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DIFFERENTIATION IN METRIC SPACES

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Abstract. Differentiation of Lipschitz maps between abstract metric spaces is discussed. Differentiability of isometries, first variation formula, and Rademacher-type theorems are studied.

§1. Introduction

1.1. The aim. This paper is devoted to the study of the first order geometry of metric spaces. Mainly, our study was motivated by the observation that whereas the advanced features of the theories of Aleksandrov spaces with upper and with lower curvature bounds are quite different, the beginnings are almost identical, at least as far as only first order derivatives are concerned (for example, tangent spaces and the first variation formula). This naturally raises the question: What are the spaces admitting the first order geometry? As it turns out, the same first order geometry exists in many other spaces that we call geometric. The class of geometric spaces contains all Hölder continuous Riemannian manifolds, sufficiently convex and smooth Finsler manifolds (see [LY]), a big class of subsets of Riemannian manifolds (e.g., the sets of positive reach; see [Fed59, Lyta]), surfaces with an integral curvature bound (see [Res93]), and extremal subsets of Aleksandrov spaces with lower curvature bound (see [PP94a]). The last case was discussed in [Pet94], and the proof of the first variation formula was a major step towards proving the deep gluing theorem [Pet94]. Moreover, the class of geometric spaces is stable under metric operations, even under the difficult operation of taking quotients. Finally, the existence of the first order geometry is a good assumption for studying features of higher order, such as gradient flows of semiconcave functions (see [PP94b, Lytc]). One of the main issues in this paper is in establishing natural and easily verifiable axioms that describe this first order geometry, and in obtaining their consequences.

Another more direct motivation comes from the question as to whether a submetry (or, more specifically, an isometry) between metric spaces must be differentiable in some suitable sense. This question was answered in the affirmative for smooth Riemannian manifolds in [BG00]. On the other hand, in [CH70] an example of a nondifferentiable isometry between Riemannian manifolds with continuous Riemannian metrics was constructed. Even for asking this question, a language is needed that allows us to speak about differentiability of Lipschitz mappings. Another main issue in this paper is the establishment of such a language.

Remark 1.1. For (special) doubling metric measure spaces, Cheeger [Che99] developed a deep theory giving a Rademacher-type theorem for such spaces. However, this approach does not permit talking about differentiability at a given (singular) point. Moreover, this approach is restricted essentially to differentiation of functions and does not apply to maps to another singular space. In [Kir94], Kirchheim developed a very interesting

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theory of metric differentiation for Lipschitz mappings of the Euclidean space to arbitrary metric spaces. The disadvantage of this theory is that it completely neglects the (possibly existing) tangent structure of the image space. Whereas Kirchheim's definition has a clear interpretation in our language, the links with the theory of Cheeger are much less clear and will not be discussed here.

In this paper we discuss the basics of the theory. In [Lytc] we studied relationships between the properties of the differentials and the map itself; in [Lytb] we applied these ideas to differentiability in Carnot–Carathéodory spaces.

1.2. The problem. The notion of a tangent cone at a point of a proper metric space was defined by Gromov by using the Gromov–Hausdorff convergence of rescaled spaces, via the requirement that an infinitesimal portion of the space at x be independent of the infinitesimal scale. In many situations this concept has been used to study the properties of the original space (see, e.g., [BGP92, Pet94, Mit85] and many other publications). Unfortunately, this definition, which is perfect for the study of infinitesimal portions of a given space, is not quite suitable for the study of differentials. The problem is that the Gromov–Hausdorff convergence of abstract metric spaces is defined only on the set of isometry classes. For example, the question about the differentiability of isometries does not make any sense in this context. Dealing with differentials, one would prefer to know what happens in a fixed direction in the tangent space. Mostow and Margulis encountered this problem as they were dealing with differentials between Carnot–Carathéodory spaces (see [MM00]).

1.3. The method. To circumvent this problem, we give a slightly different definition of the tangent cone, working with ultraconvergence instead of the Gromov–Hausdorff convergence, a notion widely used in the theory of nonpositively curved spaces. Namely, for each zero sequence $(o) = (\epsilon_i)$, which we call an (*infinitesimal*) *scale*, we can consider the blow-up $X_x^{(o)}$ of the pointed space (X, x) at the scale (o), given as the ultralimit $X_x^{(o)} = \lim_\omega(\frac{1}{\epsilon_i}X, x)$. Now we say that a tangent cone T_xX of X at the point x is a metric cone $(T, 0)$ together with a fixed choice of pointed isometries $i^{(o)} : T \to X_x^{(o)}$ for each scale (o) such that certain natural commutation relations (see Definition 6.1) are satisfied.

If the tangent space exists in the sense of Gromov, our definition merely makes the additional requirement of fixing a special choice of a metric space in the isometry class of the tangent space in the sense of Gromov (see Remark 6.2). With this definition of the tangent space, the differential of a Lipschitz map is the blow-up at the given point if this blow-up is unique.

If the tangent spaces in X and in Y exist, then they exist in a natural way in the product $X \times Y$ and in the Euclidean cone CX. Moreover, there is a natural choice (up to the tangent cones in X) of the tangent cones to subsets of X.

In general, there may be no tangent space in our sense, or there may be no natural choice (in the definition we assumed that the isometries $i^{(\epsilon_i)}$ are given somehow). However, tangent spaces exist in many important singular metric spaces. This existence is given by a (not necessarily continuous) map e from a small portion of a metric cone T to a small neighborhood of the given point x, that is, by an infinitesimal isometry at x (thus, by a very singular equivalent of the exponential map; see Subsections 3.5 and 6.1 for the precise definition). All examples of tangent cones known to the author arise in this way. One problem closely related to the question as to whether the tangent cone is defined in a natural way is that the identification of the tangent space at x with the ultraproduct T^ω (this ultraproduct is equal to T if T is proper) via this map e depends not only on e and the metric of T but also on the particular metric cone structure on T, i.e., on a particular choice of dilations (see §4). This is the reason for the pathological example of [CH70]; see Example 7.6.

Even though we use a choice of an ultrafilter ω in our definitions, the notions of differentiability and differential do not depend on ω if the tangent cones are given by a map e as above (see Subsection 7.2). For instance, for Lipschitz mappings between Banach spaces we get the usual definition of directional differentiability.

1.4. Geometric conditions. In order to obtain the tangent cones (the isometries $i^{(\epsilon_i)}$) in a natural way, we observe that each metric space naturally determines a cone C_x at each point x, which is the set of germs of unparameterized geodesics starting at x. Moreover, this cone C_x comes along with a natural family of 1-Lipschitz exponential mappings $\exp_x^{(\epsilon_i)} : C_x \to X_x^{(\epsilon_i)}$ to the different blow-ups of X at x. Now we define a generalized angle condition (A), which is satisfied by spaces with one-sided curvature bound, by strongly convex Banach spaces, and by many other spaces (see below). This condition generalizes the usual requirement that the upper and the lower angles be equal (see Example 5.3). We say that X has property (A) at x if $\lim_{t\to 0} \frac{d(\gamma_1(t),\gamma_2(st))}{t}$ exists for all $s \in \mathbb{R}^+$ and all geodesics γ_1, γ_2 starting at x.

However, even in proper geodesic spaces, geodesics may see only a small part of the blow-ups, as the example of Carnot–Carathéodory spaces shows. To guarantee the surjectivity of the exponential maps, we impose a uniformity condition (U). We say that a locally geodesic space X with property (A) at x has property (U) at x if the geodesic cone C_x is proper and $d(\gamma_1(t), \gamma_2(t)) \leq O(t, d(\gamma_1^+, \gamma_2^+))t$, where $d(\gamma_1^+, \gamma_2^+)$ is the distance between the starting directions γ_i^+ of γ_i in C_x, and O is some function going to 0 if both arguments go to 0. Given this condition, one can define a natural (however not continuous) exponential map $e : C_x \to X$ by identifying C_x with the tangent cone T_xX. Hence, in the spaces with property (U) the tangent space exist in a natural way. We say that a locally geodesic space X is *infinitesimally cone-like* if it has property (U) at each point, and each tangent cone $T_xX = C_x$ is a Euclidean cone (see Definition 6.3). In [Lytc] we proved that gradient flows of semiconcave functions exist in such spaces, which generalizes the corresponding result of [PP94b].

Finally, to be able to deal with distance functions, we need a further condition. We say that geodesics *vary smoothly* at x if small, long, and thin quadrangles with vertex at x essentially look like quadrangles in C_x (see Definition 9.2). This expresses the property that the geodesics converging pointwise to a given geodesic converge also in some better sense. For example, this is true in a continuous Riemannian metric if all geodesics are uniformly $C^{1,\alpha}$ for some $\alpha > 0$, which explains the name.

Remark 1.2. This (local) condition is almost equivalent to the global statement that the first variation formula is valid in X; see §9 for the details.

A proper geodesic space is said to be *geometric* if it has property (U) at each point, each tangent cone $T_xX = C_x$ is a uniformly convex and smooth cone (e.g., a Euclidean cone or a Banach space with a strongly convex and smooth norm; see Definitions 4.3 and 4.4) and if geodesics vary smoothly at each point (see Definition 10.1).

1.5. Results. As was already mentioned at the beginning, the class of geometric spaces is very large. The Aleksandrov spaces (see Definition 2.2), the surfaces with an integral curvature bound, the manifolds with only Hölder continuous Riemannian metrics, the sets of positive reach and some more general subsets of Riemannian manifolds are geometric and infinitesimally cone-like. A finite-dimensional Banach space is geometric if and only if its norm is strongly convex and smooth. The Finsler manifolds with Hölder continuous and pointwise smooth and sufficiently convex norms are geometric. The products and convex sets of Euclidean cones over (infinitesimally cone-like) geometric spaces are (infinitesimally cone-like) geometric. Each open subset of an infinitesimally cone-like space is infinitesimally cone-like.

Now we can state our results. A map $f : X \to Z$ of a space X with property (U) at x to another metric space Z is differentiable at x if and only if it is directionally differentiable at x, i.e., if $f \circ \gamma : [0, \epsilon) \to Z$ is differentiable at 0 for all geodesics γ starting at x. This implies the following.

Proposition 1.1. *Let $f : X \to Z$ be an isometric embedding. If X and Z are infinitesimally cone-like* (*or*, *more generally*, *have property* (U)), *then f is differentiable at all points.*

In geometric spaces the first variation formula holds, i.e., the distance functions d_S to subsets S and the metric $d : X \times X \to \mathbb{R}$ itself are differentiable, and the differential of d_S at x depends only on the set of directions in C_x of the minimal geodesics between x and S (see Proposition 9.3 for the precise formulation), where the usual angles are replaced by the corresponding Busemann functions. If the tangent spaces are Euclidean cones, we get the usual first variation formula.

Theorem 1.2. *Let X be an infinitesimally cone-like space*, *and let $x \neq z \in X$. Let γ be a geodesic between x and z with starting* (*respectively*, *ending*) *directions $\gamma^+ \in T_xX$ and $\gamma^- \in T_zX$. Then the differential of the distance $d : X \times X \to \mathbb{R}$ can be estimated by $D_{(x,z)}d(v, w) \leq -\langle \gamma^+, v\rangle - \langle \gamma^-, w\rangle$. If*, *moreover*, *$X$ is geometric*, *then $D_{(x,z)}d(v, w)$ exists and is equal to the above sum for some geodesic γ between x and z.*

Using the uniform convexity of the tangent spaces, we see that the distance functions to points in a geometric space play the role of the coordinate functions in the Euclidean space, i.e., a Lipschitz map $f : X \to Z$ of a space X to a geometric space Z is differentiable at x if for a dense sequence of points z_n in a punctured neighborhood of $f(x)$ the composition functions $d_{z_n} \circ f$ are differentiable at x. Now, the first statement of the next proposition is an easy exercise, whereas the second requires some work. It shows that our notion of geometric spaces is stable enough to survive such a difficult operation as taking quotients.

Theorem 1.3. *Let $f : X \to Y$ be a submetry. If X and Y are geometric*, *then f is differentiable at each point. Moreover*, *the assumption that X is geometric already implies that Y is geometric.*

Furthermore, it is possible to describe precisely the differential structure of a submetry, getting the usual vertical (tangent space to the fiber) and horizontal (tangent space to the union of horizontal geodesics) subspaces of the tangent space.

