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ON SELFADJOINT EXTENSIONS OF SOME DIFFERENCE OPERATOR

R. M. KASHAEV

Dedicated to L. D. Faddeev on the occasion of his 70th birthday

Abstract. A one-parameter family of selfadjoint extensions is presented for the operator

$$L = -e^{2\pi p} + 2\cosh(z\pi b q),$$

where $0 < b \le 1$ and p and q are unbounded selfadjoint operators satisfying the Heisenberg commutation relation

$$[p, q] = pq - qp = (2\pi i)^{-1}.$$

The corresponding spectral problem is also solved.

§1. Introduction

In this paper we consider the spectral problem for the linear operator

$$\mathsf{L} = -e^{2\pi b \mathsf{p}} + 2\cosh(2\pi b \mathsf{q}),$$

acting in some complex Hilbert space. In the above formula,

$$b \in\,]0, 1] \subset \mathbb{R},$$

and p and q are unbounded selfadjoint operators satisfying the Heisenberg commutation relation

$$[\mathsf{p}, \mathsf{q}] = \mathsf{pq} - \mathsf{qp} = (2\pi i)^{-1}.$$

The spectral resolution for the operator $\mathsf{q} = \int_{\mathbb{R}} x|x\rangle dx\langle x|$ allows us to treat the problem in the Hilbert space $L^2(\mathbb{R})$ of square integrable functions on the real axis, where p and q are realized by the operators of differentiation and multiplication, respectively. Symbolically, on the elements of the "continuous basis" $\{|x\rangle\}_{x\in\mathbb{R}}$ these operators act as follows:

$$\langle x|\mathsf{p} = \frac{1}{2\pi i}\frac{\partial}{\partial x}\langle x|, \quad \langle x|\mathsf{q} = x\langle x|,$$

and the operator L is realized by the difference operator

$$\langle x|\mathsf{L} = (-e^{-ib\frac{\partial}{\partial x}} + 2\cosh(2\pi b x))\langle x| = -\langle x - ib| + 2\cosh(2\pi b x)\langle x|.$$

This is an unbounded symmetric operator; as its domain we take the intersection of the domains of the summands involved, i.e.,

$$D(\mathsf{L}) = D(e^{2\pi b \mathsf{p}}) \cap D(\cosh(2\pi b \mathsf{q})) = D(e^{2\pi b \mathsf{p}}) \cap D(e^{2\pi b |\mathsf{q}|}).$$

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Thus, $f(x) \in D(\mathsf{L})$ if $f(x)$ is the restriction to $\mathbb{R} \subset \mathbb{C}$ of a function $f(z)$ defined and holomorphic in the strip $-b \leq \operatorname{Im} z \leq 0$ and both functions $e^{2\pi b|x|} f(x)$ and $f(x - ib)$ are square integrable on $\mathbb{R}$.

In this paper we present a one-parameter family of selfadjoint extensions of the operator L, together with a solution of the corresponding spectral problem. Namely, for any $\theta \in [0, 1[$ with $\theta + b^{-2}/4 =: \Delta \notin \mathbb{Z}$ there exists a selfadjoint extension L_θ of L such that the spectrum of θ consists of the continuous part $\sigma_c(\mathsf{L}_\theta) = \mathbb{R}_{\geq 2}$ independent of θ and the point part

$$\sigma_p(\mathsf{L}_\theta) = -2\cosh\left(2\pi b\sqrt{\theta - \mathbb{Z}_{<\Delta}}\right).$$

We note that the operator

$$\widetilde{\mathsf{L}} = e^{2\pi b\mathsf{p}} + 2\cosh(2\pi b\mathsf{q}),$$

which differs from L in the sign of the first term only, is selfadjoint and possesses a purely continuous spectrum [8] that coincides with the continuous part of the spectrum of L_θ,

$$\sigma(\widetilde{\mathsf{L}}) = \sigma_c(\widetilde{\mathsf{L}}) = \sigma_c(\mathsf{L}_\theta) = \mathbb{R}_{\geq 2}.$$

In the context of integrable systems, both operators L and $\widetilde{\mathsf{L}}$ arise naturally in the study of the discrete Liouville equation [4, 2, 5], which is an integrable discretization of the continuous Liouville equation (see, e.g., [9] for a state-of-the-art survey of the quantum theory of the continuous Liouville equation). The operator $\widetilde{\mathsf{L}}$ corresponds to the case of purely positive field variables in the discrete Liouville equation, while the operator L is added to $\widetilde{\mathsf{L}}$ when a more general model with real but not necessarily positive field variables is treated. At the continuous Liouville equation level, such generalization corresponds to field configurations with singularities.

