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GROTHENDIECK'S DESSINS D'ENFANTS, THEIR DEFORMATIONS, AND ALGEBRAIC SOLUTIONS OF THE SIXTH PAINLEVÉ AND GAUSS HYPERGEOMETRIC EQUATIONS

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Dedicated to Ludwig Dmitrievich Faddeev on the occasion of his 70th birthday

Abstract. Grothendieck's *dessins d'enfants* are applied to the theory of the sixth Painlevé and Gauss hypergeometric functions, two classical special functions of isomonodromy type. It is shown that higher-order transformations and the Schwarz table for the Gauss hypergeometric function are closely related to some particular Belyĭ functions. Moreover, *deformations* of the dessins d'enfants are introduced, and it is shown that one-dimensional deformations are a useful tool for construction of algebraic sixth Painlevé functions.

§1. Introduction

In this paper we report on the further development of the so-called RS-transformations method recently introduced (see [11]) in the theory of special functions of isomonodromy type (SFITs) (see [10]). In [11] it was already observed that the RS-transformations (compositions of rational transformations of the independent variable and the Schlesinger transformations acting on solutions of the ordinary differential equations related to a SFIT) are a useful tool for solving many problems in the theory of SFITs, in particular, for constructing higher-order transformations and calculating special values of these (generally speaking, transcendental) functions, say, algebraic values at algebraic points.

Technically, the most complicated problem in the method of RS-transformations is the construction of their R-parts, i.e., of rational functions with some special properties. In many cases it is not clear *a priori* whether such rational functions actually exist.

The progress achieved in this paper is based on the observation that the class of the so-called Belyĭ functions (a special class of rational functions introduced in [3]), which are important in many questions of algebraic geometry, also plays an important role in the construction of RS-transformations.[1] It is well known that in connection with the theory of the Belyĭ functions Grothendieck [7] suggested his theory of "*dessins d'enfants*". We show that for the theory of SFITs we need not only the theory of *dessins d'enfants* but also a theory of their deformations. This latter theory will help us to establish the existence of R-parts of RS-transformations. If the existence of the desired rational function is proved, the existence of the S-part of the corresponding RS-transformation follows immediately. The construction of the S-part is performed in a purely algorithmic way in terms of the coefficients of the R-part and of the original linear ODE, which is subject to the RS-transformation in question. Of course, the mere existence does not

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[1] An explicit definition of this class of rational functions is discussed below in §2.

tell how to construct the R-part explicitly; this is a separate problem. However, many theoretical consequences can still be drawn; in particular, the differential equations for the corresponding SFITs can be found. One can write, say, an *Ansatz* for an RS-transformation with unknown coefficients. The latter can be calculated numerically or studied in some other way. The exact calculation of the monodromy parameters of the SFITs constructed via the method of RS-transformations also does not require explicit expressions; it suffices to know the *Ansatz* mentioned above with the numerical values of the coefficients. In fact, the mere existence of the function R is also helpful in finding explicit formulas, because it stimulates the continuation of efforts towards obtaining such formulas, even though the first attempts might fail. It is this last comment that was important to the author in doing this work. It should be emphasized at the very beginning that the possibly most interesting part of this work concerning a relationship between deformations of *dessins d'enfants* and the existence of R-parts is only conjectured rather than proved.

In this paper, instead of dealing with the general theory of SFITs, we continue to consider the application of the method of RS-transformations to the theory of two classical one-variable SFITs, namely, the sixth Painlevé functions and Gauss hypergeometric functions. More precisely, we are interested in explicit constructions of algebraic solutions of the sixth Painlevé equation (see [2, 11]), a topic which has recently attracted considerable attention, and we also discuss some related questions for its linear analog, the Gauss hypergeometric function (see [1]).

The main new observation made in this paper about algebraic solutions of the sixth Painlevé equation is that a wide class of these solutions (maybe all?) can be constructed by a pure algebraic procedure without involving differential equations at all! The procedure reads as follows:

1. Take a proper *dessin d'enfant* (a bicolor graph).
2. Consider its one-dimensional deformations (tricolor graphs).
3. For each tricolor graph there exists a rational function that is the R-part of some RS-transformation.
4. One of the critical points of R is an algebraic solution of the sixth Painlevé equation.

As has already been mentioned, we do not have a general existence proof at step 3. In all our examples this existence is obtained by an explicit construction; of course, technically, this is the most complicated part of the above method. At the same time, there is a substantial simplification as compared to the paper [2], because to get explicit formulas for the algebraic solutions at step 4, we now do not need to find the S-parts of the corresponding RS-transformations explicitly. Surely, without S-parts we cannot obtain explicit solutions for the associated monodromy problems. However, constructing the latter is a separate issue, which is not related directly to the initial problem of finding algebraic solutions for the sixth Painlevé equation.

It is important to emphasize that, in general, a few "seed" algebraic solutions can be associated with each R-part. Only one of them can be constructed in accordance with step 4 of the above outline. For finding the other solutions, the S-parts of the corresponding RS-transformations must still be found explicitly. Note that by the seed solutions we understand those not related to one another by transformations acting on the set of general solutions for the sixth Painlevé equation. So, an RS-transformation is not equivalent to finding only one algebraic solution by its R-part and proliferating this solution via the Schlesinger transformations.

In [2] we began the classification of the RS-transformations generating the algebraic sixth Painlevé functions. As was explained at the beginning of this Introduction, the principal problem here is the classification of the R-parts of these transformations. It is important to notice that these R-parts can be used for many other RS-transformations, say, for SFITs related to isomonodromy deformations of matrix ODEs with matrix dimension exceeding 2. Therefore, the problem of classification of the R-parts goes beyond the particular problem related to the algebraic sixth Painlevé functions. The method of classification of the R-parts suggested by the deformation point of view is based on the observation that it suffices to classify the one-dimensional deformations of *dessins d'enfants*.

For the Gauss hypergeometric function, our main new result is an explicit formula for a special Belyĭ function, which allows us to construct three octic transformations that act on certain finite sets (which we call clusters) of transcendental hypergeometric functions. So far, the only known higher-order transformations of algebraic Gauss hypergeometric functions that do not belong to the standard Schwarz list (below, we refer to these standard transformations as belonging to the *Schwarz cluster*) are quadratic and cubic transformations and their compositions.

Now we give a brief overview of the contents of the paper.

In §2 we introduce the notion of deformations of *dessins*. More precisely, we begin with the presentation of the *dessins* as bicolor graphs on the Riemann sphere. Then one-dimensional deformations of the *dessins* are defined as special tricolor graphs, which can be obtained from bicolor graphs by some simple rules. After that, the main statements concerning a relationship between the tricolor graphs and a class of rational functions are formulated as conjectures. Then we establish a proposition allowing us to obtain algebraic solutions of the sixth Painlevé equation directly from these rational functions. In the remaining part of the section, we consider some special examples and discuss questions important to the theory of the sixth Painlevé equation.

In §3, deformations of the dessins for the Platonic solids are studied. Here and in §2 we obtain many different algebraic solutions of the sixth Painlevé equations. Some of them are new; the others are related to known solutions, albeit not in a straightforward way. However, it is only a secondary goal for us in this paper, as well as in the papers [11, 2], to enrich the "zoology" of algebraic solutions of the sixth Painlevé equation. Our main purpose is to study various features of the method of RS-transformations and to better understand its place in the theory of SFITs [10]. Another goal achieved in §3 is to show that all genus zero algebraic solutions of the sixth Painlevé equation, classified by Dubrovin and Mazzocco [6] in a special case, can be constructed with the help of RS-transformations. This is aimed at checking my conjecture (see [11]) that all algebraic solutions of the sixth Painlevé equation can be generated via RS-transformations and the so-called Okamoto transformation [14]. It seems that all algebraic solutions of genus zero that have appeared in the literature so far are now recovered by the method of RS-transformations or are related via certain transformations to solutions constructed by this method. However, in [6] Dubrovin and Mazzocco showed that there exists yet another genus one algebraic solution of the sixth Painlevé equation. At this stage, the author cannot confirm that this genus one algebraic solution can be produced in accordance with the above conjecture. On the other hand, some complicated *dessins* still remain to be examined in order to confirm or disprove the conjecture.

In connection with the conjecture mentioned above, it is interesting and instructive to check a closely related, though much simpler, case of the Gauss hypergeometric function, especially taking into account that a complete classification of the cases where the general solution of the Euler equation for the Gauss hypergeometric function is algebraic is known, due to H. A. Schwarz [19]. Actually, in §4 we show that the entire Schwarz list can be generated with the help of RS-transformations whose R-parts are Belyĭ functions, starting with the simplest Fuchsian ODE with two singular points. This point of view (which seems to be new) allows us to find explicit formulas for all algebraic Gauss hypergeometric functions in a straightforward, though in some cases tedious, way. We call the set of these functions the *Schwarz cluster*.

§5 is a continuation of the earlier paper [1] devoted to higher-order transformations for the Gauss hypergeometric function. In [1], some new higher-order algebraic transformations for the Gauss hypergeometric functions were found; however, all these transformations except the quadratic and cubic ones act within the Schwarz cluster. Thus, it was of interest to understand whether there exist higher-order transformations acting on transcendental Gauss hypergeometric functions and not reducing to compositions of quadratic and cubic transformations. In [1] we presented a numerical construction of an octic transformation with this property. Although we were able to find a numerical solution with a much larger number of digits than that indicated in [1], and thus there was no doubt that this transformation does exist, we did not have a mathematical existence proof. In §5, we identify this transformation with one of the Belyĭ functions, which immediately gives the desired proof. Moreover, by using a better computer, the corresponding Belyĭ function is calculated explicitly. Actually, this makes straightforward an explicit construction of three different octic *RS*-transformations. Together with the quadratic and cubic transformations, these transformations and their inverses determine three different clusters of transcendental Gauss hypergeometric functions, which are related via algebraic higher-order transformations, and, thus, have the same type of transcendency. We call them *octic clusters* and present them explicitly at the end of §5 in the corresponding tables.

Discussing various questions about particular algebraic solutions in §2 and §3, we make reference to quadratic transformations for the sixth Painlevé equation. For convenience, in the Appendix we give an overview of these transformations; in the spirit of the present paper, we use the "Belyĭ functions" viewpoint. Hopefully, even a specialist may find this outlook interesting.

Acknowledgement and comments. After the work was finished and put into the Web archive, the author received a letter from P. Boalch, who informed him about two fairly interesting papers [4, 5] that are closely related to the content of the present paper and substantially overlap it. The author would like to thank him for this very important information and the subsequent informal discussion on related issues. Below we make necessary comments concerning these papers.

The solution presented in Subsection 3.4 (which, in our scheme, corresponds to the deformation called the cross of the reduced dodecahedron *dessin*) was constructed explicitly in the recent paper [4] by Boalch. He used the method suggested in [6], which is very different from that considered here. Moreover, in [4] a relationship was established between this solution and the famous Klein quartic algebraic curve in $\mathbb{P}^2$ of genus 3 and with maximal possible number of holomorphic automorphisms.

Ch. F. Doran proved a theorem (see [5, §4, Theorem 4.5]) which is almost equivalent to Theorem 2.1 of this paper; in addition to Doran's result, Theorem 2.1 contains explicit formulas for the coefficients of the sixth Painlevé equation. Another difference is that our formulation is in terms of tricolor graphs rather than in terms of the corresponding rational functions. Essentially, Doran's work was based on the scalar second-order Fuchsian equation, and it gives a general theoretical insight based on many remarkable results known for the Belyĭ functions, Hurwitz spaces, arithmetic Fuchsian groups, etc.

However, Doran did not introduce the new concept of deformation of dessins as we do, and he also did not perform any explicit constructions of algebraic solutions as an illustration of his Theorem 4.5, leaving an opportunity for the interested reader to apply the method of J.-M. Couveignes.

Under the assumption that Conjectures 2.1 and 2.2 are valid, our deformation techniques allows us to immediately reproduce the classification results of [5], formulated there as Corollaries 4.6–4.8, and continue a systematic production of further "solvable" types of suitable R-parts. Thus, our examples are not a mere illustration of the classification in [5]: many of them, say, the deformations studied in §2 or in Subsection 3.4, go beyond Corollaries 4.6–4.8. We also call the reader's attention to the discussion of the renormalization aspect. It is clear that if Conjectures 2.1 and 2.2 are valid, they give an answer to the "inverse problem"; i.e., they imply a classification of the types of the R-parts that generate algebraic sixth Painlevé's functions (see [2] and [5, Remark 12]). Of course, this answer is not absolutely explicit, but in principle, for any given type, finitely many operations are required to check whether the corresponding tricolor graph exists or not, thus giving an answer to the inverse problem. The author hopes that further studies will shed more light on these conjectures, so that more explicit statements will become available. It should also be mentioned that all R-parts presented in this paper are found by a straightforward though complicated method, as explained in Remark 2.1.

The work by Doran raises an important question, which can be split into two: 1) The idea of using RS-transformations in the context of SFITs, in particular, the sixth Painlevé equation, and 2) the fact that some algebraic solutions of the sixth Painlevé equation are encoded in the R-parts of RS-transformations.

The method of RS-transformations was used by the author in [13] (1991) for the construction of quadratic transformations for the sixth Painlevé equation. The application of RS-transformations for the construction of higher-order transformations for SFITs and algebraic SFITs was reported by the author at the Workshop on Isomonodromic Deformations and Applications in Physics, Montreal, Canada, May 1–6, 2000, and was published in [11]. The results of [1] and [2] were reported at workshops in Strasbourg (February 2001) and in Otsu (August 2001). It should also be noted that the original method of [11] always involves both the R- and S-parts of an RS-transformation. The present paper shows clearly that this procedure is more general than that based on Theorem 2.1 (Theorem 4.5 of [5]); we were not able to reproduce some known solutions without the noted S-parts. Strictly speaking, this fact does not mean that such solutions (constructed with the help of the S-parts) cannot be obtained via Theorem 2.1 by finding some other suitable R-parts; however, even apart from the fact that the construction of R-parts is a much more complicated enterprise than that of S-parts, these different constructions may have interesting interpretations from the viewpoint of applications. Note that this is a more general context than in [5].

At the same time, the relationship between of the R-parts and the Belyĭ functions and their deformations was noticed by the author only in this work, after he decided to reproduce the results reported in [6] via the method of RS-transformations and also to explicitly construct the octic transformation for the Gauss hypergeometric function (found in [1]): these latter goals require a more detailed study of the corresponding rational functions. Thus, the second fact was first found by Doran and only rediscovered here.

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§2. Deformations of dessins d'enfants and algebraic solutions of the sixth Painlevé equation

Definition 2.1 (see, e.g., [20]). A rational function $\mathbb{CP}^1 \to \mathbb{CP}^1$ is called a Belyĭ function if it has at most three critical values.

Proposition 2.1. *Let $R : \mathbb{CP}^1 \to \mathbb{CP}^1$ be a rational function of degree n with $k \geq 3$ critical values, and let k_i denote the number of critical points corresponding to the ith critical value. Put*

$$m = \sum_{i=1}^{3} k_i - n - 2. \tag{2.1}$$

Then $m \geq 0$. Moreover, R is a Belyĭ function with three critical values if and only if $m = 0$.

Proof. The Riemann–Hurwitz formula gives $\sum_{i=1}^{k} k_i = (k-2)n+2$. Adding this to (2.1), we arrive at the relation

$$m + \sum_{i=4}^{k} k_i = (k-3)n, \tag{2.2}$$

where we assume that the sum is equal to 0 if $k < 4$. Observe that $\sum_{i=4}^{k} k_i \leq (k-3)(n-1)$; therefore, (2.2) implies $m \geq k-3 \geq 0$. Moreover, if $k = 3$, then $m = 0$ again by (2.2). $\square$

Remark 2.1. The condition $m \geq 0$ has a very lucid sense and is intuitively evident. Indeed, we can assume that the first three critical values are located at 0, 1, and ∞ and define the functions R and $R-1$ by two rational expressions with indeterminate preimages of 0, 1, and ∞ with prescribed multiplicities. Writing then the consistency condition, we arrive at a system of algebraic equations for the indeterminate preimages. In this setting, the condition $m \geq 0$ says that the number n of equations in the system should not be greater than the number $\sum_{i=1}^{3} k_i - 2$ of unknown parameters. However, this necessary condition is not sufficient for the existence of R; see the example in Remark 4.2. If $m > 0$ and the corresponding function R exists, it may depend on m parameters. Sometimes we call such functions m-dimensional deformations of the Belyĭ functions.

From the recent paper [8] the author learned that, essentially, this proposition coincides with Silverman's proof of the *abc*-theorem for polynomials. A minor difference occurs, because in our setting it is natural to count all critical points including the point at ∞, while in the *abc*-setting the ∞ point is excluded. Below in Remark 2.2 we also give a graphical interpretation of the above proposition.

