^{(Q)}$ can be approximated by the fields

$$\widetilde{\mathbf{v}}_\varepsilon^{(Q)} = -i(\mu^\varepsilon)^{-1}\mathbf{r} + \widetilde{\mathbf{v}}_\varepsilon^{(r)}, \tag{12.57}$$

$$\widetilde{\mathbf{z}}_\varepsilon^{(Q)} = -i\mathbf{r} + \widetilde{\mathbf{z}}_\varepsilon^{(r)}, \tag{12.58}$$

where $\widetilde{\mathbf{v}}_\varepsilon^{(r)}$, $\widetilde{\mathbf{z}}_\varepsilon^{(r)}$ are defined by (12.18), (12.19). We have

$$\|\mathbf{v}_\varepsilon^{(Q)} - \widetilde{\mathbf{v}}_\varepsilon^{(Q)}\|_{\mathfrak{G}} \le C\varepsilon \|\mathbf{r}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1, \tag{12.59}$$

$$\|\mathbf{z}_\varepsilon^{(Q)} - \widetilde{\mathbf{z}}_\varepsilon^{(Q)}\|_{\mathfrak{G}} \le C\varepsilon \|\mathbf{r}\|_{\mathfrak{G}}, \quad 0 < \varepsilon \le 1. \tag{12.60}$$

Next, if $(\mathbf{w}_0^{(r)}, \mathbf{z}_0^{(r)})$ is the solution of the "homogenized" system (12.11), then the pair of vectors

$$\mathbf{w}_0^{(Q)} = \mathbf{w}_0^{(r)}, \quad \mathbf{z}_0^{(Q)} = -i\mathbf{r} + \mathbf{z}_0^{(r)} \tag{12.61}$$

is the solution of the dual "homogenized" system

$$(\mathcal{M}^0 - iI)\begin{pmatrix} \mathbf{w}_0^{(Q)} \\ \mathbf{z}_0^{(Q)} \end{pmatrix} = \begin{pmatrix} \mathbf{Q} \\ 0 \end{pmatrix}, \quad \mathbf{Q} := \operatorname{curl}(\mu^0)^{-1}\mathbf{r}. \tag{12.62}$$

We put

$$\mathbf{u}_0^{(Q)} = (\eta^0)^{-1}\mathbf{w}_0^{(Q)} = \mathbf{u}_0^{(r)}, \tag{12.63}$$

$$\mathbf{v}_0^{(Q)} = (\mu^0)^{-1}\mathbf{z}_0^{(Q)} = -i(\mu^0)^{-1}\mathbf{r} + \mathbf{v}_0^{(r)}. \tag{12.64}$$

Applying Theorem 12.1 and using relations (12.53), (12.55), (12.56), (12.61), (12.63), (12.64), we arrive at the following result.

Theorem 12.4. *Let* $\mathbf{r} \in J$, *and let* $\mathbf{Q}_\varepsilon = \operatorname{curl}(\mu^\varepsilon)^{-1}\mathbf{r}$. *Let* $(\mathbf{w}_\varepsilon^{(Q)}, \mathbf{z}_\varepsilon^{(Q)})$ *be the solution of problem* (12.54), *and let* $\mathbf{u}_\varepsilon^{(Q)} = (\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon^{(Q)}$, $\mathbf{v}_\varepsilon^{(Q)} = (\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon^{(Q)}$. *Let* $(\mathbf{w}_0^{(Q)}, \mathbf{z}_0^{(Q)})$ *be the solution of the "homogenized" problem* (12.62), *and let* $\mathbf{u}_0^{(Q)} = (\eta^0)^{-1}\mathbf{w}_0^{(Q)}$, $\mathbf{v}_0^{(Q)} = (\mu^0)^{-1}\mathbf{z}_0^{(Q)}$. *Let* $\widetilde{\mathbf{v}}_\varepsilon^{(Q)}$, $\widetilde{\mathbf{z}}_\varepsilon^{(Q)}$ *be defined by* (12.57), (12.58). *Then the following is true.*

1°. *The strength of the magnetic field* $\mathbf{v}_\varepsilon^{(Q)}$ *admits an approximation in the* $\mathfrak{G}$*-norm with estimate* (12.59).

2°. *As* $\varepsilon \to 0$, $\mathbf{v}_\varepsilon^{(Q)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{v}_0^{(Q)} - i(\underline{\mu}^{-1} - (\mu^0)^{-1})\mathbf{r}$.

3°. *The magnetic displacement vector* $\mathbf{z}_\varepsilon^{(Q)}$ *admits an approximation in the* $\mathfrak{G}$*-norm with estimate* (12.60).

4°. *As* $\varepsilon \to 0$, $\mathbf{z}_\varepsilon^{(Q)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{z}_0^{(Q)}$.

5°. *As* $\varepsilon \to 0$, *the strength of the electric field* $\mathbf{u}_\varepsilon^{(Q)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{u}_0^{(Q)}$. *For* $\operatorname{curl}\mathbf{u}_\varepsilon^{(Q)} = -\mathbf{z}_\varepsilon^{(Q)}$ *we have an approximation in the* $\mathfrak{G}$*-norm* (*see* (12.60)).

6°. *As* $\varepsilon \to 0$, *the electric displacement vector* $\mathbf{w}_\varepsilon^{(Q)}$ *weakly tends in* $\mathfrak{G}$ *to* $\mathbf{w}_0^{(Q)}$.

Note that it is not quite clear if the dual problem admits a natural physical interpretation.

It is possible to combine the results of Theorems 12.1 and 12.4. Let $\mathbf{r}, \widetilde{\mathbf{r}} \in J$, and let $\widetilde{\mathbf{Q}}_\varepsilon = \operatorname{curl}(\mu^\varepsilon)^{-1}\widetilde{\mathbf{r}}$. Then, for the system

$$(\mathcal{M}_\varepsilon - iI)\begin{pmatrix} \mathbf{w}_\varepsilon \\ \mathbf{z}_\varepsilon \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{Q}}_\varepsilon \\ \mathbf{r} \end{pmatrix},$$

"nice" approximations for the (total) magnetic fields $\mathbf{v}_\varepsilon$, $\mathbf{z}_\varepsilon$ can be obtained.

Similarly, we can combine the results of Theorems 12.2 and 12.3. Let $\mathbf{q}, \widetilde{\mathbf{q}} \in J$, and let $\widetilde{\mathbf{R}}_\varepsilon = \operatorname{curl}(\eta^\varepsilon)^{-1}\widetilde{\mathbf{q}}$. Then, for the system

$$(\mathcal{M}_\varepsilon - iI)\begin{pmatrix} \mathbf{w}_\varepsilon \\ \mathbf{z}_\varepsilon \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \widetilde{\mathbf{R}}_\varepsilon \end{pmatrix},$$

"nice" approximations for the (total) electric fields $\mathbf{u}_\varepsilon$, $\mathbf{w}_\varepsilon$ can be obtained.

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Received 24/MAY/2004

Translated by THE AUTHOR