In view of the relationship between the Liouville equation and the Teichmüller theory of hyperbolic structures on two-dimensional surfaces, it should be mentioned that, in the quantum Teichmüller theory, the operator $\widetilde{\mathsf{L}}$ corresponds to the length operator for a topologically nontrivial closed geodesic (see [6, 1, 7]). It can also be shown that the classical versions of the operators L and $\widetilde{\mathsf{L}}$ arise as traces of holonomies along topologically nontrivial closed curves for flat $PSL_2\mathbb{R}$ connections on two-dimensional surfaces. In this context, it is natural to expect that L and $\widetilde{\mathsf{L}}$ will represent the quantum versions of traces of holonomies in the quantum theory of flat $PSL_2\mathbb{R}$ connections on two-dimensional surfaces.

In [10], Woronowicz thoroughly analyzed the selfadjoint extensions of the operators of the form $R + S$ with selfadjoint R and S satisfying the commutation relation $RS = e^{-i\hbar} SR$, where $\hbar \in \mathbb{R}$. In our setting, such operators are exemplified by $R + S = e^{2\pi b\mathsf{p}} - e^{2\pi b\mathsf{q}}$ with $\hbar = 2\pi b^2$. The relationship between the selfadjoint extensions of the latter operator and of the operator L treated in the present paper is of interest, in particular, from the viewpoint of the above-mentioned quantum theory of flat $PSL_2\mathbb{R}$ connections on two-dimensional surfaces.

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§2. The main result

Consider the function

$$\varrho_z(x) := \frac{\varphi_b(z + x)}{\varphi_b(z - x)} e^{-i2\pi zx} = \varrho_{-z}(x), \tag{1}$$

where $\varphi_b(z)$ is the noncompact quantum dilogarithmic function (see §3 for the definitions). With any complex z in the strip $|\operatorname{Im} z| < b^{-1}/2$, we associate the normalizable vector $|\beta_z\rangle$ defined by

$$\langle x|\beta_z\rangle = \beta_z(x) := \varrho_z(x+ib/2). \tag{2}$$

Also, we introduce the following family of nonnormalizable vectors $|\gamma_t\rangle$:

$$\langle x|\gamma_t\rangle = \gamma_t(x) := \varrho_t\left(x - i0 + i(b+b^{-1})/2\right)\sin\left(\pi\left(t^2-\theta-b^{-2}/4+ib^{-1}x\right)\right), \tag{3}$$

where t and θ are real.

Theorem 1. *For any $\theta \in [0,1[$ such that $\theta + b^{-2}/4 =: \Delta \notin \mathbb{Z}$ there exists a selfadjoint extension $\mathsf{L} \subset \mathsf{L}_\theta \subset \mathsf{L}^*$ with spectral resolution*

$$\mathsf{L}_\theta = \int_{\lambda\in\mathbb{R}} \lambda\, d\mathsf{E}(\lambda),$$

where the spectral projection $\mathsf{E}(\lambda)$ is the sum of point and continuous parts,

$$\mathsf{E}(\lambda) = \mathsf{E}_p(\lambda) + \mathsf{E}_c(\lambda),$$

that admit the following explicit formulas:

$$\mathsf{E}_p(\lambda) = \sum_{\substack{m\in\mathbb{Z}_{<\Delta} \\ -2\cosh(2\pi b s_m)\le\lambda}} |\beta_{s_m}\rangle s_m^{-1}\sinh(2\pi b s_m)\langle\beta_{s_m}|, \quad s_m := \sqrt{\theta - m} \tag{4}$$

and

$$\mathsf{E}_c(\lambda) = \int_{2\le 2\cosh(2\pi b t)\le\lambda} |\gamma_t\rangle\nu(t)\,dt\langle\gamma_t|, \quad \nu(t) := \frac{4\sinh(2\pi b t)\sinh(2\pi b^{-1}t)}{|\sin\left(\pi\left((t-ib^{-1}/2)^2-\theta\right)\right)|^2}. \tag{5}$$

§3. Technical preparation

3.1. Some properties of the quantum dilogarithm. Recall that the quantum dilogarithmic function $\varphi_b(z)$, first considered by L. D. Faddeev in [3] in the context of quantum integrable systems, is defined in the strip $|\operatorname{Im} z| < (b+b^{-1})/2$ by the explicit formula

$$\varphi_b(z) := \exp\left(\int_{i0-\infty}^{i0+\infty} \frac{e^{-i2zw}\,dw}{4\sinh(wb)\sinh(wb^{-1})w}\right)$$

and then extends analytically to the entire complex plane via the difference functional equations

$$\varphi_b(z - ib^{\pm1}/2) = \varphi_b(z+ib^{\pm1}/2)\left(1+e^{2\pi b^{\pm1}z}\right). \tag{6}$$

This function enjoys the symmetry

$$\varphi_b(z)\varphi_b(-z) = \varphi_b(0)^2 e^{i\pi z^2}, \quad \varphi_b(0) = e^{i\pi(b^2+b^{-2})/24}, \tag{7}$$

and is meromorphic on $\mathbb{C}$ with poles at the points of the countable set

$$P := \{ib(m+1/2) + ib^{-1}(n+1/2) \mid m,n\in\mathbb{Z}_{\ge 0}\},$$

with zeros at the symmetric points $-P$, and with the following asymptotic behavior at infinity along the negative real axis:

$$\lim_{x\to-\infty}\varphi_b(x) = 1. \tag{8}$$

3.2. Relationship between $\beta_z(x)$ and $\gamma_z(x)$. We shall treat the functions $\beta_z(x)$ and $\gamma_z(x)$ defined by (2) and (3) as meromorphic functions of two complex variables.