For the description of rational functions $z = R(z_1)$, we use a special symbol, which is called their *type* and is written as $R(\dots|\dots|\dots)$. In the space between two neighboring vertical lines or between the line and one of the parentheses, which we call a box, we write a partition of $\deg R$ into the sum of multiplicities of the critical points of $R(z_1)$ corresponding to one of its critical values in descending order. Normally, the total number of boxes coincides with the number of critical values of R. However, in the case of $m = 1$, which is studied in this and the next sections, we do not indicate the fourth "evident" box: $|2+1+\dots+1|$. At the same time, where it is convenient, we include the boxes for noncritical values: $|1+\dots+1|$.

On the Riemann sphere we consider bicolor connected graphs with black and white vertices and with faces homeomorphic to a circle. The valencies of the black and white vertices are defined in the usual way. These valencies can be equal to any natural number. The vertices of the same color are not connected by edges. By a *black edge* we mean a path in the graph connecting two black vertices: it contains only two black vertices, which are its end points, and one white vertex. Any two black vertices can belong to several different black edges or cycles. Any cycle should contain at least one black vertex. The loops are not allowed. We also define the *black order of a face* as the number of black edges in its boundary.

With each Belyĭ function we can associate a bicolor connected graph in the following way. The black vertices are the preimages of ∞, and the white vertices are the preimages of 1. Their valencies are equal to their respective multiplicities. Each face corresponds to the preimage of 0; the black orders of faces are equal to the multiplicities of the corresponding preimages. Conversely, for each bicolor graph with the properties stated above, there exists a Belyĭ function whose associated bicolor graph is isomorphic to the given one, and this function is unique up to a fractional linear transformation of the independent variable. When all valencies of the white vertices equal 2, we do not indicate them at all, and instead of a bicolor graph we get the usual planar graph with black edges and black orders of faces coinciding with the usual edges and orders of faces. Note that, after we dismiss white vertices, loops may appear in the resulting planar graph.

The pictures of such bicolor graphs on the Riemann sphere will be called *dessins d'enfants* or simply *dessins*. On the Riemann sphere these *dessins* determine the so-called bipartite maps [20].

Now we define tricolor graphs. These are also connected graphs on the Riemann sphere with black and white vertices, and they obey the conditions for the bicolor graphs, but with one blue vertex. Any cycle should contain at least one black or blue vertex, and any edge connects vertices of different colors. Again, no loops are allowed. The boundary of every face should contain at least one black vertex. The valency of the blue vertex equals 4. Again, if all the valencies of the white vertices are equal to 2, we do not indicate them, and instead of a tricolor graph we get a bicolor graph with all black vertices and only one blue vertex. Of course, the latter bicolor graph has nothing to do with the black-white bicolor graphs introduced above. In particular, the black-blue bicolor graph may contain loops. We keep the same notions of the black edge and the black order of a face for the tricolor graphs; the blue point is not counted in either case.

Graphically, any tricolor graph can be viewed as a result of a simple "deformation" of a bicolor graph; see examples below and in the next section. Therefore, we call them *deformation dessins*, or very often, when no confusion is possible, simply *dessins*.

Remark 2.2. At this stage, the introduction of tricolor graphs with only one blue vertex looks somewhat artificial; but this is what we need for applications to the theory of the sixth Painlevé equation. If we thought a more general rational function was needed for the classification of RS-transformations for multivariable SFITs, we would have introduced multicolor graphs. This could be done, say, via a recurrence procedure by considering one-dimensional deformations of n-color graphs, which leads to n- or $(n+1)$-color graphs. At each stage we can add a vertex either with a new color or with one of the "old" colors. We admit coalescence of vertices of the same color, so that the valencies of the color vertices would be even and ≥ 4. The vertices of the same color correspond to critical points for the same critical value of R. The multiplicities of these critical points coincide with half the valencies of the corresponding color vertices. Thus, the color vertices correspond to multiple critical points of R. The dimension m of the deformation is equal to half the sum of the valencies of the color vertices minus their total number.

Conjecture 2.1. *Under the conditions of Proposition* 2.1, *for any rational function* $z = R(z_1)$ *with* $m = 1$ *and with the first three critical values* 0, 1, *and* ∞, *there exists a tricolor graph with the following properties.*

1) There is a one-to-one correspondence between its faces, white vertices, and black vertices and the critical points of the function R *for the critical values* 0, 1, *and* ∞, *respectively.*

2) The black orders of the faces and the valencies of the vertices coincide with the multiplicities of the corresponding critical points.

Remark 2.3. As the reader will see in § 3, several seemingly different tricolor graphs can be associated with the same rational function $z = R(z_1)$ with $m = 1$. The reason is that, actually, R is a function of two variables, $R(z_1) = R(z_1, y)$, where $y \in \mathbb{CP}^1$ is a parameter. As a function of y, R has different branches, and different tricolor graphs are related to these different branches. Also, these branches will lead to some equivalence relation on the set of tricolor graphs. The examples considered in items 4 and 5 of Subsection 3.3 show that we cannot simply declare to be equivalent all tricolor graphs whose associated rational functions are of the same type, because these rational functions can be different as functions of y. Of course, until this equivalence relation is established in the cases where there are different functions $R(z_1, y)$ of y having the same type as rational functions of z_1, it is hard to relate them to proper tricolor graphs. In the examples of Subsection 3.3 referenced above, we succeeded in making such a distinction, being guided by a symmetry that exists in one of the examples. This remark explains why we do not claim uniqueness for the function-to-graph correspondence in Conjecture 2.1, as well as uniqueness modulo fractional linear transformations for the inverse correspondence in Conjecture 2.2 below.

Conjecture 2.2. *For any tricolor graph there exists a function $z = R(z_1, y)$ that is a rational function of $z_1 \in \mathbb{CP}^1$ with four critical values. Three of them are* 0, 1, *and* ∞, *and the corresponding critical points are related to a tricolor graph as stated in Conjecture* 2.1. *The variable y denotes a unique second-order critical point corresponding to the fourth critical value of $z = R(z_1, y)$. Here R is an algebraic function of y of genus zero.*

Corollary 2.1. *Under the conditions of Conjecture* 2.2, *the function R admits a representation in the form of the ratio of coprime polynomials in z_1 such that its coefficients and y allow a simultaneous rational parametrization.*

Consider a rational parametrization $y = y(s)$ and $z = R(z_1, y) \equiv R_1(z_1, s)$ with some parameter s, as in Corollary 2.1. We can interchange the roles of the variables z_1 and s; i.e., we can view $R_1(z_1, s)$ as a rational function of s and treat z_1 as an auxiliary parameter. In this case we call $R_1(z_1, s)$ the *conjugate function* with respect to $R(z_1)$. Making fractional linear transformations of R that interchange its critical points, we get new rational functions of equivalent type. However, generically, their conjugate functions have nonequivalent types (see examples below). The reason is that conjugate functions have dimensions $m \geq 1$, but, instead of m parameters, they have only one parameter z_1. In general, the critical points of conjugate functions depend on z_1. However, each function has m critical points independent of z_1; we call them *additional critical points.* The set of additional critical points of all conjugate functions coincides with the set of values of the parameter s such that the function R changes its type and, therefore, coincides with one of the Belyĭ functions.

We recall the canonical form of the sixth Painlevé equation:

$$\begin{aligned}\frac{d^2y}{dt^2} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} \\ &\quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha_6 + \beta_6\frac{t}{y^2} + \gamma_6\frac{t-1}{(y-1)^2} + \delta_6\frac{t(t-1)}{(y-t)^2}\right),\end{aligned} \tag{2.3}$$

where $\alpha_6, \beta_6, \gamma_6, \delta_6 \in \mathbb{C}$ are parameters. For convenience of comparison of the results obtained here with those from other works, we shall also use the following parametrization of the coefficients in terms of the formal monodromies $\hat{\theta}_k$:

$$\alpha_6 = \frac{(\hat{\theta}_\infty - 1)^2}{2}, \quad \beta_6 = -\frac{\hat{\theta}_0^2}{2}, \quad \gamma_6 = \frac{\hat{\theta}_1^2}{2}, \quad \delta_6 = \frac{1 - \hat{\theta}_t^2}{2}.$$

To formulate our main result, which shows a relationship between the tricolor graphs and the algebraic solutions of equation (2.3), we recall the notion of the *RS-symbol* for RS-transformations and introduce some necessary notation.

The RS-symbol is designed for the description of RS-transformations for arbitrary SFITs related to the Fuchsian ODEs. Below, instead of a general definition, we refer to a more specific situation considered in this paper. The brief notation for the RS-symbols needed here is $RS_k^2(3)$ for $k = 2, 3, 4$. This notation means that we map a (2×2)-matrix Fuchsian ODE with three singular points into an similar ODE but with k singular points. Instead of the number 3 in the parentheses, the extended notation for the RS-symbol involves three boxes, each containing two rows of numbers. In the first row of each box, we simply have a rational number, and in the second row we have the sum of the integers that represent the multiplicities of the preimages of the critical values of R, the R-part of the RS-transformation. It is assumed that the three critical values of R are located at 0, 1, and ∞. The boxes in the notation of the RS-symbol are ordered accordingly. For the transformations studied in this paper, we have no need to indicate the other critical values (if any). The entire set of preimages of the critical values 0, 1, and ∞ is split (nonuniquely!) into two sets of apparent and nonapparent points: we call them *apparent* and *nonapparent sets*, respectively.

The apparent set is the union of apparent sets of the boxes. The apparent set of the ith box, where $i = 0, 1, \infty$, consists of all points whose multiplicities are divisible by some natural number ≥ 2. We denote by $n_i \geq 2$ the *greatest common divisor* of the apparent set of the ith box. In particular, this set can be empty, in which case we formally put $n_i = 1/\theta_i$, where θ_i is a parameter. Let $N_i \geq 0$ be the number of apparent points in the ith box. If $N_i \geq 1$, then the multiplicity of the jth apparent point in the ith box can be written as $k_j^i n_i$, where $j = 1, \ldots, N_i$ and $k_j^i \in \mathbb{N}$.

A critical point of the ith critical value is nonapparent if and only if its multiplicity is not divisible by n_i. The union of such points is the nonapparent set of R. Nonapparent sets for the transformations $RS_4^2(3)$, $RS_3^2(3)$, or $RS_2^2(3)$ consist of 4, 3, or 2 points, respectively.

In general, we say that the R-part of some RS-transformation with three or more critical values is *normalized* if the set $\{0, 1, \infty\}$ is a subset of the set of critical values of R and also a subset of its nonapparent set (if the nonapparent set consists of only two preimages, then it coincides with the set $\{0, \infty\}$). The rational function R with two critical values is normalized if and only if the set of its critical values is a subset of $\{0, 1, \infty\}$ and the nonapparent set satisfies the same condition as in the preceding sentence.

Further in this section we consider only $RS_4^2(3)$-transformations. Suppose that their R-parts are normalized; then the nonapparent set consists of four points: 0, 1, ∞, and t. We denote their multiplicities by m_0, m_1, m_∞, and $m_t \in \mathbb{N}$.

Definition 2.2. An $RS_4^2(3)$-symbol is said to be special if for $i = 0, 1$, and ∞ the rational number in the first row of the ith box is $1/n_i$.

Remark 2.4. The numbers $1/n_i$ in the first rows of the boxes of the special RS-symbols are equal to the formal monodromy of the ith singular point of the initial Fuchsian ODEs.

Theorem 2.1. *Suppose that the normalized rational function $z = z(z_1)$ corresponding to a tricolor dessin is the R-part of some $RS_4^2(3)$-transformation with a special RS-symbol. Put $\varepsilon = 0$ or 1 depending on whether $\sum_{i=1}^{3}\sum_{j=1}^{N_i} k_j^i$ is even or odd.*[2] *Then the double*

[2]The usual convention $\sum_1^0 = 0$ is assumed.

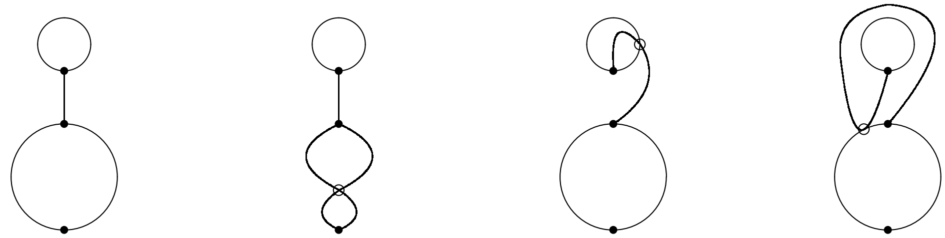


FIGURE 1. One-parameter "face" deformations of the *dessin* for the Belyĭ function $R(5+2+1|2+2+2+2|3+3+2)$. The distributions of the black orders of faces are indicated under the *dessins*.

critical point y of the function $z(z_1)$ corresponding to its fourth critical value,[3] *viewed as a function of the fourth nonapparent critical point t, is an algebraic solution of the sixth Painlevé equation* (2.3) *for the following $\hat{\theta}$-assembly*:

$$\hat{\theta}_0 = \frac{m_0}{n_{z(0)}}, \quad \hat{\theta}_1 = \frac{m_1}{n_{z(1)}}, \quad \hat{\theta}_t = \frac{m_t}{n_{z(t)}}, \quad \hat{\theta}_\infty = \varepsilon + (-1)^\varepsilon \frac{m_\infty}{n_{z(\infty)}}.$$

Proof. We assume that the reader is acquainted with how the method of RS-transformations works to produce algebraic solutions of the sixth Painlevé equation (see [2]). A suitable RS-transformation can be viewed as a composition of some R-transformation with finitely many elementary Schlesinger transformations that are applied successively to the linear (2×2)-matrix ODE associated with a matrix form of the Gauss hypergeometric equation. The key observation is that for the special RS-symbols all elementary Schlesinger transformations can be chosen to have the same upper-triangular structure. Thus, the root y of the equation $R'(z_1)=0$ is also a root of the $\{21\}$-entry of the successively transformed coefficient matrix of the associated linear ODE at each step of the application of the elementary Schlesinger transformations. □

Remark 2.5. Of course, instead of mentioning special RS-symbols, in Theorem 2.1 we can formulate all necessary conditions to be imposed on the rational function $z(z_1)$ purely graphically, in terms of the valencies of the tricolor graph. For $\varepsilon = 1$, the minus sign in the above formula for $\hat{\theta}_\infty$ is not essential because it does not influence the coefficients of equation (2.3). We keep this sign in order to get smaller absolute values for $\hat{\theta}_\infty$.

Now we consider some special deformations of *dessins* and the corresponding constructions for the algebraic solutions of the sixth Painlevé equation that are given by Theorem 2.1.

Remark 2.6. In all the figures throughout this paper we follow the convention that the blue vertices are indicated like the white ones, but with a larger diameter. If a picture contains only one "white vertex", then it is actually blue.

In Figure 1 all one-parameter "face" deformations for the Belyĭ function $R(5+2+1|2+2+2+2|3+3+2)$ are shown. This Belyĭ function is dual to the function that will appear later in the proof of Proposition 4.2 (see (4.5)). Hereafter, these and similar deformations of other *dessins* are called *twist*, *cross*, and *join*, respectively. Observe that two face distributions, namely, $2+2+2+2$ and $3+3+1+1$, which satisfy the necessary condition of Proposition 2.1, cannot be realized as deformations of the first *dessin* in Figure 1. However, the latter face distribution can be obtained as a

[3]The critical value different from $0, 1$, and ∞.

face deformation of the *dessin* for the Belyĭ function of the type $R(3+3+2|2+2+2+2|3+3+2)$. The former face distribution cannot be realized as a deformation of any *dessin*; thus, it does not determine any rational function. There are several other seemingly different deformations that lead to the same face distributions. In this case, they can be interpreted as different branches of the same algebraic solution. Below we give exact formulas for the deformations of the Belyĭ function for all three deformed *dessins* presented in Figure 1, together with the corresponding solutions $y(t)$ of the sixth Painlevé equation (2.3), calculated via Theorem 2.1.