For maps into a geometric space, the theorem of Rademacher is equivalent to the theorem of Rademacher for functions.

Proposition 1.4. *Let Z be a metric space with Borel measure μ and with tangent spaces at almost each point such that each Lipschitz function $f : Z \to \mathbb{R}$ is differentiable μ-almost everywhere. Then*, *for each geometric space X*, *each Lipschitz map $f : Z \to X$ is differentiable almost everywhere.*

Corollary 1.5. *If Z is a measurable subset of the Euclidean space $\mathbb{R}^n$ and $f : Z \to X$ is a locally Lipschitz map to a geometric space X*, *then for almost all $z \in Z$ the differential D_zf exists*, *the image $D_zf(\mathbb{R}^n) \subset T_{f(z)}X$ is a Banach space*, *and the restriction $D_zf : \mathbb{R}^n \to D_zf(\mathbb{R}^n)$ is linear.*

A final issue that we address in this paper is differentiability of maps to arbitrary spaces with one-sided curvature bound. In this situation the tangent space in our sense may fail to exist; however, the same ideas can be used by dealing with the geodesic cone C_x instead of the tangent cone. For semiconcave functions this was done in [Lytc]; here we prove the following statement.

Theorem 1.6. *Let Z be either $CAT(\kappa)$ space or a space with curvature at least κ. Let $S \subset \mathbb{R}^n$ be a measurable subset, and $f : S \to Z$ a locally Lipschitz map. Then f has a differential $D_x f : T_x S \to C_{f(x)} Z$ at almost each point.*

Remark 1.3. If Z is an Aleksandrov space in the sense of Definition 2.2, then Theorem 1.6 is a special case of Proposition 1.4.

1.6. The plan. After the preliminaries, we recall some basic notions concerning ultra-convergence of spaces and maps, a major tool in this paper. In §4 we discuss the basic issues concerning general metric cones. In §5 we start with differential issues and discuss geodesic cones and the exponential mappings. In §§6 and 7 we give the definition of tangent cones and differentials, present the main examples, and discuss condition (U) and some other related topics. In §8 we recall Kirchheim's notion of metric differentiability. In §9 we discuss the first variation formula. In §§10 and 11 we study geometric spaces. Finally, in §12 we prove Theorem 1.6.

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§2. Preliminaries and notation

2.1. Notation. By $\mathbb{R}^+$ (respectively, $\mathbb{R}^n$) we denote the set of positive real numbers (respectively, the Euclidean space). We shall denote by d the distance in metric spaces. For a subset A of a metric space X, we denote by d_A the distance function to the set A. For a positive number r, we denote by rX the set X with the metric scaled by r. By $B_r(x)$ we denote the closed ball of radius r around x. A pseudometric d on a space X is a metric for which the distance between different points may be 0. Identifying points x, z in X with $d(x, z) = 0$, we get the corresponding metric space. A map $f : X \to Y$ between metric spaces is said to be *L-Lipschitz* if for all $x, z \in X$ we have $d(f(x), f(z)) \leq Ld(x, z)$.

Example 2.1. Each distance function d_A is 1-Lipschitz, whereas the metric $d : X \times X \to \mathbb{R}$ is a $\sqrt{2}$-Lipschitz function.

An s-dilation is a bijective map $f : X \to Y$ between metric spaces with $d(f(x), f(\bar{x})) = sd(x, \bar{x})$ for all $x, \bar{x} \in X$. An isometry is a 1-dilation.

Definition 2.1. By a *scale* we shall mean a sequence $(o) = (\epsilon_i)$ of positive real numbers converging to 0.

2.2. Geodesics. For a curve γ in X we denote its length by $L(\gamma)$. A geodesic (respectively, ray; respectively, line) in X is an isometric embedding of an interval (respectively, half-line; respectively, the entire real line) in X. For disjoint subsets $S, T \subset X$, we denote by $\Gamma_{S,T}$ the set of all geodesics of length $d(S, T)$ starting in S and ending in T. The space X is said to be *geodesic* if for all points $x \neq z$ in X the set $\Gamma_{x,z}$ is not empty. Finally, we denote by Γ_x the set of all geodesics starting at x.

We say that a metric space X is *proper* if the closed bounded subsets of X are compact. For a proper geodesic space X, the set $\Gamma_{S,T}$ is compact and not empty if S is compact and T is closed.

2.3. Busemann functions. For a ray $h : [0, \infty) \to X$ in X, we denote by b_h its Busemann function $b_h(x) = \lim_{t \to \infty}(d(x, h(t)) - t)$. This limit always exists, and b_h is a 1-Lipschitz function.

Example 2.2. If $f : X \to \mathbb{R}$ is a 1-Lipschitz map with $f(h(t)) = -t$, then $f \leq b_h$. Specifically, for rays h_j converging to a ray h we have $\liminf b_{h_j} \leq b_h$.

Example 2.3. Let γ be a line determining two rays γ^+ and γ^-. Then $-b_{\gamma^-}$ is a 1-Lipschitz function satisfying $-b_{\gamma^-} = b_{\gamma^+}$ on γ. Therefore, $b_{\gamma^-} + b_{\gamma^+} \geq 0$ on X. We say that γ is *straight* if $b_{\gamma^-} + b_{\gamma^+} = 0$ in X.

Example 2.4. For $i = 1, 2$, let h_i be a ray in the space X_i. Then $h(t) = (h_1(\frac{t}{\sqrt{2}}), h_2(\frac{t}{\sqrt{2}}))$ is a ray in $X_1 \times X_2$. Let $f : X_1 \times X_2 \to \mathbb{R}$ be a $\sqrt{2}$-Lipschitz function satisfying $f(h_1(t), h_2(t)) = -2t$. Then for $(v, w) \in X_1 \times X_2$ we get the inequality $f((v, w)) \leq \sqrt{2} b_h((v, w)) = b_{h_1}(v) + b_{h_2}(w)$.

2.4. Aleksandrov spaces. We refer to [BBI01, BGP92, BH99] for the theory of spaces with one-sided curvature bound. A space X is called a $CAT(\kappa)$-*space* (respectively, a space with curvature at least κ) if it is complete and the geodesic triangles in X are not thicker (respectively, not thinner) than the triangles in the two-dimensional simply connected manifold M^2_κ of constant curvature κ.

Definition 2.2. We will call a space X an Aleksandrov space if X is a proper space that either has curvature at least κ and finite Hausdorff dimension, or is geodesically complete (i.e., each geodesic is part of an infinite locally geodesic) and contains a $CAT(\kappa)$-neighborhood of each of its points, for some $\kappa \in \mathbb{R}$.

§3. Ultralimits

3.1. Ultraconvergence of spaces. A reader not accustomed to ultrafilters and ultralimits should consult [BH99] or [KL97] for excellent accounts. Let ω denote an arbitrary nonprincipal ultrafilter on the set of natural numbers. For each sequence (x_i) in a compact Hausdorff space X, ω allows us to choose a point $\lim_\omega(x_i)$ among the limit points of this sequence. Also, it allows us to construct a limit space of a sequence of spaces, as well as limits of Lipschitz maps between them, in the following manner.

For a sequence (X_i, x_i) of pointed metric spaces, their ultralimit $(X, x) =: \lim_\omega(X_i, x_i)$ is defined to be the set of all sequences (z_i) of points $z_i \in X_i$ with $\sup\{d(z_i, x_i)\} < \infty$. On this set we consider the pseudometric $d((z_i), (y_i)) := \lim_\omega(d(z_i, y_i))$. The ultralimit (X, x) is the metric space arising from this pseudometric.

Example 3.1. Let (X_i, x_i) be a constant sequence (X, x). Then $\lim_\omega(X_i, x_i)$ is called the ultraproduct of (X, x) and is denoted by X^ω. This space contains (X, x) in a natural way $(z \to (z, z, z, \dots))$ and does not depend on the base point x. It coincides with (X, x) if and only if X is a proper space.

3.2. Relationship with the usual convergence. The following lemma allows us to replace ultralimits by limits if the statement in question concerns all sequences.

Lemma 3.1. *Let (x_j) be a sequence in a complete metric space (X, x) with uniformly bounded distances to x. If for each subsequence (x_{k_i}) of this sequence the point $z = (x_{k_i})$ in the ultraproduct $X^\omega = \lim_\omega(X, x)$ does not depend on the subsequence, then z is in X and the sequence (x_j) converges to z.*

Proof. Assume that x_i is not a Cauchy sequence. Then, replacing (x_i) by a subsequence, we may assume that $d(x_i, x_{i+1}) > \epsilon$ for all i. Consider the subsequence y_i of x_i given by $y_i = x_{i+1}$. Then in X^ω the points (y_i) and (x_i) are at a distance at least ϵ from each other. Contradiction. $\square$

The Gromov–Hausdorff topology on the set of isometry classes of pointed proper metric spaces is closely related to ultralimits. If a sequence (X_i, x_i) of proper metric spaces converges to a proper space (X, x) in the Gromov–Hausdorff topology, then $\lim_\omega(X_i, x_i)$ is in the isometry class of (X, x) (see [KL97, p. 132]).

3.3. Ultralimits of maps. In a natural way, each sequence of L-Lipschitz maps $f_j : (X_j, x_j) \to (Y_j, y_j)$ induces an ultralimit $f = \lim_\omega f_j$ that is an L-Lipschitz map between the ultralimits (X, x) and (Y, y) of the sequences (X_j, x_j) and (Y_j, y_j)), respectively; specifically, $f((z_j)) := (f_j(z_j))$. These ultralimits of maps commute with compositions.

Example 3.2. If γ_j are L-Lipschitz curves in X_j starting at x_j, then $\gamma = \lim_\omega \gamma_j$ is an L-Lipschitz curve in $(X, x) = \lim_\omega(X_j, x_j)$ starting at x. If all curves γ_j are geodesics, then so is γ. In particular, if all the spaces X_j are geodesic, then so is X. Actually X is geodesic if X_j are merely length metric spaces.

Example 3.3. The ultralimit of products of spaces is the product of the corresponding ultralimits. If (S_j, x_j) are subsets of (X_j, x_j), then the ultralimit $\lim_\omega(S_j, x_j)$ is embedded in $\lim_\omega(X_j, x_j)$ in a natural way.

Example 3.4. Let X_j be a $CAT(\kappa_j)$-space (respectively, a space with curvature at least κ_j, with $\kappa_j \to \kappa$). Then $\lim_\omega(X_j, x_j)$ is a $CAT(\kappa)$-space (respectively, a space with curvature at least κ). For spaces with upper curvature bound this was proved in [KL97]. For lower curvature bounds the statement is not completely trivial, but it follows directly from [PP94b, 1.6].

Remark 3.5. The ultralimits of sequences of spaces and maps usually depend on the choice of the ultrafilter ω. In fact, if for a sequence (X_i, x_i) of proper metric spaces the isometry class of $(X, x) = \lim_\omega(X_i, x_i)$ does not depend on the ultrafilter ω and this space X is proper, then the sequence of the isometry classes of (X_i, x_i) converges with respect to the Gromov–Hausdorff topology.

3.4. Blow-up. Let X be a metric space, and let $x \in X$. For each scale $(o) = (\epsilon_i)$ we get a blow-up $X_x^{(o)} = \lim_\omega(\frac{1}{\epsilon_i}X, x)$ at the scale (o). It is a space with a distinguished point $0 = (x, x, \dots)$. If $f : (X, x) \to (Y, y)$ is a locally Lipschitz map, we get a blown-up map $f_x^{(o)} : X_x^{(o)} \to Y_y^{(o)}$. For a subspace S of X containing x we get a subspace $S_x^{(o)}$ of $X_x^{(o)}$. In particular, a geodesic γ starting at x determines a ray $\gamma_x^{(o)}$ starting at 0.

Remark 3.6. If X is a doubling metric space near x, i.e., if for some $C > 0$, each $r \leq \frac{1}{C}$, and each point $z \in B_{\frac{1}{C}}(x)$ the ball $B_r(z)$ can be covered by C balls of radius $\frac{r}{2}$, then each blow-up $X_x^{(o)}$ is a proper metric space. For instance, this is the case if X is a doubling measure space (see [Che99]).

Example 3.7. If X is a Banach space (respectively, has lower, respectively, upper curvature bound), then for each scale (o) the blow-up $X_x^{(o)}$ is a Banach space (respectively, a nonnegatively curved space, respectively, a $CAT(0)$-space).