Lemma 1. *The following identities are true*:

$$\beta_{z_\pm}(x) = \alpha_z(x) 2i \sinh\left(\pi b^{-1}\left(x \mp z\right)\right), \quad z_\pm := z \pm ib^{-1}/2, \tag{9}$$

and

$$\gamma_z(x) = \alpha_z(x) \sin\left(\pi\left(z^2 - \Delta + ib^{-1}x\right)\right), \tag{10}$$

where

$$\alpha_z(x) := \varrho_z(x + i(b + b^{-1})/2).$$

Proof. Identity (10) is equivalent to the initial definition (3), and (9) is a consequence of the difference equation (6) with shift proportional to b^{-1}:

$$\begin{aligned}
\beta_{z_+}(x) &= \beta_{z+ib^{-1}/2}(x) \\
&= \varrho_{z+ib^{-1}/2}(x + ib/2) = \frac{\varphi_b(z + x + ib/2 + ib^{-1}/2)}{\varphi_b(z - x - ib/2 + ib^{-1}/2)} e^{-i2\pi(z+ib^{-1}/2)(x+ib/2)} \\
&= \frac{\varphi_b(z + x + ib/2 + ib^{-1}/2)}{\varphi_b(z - x - ib/2 - ib^{-1}/2)} \frac{\varphi_b(z - x - ib/2 - ib^{-1}/2)}{\varphi_b(z - x - ib/2 + ib^{-1}/2)} e^{-i2\pi(z+ib^{-1}/2)(x+ib/2)} \\
&= \alpha_z(x) e^{i2\pi z(x+ib/2+ib^{-1}/2)} \left(1 + e^{2\pi b^{-1}(z-x-ib/2)}\right) e^{-i2\pi(z+ib^{-1}/2)(x+ib/2)} \\
&= \alpha_z(x)\left(1 - e^{2\pi b^{-1}(z-x)}\right) ie^{\pi b^{-1}(x-z)} = \alpha_z(x) 2i \sinh\left(\pi b^{-1}\left(x - z\right)\right),
\end{aligned}$$

and the case of the minus sign reduces to the previous case:

$$\beta_{z_-}(x) = \beta_{z-ib^{-1}/2}(x) = \beta_{-z+ib^{-1}/2}(x) = \alpha_z(x) 2i \sinh\left(\pi b^{-1}\left(x + z\right)\right). \qquad \square$$

The meromorphic function

$$\nu(z) := \frac{4\sinh(2\pi bz)\sinh(2\pi b^{-1}z)}{\sin\left(\pi\left((z - ib^{-1}/2)^2 - \theta\right)\right)\sin\left(\pi\left((z + ib^{-1}/2)^2 - \theta\right)\right)}$$

is the analytic extension of the integration density in formula (5). We define yet another meromorphic function of two complex variables; namely, we put

$$h(x, z) := \frac{2i\sinh(2\pi bz)}{\sin(\pi(z^2 - \theta))}\beta_z(x).$$

Proposition 1. *We have*

$$\nu(z)\gamma_z(x) = h(x, z - ib^{-1}/2) - h(x, z + ib^{-1}/2). \tag{11}$$

Proof. Direct substitution of all definitions of functions and the use of (9), (10) reduce the proof of (11) to the trigonometric identity

$$\sinh(u + w)\sinh(v + w) - \sinh(u - w)\sinh(v - w) = \sinh(u + v)\sinh(2w)$$

with

$$u = \pi b^{-1}x, \quad v = i\pi(\Delta - z^2), \quad w = \pi b^{-1}z. \qquad \square$$

3.3. Difference equations for $\beta_z(x)$ and $\gamma_z(x)$.

Proposition 2. *The functions $\beta_z(x)$ and $\gamma_z(x)$ defined by formulas* (2) *and* (3) (*respectively*) *satisfy the following difference equations*:

$$\beta_z(x-ib) - 2\cosh(2\pi bx)\beta_z(x) = 2\cosh(2\pi bz)\beta_z(x), \tag{12}$$

$$\gamma_z(x-ib) - 2\cosh(2\pi bx)\gamma_z(x) = -2\cosh(2\pi bz)\gamma_z(x). \tag{13}$$

Proof. Relation (12) is an immediate consequence of the difference functional equation (6) with shift proportional to b:

$$\begin{aligned}\frac{\beta_z(x-ib)}{\beta_z(x)} &= \frac{\varrho_z(x-ib/2)}{\varrho_z(x+ib/2)} = \frac{\varphi_b(z+x-ib/2)}{\varphi_b(z-x+ib/2)}\frac{\varphi_b(z-x-ib/2)}{\varphi_b(z+x+ib/2)}e^{-2\pi bz} \\ &= \left(1+e^{2\pi b(z+x)}\right)\left(1+e^{2\pi b(z-x)}\right)e^{-2\pi bz} \\ &= e^{-2\pi bz} + e^{-2\pi bx} + e^{2\pi bx} + e^{2\pi bz} \\ &= 2\cosh(2\pi bz) + 2\cosh(2\pi bx).\end{aligned}$$