1. Twist. $RS_4^2\left(\begin{array}{c|c|c} 1/5 & 1/2 & 1/3 \\ 5+1+1+1 & \underbrace{2+\cdots+2}_{4} & 3+3+2 \end{array}\right)$:

$$\begin{aligned}
z &= \frac{2^{24}s^3(5s^2+118s+5)(s+1)^{10}}{(3s^3+95s^2+25s+5)^3(5s^3+25s^2+95s+3)^3}\\
&\quad\times\frac{(z_1-1/2-a)^5 z_1(z_1-1)(z_1-t)}{(z_1-1/2-c_1)^3(z_1-1/2-c_2)^3},\\
a &= -\frac{(s-1)(s^4+12s^3-410s^2+12s+1)}{128s(s+1)\sqrt{s(5s^2+118s+5)}},\\
c_1 \equiv c_1(s) &= \frac{5s^6+462s^5+8535s^4+3060s^3+195s^2+30s+1}{16s(3s^3+95s^2+25s+5)\sqrt{s(5s^2+118s+5)}},\\
c_2 &= -c_1(1/s),\\
t &= \frac{1}{2}-\frac{(s-1)(25(s^8+1)+760(s^7+s)+4924(s^6+s^2)+75464(s^5+s^3)+329174s^4)}{2^{11}s(s+1)^5\sqrt{s(5s^2+118s+5)}},\\
y &= \frac{1}{2}-\frac{(s-1)(5(s^6+1)+58(s^5+s)+1771(s^4+s^2)+8620s^3)\sqrt{s(5s^2+118s+5)}}{8s(s+1)(5s^3+25s^2+95s+3)(3s^3+95s^2+25s+5)},
\end{aligned}$$

$$\hat\theta_0=\frac{1}{5},\quad \hat\theta_1=\frac{1}{5},\quad \hat\theta_t=\frac{1}{5},\quad \hat\theta_\infty=\frac{1}{3}.$$

It should be noted that, applying the Okamoto transformation together with the so-called Bäcklund transformations to this solution, we can obtain algebraic solutions of equation (2.3) for the following $\hat\theta$-assemblies:

$$\left(\frac{1}{15},\frac{1}{15},\frac{1}{15},\frac{7}{15}\right)\quad\text{and}\quad\left(\frac{4}{15},\frac{4}{15},\frac{4}{15},\frac{2}{15}\right).$$

It is readily seen that the introduction of a new variable

$$z_2=(z_1-1/2)\sqrt{s(5s^2+118s+5)}$$

yields a new function $z(z_2)$ that has the same type as $z(z_1)$ but is parametrized with s rationally. However, as was explained earlier, in order to relate our rational function to a solution of the sixth Painlevé equation (2.3), we must *normalize* it by placing (via a fractional linear transformation) three of its nonapparent zeroes or poles at 0, 1, and ∞. In this example the normalization procedure leads to the emergence of a genus 1 parametrization. In many cases, Remark 2.7 helps to simplify parametrizations of the R-parts of RS-transformations when we use them for construction of solutions, regardless of whether we employ Theorem 2.1 or perform a complete construction of the corresponding RS-transformations including their S-parts. As the reader will see below, with this function $z(z_1)$ we can associate another RS-transformation to which Theorem 2.1 does not apply. Thus, we actually need to build up the S-part of the latter RS-transformation in order to find the corresponding solution of the sixth Painlevé equation.

Remark 2.7. The S-parts of the RS-transformations are symmetric functions of the apparent critical points of their R-parts.

Thus, we need to know a parametrization of the symmetric functions c_1+c_2 and c_1c_2 of the apparent critical points c_1 and c_2, rather than their individual parametrizations. Of course, this leads to simplification of the parametrization. Moreover, it is clear that, theoretically, such simplification may reduce the genus of the parametrization. In particular, in item 5 of Subsection 3.3 we give an example where this actually happens. Below we show that a simplified parametrization can be obtained with the help of the Zhukovskii type transformation $s+1/s=2^5s_1+2$, which reduces the degree of the parametrization by two. However, in this case the genus of the parametrization remains unchanged.

$$\begin{aligned}
z(z_1) &= \frac{(5s_1+4)(8s_1+1)^5}{8(30s_1^3+40s_1^2+10s_1+1)^3}\\
&\quad\times\frac{(z_1-1/2-a)^5z_1(z_1-1)(z_1-t)}{(z_1^2-(1+c_1+c_2)z_1+1/4+c_1c_2+(c_1+c_2)/2)^3},\\
a &= \frac{(8s_1^2+4s_1-3)s_1}{2\sqrt{s_1(8s_1+1)(5s_1+4)}},\\
\frac{1}{4}+c_1c_2 &= -\frac{320s_1^6+1344s_1^5+1560s_1^4+480s_1^3-60s_1^2+1}{8(30s_1^3+40s_1^2+10s_1+1)(5s_1+4)},\\
c_1+c_2 &= -\frac{(6s_1^2+4s_1-1)(4s_1+5)s_1^2(8s_1+1)}{(30s_1^3+40s_1^2+10s_1+1)\sqrt{s_1(8s_1+1)(5s_1+4)}},\\
t &= \frac{1}{2}-\frac{(800s_1^4+960s_1^3+312s_1^2+100s_1+15)s_1}{2(8s_1+1)^2\sqrt{s_1(8s_1+1)(5s_1+4)}},\\
y &= \frac{1}{2}-\frac{(40s_1^3+22s_1^2+16s_1+3)\sqrt{s_1(8s_1+1)(5s_1+4)}}{2(30s_1^3+40s_1^2+10s_1+1)(8s_1+1)}.
\end{aligned}$$

It is also worth noting that, although we have a simpler parametrization of the function $z(z_1)$, the first parametrization is still needed in the theory of SFITs with several variables, where the critical points c_1 and c_2 are included into the nonapparent set, so that the second parametrization cannot be used for the purpose of construction of RS-transformations.

Now, consider the associated conjugate functions. Clearly, in this case some of them are rational functions on the torus. Here we only consider conjugate functions rational on the Riemann sphere. To get them, we need to normalize the function $z(z_2)$ mentioned in the paragraph right after the first parametrization of $z(z_1)$. If we wish to keep a rational parametrization, the only way to do this is to use apparent critical points for normalization: the zero of order 5 and the poles of orders 3, 3, and 2, together with one nonapparent zero. A normalization is said to be *degenerate* if at least one of the critical points 0, 1, or ∞ is apparent. All in all, we can arrange 33 different degenerate normalizations of $z(z_2)$, thus getting 33 different rational conjugate functions.

For example, the function $z(z_2)$ normalized so that it has a zero of the fifth order at

0 and poles of the third order at 1 and ∞ looks like this:

(2.4)
$$\tilde{z}=\frac{\tilde{z}_1^5(\tilde{z}_1-1+s_0^2)((5s_0+1)^2\tilde{z}_1^2+(5s_0+1)(5s_0^3+49s_0^2+115s_0+23)\tilde{z}_1-24(s_0^2+6s_0+1)(s_0^2+4s_0+1))}{64(\tilde{z}_1-1)^3((3s_0^3+15s_0^2+25s_0+5)\tilde{z}_1-8(s_0+1)(s_0^2+4s_0+1))^2}.$$

This function differs from that at the beginning of this item by a fractional linear transformation of z_1 related to the normalization discussed above:

$$\tilde{z}(\tilde{z}_1)=z(z_1), \quad z_1\equiv M(\tilde{z}_1)=\frac{\tilde{t}_--c_0}{\tilde{t}_--\tilde{t}_+}\frac{\tilde{z}_1-\tilde{t}_+}{\tilde{z}_1-c_0},$$

where

$$\begin{aligned}
\tilde{t}&=1-s_0^2,\\
\tilde{t}_\pm&=-\frac{24s_0^2}{5s_0+1}-\frac{23+s_0^2}{2}\pm\frac{(s_0+5)}{2}\sqrt{\frac{(5s_0^2+22s_0+5)(s_0+5)}{5s_0+1}},\\
c_0&=\frac{8(s_0+1)(s_0^2+4s_0+1)}{(3s_0^3+15s_0^2+25s_0+5)},\\
s_0&=\frac{5-s}{5s-1},
\end{aligned}$$

and s is exactly the same as in the formulas at the beginning of this item. The functions $\tilde{t}_\pm$ are the conjugate roots of the quadratic polynomial in z_1 occurring in the numerator of the function (2.4), and c_0 is its second-order pole.

The function $\tilde{z}(\tilde{z}_1)$ has a degenerate normalization. Thus, Theorem 2.1 does not apply to it. However, unexpectedly, in this particular case the first-order zeros of $\tilde{z}(\tilde{z}_1)$ have an "apparent behavior", so that Theorem 2.1 actually works. Therefore, putting $m_0=5$, $m_1=3$, $m_t=1$, $m_\infty=3$, $\epsilon=1$, $n_0=1$, and $n_\infty=2$ and applying Theorem 2.1 formally to the function $\tilde{z}(\tilde{z}_1)$, we arrive at the conclusion that the function

$$\tilde{y}=\frac{5(1-s_0^2)(s_0^2+6s_0+1)(s_0^2+4s_0+1)}{(5s_0+1)(3s_0^3+15s_0^2+25s_0+5)},$$

viewed as a function of either of the arguments $\tilde{t}$ or $\tilde{t}_\pm$, solves the sixth Painlevé equation (2.3) for

$$\hat{\theta}_0=5, \quad \hat{\theta}_1=\frac{3}{2}, \quad \hat{\theta}_t=1, \quad \hat{\theta}_\infty=-\frac{1}{2}. \tag{2.5}$$

Actually, $y(\tilde{t})$ and $y(\tilde{t}_\pm)$ are different branches of the very same genus 0 algebraic function, so that $\tilde{t}=\tilde{t}(s_0)$, $y=y(s_0)$ is its rational parametrization.

The functions $y(t)$ and $\tilde{y}(\tilde{t})$ are related to each other via a fractional linear transformation:

$$y=M(\tilde{y}), \quad t=M(\tilde{t}).$$

Of course, this transformation can be called "fractional linear" only conventionally, because it is parametrized by the same parameter s_0 (or s) as $\tilde{t}$ and $\tilde{y}$. At this point it should be noted that the solutions $y(t)$ and $\tilde{y}(\tilde{t})$ are not related by any of the transformations that act on the set of general solutions of the sixth Painlevé equation.

Moreover, we can treat $\tilde{y}$ as a function of $\tilde{t}_1=c_0$, and as such it is a rational function,

$$\tilde{y}=\frac{5\tilde{t}_1(\tilde{t}_1-2)}{3\tilde{t}_1-8},$$

which solves the sixth Painlevé equation for the same $\hat{\theta}$-assembly (2.5).

Now we consider the function (2.4) as a rational function of s_0, treating z_1 as a parameter; i.e., we pass to the conjugate function. Its type is $R(\underbrace{1+\dots+1}_{6}|2+2+2|2+2+2)$. Therefore, this function has four *additional* critical points, located at $s_0=0$, ∞, and $-7/5\pm 2\sqrt{6}/5$. In terms of the variables z and z_1 introduced at the beginning of this item, the corresponding Belyĭ functions can be written as the values of the function $z(z_1)=z(z_1,s)$ at $s=5$, $1/5$, and $-11/5\pm 4\sqrt{6}/5$, respectively. The first two Belyĭ functions are of the same type $R(5+1|2+2+2|3+2+1)$ and are related to each other by a simple change of the argument $z_1\to 1-z_1$; the last two functions coincide and their type is $R(5+2+1|\underbrace{2+\dots+2}_{4}|3+3+2)$:

$$s=5: \quad z(z_1)=\frac{20(z_1-16/25)^5(z_1-1)}{27z_1(z_1-128/125)^3},$$
$$s=\frac{1}{5}: \quad z(z_1)=\tilde{z}(1-z_1), \text{ where } \tilde{z}(z_1) \text{ coincides with } z(z_1) \text{ for } s=5.$$
$$s=-11/5\pm 4\sqrt{6}/5: \quad z=-\frac{5(z_1-27/25)^5z_1^2(z_1-1)}{8(z_1^2+216/125z_1+729/1000)^3}.$$

For completeness, we mention that with the twist under consideration we can associate another RS-transformation, namely,

$$RS_4^2\left(\begin{array}{c|c|c} 2/5 & 1/2 & 1/3 \\ 5+1+1+1 & \underbrace{2+\dots+2}_{4} & 3+3+2 \end{array}\right).$$

However, the corresponding solution of equation (2.3) cannot be calculated via Theorem 2.1. To find it explicitly, we need to construct the S-part of the transformation, as was done in [2]. Here we only note that this solution also admits an elliptic parametrization and corresponds to the following $\hat{\theta}$-assembly:

$$\hat{\theta}_0=\frac{2}{5}, \quad \hat{\theta}_1=\frac{2}{5},$$
$$\hat{\theta}_t=\frac{2}{5}, \quad \hat{\theta}_\infty=\frac{2}{3}.$$

Again, the Okamoto transformation together with Bäcklund transformations make it possible to obtain algebraic solutions for the following $\hat{\theta}$-assemblies:

$$\left(\frac{2}{15},\frac{2}{15},\frac{2}{15},\frac{14}{15}\right)$$

and

$$\left(\frac{8}{15},\frac{8}{15},\frac{8}{15},\frac{4}{15}\right)\sim\left(\frac{7}{15},\frac{7}{15},\frac{7}{15},\frac{11}{15}\right).$$

2. Cross. $RS_4^2\left(\begin{array}{c|c|c} 1/4 & 1/2 & 1/3 \\ 4+2+1+1 & \underbrace{2+\cdots+2}_{4} & 3+3+2 \end{array}\right)$:

$$
\begin{aligned}
z &= -\frac{256s^6}{(3s^2-2)^3(s^2-6)^3}\frac{(z_1-a)^4(z_1-t)^2z_1(z_1-1)}{(z_1-c_1)^3(z_1-c_2)^3},\\
c_1 &= \frac{(s+1)^2(s^2+2s-2)^2}{4s^3(3s^2-2)},\\
a &= -\frac{1}{32s^3}(s^2-4s-2)(s^2+2s-2)^2,\\
c_2 &= \frac{(s-2)^2(s^2+2s-2)^2}{16s(s^2-6)},\\
t &= \frac{(s-2)^2(s+1)^2(s^2+2s-2)}{4s^3},\\
y(t) &= -\frac{(s+1)(s-2)(s^2+2s-2)(s^2-4s-2)}{2s(3s^2-2)(s^2-6)},\\
\hat\theta_0 &= \frac{1}{4},\quad \hat\theta_1=\frac{1}{4},\quad \hat\theta_t=\frac{1}{2},\quad \hat\theta_\infty=\frac{1}{3}.
\end{aligned}
$$

Now, as in the cases treated earlier, some transformations imply algebraic solutions for the following $\hat\theta$-assemblies:

$$\left(\frac{1}{12},\frac{1}{12},\frac{1}{6},\frac{2}{3}\right)\quad\text{and}\quad\left(\frac{5}{12},\frac{5}{12},\frac{1}{6},\frac{1}{3}\right).$$

With this deformation *dessin*, we can associate yet another RS-transformation with the following symbol:

$$RS_4^2\left(\begin{array}{c|c|c} 1/2 & 1/2 & 1/2 \\ 4+2+1+1 & \underbrace{2+\cdots+2}_{4} & 3+3+2 \end{array}\right).$$

The corresponding explicit solution for the $\hat\theta$-assembly $(1/2,1/2,3/2,-3/2)$ can easily be found by renormalization of the function $z(z_1)$ via a fractional linear transformation of z_1, as was done in the preceding item. The conjugate function is of the type

$$R(\underbrace{4+\cdots+4}_{6}+\underbrace{2+\cdots+2}_{6}\,|\,\underbrace{2+\cdots+2}_{18}\,|\,\underbrace{3+\cdots+3}_{12}).$$

A general function of this type depends on four arbitrary parameters. The conjugate function has four additional critical points: $\pm\sqrt{2}$ and $\pm i\sqrt{2}$. For these values of the parameter s, the initial function coincides with the Belyĭ functions of the types

$$R(6+1+1|\underbrace{2+\cdots+2}_{4}|3+3+2)\quad\text{and}\quad R(4+2+1+1|\underbrace{2+\cdots+2}_{4}|6+2),$$

respectively:

$$
\begin{aligned}
s=\pm\sqrt{2}:&\qquad z=\frac{(z_1-1/2)^6z_1(z_1-1)}{2(z_1^2-z_1-1/32)^3},\\
s=\sqrt{-2}:&\qquad z=\frac{(z_1-1/2-5\sqrt{-2}/4)^4(z_1-1/2+11\sqrt{-2}/2)^2z_1(z_1-1)}{128(z_1-1/2+7\sqrt{-2}/16)^6}.
\end{aligned}
$$

By fractional linear transformations of z_1, we can get other nonequivalent conjugate functions, e.g., a function of the type

$$R(\underbrace{2+\cdots+2}_{4}+\underbrace{1+\cdots+1}_{8}\,|\,\underbrace{2+\cdots+2}_{8}|6+6+2+2).$$

The latter function has six additional critical points located at 0, ∞, and $\pm 1 \pm \sqrt{3}$, and one and the same Belyĭ function of the type $R(1+1|2|2)$, namely, $z = -4z_1(z_1 - 1)$, corresponds to all of them.