Example 3.8. Let $(o) = (t_i)$ and $(\tilde{o}) = (r_i)$ be different scales. In general, there is no possibility to compare the blow-ups $X_x^{(o)}$ and $X_x^{(\tilde{o})}$. However, if the scales are comparable, i.e., if $0 < \lim_\omega(\frac{t_i}{r_i}) := s < \infty$, then the $\frac{t_i}{r_i}$-dilation $id : (\frac{1}{t_i}X, x) \to (\frac{1}{r_i}X, x)$ induces a natural s-dilation $id_{(\tilde{o})}^{(o)} : (X_x^{(o)}, 0) \to (X_x^{(\tilde{o})}, 0)$.

3.5. Infinitesimal isometries. The following definition is a metric analog of the notion of a Lebesgue point.

Definition 3.1. Let (S,x) be a subset of (X,x). We shall say that S is infinitesimally dense at x if for each scale (o) the canonical isometric embedding $i^{(o)}: S_x^{(o)} \to X_x^{(o)}$ is onto (i.e., an isometry).

The above definition simply says that for each $\epsilon > 0$ and all sufficiently small δ the ball $B_\delta(x) \subset S$ is $\epsilon\delta$-dense in the ball $B_\delta(x) \subset X$.

Example 3.9. If S is dense in a neighborhood of x in X, then S is infinitesimally dense at x. If X is a doubling metric measure space (see [Che99]) and S a measurable subset, then S is infinitesimally dense at each of its Lebesgue points.

Example 3.10. Let X be complete and geodesic. If a closed subset S of X is infinitesimally dense at each point $x \in S$, then $S = X$ (see [Lytc]).

Definition 3.2. Let $e : (X,x) \to (Y,y)$ be a not necessarily continuous map. We shall call e an infinitesimal isometric embedding (at x) if $|d(e(x_1),e(x_2)) - d(x_1,x_2)| \le o(d(x_1,x)+d(x_2,x))$ for all $x_1,x_2 \in X$ and some function $o : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{t\to 0} \frac{o(t)}{t} = 0$. We say that e is an infinitesimal isometry (at x) if, moreover, the image $e(B_\delta(x))$ of each ball $B_\delta(x)$ around x is infinitesimally dense at y in Y.

Example 3.11. A Lipschitz map $e : (X,x) \to (Y,y)$ is an infinitesimal isometry if and only if for each scale (o) the blow-up $e_x^{(o)} : X_x^{(o)} \to Y_y^{(o)}$ is an isometry.

A composition of infinitesimal isometries is again an infinitesimal isometry. The importance of this notion is due to the following easy observation.

Lemma 3.2. *Let $e : (X,x) \to (Y,y)$ be an infinitesimal isometry. Then for each scale (o) the map $e_x^{(o)} : X_x^{(o)} \to Y_y^{(o)}$ given by $e_x^{(o)}((x_i)) = (e(x_i))$ is well defined. Moreover, it is an isometry.*

Example 3.12. Let (S,x) be a subset of (X,x). Define a map $e : (X,x) \to (S,x)$ by setting $e(z) = \bar{z}$, where $\bar{z}$ is an arbitrary point in S with $d(z,\bar{z}) \le 2d(z,S)$. Then e is an infinitesimal isometry if and only if S is infinitesimally dense at x. In this case $e_x^{(o)} : X_x^{(o)} \to S_x^{(o)}$ is the canonical identification.

These (generalized) blow-ups are again compatible with compositions (of infinitesimal isometries). From Example 3.12 we see that for each infinitesimal isometry $e : (X,x) \to (Y,y)$ there is an infinitesimal isometry $\bar{e} : (Y,y) \to (X,x)$ such that $e^{(o)} \circ \bar{e}^{(o)} = id$ and $\bar{e}^{(o)} \circ e^{(o)} = id$ for each scale (o).

§4. Metric cones

4.1. Group of dilations. Let (X,x) be a pointed metric space. Consider the group $\mathrm{Dil}_x(X)$ of all dilations of X leaving the point x invariant and equip it with the topology of pointwise convergence. The natural map $P : \mathrm{Dil}_x(X) \to \mathbb{R}^+$ sending an s-dilation to the number s is a continuous homomorphism. The kernel of P is the group I_x of isometries of X fixing the point x.

Definition 4.1. A metric cone structure on the space (X,x) is a continuous section of the homomorphism P above, i.e., a continuous homomorphism $\rho : \mathbb{R}^+ \to \mathrm{Dil}_x(X)$ that sends s to some s-dilation ρ_s. A metric cone is a space with a metric cone structure. We call x the origin of the metric cone X and denote it by 0. A map $f : X \to Y$ between metric cones is said to be homogeneous if it commutes with all dilations ρ_t.

A metric space (X,x) can admit several families of dilations making it a metric cone. If X is a proper metric space, then the pointwise topology on $\mathrm{Dil}_x(X)$ coincides with the compact-open topology, and the group $\mathrm{Dil}_x(X)$ (respectively, I_x) is locally compact (respectively, compact). If a metric cone structure on (X, x) exists, the projection $P : \mathrm{Dil}_x(X) \to \mathbb{R}^+$ is surjective. On the other hand, if the map P is surjective and X is proper, then it is easy to see that the group $\mathrm{Dil}_x(X)$ splits as a direct product $I_x \times \mathbb{R}^+$ such that P becomes the projection onto the second factor. (First reduce to the connected component of $\mathrm{Dil}_x(X)$. Then use the fact that the group of outer automorphisms of I_x is totally disconnected; see [HM98, p. 512].) In particular, in this case there exists a metric cone structure on X such that all dilations ρ_s are in the center of Dil_x. Moreover, different metric cone structures are in one-to-one correspondence with different continuous homomorphisms $p : \mathbb{R}^+ \to I_x$.

4.2. Cones. The products and ultralimits of metric cones are metric cones with naturally defined dilations ρ_s. For a metric cone $(X, 0)$, the metric $d : X \times X \to \mathbb{R}$ is a homogeneous function. By the norm $|\cdot|$ we mean the homogeneous function d_0.

A ray $\gamma : [0, \infty) \to X$ starting at the origin of the cone X is said to be *radial* if it is stable under the dilations, i.e., it is a homogeneous map. If a ray γ is radial, then its Busemann function $b_\gamma : X \to \mathbb{R}$ is homogeneous.

Example 4.1. A Banach space B is a cone with dilations $\rho_t(v) = tv$. The radial rays are precisely the linear ones: $\gamma(t) = tv$. The Busemann function b_γ of such a ray γ is linear if and only if v is a smooth point of the unit sphere (see [JL01, p. 30] for the definition), i.e., if and only if the affine line in the direction of v is straight in the sense of Example 2.3.

Example 4.2. The Euclidean cone CY over a metric space Y (see [BBI01, p. 91]) is a metric cone. Each ray starting at 0 is radial and has the form $\gamma(t) = tv$ with $v \in Y$. Its Busemann function is given for $w \in Y$ by $b_\gamma(sw) = -\langle v, sw\rangle := -s\cos(d^Y(v, w))$.

4.3. Special metric cones. Cones can be arbitrarily wild in general. We shall use the following particularly nice classes of metric cones.

Definition 4.2. We say that a metric cone X is *radial* if for each $x \in X$ with $|x| = 1$ the map $t \to \rho_t(x)$ is a ray.

A cone is radial if and only if it is the union of its radial rays. Consider the unit sphere S in a radial cone X, i.e., the set of all points $v \in X$ with $|v| = 1$. Then, using only the triangle inequality, it is easy to check that the natural homogeneous map $CS \to X$ of the Euclidean cone over S to X that sends the point $tv \in CS$ to $\rho_t(v)$ is bi-Lipschitz.

Definition 4.3. A radial cone X is said to be uniformly convex if for each $\epsilon > 0$ there is some $\delta > 0$ such that for each radial ray $\gamma(t) = \rho_t(v_0)$ ($|v_0| = 1$) and each $v \in X$ with $|v| = 1$ and $d(v, v_0) \geq \epsilon$ we have $b_\gamma(v) \geq -1 + \delta$.

Definition 4.4. A metric cone X is said to be *smooth* if for each sequence of radial rays γ_j converging to a radial ray γ the Busemann functions b_{γ_j} converge pointwise to the Busemann function b_γ.

A direct product or a subcone of radial (respectively, uniformly convex, respectively, smooth) cones is radial (respectively, uniformly convex, respectively, smooth). A completion or an ultraproduct of radial (respectively, uniformly convex) cones is radial (respectively, uniformly convex). Euclidean cones are uniformly convex and smooth. Banach spaces are radial cones, and the uniform convexity (respectively, smoothness) is the usual uniform convexity (respectively, smoothness) of the norm.

Example 4.3. The Carnot groups are not radial, but it is possible to prove that they are smooth cones.

§5. Geodesic cones

5.1. Germs of geodesics. Let x be a point in a space X. Consider the set Γ_x of all geodesics starting at x and the direct product $\Gamma_x \times [0,\infty)$. We define a pseudometric $\tilde{d}$ on $\Gamma_x \times [0,\infty)$ by setting $\tilde{d}((\gamma_1, s_1),(\gamma_2, s_2)) = \limsup_{t\to 0} \frac{d(\gamma_1(s_1 t),\gamma_2(s_2 t))}{t}$. Denote by $\widetilde{C}_x$ the metric space corresponding to the pseudometric space $(\Gamma_x \times [0,\infty), \tilde{d})$ and by C_x its metric completion. The set $\Gamma_x \times \{0\}$ is identified with the point 0 in C_x. On the spaces $(\widetilde{C}_x, 0)$ and $(C_x, 0)$, the dilations of $[0,\infty)$ determine structures of metric cones.

Each geodesic $\gamma \in \Gamma_x$ gives rise to a radial ray $\bar{\gamma}(t) = (\gamma, t)$ in $\widetilde{C}_x \subset C_x$. Hence, C_x is always a radial metric cone. We denote by γ^+ the ray (γ, t) in C_x, as well as the point $\gamma^+(1) = (\gamma, 1)$. Let S_x denote the unit sphere in C_x; we call it the link at x. One can think of C_x as the space of unparameterized geodesic germs at x. However, one should be cautious, as the following example shows.

Example 5.1. Let X be a Banach space, and let $x = 0$ be its origin. If the norm of X is strongly convex, then all geodesics are straight lines, and C_x is naturally isometric to X. But if the norm is not strongly convex, then the space C_x is much larger than X; for instance, C_x is never locally compact in this case.

5.2. Exponential mappings. For each scale $(o) = (t_i)$ there is a natural map $\exp_x^{(o)}$: $\Gamma_x \times [0,\infty) \to X_x^{(o)}$ defined by $\exp_x^{(o)}((\gamma, t)) = (\gamma(tt_i)) \in X_x^{(t_i)}$. The exponential map $\exp_x^{(o)}$ goes down and determines a 1-Lipschitz map $\exp_x^{(o)} : \widetilde{C}_x \to X_x^{(o)}$, due to the upper limit in the definition of the distance in $\widetilde{C}_x$. Also, $\exp_x^{(o)}$ will denote a unique 1-Lipschitz extension $\exp_x^{(o)} : C_x \to X_x^{(o)}$.

Example 5.2. Let $(o) = (t_i)$ and $(\tilde{o}) = (r_i)$ be two scales with $0 < \lim_\omega(\frac{t_i}{r_i}) := s < \infty$. For the canonical s-dilation $id^{(o)}_{(\tilde{o})} : (X_x^{(o)}, 0) \to (X_x^{(\tilde{o})}, 0)$ we get $id^{(o)}_{(\tilde{o})} \circ \exp_x^{(o)} = \exp_x^{(\tilde{o})} \circ \rho_s$, where ρ_s is the natural s-dilation on C_x.

For each $\gamma \in \Gamma_x$, the radial ray $\gamma^+ \subset C_x$ is mapped isometrically onto the ray $\gamma_x^{(o)}$ by $\exp_x : C_x \to X_x^{(o)}$. The exponential mappings $\exp_x^{(o)} : C_x \to X_x^{(o)}$ are isometric embeddings for all scales $(o) = (t_i)$ if and only if the upper limit in the definition of the distance in C_x is always a limit. The fundamentality of this property justifies the following definition.