Obviously, with respect to x the function $h(x,z)$ satisfies the same difference equation as $\gamma_z(x)$, and by (11) the function $\beta_z(x)$ is a linear combination of solutions of one and the same homogeneous difference equation (13). □

3.4. The scalar product $\langle\beta_r|\beta_s\rangle$.

Proposition 3. *If $s, r \in \mathbb{C}$ belong to the strip $|\operatorname{Im} z| < b^{-1}/2$, then*

$$\langle\beta_r|\beta_s\rangle = \frac{b\sin(\pi(s^2-\bar{r}^2))}{\cosh(2\pi bs) - \cosh(2\pi b\bar{r})}. \tag{14}$$

Proof. Taking complex conjugation, we obtain

$$\overline{\beta_r(z)} = \bar{\beta}_r(\bar{z}),$$

where $\bar{\beta}_r(z) = \beta_{\bar{r}}(-z)$ satisfies the difference functional equation

$$\bar{\beta}_r(z+ib) - 2\cosh(2\pi bz)\bar{\beta}_r(z) = 2\cosh(2\pi b\bar{r})\bar{\beta}_r(z), \tag{15}$$

which is implied by (12). The combination

$$\bar{\beta}_r(z)(\text{equation (12)}) - (\text{equation (15)})\beta_s(z)$$

yields

$$F(z) - F(z+ib) = 2(\cosh(2\pi bs) - \cosh(2\pi b\bar{r}))\bar{\beta}_r(z)\beta_s(z), \tag{16}$$

where the function $F(z) := \bar{\beta}_r(z)\beta_s(z-ib)$ has no poles in the strip $0 \le \operatorname{Im}(z) \le b$. Therefore, by the Cauchy theorem, for any $A \in \mathbb{R}_{>0}$ we have

$$0 = \oint_{\partial R_A} F(z)\,dz = \int_{-A}^{A}(F(x) - F(x+ib))\,dx + i\int_0^b (F(A+ix) - F(-A+ix))\,dx,$$

where R_A is the rectangle $[-A, A] \times [0, b]$. We use (16) to rewrite this in the form

$$2(\cosh(2\pi bs) - \cosh(2\pi b\bar{r}))\int_{-A}^{A}\bar{\beta}_r(x)\beta_s(x)\,dx = -i\int_0^b (F(A+ix) - F(-A+ix))\,dx.$$

Finally, passing to the limit as $A \to \infty$ and using the limit values

$$\lim_{A\to+\infty} F(\pm A + ix) = e^{\pm i\pi(s^2-\bar{r}^2)},$$

which can be calculated easily with the help of the properties (7) and (8) of the quantum dilogarithm, we arrive at (14). □

3.5. The scalar product $\langle\gamma_s|\gamma_t\rangle$ for $s \neq t$.

Proposition 4. *If $s, t \in \mathbb{R}$ and $s \neq t$, then*

$$\langle\gamma_s|\gamma_t\rangle = 0, \quad s \neq t. \tag{17}$$

Proof. Essentially, the arguments repeat those in the proof of Proposition 3 (except for the last part). Since the poles of $\gamma_t(z)$ are located at the points

$$z = \pm t + ibm + ib^{-1}n, \quad m, n \in \mathbb{Z}_{\geq 0},$$

the function

$$H(z) := \gamma_t(z - i0 - ib)\gamma_s(-z - i0)$$

has no poles inside the rectangle

$$R_A := [-A, A] \times [0, b], \quad A \in \mathbb{R}_{>0}.$$

Therefore,

$$0 = \oint_{\partial R_A} H(z)\, dz = \int_{-A}^{A} (H(x) - H(x + ib))\, dx + i \int_0^b (H(A + ix) - H(-A + ix))\, dx.$$

We have

$$\lim_{A\to+\infty} H(\pm A + ix) = 1/4,$$

and (13) implies the relation

$$H(x) - H(x + ib) = 2(\cosh(2\pi bs) - \cosh(2\pi bt))\gamma_t(x - i0)\gamma_s(-x - i0).$$

Passing to the limit as $A \to +\infty$, we obtain (17). □

3.6. Yet another difference identity. We consider two meromorphic functions of three complex arguments each:

$$G(x, y, z) = \beta_z(x)\beta_z(y) \frac{\sinh(2\pi bz) \sin\left(\pi\left(z^2 - \theta + ib^{-1}(x + y)\right)\right)}{\sinh\left(\pi b^{-1}(x + y)\right) \sin\left(\pi\left(z^2 - \theta\right)\right)} \tag{18}$$

and

$$g(x, y, z) = \gamma_z(x)\gamma_z(y)\nu(z). \tag{19}$$

Proposition 5. *We have*

$$G(x, y, z - ib^{-1}/2) - G(x, y, z + ib^{-1}/2) = g(x, y, z). \tag{20}$$

The proof of this proposition is based on the next lemma.