3. Join. $RS_4^2\left(\begin{array}{c|c|c} 1/3 & 1/2 & 1/3 \\ 3+2+2+1 & \underbrace{2+\cdots+2}_{4} & 3+3+2 \end{array}\right)$:

$$z = -\frac{(s^2+3)^6}{(s^2+1)^3(s^2+9)^3}\frac{(z_1-a)^3 z_1^2 (z_1-1)^2(z_1-t)}{(z_1-c_1)^3(z_1-c_2)^3},$$

$$a = \frac{(s^2+6s+3)}{2(s^2+3)}, \quad c_1 = \frac{(s+3)^2}{2(s^2+9)}, \quad c_2 = \frac{(s+1)^2}{2(s^2+1)},$$

$$t = \frac{(s+1)^2(s+3)^2(s^2-2s+3)}{2(s^2+3)^3},$$

$$y(t) = \frac{(s+1)(s+3)(s^2-2s+3)(s^2+6s+3)}{2(s^2+1)(s^2+3)(s^2+9)},$$

$$\hat{\theta}_0 = \frac{2}{3}, \quad \hat{\theta}_1 = \frac{2}{3}, \quad \hat{\theta}_t = \frac{1}{3}, \quad \hat{\theta}_\infty = \frac{1}{3}.$$

Making the fractional linear transformation $\tilde{z}_1 = (1-t)z_1/(z_1-t)$, which takes the points $0, 1, \infty$, and t to $0, 1, 1-t$, and ∞, respectively, we obtain another algebraic solution, $\tilde{y}(\tilde{t}) = (1-t)y(t)/(y(t)-1)$ and $\tilde{t} = 1-t$. Changing the notation $\tilde{y}$ and $\tilde{t}$ back to y and t, we can write this solution as follows:

$$t = \frac{(s-1)^2(s-3)^2(s^2+2s+3)}{2(s^2+3)^3},$$

$$y(t) = \frac{(s-1)(s-3)(s^2+6s+3)}{4s(s^2+3)},$$

$$\hat{\theta}_0 = \frac{2}{3}, \quad \hat{\theta}_1 = \frac{2}{3}, \quad \hat{\theta}_t = \frac{2}{3}, \quad \hat{\theta}_\infty = \frac{2}{3}.$$

For completeness, we mention that the *dessin* under consideration generates yet another RS-transformation:

$$RS_4^2\left(\begin{array}{c|c|c} 1/2 & 1/2 & 1/2 \\ 3+2+2+1 & \underbrace{2+\cdots+2}_{4} & 3+3+2 \end{array}\right).$$

The corresponding solution solves equation (2.3) for the $\hat{\theta}$-assembly $(3/2, 1/2, 3/2, -1/2)$. Its explicit construction can be made in the standard way by renormalization of the function $z(z_1)$.

By using the Okamoto, Bäcklund, and quadratic (see the Appendix) transformations and their compositions, from the solutions presented in this item we can obtain many different algebraic solutions. Here we do not write the corresponding $\hat{\theta}$-assemblies, because care must be taken when we consider the action of transformations on $\hat{\theta}$-assemblies: transformations that work on general solutions may become degenerate on some particular ones and lead to "solutions" like $y(t) = 0$, $y(t) = 1$, $y(t) = \infty$. Whether or not this happens at some intermediate step of a chain of transformations is not clear until the actual calculation for the particular solution is complete.

The conjugate function is of the type

$$R(3+3+\underbrace{1+\cdots+1}_{6}\,|\,\underbrace{2+\cdots+2}_{6}\,|\,\underbrace{3+\cdots+3}_{4}).$$

It has four additional critical points at 0, ∞, and $\pm\sqrt{3}$, and at these values of the parameter s the initial function coincides with the Belyĭ functions of the types

$$R(2+2|2+2|2+2) \quad \text{and} \quad R(3+2+2+1|\underbrace{2+\cdots+2}_{4}|6+2),$$

respectively:

$$s=0,\infty: \quad z=-\frac{4z_1^2(z_1-1)^2}{(2z_1-1)^2},$$
$$s=\pm\sqrt{3}: \quad z=-\frac{27(z_1-1/2\mp\sqrt{3}/2)^3 z_1^2(z_1-1)^2(z_1-1/2\mp 5\sqrt{3}/18)}{64(z_1-1/2\mp\sqrt{3}/4)^6}.$$

Making a fractional linear transformation, we can get other nonequivalent conjugate functions; one of them is of the type

$$R(\underbrace{2+\cdots+2}_{8}+\underbrace{1+\cdots+1}_{4}|\underbrace{2+\cdots+2}_{10}|6+6+\underbrace{2+\cdots+2}_{4}).$$

It has six additional critical points at 0, ∞, $1\pm i\sqrt{2}$, and $-1\pm i\sqrt{2}$. The Belyĭ functions corresponding to the critical points $s=\pm1+\sqrt{-2}$ are of the same type $R(3+3+2|\underbrace{2+\cdots+2}_{4}|3+3+2)$, and the quadratic transformation $\tilde{z}=(z_1-(1\pm1)/2)^2$ reduces it to the function of the type $R(3+1|2+2|3+1)$, namely, $z=-64\tilde{z}(\tilde{z}-1)^3/(8\tilde{z}+1)^3$.

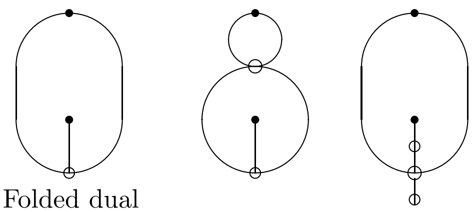
§3. Deformations of dessins for the Platonic solids and algebraic solutions of the sixth Painlevé equation

Since the Platonic solids have rich symmetry groups, their *dessins* can be obtained by the action of certain finite rotation groups on simpler *dessins* called the *reduced dessins*. Thus, the corresponding Belyĭ functions are the compositions of the Belyĭ functions for the reduced *dessins* and a monomial $z=z_1^n$ with some integer n. For the details, we refer the reader to the interesting paper [15]. As will be shown in this and the next sections, these irreducible reduced *dessins* play an important role in the theory of the algebraic sixth Painlevé and Gauss hypergeometric functions.

In the case where the $\hat{\theta}$-assembly is of the form $(0,0,0,\hat{\theta}_\infty)$, where $\hat{\theta}_\infty\in\mathbb{C}$ is a parameter, the algebraic solutions of the sixth Painlevé equations were classified by Dubrovin and Mazzocco [6]. They found five cases, namely: $\theta_\infty=-1/2$ (the so-called tetrahedron solution), $\theta_\infty=-2/3$ (the cube solution), another solution for $\theta_\infty=-2/3$ (the great dodecahedron solution), $\theta_\infty=-4/5$ (the icosahedron solution), and $\theta_\infty=-2/5$ (the great icosahedron solution). They also presented a rational parametrization for all solutions except for the great dodecahedron solution. They mentioned that the latter is an algebraic function of genus one and used computer simulations to produce an incredibly long algebraic equation for this solution, which can be found in the preprint version of their paper.[4] Here, we show that all algebraic solutions of zero genus mentioned above can be constructed via the method of RS-transformations whose R-parts are obtained as deformations of the *dessins* for the Platonic solids. Also, we introduce different kinds of vertex deformations of *dessins*, which we call *splits*. Instead of the splits, one can always consider face deformations of the dual *dessins*.

[4]http://arXiv.org/abs/math/9806056.

3.1. Deformations of the folded reduced cube (tetrahedron solution). Below we define the folded reduced cube *dessin*. This very simple *dessin* has four one-dimensional deformations: twist, join, and B- and W-splits. However, the first three of them generate equivalent RS-transformations: the B-split is dual to the twist, and the latter determines the same face distribution as the join. The last two *dessins* are continuously transformed into each other when the blue vertex passes through the white one. Thus, they correspond to two different branches of the same function $z(z_1)$ (as a function of the deformation parameter s) and give rise to the same algebraic solution of the sixth Painlevé equation. So, there are two nonequivalent tricolor *dessins*, and one (seed) algebraic solution is associated with each of them. Both solutions were constructed by the method of RS-transformations in [2, §3], where, of course, their deformation nature was not discussed.



Folded dual reduced cube   Twist W-split

The Belyĭ function corresponding to the reduced cube is of the type $R(4+1+1|2+2+2|3+3)$ (see [15]). Interchanging the colors of white and black vertices, we obtain the dual *dessin*. The folding procedure for the latter *dessin* is the transition to the depicted *dessin*, based on the following decomposition of the types of the Belyĭ functions: $R(4+1+1|3+3|2+2+2) = R(\underset{\wedge}{2}+1|3|2+\underset{\wedge}{1}) \circ R(2|2)$.

The symbol of the RS-transformation associated with the W-split reads

$$RS_4^2\begin{pmatrix}\theta_0 & 1/2 & 1/2\\ 2{+}1 & 2{+}1 & 2{+}1\end{pmatrix}.$$

The complete list of formulas related to this RS-transformation, including the corresponding solution, can be found in [2, Subsection 3.1.1]. Below we consider only the twist (which generates the tetrahedron solution of [6]), treating it with the help of Theorem 2.1. The corresponding RS-symbol is

$$RS_4^2\begin{pmatrix}\theta_0 & 1/3 & 1/2\\ 1{+}1{+}1 & 3 & 2{+}1\end{pmatrix}.$$

We have

$$z = (1-s^2)^3\frac{z_1(z_1-1)(z_1-t)}{\big((1-s^3)z_1-1\big)^2}, \quad z-1 = \frac{\big((1-s^2)z_1-1\big)^3}{\big((1-s^3)z_1-1\big)^2},$$
$$t = \frac{(2s+1)}{(1-s)(s+1)^3}, \quad y(t) = \frac{(2s+1)}{(s+1)(s^2+s+1)},$$
$$\hat\theta_0 = \hat\theta_1 = \hat\theta_t \in \mathbb{C}, \quad \hat\theta_\infty = 1/2.$$

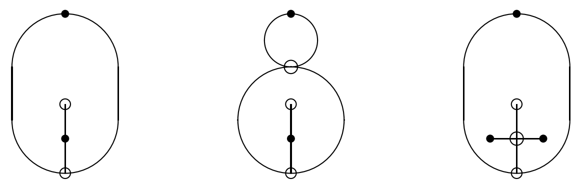
Since $\hat\theta_0 = \hat\theta_1 = \hat\theta_t$ is arbitrary and $y(t)$ does not depend on this parameter, we arrive at the following algebraic equation for $y(t)$:

$$\frac{t}{y^2} - \frac{(t-1)}{(y-1)^2} + \frac{t(t-1)}{(y-t)^2} = 0.$$

Actually, the solution found in [6] and called there the tetrahedron solution is more complicated and corresponds to a more special $\hat\theta$-assembly $\hat\theta_0 = \hat\theta_1 = \hat\theta_t = 0$, $\hat\theta_\infty = -1/2$.

This solution is related to that presented here via a Bäcklund transformation, and the associated Fuchsian linear ODE has the same monodromy group. See the details in Remark 2 in [2, Subsection 3.1.2].[5]

3.2. Deformations of the reduced tetrahedron (cube solution). There are two deformations of the reduced tetrahedron, and with each of them we can associate one solution. Both solutions were obtained in [10, 2] by the method of RS-transformations. However, here we provide the first of them with a simpler parametrization, as compared to that given in the cited papers, and we discuss application of the quadratic transformations to the second solution. The quadratic transformations are interesting in the context of finding algebraic solutions with nontrivial genus, because their application often leads to solutions with elliptic parametrization.



Dual reduced tetrahedron — Twist — B-split

The Belyĭ function for the dual reduced tetrahedron looks like this:

$$z = \frac{64z_1(z_1-1)^3}{(8z_1^2+20z_1-1)^2} \quad \text{and} \quad z-1 = -\frac{(8z_1+1)^3}{(8z_1^2+20z_1-1)^2}.$$

1. Twist. Cube solution: $RS_4^2\left(\begin{array}{c|c|c} \theta_0 & 1/3 & 1/2 \\ 2{+}1{+}1 & 3{+}1 & 2{+}2 \end{array}\right)$.

$$\begin{aligned} z &= \frac{64(z_1-t)^2 z_1(z_1-1)}{(8z_1^2-2(s+1)(s^2-s+4)z_1+(1+s)^3)^2}, \\ z-1 &= \frac{(4sz_1-(s+1)^2)^3}{(8z_1^2-2(s+1)(s^2-s+4)z_1+(1+s)^3)^2}, \\ t &= \frac{1}{4}(2-s)(s+1)^2, \\ y(t) &= \frac{1}{2s}(s+1)(2-s), \\ &\hat{\theta}_0 = \hat{\theta}_1 = \frac{1}{2}\hat{\theta}_t \in \mathbb{C}, \quad \hat{\theta}_\infty = \frac{2}{3}. \end{aligned}$$

As in Subsection 3.1, no effort is required to derive an algebraic equation for the function $y(t)$:[6]

$$\frac{t}{y^2} - \frac{(t-1)}{(y-1)^2} + \frac{4t(t-1)}{(y-t)^2} = 0.$$

Concerning the relationship between this solution and the cube solution of [6], a remark similar to that at the end of Subsection 3.1 can be made: the cube solution satisfies the sixth Painlevé equation only for

$$\hat{\theta}_0 = \hat{\theta}_1 = \frac{1}{2}\hat{\theta}_t = 0, \qquad \theta = -2/3$$

and is related to the solution constructed here with the help of a Bäcklund transformation.

[5] There is a misprint in the formula for ρ given there: the denominator in this formula should be squared.

[6] In [11] there is a misprint in the sign before the last term in this equation.

2. B**-split.** Cube solution (?): $RS_4^2\begin{pmatrix}1/3 & 1/3 & 1/2\\ 3{+}1 & 3{+}1 & 2{+}1{+}1\end{pmatrix}$.

$$z=\frac{\rho(z_1-a)^3z_1}{(z_1-c)^2(z_1-1)},\quad z-1=\frac{\rho(z_1-b)^3(z_1-t)}{(z_1-c)^2(z_1-1)},$$
$$\rho=\frac{(2s+1)^3}{(3s+1)^2(1-3s^2)^2},\quad a=\frac{(1-3s^2)}{(2s+1)}(3s^2+2s+1),$$
$$b=1-3s^2,\quad c=\frac{(1-3s^2)}{(3s+1)}(3s^2+3s+1),$$
$$t=\frac{(1-3s^2)}{(2s+1)^3}(3s^2+3s+1)^2,\quad y(t)=\frac{(3s^2+2s+1)(3s^2+3s+1)}{(2s+1)(3s+1)},$$
$$\hat\theta_0=\frac{1}{3},\quad \hat\theta_1=\frac{1}{2},\quad \hat\theta_t=\frac{1}{3},\quad \hat\theta_\infty=\frac{1}{2}.$$

We can further apply the quadratic transformation given in example 3 of Appendix A. Put $\hat\theta_t^0=2/3$ and $\hat\theta_\infty^0=0$ in that example. Then we see that the $\hat\theta$-assembly of example 3 coincides with the $\hat\theta$-assembly for the solution obtained above, so that we can make the *inverse* quadratic transformation of the above solution. Redenoting the variables t_0 and $y_0(t_0)$ back by t and $y(t)$, we arrive at the following solution:

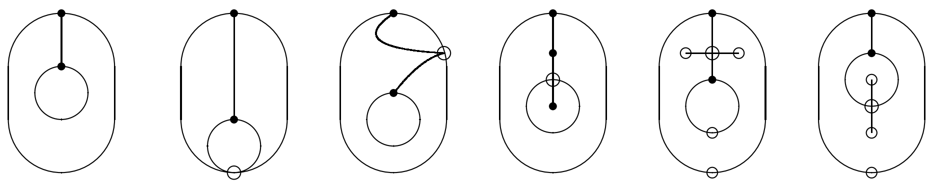
$$t=\frac{1}{2}+\frac{(3s+1)(s+1)(3s^3(3s+2)+(s+1)^2)}{4\left(\sqrt{s(2s+1)(3s+2)}\right)^3},$$
$$y(t)=\frac{1}{2}-\frac{9(s+1)^4-2(3s+2)^2}{6(s+1)(3s+1)\sqrt{s(2s+1)(3s+2)}},$$
$$\hat\theta_0=0,\quad \hat\theta_1=0,\quad \hat\theta_t=\frac{2}{3},\quad \hat\theta_\infty=0.$$

We see that the resulting solution has an elliptic parametrization. However, the study of its Puiseux expansions at the singular points $t=0,1,\infty$ shows that they are very similar in appearance to those for the cube solution. Thus, most probably, this solution can be parametrized rationally and mapped to the cube solution by a Bäcklund transformation such that $(0,0,\theta_t,0)\longmapsto(0,0,0,\theta_t)$.