Definition 5.1. We say that the space X has property (A) at x if the upper limit in the definition of the distance in C_x is always a limit.

Example 5.3. The upper angle between arbitrary geodesics starting at x coincides with the lower angle (for the definition, see [BBI01, p. 98] if and only if condition (A) is fulfilled at x and the geodesic cone is a Euclidean cone $C_x = CS_x$.

Remark 5.4. Even if X is a geodesic space with property (A) at x (so that the mappings $\exp_x^{(o)}$ determine isometric embeddings of C_x in the geodesic spaces $X_x^{(o)}$), the geodesic cone C_x need not be a geodesic space. For example, this is not the case in Carnot–Carathéodory spaces or, in general, in spaces with lower curvature bound (see [Hal00]).

Remark 5.5. If X has property (A) at x and some blow-up $X_x^{(o)}$ is a proper space (cf. Remark 3.6), then C_x is also a proper space.

5.3. Geodesic cones under one-sided curvature bounds. In a general space X with one-sided curvature bound, the lower and the upper angles between arbitrary geodesics coincide (see [BBI01]). Therefore, the geodesic cone C_x at each point $x \in X$ is a Euclidean cone, and for each scale (o) the exponential map $\exp_x^{(o)} : C_x \to X_x^{(o)}$ is an isometric embedding. We shall see later that if X is an Aleksandrov space, then $\exp_x^{(o)}$ is also onto. However, in general spaces with one-sided curvature bound this is almost never the case.

In [Nik95] it was proved that if X has an upper curvature bound, then C_x is a geodesic space; hence, it is a totally convex subset of each blow-up $X_x^{(o)}$. The example presented in [Hal00] shows that C_x is not necessarily a geodesic space if X has a lower curvature bound. However, since C_x is an isometrically embedded Euclidean cone in $X_x^{(o)}$, from the Toponogov rigidity theorem it follows that for all radial rays η_1, η_2 in C_x and an arbitrary geodesic η in $X_x^{(o)}$ connecting arbitrary points on η_1 and η_2, the triangle $\eta_1\eta\eta_2$ is Euclidean.

Remark 5.6. Let X be a space with lower curvature bound. It is not difficult to prove that if a unit vector $v \in C_x$ has an antipode $w \in X_x^{(o)}$, i.e., a point w with $d(w,0) = 1$ and $d(w,v) = 2$, then w is contained in $\exp_x^{(o)}(C_x)$, and the line determined by v and w is a Euclidean factor not only of $X_x^{(o)}$ but also of C_x, i.e., v is connected with every other point $\bar{v} \in C_x$ by a geodesic in C_x.

§6. Tangent cones

6.1. Definition and the main example. The following main definition of this paper is motivated by Example 5.2.

Definition 6.1. Let (X,x) be a metric space. We say that the *tangent space* of X at x exists if for some metric cone (T,ρ_s) and each scale $(o) = (t_i)$ an isometry $i^{(o)} : (T,0) \to (X_x^{(o)}, 0)$ is chosen such that for different scales $(o) = (t_i)$ and $(\tilde{o}) = (r_i)$ with $0 < \lim_\omega(\frac{t_i}{r_i}) := s < \infty$ the map $i^{(\tilde{o})} \circ \rho_s \circ (i^{(o)})^{-1} : X_x^{(o)} \to X_x^{(\tilde{o})}$ is the natural s-dilation (see Subsection 3.4).

This metric cone T together with the fixed isometries $i^{(o)}$ will be called the tangent space at x and denoted by T_xX.

Example 6.1. If $(T,0)$ is a metric cone, then the tangent space at 0 is naturally isometric to the ultraproduct T^ω. The isometries $i^{(t_i)} : T^\omega \to T_0^{(t_i)}$ are given by $i^{(t_i)}((x_i)) = (\rho_{t_i}(x_i))$.

If the tangent cone T_xX exists, then the isometries $i^{(o)}$ allow us to identify the points in the blow-up $X_x^{(o)}$ with points in T_xX. We shall always use this particular identification below.

The following remark relates our definition to that of Gromov.

Remark 6.2. Let (X,x) be a proper space. Assume that, as $t \to 0$, the set of all (isometry classes of) spaces $(\frac{1}{t}X, x)$ is relatively compact in the Gromov–Hausdorff topology (cf. Remark 3.6). If T_xX exists in the sense of Definition 6.1, then, as $t \to 0$, the isometry classes of $(\frac{1}{t}X, x)$ converge to the isometry class of T_xX (see Subsection 3.2). On the other hand, if the convergence occurs, we know that for each scale $(o) = (t_i)$ there is an isometry $i^{(o)} : T_xX \to X_x^{(o)}$. The natural s-dilations between blow-ups give rise to s-dilations on T_xX. It is possible to choose a metric cone structure on T_xX (see Subsection 4.1) and to change the isometries $i^{(o)}$ so that the commutation relations of Definition 6.1 be satisfied.

Most of tangent spaces arise from Example 6.1 and the following example.

Example 6.3. Let $e : (X,x) \to (Y,y)$ be a (not necessarily continuous) infinitesimal isometry. If T_xX exists, then T_yY also exists and is naturally isometric to T_yY. Namely, the isometries $i^{(o)} : T_xX \to X_x^{(o)}$ give rise to isometries $\tilde{i}^{(o)} : T_xX \to Y_y^{(o)}$ defined by $\tilde{i}^{(o)} = e_x^{(o)} \circ i^{(o)}$, where the isometries $e_x^{(o)} : X_x^{(o)} \to Y_y^{(o)}$ are given by Lemma 3.2.

Combining Examples 6.1, 6.3, and 3.12, we obtain the following statement.

Lemma 6.1. *Let $(T, 0)$ be a metric cone, let $(D, 0)$ be a subset of T infinitesimally dense at* 0, *and let $e : (D, 0) \to (X, x)$ be an infinitesimal isometry. Then T_xX exists and is naturally isometric to the ultraproduct T^ω.*

Remark 6.4. The identification of T_xX with T^ω as above depends on the metric cone structure of T in an essential way (see Example 6.1). Thus, changing the cone structure on T (and leaving e and the metric on T fixed), we get a different tangent cone structure at x.

Remark 6.5. If the cone T in Lemma 6.1 is proper, then the tangent cone T_xX is isometric to T; in particular, it does not depend on the choice of the ultrafilter ω.

Example 6.6. Let $(M, |\cdot|)$ be a smooth manifold with a continuous Finsler metric. For $x \in M$, let $(T_xM, |\cdot|_x)$ be the usual tangent space at x. Then each chart $e : T_xM \to M$ with $e(0) = x$ and such that its differential at 0 is the identity, satisfies the assumptions of Lemma 6.1. Therefore, the tangent space in the sense of Definition 6.1 coincides with the Banach space $(T_xM, |\cdot|_x)$. We note that the identification (i.e., the maps $i^{(o)}$) does not depend on the choice of the chart e. Moreover, the topology, the metric cone structure, and the identification map e only depend on the manifold structure of X. Only the metric (norm) on T_xM depends on the Finsler metric $|\cdot|$.

Example 6.7. In [Bel96], for each Carnot–Carathéodory space M, Bellaiche constructed an almost isometry $e : G_x \to M$ from the nilpotentization G_x of M at x to M, identifying T_xM with G_x.

6.2. Metric operation. Let X and Y be metric spaces. If T_xX and T_yY exist, then $T_{(x,y)}X \times Y$ exists and is naturally isometric to $T_xX \times T_yY$. The tangent space at x to the rescaled space tX is naturally isometric to $tT_xX = T_xX$. If X and Y are geodesic spaces and $f : X \to \mathbb{R}^+$ is a continuous function, then $T_xX \times T_yY$ is the tangent cone of the warped product $X \times_f Y$ at (x, y) (see [BBI01, p. 95]). (Use Example 6.3 for the identity map between $X \times_f Y$ and $X \times_{\tilde{f}} Y$ for the constant function $\tilde{f} = f(x)$.) In particular, if T_xX exists, then for each $t \in \mathbb{R}^+$ the tangent space $T_{tx}CX$ to the Euclidean cone exists and is naturally isometric to $T_xX \times \mathbb{R}$.

Let S be a subset of X, let $x \in S$, and let T_xX exist. We say that S has a tangent cone at x if the subset $S_x^{(o)} \subset X_x^{(o)} = T_xX$ does not depend on the scale (o). If S_1, S_2 are subsets of X both containing x with tangent cones $T_xS_1, T_xS_2 \subset T_xX$, then the union $S_1 \cup S_2$ admits the tangent cone $T_xS_1 \cup T_xS_2$ at x.

Example 6.8. If the subset (S, x) of (X, x) is infinitesimally dense at x, then T_xS exists and is equal to T_xX. In particular, if X is a doubling metric measure space such that T_xX exists for almost all $x \in X$, then for every measurable subset $S \subset X$ and almost every $x \in S$ (with respect to the induced measure) the tangent space $T_xS \subset T_xX$ exists and coincides with T_xX.

Example 6.9. Let S be an extremal subset of an Aleksandrov space X of curvature at least k (see ([PP94a]). Then at each point $x \in S$ the tangent space $T_xS \subset T_xX$ exists and is an extremal subset of T_xX.

6.3. Property (U)**.** The following condition seems to be very natural. It is a very rough generalization of the lower curvature bound condition.

Definition 6.2. Let X be a space, and let $x \in X$. Assume that the union of all geodesics starting at x contains a neighborhood of x, property (A) is fulfilled at x, and the geodesic cone C_x is proper. We say that X has property (U) at x if for each $\epsilon > 0$ there is $\rho > 0$ such that $d(\gamma(t), \eta(t)) \leq \epsilon t$ for all $t < \rho$ and all $\gamma, \eta \in \Gamma_x$ with $d(\gamma^+, \eta^+) < \rho$.

Example 6.10. If X has property (U) at x, then so does each subset S of X that is a union of geodesics starting at x.

Example 6.11. A complete metric cone T has property (U) at the origin if and only if it is proper and the metric cone structure can be changed so that T become radial and the only geodesics starting at 0 be parts of radial rays. The "if" direction is clear and the "only if" implication follows from the fact (proved below) that under condition (U) the geodesic cone C_0 is isometric to the ultraproduct T^ω. In particular, all proper Euclidean cones and all proper uniformly convex Banach spaces have property (U).

Property (U) allows us to compare distances in C_x and in X.

Proposition 6.2. *Let X be a space with property (U) at x. Then for each $\epsilon > 0$ there is $\rho > 0$ such that for all $r \leq t \leq \rho$ and all $\gamma, \eta \in \Gamma_x$ we have $|d(\gamma(r), \eta(t)) - d((\gamma, r), (\eta, t))| \leq \epsilon t$.*

Proof. Assume that there are sequences $\gamma_i, \eta_i \in \Gamma_x$ and zero sequences $r_i \leq t_i \to 0$ violating the above inequality. Choosing a subsequence, we may assume that γ_i^+ and η_i^+ are Cauchy sequences and the $\frac{r_i}{t_i}$ converge to a number s with $0 \leq s \leq 1$. Moreover, it may be assumed that $r_i = st_i$ and that the sequence t_i is monotone nonincreasing.

For an arbitrarily small $\rho > 0$, we can choose i so large that $d(\eta_i^+, \eta_j^+) + d(\gamma_i^+, \gamma_j^+) < \rho < \frac{\epsilon}{5}$ for all $j \geq i$. Using property (U), increasing i if necessary, and fixing ρ sufficiently small, we get $d(\gamma_i(t), \gamma_j(t)) + d(\eta_i(t), \eta_j(t)) \leq \frac{\epsilon t}{5}$ for all $t \leq t_i$. Hence, $|d(\gamma_j(st_j), \eta_j(t_j)) - d(\gamma_i(st_j), \eta_i(t_j))| + |d((\gamma_j, st_j), (\eta_j, t_j)) - d((\gamma_i, st_j), (\eta_i, t_j))| \leq \frac{4\epsilon}{5}$. Therefore, we obtain the inequality $|d(\gamma_i(st_j), \eta_i(t_j)) - d((\gamma_i, st_j), (\eta_i, t_j))| \geq \frac{\epsilon t_j}{5}$ for all $j \geq i$. This contradicts property (A). □

Corollary 6.3. *Suppose X has property (U) at x, and let γ_i be a sequence in Γ_x converging pointwise to a geodesic γ of positive length. Then the γ_i^+ converge to γ^+ in C_x. For each $\epsilon > 0$ there is $\rho > 0$ such that for each z with $d(z, x) < \rho$ the inequality $d(\gamma^+, \eta^+) < \epsilon$ is fulfilled for all geodesics $\gamma, \eta \in \Gamma_{x,z}$.*

Now we can use Proposition 6.2 and Lemma 6.1 to identify C_x with T_xX. Namely, we consider the logarithmic map $h : X \to C_x$ that sends a point $z \in X$ to some pair $(\gamma, t) \in C_x$ with $t = d(x, z)$ and $\gamma \in \Gamma_{x,z} \subset \Gamma_x$. By assumption, h is defined on the neighborhood $\bigcup_{\gamma \in \Gamma_x}$ of x. Proposition 6.2 shows that h is an infinitesimal isometry; consequently, each map $e : h(X) \subset C_x \to X$ satisfying $e \circ h = \mathrm{id}$ possesses the properties employed in Lemma 6.1 to identify C_x with T_xX. The identification between $X_x^{(o)}$ and C_x given by Lemma 6.1 is precisely the exponential map $\exp_x^{(o)}$.