Lemma 2. *The following trigonometric identity is fulfilled for any complex x, y, z, and u:*

$$\begin{aligned} &\sin(x + u)\sin(y + u)\sin(z + u)\sin(x + y + z - u) \\ &\quad - \sin(x - u)\sin(y - u)\sin(z - u)\sin(x + y + z + u) \\ &= \sin(x + y)\sin(x + z)\sin(y + z)\sin(2u). \end{aligned} \tag{21}$$

Proof. The left-hand side of (21), which will be denoted by $A(x, y, z, u)$, is a totally symmetric trigonometric polynomial of degree two with respect to the variables x, y, z, and it vanishes on the plane $x + y = 0$. Therefore, necessarily,

$$A(x, y, z, u) = \sin(x + y)\sin(x + z)\sin(y + z)\, B(u)$$

for some univariate function $B(u)$. Putting $u = x$ and using the initial definition of $A(x, y, z, u)$, we immediately see that $B(u) = \sin(2u)$. □

Proof of Proposition 5. On the one hand, relations (9) allow us to write

$$\begin{aligned}
&\frac{(G(x,y,z_-)-G(x,y,z_+))}{4\alpha_z(x)\alpha_z(y)}\frac{\sinh\left(\pi b^{-1}(x+y)\right)}{\sinh(2\pi bz)}\\
&\quad=\frac{\beta_{z_-}(x)\beta_{z_-}(y)}{4\alpha_z(x)\alpha_z(y)}\frac{\sinh(2\pi bz_-)\sin\left(\pi\left(z_-^2-\theta+ib^{-1}(x+y)\right)\right)}{\sinh(2\pi bz)\sin\left(\pi\left(z_-^2-\theta\right)\right)}\\
&\qquad-\frac{\beta_{z_+}(x)\beta_{z_+}(y)}{4\alpha_z(x)\alpha_z(y)}\frac{\sinh(2\pi bz_+)\sin\left(\pi\left(z_+^2-\theta+ib^{-1}(x+y)\right)\right)}{\sinh(2\pi bz)\sin\left(\pi\left(z_+^2-\theta\right)\right)}\\
&\quad=\sinh\left(\pi b^{-1}(z+x)\right)\sinh\left(\pi b^{-1}(z+y)\right)\frac{\sin\left(\pi\left(z_-^2-\theta+ib^{-1}(x+y)\right)\right)}{\sin\left(\pi\left(z_-^2-\theta\right)\right)}\\
&\qquad-\sinh\left(\pi b^{-1}(z-x)\right)\sinh\left(\pi b^{-1}(z-y)\right)\frac{\sin\left(\pi\left(z_+^2-\theta+ib^{-1}(x+y)\right)\right)}{\sin\left(\pi\left(z_+^2-\theta\right)\right)}.
\end{aligned}$$

On the other hand, we can use (10) to obtain

$$\begin{aligned}
&\frac{g(x,y,z)}{4\alpha_z(x)\alpha_z(y)}\frac{\sinh\left(\pi b^{-1}(x+y)\right)}{\sinh(2\pi bz)}\\
&\quad=\frac{\gamma_z(x)\gamma_z(y)}{\alpha_z(x)\alpha_z(y)}\frac{\sinh(2\pi b^{-1}z)\sinh\left(\pi b^{-1}(x+y)\right)}{\sin\left(\pi\left(z_-^2-\theta\right)\right)\sin\left(\pi\left(z_+^2-\theta\right)\right)}\\
&\quad=\sin\left(\pi\left(z^2-\Delta+ib^{-1}x\right)\right)\sin\left(\pi\left(z^2-\Delta+ib^{-1}y\right)\right)\\
&\qquad\times\frac{\sinh(2\pi b^{-1}z)\sinh\left(\pi b^{-1}(x+y)\right)}{\sin\left(\pi\left(z_-^2-\theta\right)\right)\sin\left(\pi\left(z_+^2-\theta\right)\right)}.
\end{aligned}$$

Now, equating the right-hand sides of the two identities obtained, and substituting

$$x\mapsto ibx/\pi,\quad y\mapsto iby/\pi,\quad z\mapsto ibu/\pi,\quad \theta\mapsto z/\pi-b^{-2}/4-(ub/\pi)^2,$$

we get

$$\begin{aligned}
&-\sin(x+u)\sin(y+u)\frac{\sin(x+y+z-u)}{\sin(z-u)}\\
&\qquad+\sin(x-u)\sin(y-u)\frac{\sin(x+y+z+u)}{\sin(z+u)}\\
&\quad=-\sin(x+z)\sin(y+z)\frac{\sin(2u)\sin(x+y)}{\sin(z-u)\sin(z+u)},
\end{aligned}$$

which is equivalent to (21). □

§4. Proof of Theorem 1

4.1. Scalar products. The fact that the operators $\mathsf{E}(\lambda)$ are projections of the spectral resolution of a selfadjoint operator is proved by calculation of all scalar products of the vectors $|\beta_{s_m}\rangle$ and $|\gamma_t\rangle$.