So, here we have described a mechanism of the emergence of elliptic parametrization for algebraic solutions in the process of the application of quadratic transformations. Seemingly, this mechanism is different from the normalization procedure explained in §2. At the moment, it is not clear whether such mechanisms could really produce an elliptic algebraic solution. As will be shown in Appendix A, the quadratic transformations are also generated by RS-transformations. However, to get the complete list of quadratic transformations by applying the method of RS-transformations to the (2×2)-matrix Fuchsian system with four singularities in Jimbo–Miwa parametrization, we need the Okamoto transformation, which can be treated as a reparametrization of the Fuchsian system. Therefore, the method of RS-transformations can also be applied to the Fuchsian system in the Okamoto parametrization in order to produce algebraic solutions. In the latter setting, we do not need to apply separately any quadratic transformations: we simply apply the same RS-transformations to two copies of the Fuchsian system in different parametrizations. Therefore, if an algebraic solution with a nontrivial genus can actually be produced by the method of RS-transformations, then its emergence can be "explained" as a special case of the first normalization mechanism of §2.

Since throughout the paper we often refer to the Okamoto transformation, it should probably be noted that this transformation does not change t and is rational in $y(t)$, $y'(t)$, and t. Thus, it cannot change the genus of our algebraic solutions.

3.3. Deformations of the reduced cube. In the pictures below all nonequivalent one-dimensional deformations of the reduced cube are presented. In the first four *dessins* the white vertices are not indicated. It is of interest that here we have two nonequivalent W-splits that determine the same face distributions.



Reduced cube $\quad$ CC-join $\quad$ LC-join $\quad$ B-split $\quad$ LW-split $\quad$ CW-split

1. CC**-join.** As a function of y, the function $z(z_1)$ has two branches. The *dessin* corresponding to the second branch has an edge going from the upper B-vertex around the inner circle, joining with itself, and finally connecting with the B-vertex on the inner circle. Two RS-transformations can be associated with this dessin:

$$RS_4^2\left(\begin{array}{c|c|c} 1/2 & 1/2 & \theta \\ 2{+}2{+}1{+}1 & 2{+}2{+}2 & 3{+}3 \end{array}\right), \quad RS_4^2\left(\begin{array}{c|c|c} \theta & 1/2 & 1/3 \\ 2{+}2{+}1{+}1 & 2{+}2{+}2 & 3{+}3 \end{array}\right).$$

Note that the R-type of these transformations can be presented as a composition of R-types of degrees 3 and 2, $R(2+2+1+1|2+2+2|3+3) = R(2+1|2+\underset{\wedge}{1}|3) \circ R(2)$. This implies that both transformations generate precisely the same solution as $RS_4^2\left(\begin{array}{c|c|c} \theta_0 & 1/2 & \theta_\infty \\ 1{+}1 & 2 & 1{+}1 \end{array}\right)$, but formally they give a different and more restricted $\hat{\theta}$-assembly than the latter. The reader can find this solution in [2, §2].

2. LC**-join.** As a function of y, the function $z(z_1)$ has three branches. Two other branches correspond to two crosses: one of the inner circle and another of the external circle. The corresponding RS-transformation looks like this:

$$RS_4^2\left(\begin{array}{c|c|c} \theta_0 & 1/2 & 1/3 \\ 3{+}1{+}1{+}1 & 2{+}2{+}2 & 3{+}3 \end{array}\right);$$

$$z = -\frac{4(p_1-p_3)p_1^4p_3^4}{3^6(4s+1)^9(10s+1)^9}\frac{z_1(z_1-1)(z_1-t)}{(z_1-c_1)^3(z_1-c_2)^3},$$

$$z-1 = -\frac{(z_1^3+b_2z_1^2+b_1z_1+b_0)^2}{(z_1-c_1)^3(z_1-c_2)^3},$$

$$b_0 = \frac{s^3(11s+2)^3p_3^3}{(4s+1)^6(10s+1)^6}, \quad b_1 = \frac{p_3^2(p_1^4+p_2^4)}{3^3(4s+1)^6(10s+1)^6}, \quad b_2 = -\frac{p_3(p_1^2+p_2^2)}{3(4s+1)^3(10s+1)^3},$$

$$c_1 = \frac{(11s+2)^2p_3}{3^2(4s+1)^3(10s+1)}, \quad c_2 = \frac{3^2s^2p_3}{(4s+1)(10s+1)^3}, \tag{3.1}$$

$$t = \frac{3^3s^3(11s+2)^3p_3}{(p_1-p_3)(4s+1)^3(10s+1)^3}, \quad y(t) = \frac{3^2s^2(11s+2)^2}{(p_1-p_3)(4s+1)(10s+1)},$$

$$\hat{\theta}_0 = \hat{\theta}_1 = \hat{\theta}_t = \theta_0 \in \mathbb{C}, \quad \hat{\theta}_\infty = 3\theta_0 + 1,$$

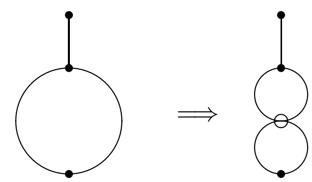
where the p_k, $k=1,2,3$, are quadratic polynomials:

$$p_1 = 73s^2+20s+1, \quad p_2 = 37s^2+11s+1, \quad p_3 = 47s^2+22s+2.$$

They satisfy the following relations: $p_1+p_2=(11s+2)(10s+1)$, $p_1-p_2=9s(4s+1)$, $p_3+p_2=3(4s+1)(7s+1)$, $p_3-p_2=(s+1)(10s+1)$. Since θ_0 is arbitrary, we find the following algebraic equation for the function $y(t)$:

$$\frac{t}{y^2}-\frac{t-1}{(y-1)^2}+\frac{t(t-1)}{(y-t)^2}=9.$$

3. B**-split.** This *dessin* gives rise to a function $z(z_1)$ of the type $R(4+1+1|2+2+2|3+2+1)$. It is of interest to note that, as a function of y, it has three branches. Only two of them can be obtained as B-splits of the reduced cube. The third branch is determined by the twist of another Belyĭ function that is of the type $R(4+2|2+2+2|3+2+1)$, which is not related to the Platonic solids (see the picture). With the B-split we can associate four (seed) RS-transformations:



(3.2)
$$\begin{gathered}
RS_4^2\left(\begin{array}{c|c|c} 1/4 & 1/2 & 1/3 \\ 4{+}1{+}1 & 2{+}2{+}2 & 3{+}2{+}1\end{array}\right),\\
RS_4^2\left(\begin{array}{c|c|c} 1/4 & 1/2 & 1/2 \\ 4{+}1{+}1 & 2{+}2{+}2 & 3{+}2{+}1\end{array}\right),\\
RS_4^2\left(\begin{array}{c|c|c} 1/2 & 1/2 & 1/3 \\ 4{+}1{+}1 & 2{+}2{+}2 & 3{+}2{+}1\end{array}\right),\\
RS_4^2\left(\begin{array}{c|c|c} 1/2 & 1/2 & 1/2 \\ 4{+}1{+}1 & 2{+}2{+}2 & 3{+}2{+}1\end{array}\right).
\end{gathered}$$

To find the solutions corresponding to the last two RS-transformations, we need to construct their Schlesinger part also. Not dwelling on this here, we note that the corresponding algebraic solutions have genus 0 and solve equation (2.3) for the following $\hat{\theta}$-assemblies:

$$\Big(\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{1}{3}\Big), \quad \Big(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\Big).$$

We observe that we can apply the quadratic transformation A.1 (see Appendix A) to the solution corresponding to the first of these $\hat{\theta}$-tuples to get a solution for the $\hat{\theta}$-assembly $(1/3,1/3,1/3,1/3)$. Further application of the Okamoto transformation produces a solution for $(0,0,0,2/3)$. This construction should be examined in connection with the great dodecahedron solution.

The solution associated with the first two RS-transformations can be found with the help of Theorem 2.1:

$$z=\rho\frac{(z_1-a)^4 z_1(z_1-1)}{(z_1-c)^3(z_1-t)}, \quad z-1=\rho\frac{(z_1^3+b_2z_1^2+b_1z_1+b_0)^2}{(z_1-c)^3(z_1-t)},$$
$$c=\frac{(s+1)^4}{2^3s(s^2+1)}, \quad \rho=-\frac{2^6s^6}{(3s^2-1)(s^2+1)^3},$$
$$a=\frac{(3s-1)(s+1)^3}{2^4s^3}, \quad b_0=-\frac{(2s-1)(s+1)^8}{2^9s^6},$$
$$b_2=-\frac{(3s^4+12s^3+6s^2-1)}{2^3s^3}, \quad b_1=\frac{(15s^4+36s^3-22s^2+4s-1)(s+1)^4}{2^8s^6},$$

$$t=\frac{(s+1)^4(2s-1)^2}{8s^3(3s^2-1)},\quad y(t)=\frac{(3s-1)(2s-1)(s+1)^3}{4s(3s^2-1)(s^2+1)},$$
$$\hat{\theta}_0=\frac{1}{4},\quad \hat{\theta}_1=\frac{1}{4},\quad \hat{\theta}_t=\frac{1}{3},\quad \hat{\theta}_\infty=\frac{1}{3}.$$

Starting with this solution and using quadratic transformations and Okamoto transformation (see Appendix A), we can obtain algebraic solutions for the following $\hat{\theta}$-assemblies:

$$\left(\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{1}{4}\right),\quad \left(0,0,\frac{1}{12},\frac{7}{12}\right),\quad \left(\frac{1}{24},\frac{1}{24},\frac{5}{24},\frac{19}{24}\right),\quad \left(\frac{7}{24},\frac{7}{24},\frac{11}{24},\frac{13}{24}\right).$$

The solution corresponding to the second RS-transformation (3.2) is a renormalization of the function $z(z_1)$:

$$\tilde{z}=z(M^{-1}(\tilde{z}_1)),\quad \tilde{z}_1=M(z_1)\equiv\frac{(1-c)z_1}{z_1-c};\quad \tilde{t}=M(t),\quad \tilde{y}=M(y).$$

Redenoting back $\tilde{t}\mapsto t$ and $\tilde{y}\mapsto y$, we get

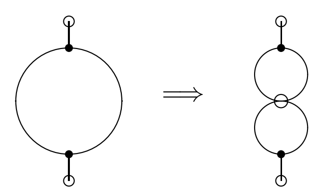
$$t=-\frac{(2s-1)^2}{8s},\quad y(t)=\frac{(s-1)(3s-1)(2s-1)}{12s(s^2+1)},$$
$$\hat{\theta}_0=\frac{1}{4},\quad \hat{\theta}_1=\frac{1}{4},\quad \hat{\theta}_t=\frac{1}{2},\quad \hat{\theta}_\infty=-\frac{1}{2}.$$

All kinds of known transformations are applicable to the general solution for the latter $\hat{\theta}$-assembly, which generates solutions for the following $\hat{\theta}$-assemblies:

$$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right),\quad \left(0,0,\frac{1}{2},0\right),\quad \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{4}\right),$$
$$\left(0,0,\frac{1}{4},\frac{3}{4}\right),\quad \left(\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{7}{8}\right),\quad \left(\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{5}{8}\right).$$

Care should be taken to check that this particular solution will not degenerate after application of some transformations; i.e., this solution is not mapped to 0, 1, or ∞.

4. LW**-split.** As a function of y, the function $z(z_1)$ has two branches. The second branch corresponds to the twist of the *dessin* that is a product of the folded reduced cube and the segment corresponding to the following composition of the types for the associated Belyĭ functions: $R(4{+}2|2{+}2{+}1{+}1|3{+}3)=R(\underset{\wedge}{2}{+}\underset{\wedge}{1}|2{+}1|3)\circ R(2|2)$ (see the picture). The type of the resulting *dessin* is also a composition of types of degrees 3 and 2, $R(4+1+1|2+2+1+1|3+3)=R(\underset{\wedge}{2}{+}1|2{+}1|3)\circ R(2)$. Therefore, the LW-split *dessin* generates precisely the same solution as that in item 1 (CC-join), and we refer again to §2 of [2].



The RS-transformation associated with the LW-split is

$$RS_4^2\left(\begin{matrix}1/4\\4{+}1{+}1\end{matrix}\;\middle|\;\begin{matrix}1/2\\2{+}2{+}1{+}1\end{matrix}\;\middle|\;\begin{matrix}1/3\\3{+}3\end{matrix}\right).$$

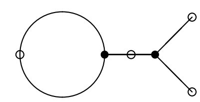
Therefore, this solution corresponds to the $\hat{\theta}$-assembly $(1/4,1/4,1/2,1/2)$. However, since the same solution can be produced by a simpler RS-transformation, it solves equation (2.3) for a more general $\hat{\theta}$-assembly. In fact, it is of interest to compare this solution with that for the CW-split (see the next item), because the latter has precisely the same RS-symbol. Therefore, below we examine this RS-transformation in detail. Our solution differs from the solution indicated in [2] by fractional linear transformations of y and t.

$$z=\frac{(z_1-a)^4z_1(z_1-t)}{(z_1-c_1)^3(z_1-c_2)^3},\quad z-1=-\frac{3(s^2{+}4s{+}1)^2}{s^2(s{+}2)^2}\frac{(z_1^2{+}b_1z_1{+}b_0)^2(z_1-1)}{(z_1{-}c_1)^3(z_1{-}c_2)^3},$$

$$a = \frac{(s^2+4s+1)}{(s+2)s}, \quad c_1 = \frac{(s^2+4s+1)}{(s+2)^2}, \quad c_2 = -\frac{(s^2+4s+1)}{3s^2},$$
$$b_0 = \frac{(s^2+4s+1)^2}{9s^2(s+2)^2}, \quad b_1 = -\frac{2(5s^2+2s-4)(s^2+4s+1)}{9s^2(s+2)^2},$$
$$t = \frac{(s^2-1)(s^2+4s+1)}{s^2(s+2)^2}, \quad y(t) = \frac{(s^2-1)}{(s+2)s},$$
$$\hat{\theta}_0 = \hat{\theta}_t \in \mathbb{C}, \hat{\theta}_1 = \hat{\theta}_\infty - 1 \in \mathbb{C}.$$

The algebraic equation for $y(t)$ is very simple: $(y-1)^2 = 1-t$.

5. ***CW*-split.** Precisely the same deformation as CW-split, as well as some deformations related to different branches of the same function $z(z_1)$, can also be obtained as a cross and joins of the dessin shown in the picture. The RS-symbol of the transformation associated with the CW-split coincides with that for the LW-split:



$$RS_4^2\left(\begin{array}{c|c|c} 1/4 & 1/2 & 1/3 \\ 4{+}1{+}1 & 2{+}2{+}1{+}1 & 3{+}3 \end{array}\right).$$

However, its R-part, i.e., the function $z(z_1)$ viewed as a function of y, is different! It is important to mention that now we keep the denominator of $z(z_1)$ in the nonfactorized form. If we factorize it, as in the previous item, then only an elliptic parametrization of the function $z(z_1)$ is possible.

$$z = \frac{(z_1-a)^4 z_1(z_1-t)}{(z_1^2+c_1z_1+c_0)^3}, \quad z-1 = \frac{3(s^2-2)^2(s^2-4s-2)^2}{16(s+1)^3(s-1)^3}\frac{(z_1^2+b_1z_1+b_0)^2(z_1-1)}{(z_1^2+c_1z_1+c_0)^3},$$

$$a = \frac{(s^2-2s+2)(s^2-2)}{4(s-1)^3}, \quad c_1 = -\frac{((s^2+2)^2-4s(s-2)^2)(s^2-2)}{12(s-1)^3(s+1)}, \quad c_0 = \frac{(s^2-2)^2(s^2+2)^2}{48(s+1)(s-1)^5},$$

$$b_0 = -\frac{(s^2-2)^2(s^2+2)^3}{144(s^2-4s-2)(s-1)^6}, \quad b_1 = -\frac{(s^2-2)(s^5-6s^4+10s^3-32s^2+12s-20)}{18(s^2-4s-2)(s-1)^3},$$

$$t = \frac{(s^2-2)(s^2+2)^3}{16(s+1)^3(s-1)^3}, \quad y(t) = -\frac{(s^2-2s+2)(s^2+2)^2}{4(s+1)(s^2-4s-2)(s-1)^2},$$

$$\hat{\theta}_0 = \hat{\theta}_t = \frac{1}{4}, \quad \hat{\theta}_1 = \hat{\theta}_\infty = \frac{1}{2}.$$

We can further apply to this solution a quadratic transformation, namely, the inverse to that in example 3 of Appendix A. As a result, we get a new solution, again denoted by $y(t)$:

$$t = \frac{1}{2} + \frac{(s^8-28s^6+96s^4-112s^2+16)}{16s\left(\sqrt{(1-s^2)(s^2-4)}\right)^3},$$
$$y(t) = \frac{1}{2} + \frac{s(7s^4-44s^2+28)}{2((s^2-2)^2-16s^2)\sqrt{(1-s^2)(s^2-4)}},$$
$$\hat{\theta}_0 = 0, \quad \hat{\theta}_1 = 0, \quad \hat{\theta}_t = \frac{1}{2}, \quad \hat{\theta}_\infty = 0.$$

Due to the obvious symmetry, we can use the Zhukovskii transformation to parametrize this solution rationally:

$$\frac{s^2}{2} + \frac{2}{s^2} = \frac{5-s_1^2}{2}: \qquad t = \frac{(s_1-1)(s_1+3)^3}{16s_1^3}, \quad y(t) = \frac{(s_1+1)(s_1+3)^2}{2s_1(15+s_1^2)}.$$

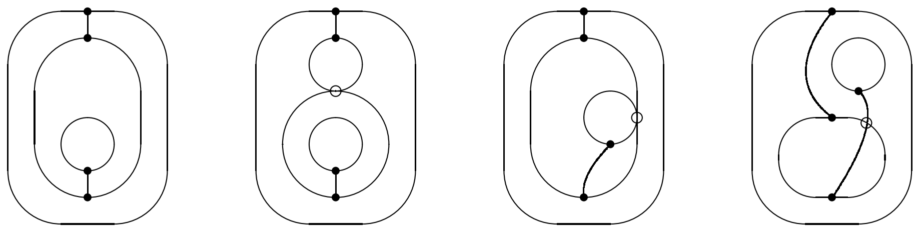
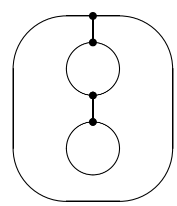


FIGURE 2. Some face deformations of the *dessin* for the Belyĭ function of the reduced dodecahedron $R(5{+}5{+}1{+}1|2{+}2{+}2{+}2{+}2{+}2|3{+}3{+}3{+}3)$.