Remark 6.12. The above construction is well known in many cases. In the case of Riemannian manifolds, the logarithmic map h as above is merely the inverse of the usual exponential map. If X is an Aleksandrov space of nonpositive curvature, then h is uniquely determined, surjective, and 1-Lipschitz. If X is an Aleksandrov space of nonnegative curvature, then the almost inversion $e : C_x \to X$ can be defined to be surjective and 1-Lipschitz (see [PP94b]).

Definition 6.3. A space X is said to be *infinitesimally cone-like* if it is locally geodesic, property (U) is fulfilled at each point $x \in X$, and each tangent cone $T_xX = C_x$ is a Euclidean cone.

§7. Differentials

7.1. Generalities. Let $f : (X, x) \to (Y, y)$ be a locally Lipschitz map and assume that $T_x X$ and $T_y Y$ exist. For each scale (o), the blow-up $f_x^{(o)} : X_x^{(o)} \to Y_y^{(o)}$ provides a map between the tangent spaces.

Definition 7.1. Let $f : X \to Y$ be as above. We say that f is differentiable at x if the blow-up $f_x^{(o)} : T_x X \to T_y Y$ does not depend on the scale (o). In this case we denote this uniquely determined map by $D_x f$.

Example 7.1. If $f : (X, 0) \to (Y, 0)$ is a homogeneous map between cones, then f is differentiable at 0, and the differential $D_0 f$ is the ultraproduct $f^\omega = \lim_\omega f : X^\omega \to Y^\omega$ of f.

Example 7.2. Let $f : X \to Y$ be an isometry. If $T_x X$ admits no nontrivial isometry fixing the origin 0, then f is differentiable at x, because for each scale (o) the map $f_x^{(o)} : T_x X \to T_y Y$ is an origin preserving isometry.

Example 7.3. Let S be a subset of X, and let $x \in S$. If $T_x X$ and $T_x S \subset T_x X$ exist (see Subsection 6.2), then the inclusion $I : S \to X$ is differentiable at x, and the differential is the natural embedding $I_x : T_x S \to T_x X$.

Example 7.4. Let $f : X \to Y$ be a bi-Lipschitz embedding. If f is differentiable at x, then $f(X)$ has a tangent cone at $f(x)$ given by $T_{f(x)} f(X) = D_x f(T_x X) \subset T_{f(x)} Y$. On the other hand, if $f : X \to Y$ is a differentiable C-open map (see [Lytc]) and S is a subset of Y admitting a tangent cone at $f(x)$, then $f^{-1}(S)$ has a tangent cone at x given by $T_x f^{-1}(S) = (D_x f)^{-1}(T_{f(x)} S) \subset T_x X$.

Example 7.5. If $T_x X$ and $T_y Y$ exist, then the projection $p : X \times Y \to X$ is differentiable, and the differential is equal to the projection. If $T_x X$ exists, then the metric $d : X \times X \to \mathbb{R}$ is differentiable at each point (x, x) on the diagonal, and the differential coincides with the metric on $T_x X$. The distance function $d_x : X \to \mathbb{R}$ is differentiable at x, with differential $D_x d_x(v) = |v|$. The differentiability of the metric at points outside the diagonal will be discussed in §9.

Example 7.6. Let $(T, 0)$ be a proper metric cone with dilations lying in the center of Dil_0. Let $(\widetilde{T}, 0)$ be the same space with a different metric cone structure given by a continuous homomorphism $p : \mathbb{R}^+ \to I_0$ (see Subsection 4.1). Let $f : (T, 0) \to (\widetilde{T}, 0)$ be the identity. Then $f_0^{(t_i)}$ is precisely the isometry $\lim_\omega(p(t_i))$. Hence, f is differentiable at 0 if and only if $\lim_{t \to 0} p(t)$ exists. However, this can happen only if p is the trivial map. This suggests that there is at most one natural tangent cone structure. Considering $T = \mathbb{R}^2$ leads, in essence, to the counterexample of [CH70].

Since ultralimits commute with compositions, we immediately obtain the following.

Lemma 7.1. *Let $f : X \to Y$ and $g : Y \to Z$ be Lipschitz maps with $f(x) = y, g(y) = z$. If f is differentiable at x and g is differentiable at y, then $g \circ f$ is differentiable at x with differential $D_x(g \circ f) = D_y g \circ D_x f$.*

Example 7.7. If $f : X \to Y$ is differentiable at x and S is a subset of X such that $T_x S \subset T_x X$ exists, then $f : S \to Y$ is differentiable at x, and the differential $D_x f : T_x S \to T_{f(x)} Y$ is the restriction of $D_x f : T_x X \to T_{f(x)} Y$. On the other hand, if $T_x S = T_x X$ and the restriction $f : S \to X$ is differentiable at x, then $f : X \to Y$ is also differentiable at x.

7.2. Comparison with the usual differentiability. If the tangent spaces are given by Lemma 6.1, we get the usual definition of differentiability. Namely, let $f : (X, x) \to (Y, y)$ be a Lipschitz map, let $(T_1, 0)$ (respectively, $(T_2, 0)$) be metric cones, and let $e_1 : T_1 \to X$ (respectively, $e_2 : T_2 \to Y$) be maps as in Subsection 6.1. (If the e_i are defined only on infinitesimally dense subsets $(D_i, 0) \subset (T_i, 0)$, we may extend them as in Example 3.12.) If $A : T_1 \to T_2$ is a homogeneous Lipschitz map such that $\lim_{|v|\to 0} \frac{d(f(e_1(v)), e_2(A(v)))}{|v|} = 0$ for $v \in T_1$, then the differential of f at x exists and is equal to the ultraproduct $A^\omega : T_1^\omega \to T_2^\omega$. On the other hand, we can use Lemma 3.1 to show that if the differential $D_x f$ exists, then the image of $D_x f(T_1) \subset T_2^\omega$ must be contained in $T_2 \subset T_2^\omega$; therefore, the existence of a map A as above is also necessary in this case.

In particular, differentiability does not depend on the ultrafilter ω! Moreover, if T_1 and T_2 are proper, then the differential also does not depend on the ultrafilter.

7.3. Separating maps. Let (Y, y) be a metric space, and let $\{f_j : Y \to Y_j\}$ be a set of Lipschitz maps differentiable at y and separating the points in $T_y Y$, i.e., for $v_1 \neq v_2 \in T_y Y$ there is j such that $D_y f_j(v_1) \neq D_y f_j(v_2)$. Since ultralimits of maps commute with compositions, we see that a map $g : (X, x) \to (Y, y)$ is differentiable at x if and only if for each j the map $f_j \circ g$ is differentiable at x. For example, a bi-Lipschitz map $f_0 : Y \to Y_0$ differentiable at y satisfies the above conditions. In particular, its inverse must be differentiable at $f_0(y)$.

7.4. Differentiating curves. Let $\gamma : [0, a] \to X$ be a Lipschitz curve with $\gamma(0) = x$. If γ is differentiable at 0, then the differential is a homogeneous map h of the half-line $[0, \infty)$ to $T_x X$. Since this map is uniquely determined by $h(1)$, we shall call the point $h(1)$ the right-hand side differential of γ at 0 and denote it by γ^+. In the same way we define γ^- if γ is differentiable at a. The differential exists at an interior point $t \in (0, a)$ if and only if γ^+ and γ^- exist at t.

7.5. Differentiating geodesics. The most natural and basic maps into a metric space are geodesics. One can hope to get a rich theory of differentiation only if many geodesics are differentiable. A geodesic $\gamma : [0, a] \to X$ starting at 0 is differentiable at x if and only if the ray $\gamma_x^{(o)} \subset X_x^{(o)} = T_x X$ does not depend on the scale (o). In this case we get a unique radial ray $\gamma_x \subset T_x X$. We see that all geodesics are differentiable at x if and only if the exponential mappings $\exp_x^{(o)} : C_x \to X_x^{(o)} = T_x X$ do not depend on the scale (o), i.e., if and only if C_x is naturally embedded in $T_x X$ via the exponential mappings. Then X has property (A) at x. For example, this is always the case if X has property (U) at x. However, in general, this may fail even in quite tame spaces; see [CH70] or Example 7.6.

7.6. Directional derivatives. Suppose $f : (X, x) \to (Z, z)$ is a locally Lipschitz map and $T_z Z$ exists. We say that f has directional derivatives at x if the restriction $f \circ \gamma$ to each geodesic $\gamma \in \Gamma_x$ is differentiable at 0. In this case we obtain a well-defined homogeneous map $D_x f : C_x \to T_z Z$ of the geodesic cone C_x to the tangent cone $T_z Z$. For each scale (o) we have $f_x^{(o)} \circ \exp_x^{(o)} = D_x f$. In particular, $D_x f$ inherits the Lipschitz constant of f.

Example 7.8. Each locally Lipschitz semiconcave function $f : X \to \mathbb{R}$ has directional derivatives at all points (see [Lytc] for more on this).

If X has a tangent cone at x and the geodesics are differentiable at x, then each Lipschitz map $f : (X, x) \to (Z, z)$ differentiable at x is also directionally differentiable, and $D_x f : C_x \to T_z Z$ is the restriction of $D_x f : T_x X \to T_z Z$ to C_x. On the other hand, if all geodesics are differentiable at x and the map $f : X \to Z$ is directionally differentiable at x, then the restriction of $f_x^{(o)} : T_xX \to T_zZ$ to the subset $C_x \subset T_xX$ is independent of the scale (o). This implies Proposition 1.1.

Actually, we can deduce that the isometries are a little smoother.

Corollary 7.2. *Suppose X has property (U) at x. Let $f_i : (X, x) \to (X, x)$ be isometries fixing x and converging pointwise to an isometry f. Then the isometries D_xf_i of T_xX converge to the isometry D_xf.*

Proof. Composing the isometries f_i with f^{-1}, we may assume that $f = \mathrm{id}$. Then for each geodesic γ the geodesics $\gamma_i = f_i(\gamma)$ converge to γ. By Corollary 6.3, for the starting direction γ^+ of γ the directions $D_xf_i(\gamma^+)$ converge to γ^+. □

7.7. Strong differentiability. Suppose $f : (X, x) \to (Z, z)$ is a locally Lipschitz map, T_xX exists, and Z has property (A) at z. We say that a strong differential $D_xf : T_xX \to C_z$ exists if $f_x^{(o)} = \exp_z^{(o)} \circ D_xf$ for each scale (o). If T_zZ exists and the geodesics are differentiable at z, then a map f is strongly differentiable at x if and only if f is differentiable and the differential $D_xf : T_xX \to T_zZ$ satisfies $D_xf(T_xX) \subset C_z$. Note that the strong differential (if it exists) is a homogeneous Lipschitz map.

§8. Metric Differentials

Example 7.5 gives rise to the following definition (see [Kir94]).

Definition 8.1. Suppose $f : X \to Y$ is a Lipschitz map and T_xX exists. We say that f has a metric differential at x if the composition $d \circ (f \times f) : X \times X \to \mathbb{R}$ is differentiable at (x, x). In this case the differential is a homogeneous pseudometric on T_xX; we denote it by mD_xf. We say that f has a weak metric differential at x if the map $d_{f(x)} \circ f : X \to \mathbb{R}$ is differentiable at x. Again, by mD_xf we denote the differential of this map.