Proposition 6. *For $m,n\in\mathbb{Z}_{\le\Delta}$ and $s,t\in\mathbb{R}_{>0}$ we have*

(22) $$\langle\beta_{s_m}|\beta_{s_n}\rangle=s_m(\sinh(2\pi bs_m))^{-1}\delta_{m,n},$$

(23) $$\langle\beta_{s_m}|\gamma_t\rangle=0,$$

(24) $$\langle\gamma_s|\gamma_t\rangle=(\nu(t))^{-1}\delta(s-t).$$

Proof. Formulas (22) and (23) follow directly from equation (14). For $s \neq t$ relation (24) coincides with formula (17), which was proved in Subsection 3.5. It remains to settle the case where $s = t$.

The singularity as $s \to t$ of the scalar product

$$\langle \gamma_s | \gamma_t \rangle = \int_{\mathbb{R}} \gamma_t(x - i0)\gamma_s(-x - i0)\, dx$$

coincides with that of the expression

$$2\pi i(\mathrm{res}_{z=t+i0} f_{s,t}(z) - \mathrm{res}_{z=-s-i0} f_{s,t}(z)), \quad f_{s,t}(z) := \gamma_t(z - i0)\gamma_s(-z - i0).$$

Easy calculations lead to the following formulas for singularities:

$$\mathrm{res}_{z=t+i0} f_{s,t}(z) \simeq \frac{1}{4\pi^2 \nu(t)(t - s + i0)},$$
$$\mathrm{res}_{z=-s-i0} f_{s,t}(z) \simeq \frac{1}{4\pi^2 \nu(t)(t - s - i0)}.$$

Recalling the well-known relation

$$\frac{1}{x \pm i0} = \mp i\pi\delta(x) + P\frac{1}{x},$$

we immediately deduce (24). □

4.2. Completeness. Here, for the projections $\{\mathsf{E}(\lambda)\}_{\lambda \in \mathbb{R}}$ we prove the completeness property, which is expressed by the relation

$$\langle x | \mathsf{E}(+\infty) | y \rangle = \langle x | y \rangle = \delta(x - y). \tag{25}$$

For this, for any $\lambda \in \mathbb{R}$ we introduce the following functions of two complex variables:

$$\mathsf{E}_p(x, y; \lambda) := \sum_{\substack{m \in \mathbb{Z}_{<\Delta} \\ -2\cosh(2\pi b s_m) \leq \lambda}} \beta_{s_m}(x)\beta_{s_m}(y) s_m^{-1} \sinh(2\pi b s_m), \quad s_m := \sqrt{\theta - m}$$

and

$$\mathsf{E}_c(x, y; \lambda) := \int_{2 \leq 2\cosh(2\pi b t) \leq \lambda} g(x, y, t)\, dt.$$

Note that, by construction, these functions are symmetric with respect to the interchange of the variables x and y and are meromorphic in each of them. Next, obviously,

$$\mathsf{E}_p(x, y; \lambda) = \mathsf{E}_p(x, y; 2), \quad \lambda \in \mathbb{R}_{\geq 2}.$$

It is easily seen that the functions in question are analytic extensions of the kernels of the spectral projections $\mathsf{E}_p(\lambda)$ and $\mathsf{E}_c(\lambda)$ in the sense that

$$\langle x | \mathsf{E}_p(\lambda) | y \rangle = \mathsf{E}_p(x, -y; \lambda), \tag{26}$$

$$\langle x | \mathsf{E}_c(\lambda) | y \rangle = \mathsf{E}_c(x - i0, -y - i0; \lambda), \quad x, y, \lambda \in \mathbb{R}. \tag{27}$$

Lemma 3. *Suppose $x, y \in \mathbb{C}$ and $A \in \mathbb{R}_{>0} \setminus \sqrt{\theta + \mathbb{Z}_{\geq 0}}$. Then the following identity for meromorphic functions is fulfilled*:

$$\mathsf{E}_p(x, y; +\infty) - \mathsf{E}_p(x, y; -\lambda_A) + \mathsf{E}_c(x, y; \lambda_A) = -i \int_{-b^{-1}/2}^{b^{-1}/2} G(x, y, A + it)\, dt, \tag{28}$$

where $\lambda_A := 2\cosh(2\pi b A)$ and $G(x, y, z)$ is as in (18).