This solution can be mapped further to produce the tetrahedron solution considered in Subsection 3.1. The $\hat{\theta}$-assembly for the LW-split solution (see the preceding item) is such that one can also try to apply to it the same quadratic transformation; however, this is one of the exceptional solutions for which this transformation fails: this solution is taken to the critical values 0 or 1 of the sixth Painlevé equation (2.3), depending on the choice of a branch.

3.4. Deformations of the reduced dodecahedron (icosahedron solutions). The reduced icosahedron and its dual solid (reduced dodecahedron), together with their Belyĭ functions were introduced by Magot and Zvonkin in [15] in connection with the study of the Belyĭ functions of Archimedean solids. Here we consider only face deformations of the corresponding *dessins*. They are indicated in Figure 2.

Actually, the last *dessin* in Figure 2 is the join of the dessin for the Belyĭ function of the type $R(8+2+1+1|\underbrace{2+\cdots+2}_{6}|\underbrace{3+\cdots+3}_{4})$ (see the picture), rather than the cross of reduced dodecahedron. However, we consider it here, because it is of interest to get a solution for the $\hat{\theta}$-assembly proportional to 1/7.



1. Twist. This deformation produces both the icosahedron and the great icosahedron solutions of [6] and also yet another solution related to the tetrahedron solution of [6].

$$z = \frac{3^3(s^2+3)^5(s^2-5)^5(s^2+4s-1)^5(s^2-4s-1)}{2^{10}(s+3)^{12}(s-1)^{20}}$$

$$\times \frac{(z_1-a)^5 z_1(z_1-1)(z_1-t)}{(z_1^4+c_3z_1^3+c_2z_1^2+c_1z_1+c_0)^3},$$

$$a = \frac{2^4(s^2-5)}{(s-1)(s^2+3)(s+3)^3},$$

$$c_3 = -\frac{(s^2-5)(s^6+4s^5-3s^4-8s^3+115s^2-60s+15)}{(s-1)^5(s+3)^3},$$

$$c_2 = \frac{(s^2-5)^2}{2^4(s-1)^{10}(s+3)^6}$$

$$\times \big(s^{12}+8s^{11}+10s^{10}-40s^9+1135s^8+3408s^7-10036s^6$$

$$-14160s^5+71055s^4-78040s^3+39050s^2-9480s+1185\big),$$

$$c_0 = \frac{2^8 s^2 (s^2-5)^4}{(s+3)^{10}(s-1)^{10}},$$

$$c_1 = \frac{8(s^2-5)^3(15-105s+525s^2-705s^3+107s^4+461s^5+183s^6+29s^7+2s^8)}{(s-1)^{10}(s+3)^9}.$$

Consider the following RS-symbol:

$$RS_4^2\left(\begin{array}{c|c|c} 1/5 & 1/2 & 1/3 \\ 5+4+1+1+1 & \underbrace{2+\cdots+2}_{6} & \underbrace{3+\cdots+3}_{4} \end{array}\right).$$

The corresponding solution found with the help of Theorem 2.1 looks like this:

$$t = -\frac{2^8 s^3 (s^2-5)}{(s-1)^5(s+3)^3(s^2-4s-1)},$$
$$y(t) = -\frac{2^6 s^2}{(s-1)(s+3)(s^2+3)(s^2-4s-1)},$$
$$\hat{\theta}_0 = \hat{\theta}_1 = \hat{\theta}_t = \hat{\theta}_\infty = \frac{1}{5}.$$

By the Okamoto transformation (see Appendix A), this solution can be mapped to the solution for the $\hat{\theta}$-assembly $(0,0,0,2/5)$ and then, by a Bäcklund transformation, to the great icosahedron solution of [6] corresponding to the $\hat{\theta}$-assembly $(0,0,0,-2/5)$.

Another RS-transformation associated with this deformation is

$$RS_4^2\left(\begin{array}{c|c|c} 2/5 & 1/2 & 1/3 \\ 5+4+1+1+1 & \underbrace{2+\cdots+2}_{6} & \underbrace{3+\cdots+3}_{4} \end{array}\right).$$

To find explicit formulas for the corresponding solution of the sixth Painlevé equation, we need to find the S-part of this RS-transformation, as in the examples in [2]. The corresponding $\hat{\theta}$-assembly is $(2/5, 2/5, 2/5, 2/5)$. By the same transformations as above, this solution can be mapped to the icosahedron solution of [6] corresponding to the $\hat{\theta}$-assembly $(0,0,0,-4/5)$.

We can construct yet another RS-transformation related to the symbol

$$RS_4^2\left(\begin{array}{c|c|c} 1/4 & 1/2 & 1/3 \\ 5+4+1+1+1 & \underbrace{2+\cdots+2}_{6} & \underbrace{3+\cdots+3}_{4} \end{array}\right). \tag{3.3}$$

For this purpose, we need to rearrange the normalization of the function $z(z_1)$, i.e., to use the function $\tilde{z}(\tilde{z}_1) = z\left(M^{-1}(\tilde{z}_1)\right)$, where the fractional linear transformation M is defined by $\tilde{z}_1 \equiv M(z_1) = \frac{(1-a)z_1}{z_1-a}$. Then the function $\tilde{y}(\tilde{t})$, where $\tilde{t} = M(t)$ and $\tilde{y} = M(y)$, is a new solution of the sixth Painlevé equation (2.3). Omitting the sign $\sim$ in the notation of this new solution, we obtain

$$t = \frac{2^4 s^3}{(s+3)^3(s-1)}, \quad y(t) = \frac{4(s^2+4s-1)s^2}{5(s+3)(s^2+3)(s-1)},$$
$$\hat{\theta}_0 = \hat{\theta}_1 = \hat{\theta}_t = \frac{1}{4}, \quad \hat{\theta}_\infty = -\frac{1}{4}.$$

The Okamoto transformation reshapes this solution to that for the $\hat{\theta}$-assembly $(0,0,0,\frac{1}{2})$, and in accordance with [6], the latter must coincide with the tetrahedron solution. There is one more solution associated with this deformation *dessin* and corresponding to the RS-symbol (3.3) with $1/2$ in place of $1/4$ in the first row of the first box. To get an explicit formula for this solution, we must again construct the S-part of the latter RS-transformation explicitly. We only mention the corresponding $\hat{\theta}$-assembly: $\hat{\theta}_0 = \hat{\theta}_1 = \hat{\theta}_t = \hat{\theta}_\infty = 1/2$.

2. Join. This *dessin* also produces several algebraic solutions of the sixth Painlevé equation. The first transformation is generated by the symbol $RS_4^2\left(\begin{array}{c|c|c} 1/5 & 1/2 & 1/3 \\ 5+3+2+1+1 & \underbrace{2+\cdots+2}_{6} & \underbrace{3+\cdots+3}_{4} \end{array}\right)$, whose R-part looks like this:

$$z = -\frac{2^5 3^3 (s^2-5)^5}{(s+3)^9(s-2)^6} \frac{(z_1-a)^5(z_1-1)^2 z_1(z_1-t)}{(z_1^4+c_3z_1^3+c_2z_1^2+c_1z_1+c_0)^3}, \quad a = \frac{(s-1)(s^2-5)}{2^3(s-2)^2},$$

$$c_0 = \frac{s^2(s^2-5)^4}{2^4(s+3)^4(s-2)^6}, \quad c_1 = -\frac{(8s^4-35s^3+65s^2-75s+45)(s^2-5)^3}{4(s+3)^4(s-2)^6},$$

$$c_2 = \frac{5(s-1)(2s^3+2s^2-3s-9)(s^2-5)^2}{2(s-2)^4(s+3)^4},$$

$$c_3 = -\frac{2(s^2-5)(2s^3+5s^2-15)}{(s+3)^3(s-2)^2}.$$

Then Theorem 2.1 provides the corresponding solution,

$$t = \frac{2s^3(s^2-5)}{(s-2)^2(s+3)^3}, \quad y(t) = \frac{s^2(s-1)}{3(s-2)(s+3)}, \tag{3.4}$$
$$\hat{\theta}_0 = \frac{1}{5}, \quad \hat{\theta}_1 = \frac{2}{5}, \quad \hat{\theta}_t = \frac{1}{5}, \quad \hat{\theta}_\infty = \frac{2}{5}.$$

With this R-part we can associate another RS-transformation by choosing $2/5$ instead of $1/5$ in the first box of the RS-symbol written above. However, to find the corresponding solution of the sixth Painlevé equation, we need to construct the S-part of this transformation explicitly. The $\hat{\theta}$-assembly for the latter solution is $(6/5, 4/5, 2/5, 2/5)$. By some transformations, this solution can be mapped to that corresponding to precisely the same $\hat{\theta}$-assembly as for the first solution $y(t)$. The natural question as to whether these solutions are different requires further investigation. It should be noted that the Okamoto transformation maps the solution (3.4) to a solution for the $\hat{\theta}$-assembly $(0, 0, 1/5, 3/5)$. By the quadratic transformation given in item 3 of Appendix A, the latter solution can be further reshaped to solutions for the following $\hat{\theta}$-assemblies:

$$\left(\frac{1}{10}, -\frac{1}{5}, \frac{1}{10}, \frac{4}{5}\right) \quad \text{and} \quad \left(\frac{3}{10}, -\frac{2}{5}, \frac{3}{10}, \frac{3}{5}\right).$$

Another transformation associated with this join is determined by the symbol

$$RS_4^2\left(\begin{array}{c|c|c} 1/3 & 1/2 & 1/3 \\ 5+3+2+1+1 & \underbrace{2+\cdots+2}_{6} & \underbrace{3+\cdots+3}_{4} \end{array}\right).$$

Its R-part $\tilde{z}(\tilde{z}_1)$ is obtained from the R-part of the first RS-transformation by the following renormalizing fractional linear transformation:

$$\tilde{z}(\tilde{z}_1) = z\left(M^{-1}(\tilde{z}_1)\right), \quad \tilde{z}_1 = M(z_1) \equiv \frac{(1-a)z_1}{z_1-a}.$$

The solution is $\tilde{y}(\tilde{t})$, where $\tilde{t} = M(t)$ and $\tilde{y} = M(y(t))$, with t and $y(t)$ defined by equations (3.4). As usual, omitting the sign $\sim$, we have the following explicit parametric form for $\tilde{y}(\tilde{t})$:

$$t = \frac{2s^3}{(s+1)(s-2)^2}, \quad y(t) = -\frac{(s-3)s^2}{5(s+1)(s-2)},$$
$$\hat{\theta}_0 = \frac{1}{3}, \quad \hat{\theta}_1 = \frac{2}{3}, \quad \hat{\theta}_t = \frac{1}{3}, \quad \hat{\theta}_\infty = -\frac{2}{3}.$$

Finally, there is yet another normalization of the function $z(z_1)$, which produces the RS-transformation

$$RS_4^2\left(\begin{array}{c|c|c} 1/2 & 1/2 & 1/3 \\ 5+3+2+1+1 & \underbrace{2+\cdots+2}_{6} & \underbrace{3+\cdots+3}_{4} \end{array}\right).$$

The normalization transformation is

$$\tilde{z}(\tilde{z}_1) = z\left(M^{-1}(\tilde{z}_1)\right), \quad \tilde{z}_1 = M(z_1) \equiv \frac{z_1}{z_1 - a}.$$

The corresponding solution of the sixth Painlevé equation (2.3) is given in terms of this fractional linear transformation M, and the solution $y(t)$ (see (3.4)) is given precisely by the same formulas as those for $\tilde{y}(\tilde{t})$ in the preceding paragraph,

$$t = -\frac{16s^3}{(s+1)(s-3)^3}, \quad y = \frac{8s^2(s-2)}{5(s+1)(s-3)^2},$$
$$\hat{\theta}_0 = \frac{1}{2}, \quad \hat{\theta}_1 = \frac{3}{2}, \quad \hat{\theta}_t = \frac{1}{2}, \quad \hat{\theta}_\infty = -\frac{3}{2}.$$

3. Cross. With this *dessin* we can associate four RS-transformations. The first is $RS_4^2\left(\begin{array}{c|c|c} 1/7 & 1/2 & 1/3 \\ 7+2+1+1+1 & \underbrace{2+\cdots+2}_{6} & \underbrace{3+\cdots+3}_{4} \end{array}\right)$. Its R-part and the corresponding algebraic solution are as follows:

$$z = -\frac{3^3(3s^2+1)^7}{(7s^2+1)^4}\frac{(z_1-a)^7 z_1(z_1-1)(z_1-t)}{(z_1^4+c_3z_1^3+c_2z_1^2+c_1z_1+c_0)^3},$$
$$a = -\frac{(s-1)(2s^2+s+1)^2}{2(3s^2+1)},$$
$$c_3 = \frac{2(s-1)(57s^6+57s^5+71s^4+22s^3+15s^2+s+1)}{(7s^2+1)^2},$$
$$c_2 = -\frac{(42s^6-42s^5+161s^4-44s^3+16s^2-10s+5)(2s^2+s+1)^2}{2(7s^2+1)^2},$$
$$c_1 = -\frac{7(s-1)(3s^4-3s^3+6s^2-3s+1)(2s^2+s+1)^4}{2(7s^2+1)^2},$$
$$c_0 = \frac{(3s-1)^2(2s^2+s+1)^6}{16(7s^2+1)^2},$$

$$t = -\frac{(2s^2+s+1)^2(3s-1)^3}{2(7s^2+1)^2}, \quad y(t) = -\frac{(s-1)(2s^2+s+1)(3s-1)^2}{2(7s^2+1)(3s^2+1)}, \tag{3.5}$$
$$\hat{\theta}_0 = \hat{\theta}_1 = \hat{\theta}_t = \frac{1}{7}, \quad \hat{\theta}_\infty = \frac{5}{7}.$$

Two other RS-transformations are based on the same R-part, but the first box of their RS-symbol contains $2/7$ and $3/7$ instead of $1/7$. To get explicit formulas for the corresponding solutions, we need to construct the S-part of these transformations. Here we only mention that these solutions correspond to the following $\hat{\theta}$-assemblies:

$$\left(\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{4}{7}\right) \quad \text{and} \quad \left(\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{1}{7}\right).$$

Remark 3.1. It is of interest to note that if we apply the Okamoto transformation and some other transformations to the solution (3.5), we can get algebraic solutions for both $\hat{\theta}$-assemblies written above! However, it should be checked whether the solutions constructed via the Okamoto transformation and RS-transformations coincide.