If f is differentiable at x, then it also has a metric differential given by $mD_xf(v, w) = d(D_xf(v), D_xf(w))$. If f is metrically differentiable at x, then it is also weakly metrically differentiable, with weak metric differential $mD_x(v) = mD_x(0, v)$. On the other hand, let $f : X \to Y$ be a bi-Lipschitz map with $f(x) = y$. If f is metrically differentiable at x, then a tangent space T_yY can be defined uniquely so that f become differentiable at x. An isometric embedding $I : X \to Z$ is metrically differentiable at each point x where T_xX exists, and the metric differential coincides with the metric $mD_xI = d : T_xX \times T_xX \to \mathbb{R}$.

Example 8.1. The space X has property (A) at x if and only if for each pair of geodesics $\gamma_1, \gamma_2 \in \Gamma_x$ the map $\gamma : (-\epsilon, \epsilon) \to X$ given by $\gamma(t) = \gamma_1(t)$ for $t \leq 0$ and $\gamma(t) = \gamma_2(t)$ for $t \geq 0$ is metrically differentiable at 0.

Using this example, we immediately obtain the following statement.

Lemma 8.1. *Let $f : (X, x) \to (Z, z)$ be a Lipschitz map that is an infinitesimal isometric embedding at x. Assume that T_zZ exists. If the image $f \circ \gamma$ of each geodesic $\gamma \in \Gamma_x$ is differentiable at* 0*, then X has property (A) at x, the map f is directionally differentiable at x, and the differential $D_xf : C_x \to T_zZ$ is an isometric embedding.*

Example 8.2. Let M be a Finsler manifold. If each geodesic $\gamma \in \Gamma_x$ is differentiable at 0, then M has property (A) at x.

The following deep theorem was proved in [Kir94].

Theorem 8.2. *Let K be a measurable subset of $\mathbb{R}^n$, and $f : K \to Y$ a Lipschitz map. Then f has a metric differential at almost every point, this metric differential is almost everywhere a seminorm, and the map $x \to mD_x = |\cdot|_x$ is measurable.*

Example 8.3. Let $\gamma : [p, q) \to Y$ be a Lipschitz map. If the weak metric differential of γ at p exists, we denote by mD_p^+ the number $mD_p\gamma(1)$, i.e., $mD_p^+ = \lim_{t\to 0} \frac{d(\gamma(p+t),\gamma(p))}{t}$. The fact that the metric differential exists and is a seminorm amounts to the much stronger statement that

$$\lim_{t\to 0} \frac{d(\gamma(p+s_1t), \gamma(p+s_2t))}{t} = |s_2 - s_1| mD_p^+$$

for all $s_1, s_2 > 0$.

§9. Differentiation of distance functions

9.1. Generalities. We start with the following paradigmatic example.

Example 9.1. Suppose T is a proper metric cone, h is a radial ray, and $x = h(1)$. Then the distance function d_x is differentiable at the origin 0, and the differential is given by $D_0 d_x(v) = b_h(v)$, because

$$D_0 d_x(v) = \lim_{t\to 0} \frac{d(x, \rho_t(v)) - d(x, 0)}{t} = \lim_{t\to\infty} (d(\rho_t(x), v) - t) = b_h(v).$$

To state our results, we need the following extension of Definition 6.1. Suppose $f : X \to Y$ is a Lipschitz map, and T_xX and T_yY exist. We say that D_xf has some property (even if D_xf does not exist) if each blow-up $f_x^{(o)} : T_xX \to T_yY$ has this property.

Now, let X be a space, and let $x \neq z$ be points in X such that T_xX and T_zX exist. Assume that each geodesic $\gamma \in \Gamma_{x,z}$ is differentiable at x and at z, thus determining radial rays $\gamma^+ \subset T_xX$ and $\gamma^- \subset T_zX$.

Lemma 9.1. *In the above notation, for the differential $D_{(x,z)}d : T_xX \times T_zX \to \mathbb{R}$ of the metric $d : X \times X \to \mathbb{R}^+$ we have $D_{(x,z)}d(v, w) \leq \inf_{\gamma\in\Gamma_{x,z}} (b_{\gamma^+}(v) + b_{\gamma^-}(w))$.*

Proof. Choose some geodesic $\gamma : [a_1, a_2] \to X$ connecting x and z. Then

$$d(\gamma(a_1 + s), \gamma(a_2 - s)) = d(x, z) - 2s.$$

Hence, each blow-up $d_{(x,z)}^{(o)} : T_xX \times T_zX \to \mathbb{R}$ satisfies $d_{(x,z)}^{(o)}(\gamma^+(s), \gamma^-(s)) = -2s$. We are done by Example 2.4. □

In the same way, using Example 2.2 in place of Example 2.4, we prove the following statement.

Lemma 9.2. *Suppose X is a space, S is a closed subset of X, and $x \in X \setminus S$. Assume that T_xX exists and each geodesic $\gamma \in \Gamma_{x,S}$ connecting x and S is differentiable at 0. Then for the differential of the distance function d_S we have $D_x d_S(v) \leq \inf_{\gamma\in\Gamma_{x,S}} b_{\gamma^+}(v)$.*

Remark 9.2. Even if T_xX and T_zX do not exist, we can work with the directional differentials $D_{(x,z)} : C_x \times C_z \to \mathbb{R}$ and get (for the same reason) the same estimates as in Lemma 9.1.

9.2. First variation formula. As in Riemannian geometry, one would like to have equalities in the last two lemmas.

Definition 9.1. We say that the first variation formula is valid for $S \subset X$ and $x \in X \setminus S$ if equality occurs in the statement of Lemma 9.2.

Example 9.3. Let $\gamma \in \Gamma_{z,S}$ be a geodesic, and let $x = \gamma(t)$ be an interior point of γ. If γ is differentiable at x, it defines a homogeneous line $\widetilde{\gamma}$ in T_xX. Moreover, $D_t(d_S \circ \gamma) = -\operatorname{id}$. If $\widetilde{\gamma}$ is straight in the sense of Example 2.3, then the first variation formula is valid for S and x.

The validity of the first variation formula is closely related to the question as to whether geodesics vary smoothly in X.

Definition 9.2. Let X be a space with property (U) at x. We say that geodesics *vary smoothly* at x if for all geodesics γ and η in Γ_x and each sequence of geodesics γ_n with $\gamma_n(0) = \eta(t_n)$ converging to γ, the following condition is fulfilled. For each $\epsilon > 0$ there are $n > 0$ and $\rho_n > 0$ such that $\rho_n - d(x, \gamma_n(\rho_n)) \geq (b_{\gamma^+}(v) - \epsilon)t_n$, where $v = \eta^+(1)$ is the starting direction of η.

If this condition is satisfied for all x and all $\gamma \in \Gamma_x$, we say that geodesics vary smoothly in X.

Remark 9.4. Since we require that the above condition be fulfilled for all convergent sequences of geodesics, first of all we see that the number ρ_n as above exists for all sufficiently large n. Moreover, the numbers ρ_n can be chosen so that $\rho_n \to 0$. Finally, if the inequality in Definition 9.2 is true for some ρ_n, it is also true for all $\rho \geq \rho_n$. Consequently, we may choose all ρ_n to be equal to a small constant ρ depending on γ and ϵ.

The relationship between Definition 9.2 and the first variation formula is provided by the next three results.

Proposition 9.3. *Let X be a proper geodesic space, and let $x, z \in X$ be points at which X has property (U). Assume that geodesics vary smoothly at x and at z. Then equality occurs in Lemma* 9.1.

Proof. Since $X \times X$ has property (U) at (x, z), it suffices to prove that equality occurs for the originating direction of each geodesic $\tilde{\eta}$ in $X \times X$ starting at (x, z). Hence, it suffices to prove that for arbitrary geodesics $\eta_1 \in \Gamma_x$ and $\eta_2 \in \Gamma_z$ with starting directions $v \in T_xX$ and $w \in T_zX$ we have $\liminf_{t\to 0} \frac{d(\eta_1(t),\eta_2(st))}{t} \geq \inf_{\gamma\in\Gamma_{x,z}}(b_{\gamma^+}(v) + b_{\gamma^-}(sw))$ for all $s \geq 0$. Assume the contrary and choose a zero sequence (t_n) violating the above inequality. Let γ_n be a geodesic from $\eta_1(t_n)$ to $\eta_2(st_n)$. Passing to a subsequence, we may assume that the γ_n converge to a geodesic $\gamma \in \Gamma_{x,z}$. Given $\epsilon > 0$, for all n sufficiently large we can find numbers ρ^+ and ρ^- such that $d(x, \gamma_n(\rho^+)) \leq \rho^+ - (b_{\gamma^+}(v) - \epsilon)t_n$ and $d(x, \gamma_n(L(\gamma_n) - \rho^-)) \leq \rho^- - (b_{\gamma^-}(sw) - \epsilon)t_n$. But since γ is a geodesic, we get $L(\gamma) \leq d(x, \gamma_n(\rho^+)) + (L(\gamma_n) - \rho^+ - \rho^-) + d(z, \gamma_n(L(\gamma_n) - \rho^-))$. Consequently, $L(\gamma) - L(\gamma_n) \leq (2\epsilon + b_{\gamma^+}(v) + b_{\gamma^-}(sw))t_n$. This proves the result. ∎

In the same way we obtain the next statement.

Proposition 9.4. *Let X be a proper geodesic space with property (U) at x, and let $S \subset X$ be a closed subset not containing x. Assume that geodesics vary smoothly at x. Then the first variation formula is valid for S and x.*

Example 9.5. Under the assumptions of Proposition 9.3, assume that, moreover, the tangent cones T_xX and T_zX are smooth (Definition 4.4). For $v \in T_xX, w \in T_zX$ choose a sequence $\gamma_i \in \Gamma_{x,z}$ such that $D_{(x,z)}d(v, w) = b_{\gamma_i^+}(v) + b_{\gamma_i^-}(w)$. Let $\gamma \in \Gamma_{x,z}$ be a pointwise limit of γ_i. Then the γ_i^+ (respectively, γ_i^-) converge to γ^+ (respectively, γ^-) by Corollary 6.3, and the smoothness of the tangent cones implies $D_{(x,z)}d(v, w) = b_{\gamma^+}(v) + b_{\gamma^-}(w)$. This finishes the proof of Theorem 1.2.

Lemma 9.5. *Let X be a proper geodesic space with property (U) at x. Assume that the first variation formula is valid at x for each closed subset S not containing x. If the tangent cone T_xX is smooth, then geodesics vary smoothly at x.*

Proof. Let $\gamma, \eta, \gamma_n \to \gamma$ be as in Definition 9.2; we set $v = \eta^+$. For sufficiently small δ and each geodesic $\tilde{\gamma} \in \Gamma_{x,\gamma(\delta)}$, from Corollary 6.3 we deduce that $d(\gamma^+, \tilde{\gamma}^+) \leq \epsilon_1$, with

$\epsilon_1 \to 0$ as $\delta \to 0$. By the smoothness of the tangent cone, we have $|b_{\gamma^+}(v) - b_{\tilde{\gamma}^+}(v)| \leq \epsilon$, with $\epsilon \to 0$ as $\delta \to 0$.

Now, let ϵ be given. We choose δ to be sufficiently small, and consider δ_n such that $d(\gamma_n(\delta_n), x) = \delta + t_n^2$. Let S be the closed subset of X that consists of the sequence $\gamma_n(\delta_n)$ and the point $z = \gamma(\delta)$. The first variation formula yields $D_x d_S(v) = D_x d_z(v) = \inf_{\tilde{\gamma} \in \Gamma_{x,z}} b_{\tilde{\gamma}^+}(v)$. This and the above estimate of $b_{\tilde{\gamma}^+}(v)$ directly imply the inequality in Definition 9.2. $\square$

At the beginning of this section we saw that in the Banach spaces the first variation formula is always valid for the distance functions to points. The situation for the distance functions to subsets is more complicated.