Proof. It suffices to prove this statement under the condition $\operatorname{Im} x, \operatorname{Im} y \in \mathbb{R}_{<0}$. Then all poles of $G(x, y, z)$ with respect to z that are located in the strip $|\operatorname{Im} z| < b^{-1}/2$ form the set $\{\pm s_m\}_{m\in\mathbb{Z}_{<\Delta}}$, where $s_m := \sqrt{\theta - m}$, and the corresponding residues are:

$$\operatorname{res}_{z=s_m} G(x, y, z) = \frac{i}{2\pi} \beta_{s_m}(x)\beta_{s_m}(y) s_m^{-1} \sinh(2\pi b s_m).$$

Thus, using the Cauchy theorem for the rectangle

$$R_A = [-A, A] \times [-b^{-1}/2, b^{-1}/2],$$

we obtain

$$\begin{aligned} \frac{1}{2}\oint_{\partial R_A} G(x, y, z)\, dz &= \pi i \sum_{\pm s_m \in R_A} \operatorname{res}_{z=\pm s_m} G(x, y, z) \\ &= 2\pi i \sum_{m\in\mathbb{Z}\cap]\theta-A^2,\Delta[} \operatorname{res}_{z=s_m} G(x, y, z) \\ &= \mathsf{E}_p(x, y; -\lambda_A) - \mathsf{E}_p(x, y; +\infty). \end{aligned} \tag{29}$$

We have used the fact that $G(x, y, z)$ is odd relative to z and that for any odd function $f(z)$ its residues at symmetric points are equal:

$$f(-z) = -f(z) \implies \operatorname{res}_{z=z_0} f(z) = \operatorname{res}_{z=-z_0} f(z).$$

On the other hand, splitting the contour integral in accordance with the components of the contour, we can write

$$\begin{aligned} &\frac{1}{2}\oint_{\partial R_A} G(x, y, z)\, dz \\ &\quad= \frac{1}{2}\int_{-A}^{A} \left(G(x, y, t - ib^{-1}/2) - G(x, y, t + ib^{-1}/2)\right) dt \\ &\qquad+ \frac{i}{2}\int_{-b^{-1}/2}^{b^{-1}/2} \left(G(x, y, A + it) - G(x, y, -A + it)\right) dt \\ &\quad= \int_0^A g(x, y, t)\, dt + i\int_{-b^{-1}/2}^{b^{-1}/2} G(x, y, A + it)\, dt \\ &\quad= \mathsf{E}_c(x, y; \lambda_A) + i\int_{-b^{-1}/2}^{b^{-1}/2} G(x, y, A + it)\, dt \end{aligned} \tag{30}$$

(we have used the difference identity (20) and the fact that $g(x, y, z)$ is odd and $G(x, y, z)$ is even with respect to z).

Combining equations (29) and (30), we arrive at (28). □

Lemma 4. *Suppose $x, y \in \mathbb{C}$, $x + y \neq 0$. Then*

$$\lim_{A\to+\infty} e^{-2\pi i A(x+y)} \int_{-b^{-1}/2}^{b^{-1}/2} G(x, y, A + it)\, dt = \frac{1}{2\pi(x + y)}. \tag{31}$$

Proof. Let $\operatorname{Im} z \neq 0$. By using the asymptotics of the quantum dilogarithm at infinity, it is easy to show that

$$\lim_{\operatorname{Re} z\to+\infty} 2G(x, y, z)e^{-2\pi i z(x+y)} = \coth(\pi b^{-1}(x + y)) + \operatorname{Im} z/|\operatorname{Im} z|.$$

This means that

$$G(x, y, z) = \frac{1}{2} e^{-2\pi i z(x+y)} \left(\coth(\pi b^{-1}(x + y)) + \operatorname{Im} z/|\operatorname{Im} z|\right) + o(1), \quad \operatorname{Re} z \to +\infty.$$

Now, integration leads to (31). □

Proof of formula (25). Obviously, the projection $E(+\infty)$ can be written as the following limit:

$$\mathsf{E}(+\infty) = \lim_{A\to+\infty} (\mathsf{E}_p(+\infty) - \mathsf{E}_p(-\lambda_A) + \mathsf{E}_c(\lambda_A)).$$

Accordingly, for the kernel we have

$$\begin{aligned}\langle x|\mathsf{E}(+\infty)|y\rangle &= \lim_{A\to+\infty} (\langle x|\mathsf{E}_p(+\infty)|y\rangle - \langle x|\mathsf{E}_p(-\lambda_A)|y\rangle + \langle x|\mathsf{E}_c(\lambda_A)|y\rangle) \\ &= \lim_{A\to+\infty} (\mathsf{E}_p(x,-y;+\infty) - \mathsf{E}_p(x,-y;-\lambda_A) + \mathsf{E}_c(x-i0,-y-i0;\lambda_A)) \\ &= -i \lim_{A\to+\infty} \int_{-b^{-1}/2}^{b^{-1}/2} G(x-i0,-y-i0,A+it)\,dt = \lim_{A\to+\infty} \frac{e^{2\pi i A(x-y)}}{2\pi i(x-y-i0)} \\ &= \lim_{A\to+\infty} \int_{-\infty}^{A} e^{2\pi i t(x-y)}\,dt = \int_{-\infty}^{+\infty} e^{2\pi i t(x-y)}\,dt = \delta(x-y),\end{aligned}$$

where we have used equations (26) and (27) and also Lemmas 3 and 4. □

4.3. The action of L_θ on $D(\mathsf{L}_\theta)$.

Proposition 7. *For $f \in D(\mathsf{L}_\theta)$ we have*

$$\langle x|\mathsf{L}_\theta|f\rangle = -\langle x-ib|f\rangle + 2\cosh(2\pi bx)\langle x|f\rangle. \tag{32}$$

In particular, $\mathsf{L} \subset \mathsf{L}_\theta \subset \mathsf{L}^*$.