The fourth RS-transformation is

$$RS_4^2\left(\begin{array}{c|c|c} 1/2 & 1/2 & 1/3 \\ 7+2+1+1+1 & \underbrace{2+\cdots+2}_{6} & \underbrace{3+\cdots+3}_{4} \end{array}\right).$$

Its R-part is the following renormalization $\tilde{z}(\tilde{z}_1)$ of the function $z(z_1)$:

$$\tilde{z}(\tilde{z}_1) = z\left(M^{-1}(\tilde{z}_1)\right), \quad \tilde{z}_1 = M(z_1) \equiv \frac{(1-a)z_1}{z_1-a}.$$

The new solution $\tilde{y}(\tilde{t})$ is given by the usual formulas $\tilde{t} = M(t)$, $\tilde{y} = M(y(t))$, where t and $y(t)$ are given by equations (3.5). The explicit form for $\tilde{y}(\tilde{t})$ (we omit the sign $\sim$) looks like this:

$$t = \frac{(3s-1)^3(s+1)}{16s}, \quad y(t) = -\frac{(2s^2-s+1)(3s-1)^2}{14s(3s^2+1)},$$
$$\hat{\theta}_0 = \hat{\theta}_1 = \hat{\theta}_t = \frac{1}{2}, \quad \hat{\theta}_\infty = -\frac{5}{2}.$$

§4. The Schwarz cluster

We recall the Euler equation for the Gauss hypergeometric function:

$$z(1-z)\frac{d^2u}{dz^2} + (c-(a+b+1)z)\frac{du}{dz} - abu = 0. \tag{4.1}$$

We put

$$\lambda = 1-c, \quad \mu = b-a, \quad \nu = c-a-b,$$

and on the set of triplets (λ,μ,ν) we introduce the following equivalence relation: two triplets are equivalent if and only if one of them can be transformed into the other by permutation and a transformation of the form

$$\lambda \to l \pm \lambda, \quad \mu \to m \pm \mu, \quad \nu \to n \pm \nu,$$

where the integers l, m, n are such that $l+m+n$ is an even number. If a triplet is equivalent to some triplet satisfying $\lambda+\mu+\nu = 1$, then it is said to be degenerate. In the degenerate case one independent solution is an elementary function, and the other can be expressed in terms of elementary functions and incomplete Beta functions by application of a finite sequence of simple transformations.

In [19], H. A. Schwarz proved that in the nondegenerate case the general solution of equation (4.1) is an algebraic function if and only if the corresponding parameters λ, μ, ν can be reduced to one of the fifteen cases listed in Table 1.

In Table 1, the integers p and n are such that $2p \leq n$. In what follows, instead of the scalar form of the hypergeometric equation (4.1), we refer to its matrix form

$$\frac{d}{dz}\Psi = \left(\frac{A_0}{z} + \frac{A_1}{z-1}\right)\Psi, \tag{4.2}$$

TABLE 1. The Schwarz list.

N	λ	μ	ν	N	λ	μ	ν	N	λ	μ	ν	N	λ	μ	ν	N	λ	μ	ν
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{p}{n}$	4	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	7	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3}$	10	$\frac{3}{5}$	$\frac{1}{3}$	$\frac{1}{5}$	13	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	5	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	8	$\frac{2}{3}$	$\frac{1}{5}$	$\frac{1}{5}$	11	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	14	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$
3	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	6	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	9	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{5}$	12	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	15	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{3}$

where A_0 and $A_1 \in sl_2(\mathbb{C})$. We also assume that $A_0 + A_1 = -\frac{\theta_\infty}{2}\sigma_3$, with $\theta_\infty \in \mathbb{C}$ and $\sigma_3 = \operatorname{diag}\{1, -1\}$. Due to these conditions and the freedom in normalization, $\Psi \mapsto \exp\{c\sigma_3\}\Psi$, $c \in \mathbb{C}$, the matrices A_0 and A_1 can be parametrized by the corresponding formal monodromies θ_0, θ_1, and θ_∞, where $\pm\theta_k/2$ are the eigenvalues of the matrices A_k. Here we assume that $\theta_\infty \neq 0$ and that A_0 *is not a diagonal matrix*. Under these assumptions, the fundamental solution Ψ can be represented in terms of independent solutions of equation (4.1) for the parameters

$$\lambda = \theta_\infty - 1, \quad \mu = \theta_0, \quad \nu = \theta_1. \tag{4.3}$$

The parametrization mentioned above is not used here, and we do not write it out; it can be found, e.g., in [1]. In the sequel, instead of the triplets (λ, μ, ν), we always use θ-triplets $(\theta_0, \theta_1, \theta_\infty)$ (note -1 in equations (4.3)). On the set of θ-triplets, we consider the same equivalence relation as for the (λ, μ, ν)-triplets, and we make no distinction between the triplets belonging to one and the same equivalence class.

In the matrix framework, the degenerate case of equation (4.1) is described in the following way. Suppose the matrix A_0 (and, hence, A_1) is triangular, but not diagonal. The lower triangular case corresponds to $a = 0$, and the upper triangular case corresponds to $b = 0$ in the parametrization considered in [1]. The triangular structure implies the following relation for a θ-triplet: $\theta_0 + \theta_1 + \theta_\infty = 0$. The general degenerate case is obtained from one of the triangular cases mentioned above by application of finitely many Schlesinger transformations. These transformations change a θ-triplet so that they satisfy $\theta_0+\theta_1+\theta_\infty = 2k$ with some $k \in \mathbb{Z}$. If $k \neq 0$, then the corresponding equation (4.2) fails to have a triangular structure; however, its monodromy group remains isomorphic to the group of the triangular equation, and, in particular, has the same triangular representation under a proper normalization. In the degenerate case, the general solution can also be algebraic in an infinite number of cases, which can be deduced from some cases where the incomplete Beta function is algebraic.

Under our conditions, equation (4.2) has the three singular points 0, 1, and ∞.

Theorem 4.1. *All equations* (4.2) *corresponding to the parameters on the Schwarz list* (*Table* 1) *can be obtained as inverse RS-transformations or compositions of RS-transformations and inverse RS-transformations of the* (2×2)*-matrix Fuchsian ODE with two singular points*,

$$\frac{d}{dz}\Phi = \frac{A}{z}\Phi, \tag{4.4}$$

where $A \in sl_2(\mathbb{C})$ has rational eigenvalues. Moreover, the R-parts of the RS-transformations are Belyĭ functions.

Proof of Theorem 4.1. We split the 15 cases of Table 1 into two sets; the first contains all cases with $\lambda = 1/2$ except case 9, and the second contains all the other cases. In Proposition 4.1 below, we show how to construct RS-transformations taking equation (4.2) to equation (4.4). This proves that every equation (4.2) corresponding to the first set is RS-inverse to equation (4.4). In Proposition 4.2 we shall prove that there exist RS-transformations that take equations (4.2) corresponding to the first set to equations (4.2) for the second set. □

Remark 4.1. The fundamental solution of equation (4.4) is $\Phi = Gz^{r\sigma_3}C$, where G, $C \in SL(2,\mathbb{C})$ and r is a rational number. All RS-transformations occurring in Propositions 4.1–4.3 can be constructed explicitly; therefore, in fact, our proof provides an explicit construction of all general algebraic solutions of equation (4.1).

Proposition 4.1. *If a θ-triplet that determines equation* (4.2) *coincides with one of the θ-triplets corresponding to the rows of Table* 1 *with $\lambda = 1/2$ except row* 9 (*i.e., it corresponds to rows* 1, 2, 4, 6, *and* 14), *then there exists an RS-transformation such that its R-part is a Belyĭ function and it maps equation* (4.2) *to the Fuchsian equation* (4.4).

Proof. Below we consider each of the five cases of equation (4.2) in the corresponding item.

1. Case 1. Actually, the general solution of equation (4.1) for the triplet $(1/2, \theta_1, 1/2)$ with an arbitrary $\theta_1 \in \mathbb{C}$ is an elementary function. This function is algebraic for rational values of θ_1. These facts can be observed by application of the transformation $RS_2^2\left(\begin{array}{c|c|c} 1/2 & \theta_1 & 1/2 \\ 2 & 1+1 & 2 \end{array}\right)$ with the R-part $z = z_1^2$, which maps equation (4.2) to equation (4.4).
2. Case 2. Here we use the transformation
$$RS_2^2\left(\begin{array}{c|c|c} 1/3 & 1/2 & 1/3 \\ 3+1 & 2+2 & 3+1 \end{array}\right),$$
with the Belyĭ function $z = -\frac{64(z_1+1)^3}{(z_1(z_1-8))^3}$ corresponding to the reduced tetrahedron (see [15]).
3. Case 4. The required transformation is
$$RS_2^2\left(\begin{array}{c|c|c} 1/4 & 1/2 & 1/3 \\ 4+1+1 & 2+2+2 & 3+3 \end{array}\right),$$
with the Belyĭ function $z = -\frac{108(z_1+1)^4 z_1}{(z_1^2-14z_1+1)^3}$ corresponding to the reduced cube (see [15]).
4. Cases 6 and 14. The transformations are
$$RS_2^2\left(\begin{array}{c|c|c} 1/3 & 1/2 & 1/5 \\ \underbrace{3+\cdots+3}_{4} & \underbrace{2+\cdots+2}_{6} & 5+5+1+1 \end{array}\right)$$
and
$$RS_2^2\left(\begin{array}{c|c|c} 1/3 & 1/2 & 2/5 \\ \underbrace{3+\cdots+3}_{4} & \underbrace{2+\cdots+2}_{6} & 5+5+1+1 \end{array}\right),$$
with the Belyĭ function of the reduced icosahedron [15]:
$$z = \frac{(z_1^4 + 228z_1^3 + 494z_1^2 - 228z_1 + 1)^3}{1728z_1(z_1^2 - 11z_1 - 1)^5}.$$ □

Proposition 4.2. *Fundamental solutions of equation* (4.2) *corresponding to the θ-triplets for the rows* 3, 5, 7, 8, 9, 10, 11, 12, 13, *and* 15 *of Table* 1 *can be constructed as RS-transformations of the fundamental solutions of equation* (4.2) *corresponding to the θ-triplets of the remaining rows of Table* 1, *i.e., all rows with $\lambda = 1/2$ different from the* 9*th row. Moreover, the R-parts of these transformations are Belyĭ functions.*

Proof. At the end of Subsection 6.2 in [1] it was shown that there are quadratic, cubic, and sextic RS-transformations that map equation (4.2) corresponding to row 6 of Table 1 into those for the rows 7, 8, 9, 11, 12, and 13. At the end of Subsection 6.3 of the same paper it was explained that, by using quadratic, cubic, and sextic RS-transformations, one can also map "case" 14 of Table 1 to cases 7, 9, 11, 12, 13, and 15. To complete the proof, it remains to show how to obtain cases 3, 5, and 10.

To get case 3, consider the quartic transformation

$$RS_3^2\left(\begin{array}{c|c|c} \theta_0 & 1/2 & 1/4 \\ 2+1+1 & 2+2 & 4 \end{array}\right),$$

where θ_0 is arbitrary. An explicit form of the R-part of this transformation, which is a Belyĭ function, can be found in [2, 4.2.1.A]. This RS-transformation can also be obtained by the composition of two quadratic transformations (cf. [1]). The θ-triplet of the resulting equation (4.2) is $(\theta_0, \theta_0, 2\theta_0 - 1)$. Thus, choosing $\theta_0 = 1/3$, we obtain case 3 from case 4.

The Belyĭ function that allows us to build an RS-transformation taking case 6 of Table 1 to case 10 is of the type

$$RS_3^2\left(\begin{array}{c|c|c} 1/3 & 1/2 & 1/5 \\ 3+3+2 & \underbrace{2+\cdots+2}_{4} & 5+2+1 \end{array}\right);$$

explicitly, it reads

$$\begin{aligned} z &= -\frac{(1000z_1^2 - 1728z_1 + 729)^3}{64(z_1-1)z_1^2(25z_1-27)^5}, \\ z - 1 &= -\frac{(25000z_1^4 - 80000z_1^3 + 105300z_1^2 - 69984z_1 + 19683)^2}{64(z_1-1)z_1^2(25z_1-27)^5}. \end{aligned} \tag{4.5}$$

This is the dual function for the first *dessin* in Figure 1.

The transformation

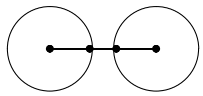
$$RS_3^2\left(\begin{array}{c|c|c} 1/3 & 1/2 & 1/4 \\ 4+3+3 & \underbrace{2+\cdots+2}_{5} & 4+4+1+1 \end{array}\right)$$

takes case 4 to case 5. Its R-part is the following Belyĭ function:

$$\begin{aligned} z &= \frac{(320z_1^2 - 320z_1 - 1)^3}{4z_1(z_1-1)(128z_1^2 - 128z_1 + 5)^4}, \\ z - 1 &= -\frac{(2z_1-1)^2(16384z_1^4 - 32768z_1^3 + 15616z_1^2 + 768z_1 - 1)^2}{4z_1(z_1-1)(128z_1^2 - 128z_1 + 5)^4}. \end{aligned} \tag{4.6}$$

□

Remark 4.2. The Belyĭ function (4.5) is easy to find with the help of Maple 8. Although at first glance the function (4.6) is only a little more complicated than (4.5), the author was not able to get it by analyzing the corresponding system of algebraic equations (see Remark 2.1), as was done when we found the function (4.5). Observe that the factor $(2z_1 - 1)$ in the second equation in (4.6) is not *a priori* obvious, because the *Anzats* for its numerator is a square of a general polynomial of the fifth order with indeterminate coefficients. Moreover, there is a simpler *Ansatz* for a function of degree 8 with a proper number of parameters, which seems to produce a simpler Belyĭ function. It is Grothendieck's theory of "*dessins d'enfants*" that helped to find the correct degree and symmetry of the Belyĭ function (see the explanation in Figure 3).



$$\Longrightarrow z = \frac{500}{729}\frac{(z_1^2-6/5)^3}{(z_1^2-1)^4(z_1^2-32/27)},\ z-1 = -\frac{z_1^2(z_1^4-70/27z_1^2+5/3)^2}{(z_1^2-1)^4(z_1^2-32/27)}.$$

FIGURE 3. The dessin for the Belyĭ function of the type $R(4+3+3|2+2+2+2+2|4+4+1+1)$.

Remark 4.3. Of course, there are some other transformations that map equations (4.2) corresponding to different cases of Table 1 to each other, including some autotransformations. In Proposition 4.3 we give an example of a transformation taking case 2 to case 3. Due to the presence of an arbitrary parameter s, this transformation allows us to get a fundamental solution of equation (4.2) for case 3 in terms of the corresponding Schlesinger transformation without using the explicit form of the initial fundamental solution of equation (4.2) for case 2.

Proposition 4.3. *The transformation*

$$RS_3^2\left(\begin{array}{c|c|c} 1/3 & 1/2 & 1/3 \\ 3+1+1+1 & 2+2+2 & 3+3 \end{array}\right)$$

maps equation (4.2) *corresponding to the third row of Table* 1 *to equation* (4.2) *for the second row.*

Proof. For the proof, it suffices to present a rational function of the type $R(3+1+1+1|2+2+2|3+3)$, which is given by (3.1). □

§5. AN IRREDUCIBLE OCTIC TRANSFORMATION

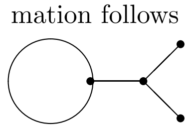
In [1] it was indicated that the Belyĭ function of the type $R(7+1|2+2+2+2|3+3+1+1)$, i.e.,

$$\begin{aligned} z &= \frac{\rho z_1(z_1-a)^7}{(z_1-1)(z_1-c_1)^3(z_1-c_2)^3}, \\ z-1 &= \frac{\rho(z_1-b_1)^2(z_1-b_2)^2(z_1-b_3)^2(z_1-b_4)^2}{(z_1-1)(z_1-c_1)^3(z_1-c_2)^3}, \end{aligned} \tag{5.1}$$

determines three irreducible octic transformations of the hypergeometric function; in terms of θ-triplets they are as follows:

$$\left(\frac{1}{2},\frac{1}{3},\frac{1}{7}\right) \to \left(\frac{1}{3},\frac{2}{3},\frac{1}{7}\right), \quad \left(\frac{1}{2},\frac{1}{3},\frac{2}{7}\right) \to \left(\frac{1}{3},\frac{1}{3},\frac{2}{7}\right), \quad \left(\frac{1}{2},\frac{1}{3},\frac{3}{7}\right) \to \left(\frac{1}{3},\frac{2}{3},\frac{3}{7}\right). \tag{5.2}$$

In [1], the function (5.1) was not identified as a Belyĭ function; therefore, strictly speaking, its existence was not established. However, there the authors were able to calculate its coefficients numerically with fairly high accuracy. Now, the existence of this transformation follows from the *dessin* presented in the following picture.