Lemma 9.6. *Let B be a finite-dimensional uniformly convex Banach space. Then the first variation formula is valid in B for all distance functions if and only if the norm of B is smooth.*

Proof. Assume that the norm is smooth. Let γ_n be a sequence of geodesics converging to a geodesic γ. We may assume that $\gamma(s) = sh$, $\gamma_n(s) = t_n v + s h_n$ with $t_n \to 0$ and with some unit vectors v, h_n, and h, where the h_n converge to h. Fix some positive ϵ. By the smoothness of the norm, for large positive C we get $|Ch + v| + |Ch - v| - 2C < \epsilon$. But $d(0, \gamma_n(Ct_n)) = |t_n v + Ct_n h_n| = t_n |v + Ch_n|$. Choosing n such that Ch_n is very close to Ch, we obtain $Ct_n - d(0, \gamma_n(Ct_n)) \geq t_n(C - |v + Ch_n|) \geq t_n(|Ch - v| - C - \frac{\epsilon}{2}) \geq t_n(b_h(v) - \epsilon)$. Therefore, geodesics vary smoothly in X, and we are done by Proposition 9.4.

If the norm is not smooth, we can consider a nonsmooth point x of the unit sphere in B. Let H be a support hyperplane at x. It is easily seen that for the distance function d_H the first variation formula fails at the origin. We leave the details to the reader. $\square$

§10. The class of geometric spaces

We recall the following definition (see the Introduction).

Definition 10.1. A proper geodesic space X is said to be *geometric* if it has property (U) at each point, each tangent space $T_x X = C_x$ is uniformly convex and smooth, and geodesics vary smoothly in X.

We are going to show that many important spaces are geometric.

10.1. Aleksandrov spaces. Let X be an Aleksandrov space. The upper angle coincides with the lower angle for each pair of geodesics starting at the same point. Hence, X has property (A) at each point, and each geodesic cone is Euclidean. If X has a lower curvature bound, then the geodesic cone C_x is proper by [BGP92], and property (U) is fulfilled by the very definition of a lower curvature bound. For Aleksandrov spaces with upper curvature bound, property (U) easily follows from geodesic completeness (see [OT]). Thus, the Aleksandrov spaces are infinitesimally cone-like.

In order to prove that they are geometric, consider geodesics γ and η that start at x and form angle α, and a sequence of geodesics γ_i converging to γ with $\gamma_i(0) = \eta(t_i)$. If X has a lower curvature bound, then, by the semicontinuity of angles, the angle between γ_i and η^+ is greater than or equal to $\alpha - \epsilon$ for an arbitrarily small ϵ and all sufficiently large i. Hence, the angle between γ_i and η^- is at most $\pi - \alpha + \epsilon$. Now, using the comparison triangle for $x\eta(t_i)\gamma_i(\rho)$, we get the required upper bound for $d(x, \gamma_i(\rho))$. If X has an upper curvature bound, then the angle between η and the geodesic connecting x with $\gamma_i(\rho)$ is at least $\alpha - \epsilon$. Again, the comparison triangle for $x\eta(t_i)\gamma_i(\rho)$ provides the required upper bound for $d(x, \gamma_i(\rho))$.

10.2. Extremal subsets. In [Pet94] Petrunin proved that an extremal subset of an Aleksandrov space with lower curvature bound is infinitesimally cone-like and geometric with respect to the intrinsic metric.

10.3. Surfaces with an integral curvature bound. We assume that the reader is familiar with the notion of a two-dimensional surface with an integral curvature bound; see [Res93] for the definition and an excellent survey. Let M be a surface with an integral curvature bound. By [Res93, Theorem 8.2.3] the upper and the lower angle between each pair of geodesics coincide, so that at each point, M has property (A) and the geodesic cone is Euclidean. We denote by Ω^+ (respectively, Ω^-) the Borel measures that describe the positive (respectively, the negative) part of the curvature. We use Theorem 8.2.2 of [Res93] saying the following: let T be a triangle in M such that the concatenation of its sides is a simple closed curve and its interior part T^0 is homeomorphic to a ball. Let α be the angle between two sides of T, and let $\tilde{\alpha}$ be the corresponding angle in the comparison triangle in the Euclidean plane. Then $\alpha - \widetilde{\alpha} \leq \Omega^+(T^0)$. Now, let $x \in M$ be an arbitrary point. Since the intersection of the punctured balls $B_r^0(x) := B_r(x) \setminus \{x\}$ is empty, for each $\epsilon \geq 0$ we can find $r > 0$ such that $\Omega^+(B_r^0(x)) + \Omega^-(B_r^0(x)) \leq \epsilon$. Consequently, for each triangle T as above with a vertex in x and with sidelengths not exceeding r, we see that each angle of T differs from the corresponding angle of the comparison triangle by at most 3ϵ.

Consider two geodesics γ_1, γ_2 of length $t \leq r$ that start at x and form an angle not exceeding ϵ. In order to verify property (U), we need to estimate $\frac{d(\gamma_1(t),\gamma_2(t))}{t}$ from above. If γ_1 and γ_2 intersect at $\gamma_1(t_0) = \gamma_2(t_0)$, then the angle between γ_1^+ and γ_2^+ at $\gamma_1(t_0)$ is at most ϵ. Hence, we may assume that γ_1 and γ_2 do not intersect. Now it is easy to show that no geodesic η between $\gamma_1(t)$ and $\gamma_2(t)$ intersects $\gamma_1[0,t) \cup \gamma_2[0,t)$. Applying the above remark to the triangle $\gamma_1\eta\gamma_2$, we get the required estimate for the length of η. Thus, M is infinitesimally cone-like.

In order to prove that geodesics vary smoothly at x, we consider two geodesics $\gamma, \eta \in \Gamma_x$ enclosing a positive angle α at x. Let γ_n be a sequence of geodesics converging to γ with $\gamma_n(0) = \eta(t_n)$. Consider a geodesic ν_n between x and $\gamma_n(r)$. The above consideration implies that the angle between ν_n and γ is at most 2ϵ for large n. Hence, the angle between η and ν_n is at least $\alpha - 2\epsilon$. Using the triangle $\eta\nu_n\gamma_n$, we arrive at the desired upper bound for the length of ν_n.

10.4. Metric operations. If X and Y are geometric, then so is the product $X \times Y$. If X is geometric and C is a closed convex subset of X, then C is geometric. Moreover, the Euclidean cone CX is geometric. The proofs are straightforward; we leave them to the reader.

10.5. A class of interesting subsets of manifolds. Let M be a smooth manifold with a continuous Finsler metric. Let $K \subset M$ be a closed subset such that the intrinsic metric on K is bi-Lipschitz equivalent to the induced one, i.e., each two points $x, z \in K$ are connected in K by a curve of length at most $Ld(x,z)$. Assume further that all geodesics in K with respect to the intrinsic metric have uniformly bounded $C^{1,\alpha}$-norms for a fixed $0 < \alpha \leq 1$.

Remark 10.1. In [Lyta] it was shown that the above conditions are satisfied by sets of positive reach ($\alpha = 1$) and by similar wide classes of subsets in smooth Riemannian manifolds. Moreover, they are satisfied if $K = M$ and the Finsler metric on M is Hölder continuous and sufficiently convex (see [LY]).

Now we are going to prove that K with its intrinsic metric has property (U) at each point, and that it has continuously varying geodesics if all norms $|\cdot|_x$ are strongly convex and smooth. We denote by d^K (respectively, by d) the intrinsic (respectively, the induced) metric on K. Since the question is local, we may assume that M is a chart $U \subset \mathbb{R}^n$ and the Finsler structure is uniformly continuous. We denote by $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^n$ and by $|\cdot|_x$ the norm determined by the Finsler structure at x. For each K-geodesic γ in U, we have $\|\gamma'(t)-\gamma'(0)\| \le Lt^\alpha$ for some fixed constant L. Moreover, $\big||\gamma(0)-\gamma(t)|_{\gamma(0)}-t\big| \le o(t)$, where the function $o(t)$ depends only on U and satisfies $\lim_{t\to 0}\frac{o(t)}{t} \to 0$. This implies the inequality $d^K(x,z) \le d(x,z)+o(d(x,z))$ (cf. [LY]). By Lemma 8.1, the space K has property (A) at each point.

If γ_1,γ_2 are two geodesics starting at x, then $|d^K(\gamma_1(t),\gamma_2(t))-|\gamma_1(t)-\gamma_2(t)|_x| \le o(t)$. Since $|\gamma_i(t)-t\gamma_i^+(0)|_x \le o(t)$, we conclude that K has property (U) at x.

Finally, let γ and η be geodesics starting at x, and let γ_n be a sequence of geodesics converging to γ with $\gamma_n(0)=x_n=\eta(t_n)$. Let v be the starting direction of η, and let h (respectively, h_n) be the starting directions of γ (respectively, of γ_n). From the uniform $\mathcal{C}^{1,\alpha}$ bound of γ_n we see that the h_n converge to h. We fix some $\epsilon>0$, choose a sufficiently large $C=C(\epsilon)>0$, and consider the triangle $x\gamma_n(0)\gamma_n(Ct_n)$.

We have $d(x,\gamma_n(Ct_n)) \le |\gamma_n(Ct_n)-x|_x+o(t_n)$. On the other hand, $|\gamma_n(Ct_n)-x|_x \le t_n|v+Ch_n|_x+o(t_n)$. Consequently, geodesics in X vary continuously at x if this is true in the Banach space T_xM, i.e., if the norm of T_xM is smooth and uniformly convex (Lemma 9.6).

Finally, we note that if each norm $|\cdot|_x$ is a Euclidean norm, then K is infinitesimally cone-like.

§11. Differentiation in geometric spaces

11.1. Basics. Let X be a geometric space, let F be a closed subset of X, and let $x \in X \setminus F$. The uniform convexity of T_xX and the first variation formula show that $D_xd_F(v) \le -1+\delta$ for a vector $v \in S_x \subset C_x$, which implies that $d(v,\gamma^+) \le \epsilon$ for some $\gamma \in \Gamma_{x,F}$ and $\epsilon=\epsilon(\delta)=\epsilon(x,\delta)$ with $\lim_{\delta\to 0}\epsilon(\delta)=0$. In particular, $D_xd_F(v)=-1$ if and only if v is the starting direction γ^+ of some $\gamma\in\Gamma_{x,F}$.

Now we choose a dense countable subset S of a punctured neighborhood of x. For each $z\in S$ the function d_z is differentiable at x, and the differential is given by the first variation formula. For each unit vector $v\in T_xX$ and each $\epsilon>0$ we can find a point $z\in S$ such that $D_xd_z=\inf_{\gamma\in\Gamma_{x,z}} b_{\gamma^+}$, where γ^+ runs over some radial rays h with $d(v,h(1))<\epsilon$, i.e., z lies almost in the direction v from x. The uniform convexity of T_xX shows that the following is true.

Lemma 11.1. *The differentials $\{D_xd_z|z\in S\}$ of the distance functions d_z separate the points in T_xX, i.e., the functions d_z satisfy the conditions of Subsection* 7.3.

Recalling Subsection 7.3, we obtain the following statement.

Corollary 11.2. *Let $f: Z\to X$ be a Lipschitz map. Assume that T_zZ exists and that X is geometric. The map f is differentiable at z if and only if the compositions $d_{x_n}\circ f: Z\to\mathbb{R}$ are differentiable at z for all points x_n in a dense countable subset D of X.*

This implies Proposition 1.4, and from Theorem 8.2 we deduce Corollary 1.5.

11.2. Differentiating submetries. We recall some facts about submetries, a notion invented in [Ber87] (see also [BG00]).

Definition 11.1. A map $f: X\to Y$ is a submetry if $f(B_r(x))=B_r(f(x))$ for all $x\in X$ and $r\in\mathbb{R}^+$.

If $f : X \to Y$ is a submetry, and X is proper (respectively, geodesic), then so is Y. For each closed subset $A \subset Y$ we have $d_A \circ f = d_{f^{-1}(A)}$. Two points $x, \bar{x}$ in X are said to be *near* with respect to f if $d(x, \bar{x}) = d(f(x), f(\bar{x}))$. Let N_x denote the set of all points near x. The restriction $f : N_x \to Y$ is a surjective map. Each geodesic γ between near points (called a horizontal geodesic) is mapped isometrically onto its image, which itself is a geodesic. If X is geodesic, then the set N_x is the union of the horizontal geodesics starting at x, and each geodesic in $\Gamma_{f(x)}$ has a horizontal lift in Γ_x.