Proof. The poles of the function $\beta_s(x)$ are located at the points

$$x = \pm s + ibm + ib^{-1}(n+1/2), \quad m,n \in \mathbb{Z}_{\geq 0}.$$

This means that $\beta_s(x)$ is analytic in the x-half-plane bounded from above by the line $b^{-1}/2 - |\operatorname{Im} s|$. In particular, it is analytic in the lower x-half-plane provided $|\operatorname{Im} s| < b^{-1}/2$.

Similarly, the poles of the function $\gamma_t(x)$ are at the points

$$x = \pm t + ibm + ib^{-1}n, \quad m,n \in \mathbb{Z}_{\geq 0},$$

which implies that $\gamma_t(x)$ is analytic in the lower x-half-plane. Thus, the diagonal action of L_θ on the vectors $|\beta_{s_m}\rangle$, $|\gamma_t\rangle$, which follows from the definition of L_θ and the formulas for scalar products, together with the difference equations (12) and (13), results in formula (32) in the case where $|f\rangle$ coincides with one of the above vectors.

The domain $D(\mathsf{L}_\theta)$ consists of the restrictions to the real axis of the functions that are defined on the strip $-b \leq \operatorname{Im} z \leq 0$, are holomorphic inside that strip, and are such that their images belong to $L^2(\mathbb{R})$. Thus, formula (32) remains true in the general case. Finally, the completeness condition and the self-conjugacy of L_θ allow us to conclude that $\mathsf{L} \subset \mathsf{L}_\theta = \mathsf{L}_\theta^* \subset \mathsf{L}^*$. □

References

[1] L. O. Chekhov and V. V. Fock, *Quantum Teichmüller spaces*, Teoret. Mat. Fiz. **120** (1999), no. 3, 511–528; English transl., Theoret. and Math. Phys. **120** (1999), 1245–1259. MR1737362 (2001g:32034)

[2] L. D. Faddeev, *Quantum symmetry in conformal field theory by Hamiltonian methods*, New Symmetry Principles in Quantum Field Theory (Cargèse, 1991) (J. Frölich et al., eds.), NATO Adv. Sci. Inst. Ser. B Phys., vol. 295, Plenum Press, New York, 1992, pp. 159–175. MR1204454 (93k:81094)

[3] ———, *Discrete Heisenberg–Weyl group and modular group*, Lett. Math. Phys. **34** (1995), 249–254. MR1345554 (96i:46075)

[4] L. D. Faddeev and L. A. Takhtajan, *Liouville model on the lattice*, Field Theory, Quantum Gravity and Strings (Meudon/Paris, 1984/85), Lecture Notes in Phys., vol. 246, Springer, Berlin, 1986, pp. 166–179. MR0848618 (87h:81213)

[5] L. D. Faddeev and A. Yu. Volkov, *Algebraic quantization of integrable models in discrete space-time*, Discrete Integrable Geometry and Physics (Vienna, 1996), Oxford Lecture Ser. Math. Appl., vol. 16, Oxford Univ. Press, New York, 1999, pp. 301–319; hep-th/97010039. MR1676602 (2000h:81097)

[6] V. V. Fock, *Dual Teichmüller spaces*, Preprint dg-ga/9702018.

[7] R. M. Kashaev, *Quantization of Teichmüller spaces and the quantum dilogarithm*, Lett. Math. Phys. **43** (1998), 105–115; q-alg/9705021. MR1607296 (99m:32021)

[8] ———, *The quantum dilogarithm and Dehn twists in quantum Teichmüller theory*, Integrable Structures of Exactly Solvable Two-Dimensional Models of Quantum Field Theory (Kiev, 2000), NATO Sci. Ser. II Math. Phys. Chem., vol. 35, Kluwer Acad. Publ., Dordrecht, 2001, pp. 211–221. MR1873573 (2003b:32017)

[9] J. Teschner, *Liouville theory revisited*, Classical Quantum Gravity **18** (2001), no. 23, R153–R222. MR1867860 (2003f:81230)

[10] S. L. Woronowicz, *Quantum exponential function*, Rev. Math. Phys. **12** (2000), 873–920. MR1770545 (2001g:47039)

Université de Genève, Section de mathématiques, 2-4, rue du Lièvre, CP 240, 1211 Genève 24, Suisse

E-mail address: Rinat.Kashaev@math.unige.ch

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