Moreover, Maple 8 and a substantially better computer than the one used in calculations for [1] allowed us to find them explicitly. To present the result, we rewrite the function (5.1) in the following form:

$$z = \frac{\rho z_1(z_1-a)^7}{(z_1-1)(z_1^2+\hat{c}_1z_1+\hat{c}_0)^3}, \quad z-1 = \frac{\rho(z_1^4+\hat{b}_3z_1^3+\hat{b}_2z_1^2+\hat{b}_1z_1+\hat{b}_0)^2}{(z_1-1)(z_1^2+\hat{c}_1z_1+\hat{c}_0)^3}, \tag{5.3}$$

TABLE 2. Three octic clusters.

The first octic cluster															
N	θ_0	θ_1	θ_∞	N	θ_0	θ_1	θ_∞	N	θ_0	θ_1	θ_∞	N	θ_0	θ_1	θ_∞
1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{7}$	4	$\frac{1}{2}$	$\frac{1}{7}$	$\frac{1}{14}$	7	$\frac{1}{3}$	$\frac{1}{7}$	$\frac{1}{7}$	10	$\frac{6}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{14}$	5	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{7}$	8	$\frac{1}{3}$	$\frac{1}{14}$	$\frac{1}{14}$	11	$\frac{5}{7}$	$\frac{2}{7}$	$\frac{2}{7}$
3	$\frac{1}{2}$	$\frac{2}{7}$	$\frac{1}{7}$	6	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{6}{7}$	9	$\frac{3}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	12	$\frac{5}{7}$	$\frac{1}{14}$	$\frac{1}{14}$
The second octic cluster															
N	θ_0	θ_1	θ_∞	N	θ_0	θ_1	θ_∞	N	θ_0	θ_1	θ_∞	N	θ_0	θ_1	θ_∞
1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{7}$	4	$\frac{1}{2}$	$\frac{2}{7}$	$\frac{5}{14}$	7	$\frac{1}{3}$	$\frac{2}{7}$	$\frac{2}{7}$	10	$\frac{3}{7}$	$\frac{5}{14}$	$\frac{5}{14}$
2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{5}{14}$	5	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{3}{7}$	8	$\frac{1}{3}$	$\frac{5}{14}$	$\frac{5}{14}$	11	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{2}{7}$
3	$\frac{1}{2}$	$\frac{2}{7}$	$\frac{3}{7}$	6	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{7}$	9	$\frac{3}{7}$	$\frac{3}{7}$	$\frac{3}{7}$	12	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{1}{7}$
The third octic cluster															
N	θ_0	θ_1	θ_∞	N	θ_0	θ_1	θ_∞	N	θ_0	θ_1	θ_∞	N	θ_0	θ_1	θ_∞
1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{3}{7}$	4	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{1}{7}$	7	$\frac{1}{3}$	$\frac{3}{7}$	$\frac{3}{7}$	10	$\frac{4}{7}$	$\frac{3}{7}$	$\frac{2}{7}$
2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{3}{14}$	5	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{4}{7}$	8	$\frac{1}{3}$	$\frac{3}{14}$	$\frac{3}{14}$	11	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
3	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{3}{14}$	6	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{7}$	9	$\frac{4}{7}$	$\frac{3}{7}$	$\frac{3}{7}$	12	$\frac{1}{7}$	$\frac{3}{14}$	$\frac{3}{14}$

where

$$\rho = \frac{1 - i3\sqrt{3}}{112}, \qquad a = \frac{27 - i39\sqrt{3}}{98},$$
$$\hat{c}_0 = \frac{-5697 + i2349\sqrt{3}}{268912}, \quad \hat{c}_1 = -\frac{513 + i435\sqrt{3}}{784}, \tag{5.4}$$

$$\hat{b}_0 = \frac{60507 + i142803\sqrt{3}}{13176688}, \quad \hat{b}_1 = \frac{249399 - i38313\sqrt{3}}{134456},$$
$$\hat{b}_2 = \frac{-4293 + i28251\sqrt{3}}{5488}, \quad \hat{b}_3 = -\frac{83 + i129\sqrt{3}}{28}. \tag{5.5}$$

Of course, it is straightforward to find explicit formulas for the roots b_1, b_2, b_3, b_4 and c_1, c_2. We omit these formulas because for b_k, $k = 1, 2, 3, 4$, they are very bulky. Moreover, the corresponding RS-transformations are actually symmetric functions of these roots; i.e., they can be expressed in terms of the coefficients (5.4) and (5.5).

Together with the quadratic and cubic transformations and their inverses (cf. [10]), each of the octic transformations (5.2) generates a cluster of hypergeometric functions that have the same type of transcendency. In Table 2 we present these clusters in terms of the corresponding θ-triplets.

Note that the first cluster contains the hypergeometric functions corresponding to the triplets $\left(\frac{5}{7}, \frac{5}{7}, \frac{5}{7}\right)$ and $\left(\frac{6}{7}, \frac{6}{7}, \frac{6}{7}\right)$, the second corresponds to $\left(\frac{2}{7}, \frac{2}{7}, \frac{2}{7}\right)$ and $\left(\frac{3}{7}, \frac{3}{7}, \frac{3}{7}\right)$, and the third corresponds to $\left(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}\right)$ and $\left(\frac{4}{7}, \frac{4}{7}, \frac{4}{7}\right)$.

APPENDIX A. ON QUADRATIC TRANSFORMATIONS FOR THE SIXTH PAINLEVÉ EQUATION

The existence of quadratic transformations for the sixth Painlevé equation was discovered in [12] via an artificial transformation found for the similarity reductions of the so-called three-wave resonant system. In the subsequent paper [13] one quadratic transformation was obtained explicitly with the help of the method of RS-transformations.

The latter quadratic transformation acts on the θ-assemblies as follows:

$$\left(\frac{1}{2}, \hat{\theta}^0_1, \hat{\theta}^0_t, \pm\frac{1}{2}\right) \longmapsto \left(\hat{\theta}^0_t, \hat{\theta}^0_1, \hat{\theta}^0_t, \hat{\theta}^0_1\right). \tag{A.1}$$

For the solution $y(t)$ corresponding to the right $\hat{\theta}$-assembly and depending on $t = 4\sqrt{t_0}/(1+\sqrt{t_0})^2$, where t_0 is the independent variable for the initial solution $y_0(t_0)$ corresponding to the left $\hat{\theta}$-assembly, an explicit though complicated formula was obtained in terms of $y_0(t_0)$. Recently, K. Okamoto and his collaborators found that, in fact, this formula can be simplified substantially. Some time ago, Manin [16] rediscovered the so-called elliptic form of the sixth Painlevé equation, given for the first time by Painlevé. Then, Manin applied the Landen transformation for elliptic functions and found another transformation in terms of the "elliptic variables" for the sixth Painlevé equation. Later, Okamoto mentioned that it is also a quadratic transformation; however, it is not equivalent to that given above. Actually, Manin's transformation cannot be obtained by the method of RS-transformations if we apply it to the associated linear Fuchsian (2×2)-matrix ODE in the Jimbo–Miwa parametrization [9]; in this parametrization we get only the quadratic transformation (A.1) and its equivalent forms. The Jimbo–Miwa parametrization, which turned out to be very helpful for the study of almost all questions related to the theory of the sixth Painlevé equation, has one drawback: the complete group of symmetries for this equation (see [18]) cannot be realized as a group acting on the solutions of the Fuchsian ODE. Therefore, if we want to build the entire theory of the sixth Painlevé equation based on the isomonodromy problem for the (2×2)-matrix linear Fuchsian ODE, then, for analyzing some questions, especially those related to symmetries and transformations, we must consider an alternative parametrization of this Fuchsian ODE, employing one special transformation found by Okamoto [18]. We call the latter transformation the *Okamoto transformation*, and we call the corresponding parametrization the *Okamoto parametrization* (see [14], (A.5)).[7] Recently, in the paper [17], a representation of the complete group of Okamoto transformations as the symmetries for solutions of a linear ODE in matrix dimension greater than 2 was obtained.

If we apply the method of RS-transformations to the 2×2 Fuchsian ODE with four singular points in the Okamoto parametrization, then we can obtain the Manin transformation for the canonical form of the sixth Painlevé equation. Here we shall not give an explicit form of the Okamoto parametrization and the "RS-derivation" of the Manin transformation. Instead, we present the final answer in a spirit close to the presentation in the main part of this paper.

We show how to construct quadratic transformations of Manin type. Consider the following quadratic Belyĭ functions $z(z_1)$, which map $\mathbb{CP}^1$ into itself:

$$z = z_1^2, \quad z = -4z_1(z_1-1), \quad z = \frac{z_1^2}{4(z_1-1)}. \tag{A.2}$$

Remark A.1. Of course, these functions have only two critical points on the Riemann sphere. However, Proposition 2.1 applies to them: if we formally put the multiplicity of the absent third critical point equal to zero, then we obtain $m=0$, as it should be for the Belyĭ functions.

Here we study transformations of the type $RS^2_4(4)$; therefore, in our notation we incorporate yet another point $t \in \mathbb{CP}^1 \setminus \{0,1,\infty\}$. Thus, the types of these functions are, respectively, as follows: $R(2|1+1|1+1|2)$, $R(|1+1|2|1+1|2)$, and $R(2|2|1+1|1+1)$, where the third box is for t and the last is for the point at ∞.

[7] There is a misprint in equation (A.6) in [14]; in place of y_0^2 in the numerator there should be y_0.

Remark A.2. In this case it is also convenient to treat all four points 0, 1, t, ∞ on "equal footing" rather than normalizing the critical values of the Belyĭ functions necessarily at 0, 1, and ∞. Finally, the latter ambiguity is absorbed by the corresponding transformations for the sixth Painlevé equation. However, instead of making these transformations afterwards, it is easier to prepare the desired quadratic transformation from the very beginning, since it does not require any special effort. For example, we also consider the function $z = 1 + (1-t)(z_1-1)^2/(4z_1)$, which is of the type $R(1+1|2|2|1+1)$. Clearly, we have six different quadratic Belyĭ functions.

For a given type of a Belyĭ function, consider the $\hat{\theta}$-assembly whose members are denoted by $\hat{\theta}^0_k$, $k = 0, 1, t_0, \infty$. The parameters $\hat{\theta}^0_k$ corresponding to the boxes of the type symbol with 2 vanish if $k = 0, 1, t_0$, or $\hat{\theta}^0_k = 1$ for $k = \infty$, and the other two members of the $\hat{\theta}$-assembly are arbitrary. Now, let $y_0(t_0)$ denote any solution of the sixth Painlevé equation (2.3) with $t = t_0$ and the $\hat{\theta}$-assembly defined above. The set of preimages $\{z^{-1}(0), z^{-1}(1), z^{-1}(t_0), z^{-1}(\infty)\}$ contains four nonapparent points; namely, these are the preimages of the points with $1+1$ boxes in the type of $z(z_1)$. We denote by $\tilde{z}(z_1)$ any of the 24 fractional linear transformations that map the set of nonapparent preimages of $z(z_1)$ into sets of the form $\{0, 1, t, \infty\}$ with some t.

Proposition A.1. *Let $\tau \in \{z^{-1}(0),\ z^{-1}(1),\ z^{-1}(t_0),\ z^{-1}(\infty)\}$ be nonapparent and such that $\tilde{z}(\tau) \notin \{0, 1, \infty\}$. Define*

$$t = \tilde{z}(\tau), \quad y(t) = \tilde{z}(z^{-1}(y_0(t_0))), \quad i_k = z(\tilde{z}^{-1}(k)), \ \ k = 0, 1, t, \infty.$$

The function $y(t)$ solves the sixth Painlevé equation for the $\hat{\theta}$-assembly with the members $\hat{\theta}_k = \frac{1}{2}\hat{\theta}^0_{i_k}$ if $k \neq \infty$ and $i_k \neq \infty$, $\hat{\theta}_k = \frac{1}{2}(\hat{\theta}^0_\infty - 1)$ if $k \neq \infty$ and $i_k = \infty$, and $\hat{\theta}_\infty - 1 = \frac{1}{2}(\hat{\theta}^0_\infty - 1)$.

Remark A.3. The entire construction is completely invertible, so that for every quadratic transformation we can define the inverse transformation, which is also called a quadratic transformation. All in all, this construction gives $6 \cdot 24 = 144$ quadratic transformations, without counting their inverses. However, of course, if we consider them modulo fractional linear transformations interchanging $0, 1, t, \infty$ for both the initial equation and for the final one, then we have actually only two seed nonequivalent transformations, because there is a difference between two cases, namely, the cases where ∞ is or is not set as a critical value of the Belyĭ function. If we further factorize them modulo Bäcklund transformations, then we are left with only one seed transformation. This seed transformation can be derived from the quadratic transformation (A.1) found by the author by application of the Okamoto transformation. Therefore, all in all, there is only one seed quadratic transformation for the sixth Painlevé equation. However, if an explicit formula is needed, it is convenient to use Proposition A.1 directly because, in many situations, it makes it possible to avoid compositions of cumbersome transformations.

Now we consider examples of quadratic transformations constructed with the help of Proposition A.1 for the Belyĭ functions (A.2). Example 3 below is the algebraic form of the transformation obtained by Manin.

1. $$z(z_1) = z_1^2, \quad \tilde{z}(z_1) = \frac{2}{1+\sqrt{t_0}}\frac{z_1+\sqrt{t_0}}{z_1+1}, \quad \tau = \sqrt{t_0},$$
$$t = \frac{4\sqrt{t_0}}{(1+\sqrt{t_0})^2}, \quad y(t) = \frac{2}{1+\sqrt{t_0}}\frac{\sqrt{y_0(t_0)}+\sqrt{t_0}}{\sqrt{y_0(t_0)}+1},$$
$$\left(0, \hat{\theta}^0_1, \hat{\theta}^0_{t_0}, 1\right) \longmapsto \left(\frac{1}{2}\hat{\theta}^0_{t_0}, \frac{1}{2}\hat{\theta}^0_1, \frac{1}{2}\hat{\theta}^0_{t_0}, 1 + \frac{1}{2}\hat{\theta}^0_1\right).$$

2. $z(z_1) = -4z_1(z_1-1)$, $\tilde{z}(z_1) = \dfrac{\left(1-\sqrt{1-t_0}\right) z_1}{2z_1 - 1 - \sqrt{1-t_0}}$, $\tau = \dfrac{1}{2}\left(1-\sqrt{1-t_0}\right)$,

$$t = -\frac{\left(1-\sqrt{1-t_0}\right)^2}{4\sqrt{1-t_0}}, \quad y(t) = -\frac{\left(1-\sqrt{1-t_0}\right)\left(1-\sqrt{1-y_0(t_0)}\right)}{2\left(\sqrt{1-t_0}+\sqrt{1-y_0(t_0)}\right)},$$

$$\left(\hat{\theta}_0^0, 0, \hat{\theta}_{t_0}^0, 1\right) \longmapsto \left(\frac{1}{2}\hat{\theta}_0^0, \frac{1}{2}\hat{\theta}_0^0, \frac{1}{2}\hat{\theta}_{t_0}^0, 1+\frac{1}{2}\hat{\theta}_{t_0}^0\right).$$

3. $z(z_1) = \dfrac{z_1^2}{4(z_1-1)}$,

$$\tilde{z}(z_1) = \frac{\left(z_1 - 2t_0 + 2\sqrt{t_0^2-t_0}\right)}{\left(1-2t_0+2\sqrt{t_0^2-t_0}\right)}, \quad \tau = 2\left(t_0 + \sqrt{t_0^2-t_0}\right),$$

$$t = \frac{4\sqrt{t_0^2-t_0}}{\left(1-2t_0+2\sqrt{t_0^2-t_0}\right)},$$

$$y(t) = \frac{2\left(y_0(t_0)+\sqrt{y_0(t_0)^2-y_0(t_0)}-t_0+\sqrt{t_0^2-t_0}\right)}{\left(1-2t_0+2\sqrt{t_0^2-t_0}\right)},$$

$$\left(0, 0, \hat{\theta}_{t_0}^0, \hat{\theta}_\infty^0\right) \longmapsto \left(\frac{1}{2}\hat{\theta}_{t_0}^0, \frac{1}{2}(\hat{\theta}_\infty^0-1), \frac{1}{2}\hat{\theta}_{t_0}^0, 1+\frac{1}{2}(\hat{\theta}_\infty^0-1)\right).$$

Remark A.4. In all formulas 1–3 above, the branches of the square roots with $y_0(t_0)$ can be taken independent of the branches of the square roots with t_0; this does not change the mapping between the $\hat{\theta}$-assemblies. All transformations are invertible.

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