Proposition 11.3. *Let $f : X \to Y$ be a submetry between geometric spaces. Then f is differentiable at each point, and the differential $D_x f : T_x X \to T_{f(x)} Y$ is a homogeneous submetry.*

Proof. Let $x \in X$, and let $y = f(x)$. For each $\bar{y} \neq y$ the function $d_{\bar{y}} \circ f$ is the distance function $d_{F_{\bar{y}}}$ to the fiber $F_{\bar{y}} = f^{-1}(\bar{y})$; therefore, it is differentiable at x. By Corollary 11.2, the map f is differentiable at x. Being an ultralimit of submetries, the differential $D_x f$ is a submetry. □

Since, under convergence of submetries, fibers converge to fibers, the tangent space to each fiber exists and is given by $T_x(f^{-1}(f(x))) = (D_x f)^{-1}(0) := V_x$ (cf. Example 7.4).

The subset N_x of all points near x is the union of all horizontal geodesics starting at x. Therefore, by Example 6.10, the space N_x has property (U) at x. Hence, the tangent space to N_x at x exists and is given by the closure of the union of the radial rays in the tangent cone $T_x X$ that correspond to horizontal geodesics. In particular, $T_x N_x$ is contained in the horizontal subcone $H_x = \{h \in T_x X \,||h| = |D_x f(h)|\}$.

Now we take an arbitrary unit direction $h \in H_x$ and consider $w = D_x f(h)$. Choose a sequence y_j converging to y from the direction w. Then $D_y d_{y_j}(w)$ goes to -1. Therefore, $D_x d_{F_j}(h)$ also goes to -1, where F_j is the fiber $f^{-1}(y_j)$. Thus, the vector h is the limit of initial directions h_j corresponding to some geodesics $\gamma_j \in \Gamma_{x,F_j}$. But each geodesic γ in Γ_{x,F_j} is horizontal. Thus, we have proved that $T_x N_x = H_x = \{h \in T_x X \,||h| = |D_x f(h)|\}$. Moreover, the proof shows that for each $h \in H_x$ and each geodesic γ in Y starting at y in the direction $D_x f(h)$ there is a horizontal lift $\bar{\gamma}$ of γ starting at x in the direction h.

11.3. More on submetries. Our aim in this subsection is to sketch the proof of the following statement.

Proposition 11.4. *Let X be a geometric space, and let $f : X \to Y$ be a submetry. Then Y is geometric.*

Proof. Choose $x \in X$ and set $y = f(x)$. The set N_x of points near x still has property (U). We denote by $H_x = C_x(N_x) = T_x N_x \subset C_x = T_x X$ the tangent space to N_x. Each geodesic in N_x starting at x is mapped isometrically onto a geodesic in Y. Thus, we get a natural surjective map $D_x f : H_x \to C_y$, which is 1-Lipschitz and maps radial rays isometrically. In particular, the geodesic cone C_y must be proper. For each radial ray h and each point v in H_x, we get the following inequality for the Busemann functions: $b_{D_x f(h)}(D_x f(v)) \leq b_h(v)$ (see Example 2.2).

In order to prove property (A) at y, we consider two geodesics γ_1 and γ_2 starting at y and denote by $\bar{\gamma}_1, \bar{\gamma}_2 \in \Gamma_x$ their horizontal lifts. Let F_r (respectively, G_r) be the fiber $f^{-1}(\gamma_1(r))$ through $\bar{\gamma}_1(r)$ (respectively, the fiber $f^{-1}(\gamma_2(r))$ through $\bar{\gamma}_2(r)$). Then $\lim_\omega \frac{d(\gamma_1(t_i), \gamma_2(st_i))}{t_i} = \lim_\omega \frac{d(F_{t_i}, G_{st_i})}{t_i}$. Hence, it suffices to prove that the equidistant decomposition of $T_x X$ defined by the submetry $f_x^{(t_i)} : X_x^{(t_i)} \to Y_y^{(t_i)}$ is independent of the scale (t_i). However, using the first variation formula and the uniform convexity of $T_x X$, it is possible to show that $v, w \in T_x X$ are in one and the same fiber of $f_x^{(t_i)}$ if and only if $D_x d_F(v) = D_x d_F(w)$ for each fiber $F = f^{-1}(\bar{y})$ with $\bar{y} \neq y$. In fact, this shows that f is metrically differentiable at x.

Now, let $\gamma \in \Gamma_y$ be a geodesic and $\bar{\gamma} \in \Gamma_x$ a horizontal lift of γ. We take an arbitrary point $\bar{y} \neq y$ and set $F = f^{-1}(\bar{y})$. Then $d_{\bar{y}} \circ \gamma = d_F \circ \bar{\gamma}$. Therefore, the differentials of these two maps at 0 coincide. If we denote by v the unit vector $\bar{\gamma}^+ \in H_x$, by the first variation formula we get $D_0(d_F \circ \gamma) = b_{\eta^+}(v)$, for some geodesic $\eta \in \Gamma_{x,F}$.

The geodesics in $\Gamma_{x,F}$ are mapped isometrically by f onto geodesics in $\Gamma_{y,\bar{y}}$. Set $w = D_x f(v)$. Then, as in Lemma 9.2, we get $D_0(d_{\bar{y}} \circ \gamma) \leq b_{D_x f(\eta^+)}(w)$. But $b_{D_x f(\eta^+)} \circ D_x f \leq b_{\eta^+}$. Thus, $b_{D_x f(\eta^+)}(w) = b_{\eta^+}(v)$.

Using once again the uniform convexity of C_x, the above identity, and the fact that X has property (U), we get the following: if $d(\gamma_1^+, \gamma_2^+) < \rho$ for some $\gamma_1, \gamma_2 \in \Gamma_y$, then $d(\bar{\gamma}_1(t), \bar{\gamma}_2(t)) < \epsilon t$ for all $t < \rho$, each horizontal lift $\bar{\gamma}_1$ of γ_1, and some horizontal lift $\bar{\gamma}_2$ of γ_2 starting at x. This verifies property (U) at y. Moreover, the identity for Busemann functions proved before implies that for each $v \in H_x$ and each radial ray $\eta \subset C_y$ there is at least one radial ray $\bar{\eta} \subset H_x$ with $D_x f(\bar{\eta}) = \eta$ and $b_{\bar{\eta}}(v) = b_\eta(D_x f(v))$.

Should both H_x and C_y be Euclidean cones, this would imply that the differential $D_x f : H_x \to C_y$ is a submetry. In general, we do not know if this must be true. However, the uniform convexity and the smoothness of H_x imply that the cone C_y is uniformly convex and smooth.

Finally, the above equality of Busemann functions in C_x and in H_y shows that the first variation formula is valid at y. From Lemma 9.5 we deduce that Y is geometric. $\square$

§12. Rademacher theorem

Now we prove Theorem 1.6.

Proof. Since the problem in question is local, we may assume that S is compact. Suppose we already know the result for $n = 1$. Then we can deduce it for arbitrary n by standard arguments (see [Kir94, MM00]). Namely, let v be a unit vector in $\mathbb{R}^n$. For each $x \in \mathbb{R}^n$, consider the line γ_x through x in the direction v. The restriction of f to $\gamma_x \cap S$ is Lipschitz, and by assumption this restriction is differentiable a.e. on $\gamma_x \cap S$. We denote by G^v the set of all $x \in S$ such that the restriction of f to $\gamma_x \cap S$ is differentiable at x. Then G^v has full measure in S by the Fubini theorem (see [MM00]). Put $G = \bigcap_{v \in D} G^v$, where v runs over a countable dense subset of the unit sphere. The set G also has full measure in S, and f is differentiable at each point of G.

So, let $S \subset I$ be a compact subset of an interval, and let $f : S \to Z$ be a Lipschitz map. Since Z is geodesic, f extends to a Lipschitz curve $\gamma : I \to Z$. Reparameterizing γ, we may assume that it is parameterized by the arclength. Then γ is 1-Lipschitz, and by Theorem 8.2 there is a subset $\widetilde{G} \subset I$ of full measure in I such that for all $s \in \widetilde{G}$ the metric differential $mD_s\gamma$ exists and is the canonical metric d on $\mathbb{R} = T_s I$.

We set $x_t = \gamma(t)$ and let $h_t : I \to \mathbb{R}$ be the nonnegative 1-Lipschitz function $h_t(s) = d_{x_t}(\gamma(s)) = d(x_t, x_s)$. Let T be a dense countable subset in I. By the usual Rademacher theorem the set G of all points $x \in \widetilde{G}$ such that $h'_t(x)$ exists and is linear for all $t \in T$ has full measure in I. We denote by N_ϵ^+ (respectively, N_ϵ^-) the set of all points $x \in G$ such that $h'_t(x) < 1 - \epsilon$ for all $t \in T$ with $t < s$ (respectively, $h'_r(x) > -1 + \epsilon$ for all $r \in T$ with $r > s$). The set N_ϵ^+ is measurable. Assume that it has positive measure and take a Lebesgue point s in N_ϵ^+.

Let ρ be such that for all $t \in G$ with $s - \rho < t < s$ we have $d(x_t, x_s) \geq (1 - \epsilon^2)|t - s|$ and $\mu(N_\epsilon^+ \cap [t, s]) \geq (1 - \epsilon^2)|s - t|$.

Recall that T is dense in I by assumption. Hence, we can choose $t \in T$ with $s - \rho < t < s$, obtaining $h_t(s) - h_t(t) \geq (s - t)(1 - \epsilon^2)$. On the other hand, the differential of the

1-Lipschitz function h_t on the subset $N_\epsilon^+ \cap [t,s]$ is bounded from above by $1-\epsilon$. Since this subset has measure at least $(s-t)(1-\epsilon^2)$, we see that $h_t(s)-h_t(t) \leq (s-t)(\epsilon^2) + (s-t)(1-\epsilon^2)(1-\epsilon) = (s-t)(1-\epsilon+\epsilon^3)$. For small ϵ, this contradicts the inequality $h_t(s)-h_t(t) \geq |s-t|(1-\epsilon^2)$.

In the same way we see that N_ϵ^- has measure 0 in G. Consequently, the sets $N_\epsilon = N_\epsilon^+ \cup N_\epsilon^-$ and $N = \bigcup_{\epsilon>0} N_\epsilon$ also have measure 0 in I. Thus, the complement $G^0 = G \setminus N$ has full measure in I. Until now we have not used the curvature assumptions, and now they will imply the result.

Let $s \in G^0$ be arbitrary, and let $z = \gamma(s)$. We choose sequences t_n and r_n with $t_n < s < r_n$ and such that $h'_{t_n}(s) \to 1$ and $h'_{r_n}(s) \to -1$. Let v_n (respectively, w_n) be the starting vectors of some geodesics from z to $\gamma(t_n)$ (respectively, from z to $\gamma(r_n)$).

We are going to prove that the v_n and w_n converge in C_x to γ^+ (respectively, to γ^-). For this, consider an arbitrary sequence $\epsilon_j \to 0$ and the point $w = (\gamma(s+\epsilon_j)) \in Z_z^{(\epsilon_j)}$.

First, assume that Z is a $CAT(\kappa)$-space. Then from the comparison triangle for $x_{r_n} x_s x_{s+\epsilon_j}$ with $\epsilon_j \ll r_n - s$ we see that the condition $h^-_{r_n}(s) \to -1$ implies that the distance between w_n and w tends to 0 as $n \to \infty$. This completes the proof in the case of an upper curvature bound.

Now, let Z be a space with curvature at least κ. Then the comparison triangle for $x_{t_n} x_s x_{s+\epsilon_j}$ with $\epsilon_j \ll s - t_n$ shows that the distance between v_n and w tends to 2 as n goes to ∞. But $Z_z^{(\epsilon_j)}$ is a nonnegatively curved space. Therefore, the v_n converge to a unique point $\bar{v} \in C_z \subset Z_z^{(\epsilon_j)}$ with $|\bar{v}| = 1$ and $d(\bar{v}, w) = 2$. But since the metric differential of γ at s is the usual metric on $\mathbb{R} = T_s I$, we see that the point $v = (\gamma(s-\epsilon_j)) \in Z_z^{(\epsilon_j)}$ also satisfies $d(v,0) = 1$ and $d(v,w) = 2$. Since in the nonnegatively curved space $Z_z^{(\epsilon_j)}$ geodesics cannot branch, v and $\bar{v}$ coincide. This finishes the proof in the case of a lower curvature bound. □

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