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# SHARP JACKSON-TYPE INEQUALITIES FOR APPROXIMATIONS OF CLASSES OF CONVOLUTIONS BY ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

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Abstract. In this paper, a new method is introduced for the proof of sharp Jackson-type inequalities for approximation of convolution classes of functions defined on the real line. These classes are approximated by linear operators with values in sets of entire functions of exponential type. In particular, a sharp Jackson-type inequality for the even-order derivatives of the conjugate function is proved. For the uniform and the integral norm, the estimates are sharp even if their left-hand sides are replaced by the best approximation. Sharp inequalities for approximations of periodic functions by trigonometric polynomials and of almost-periodic functions by generalized trigonometric polynomials are special cases of the inequalities mentioned above.

## §1. Introduction

The problem of finding sharp estimates in terms of norms for approximations of convolutions is well studied, and general methods are developed for its solution. To find sharp constants in Jackson-type inequalities (i.e., in estimates of approximations in terms of moduli of continuity) is significantly harder; these problems require an individual approach. Sharp inequalities involving high-order moduli of continuity are most difficult to obtain.

In this paper, we consider approximation of convolution classes of functions defined on the real line by entire functions of exponential type. In this context, we develop a new method of obtaining Jackson-type inequalities sharp for the uniform and the integral norm. This method can be applied to a wide class of convolutions, including convolutions with "classical kernels" such as the Poisson kernel, the heat conduction kernel, the kernels of some differential operators, and the kernels conjugate to those mentioned above. In particular, a sharp Jackson-type inequality for the even-order derivatives of the conjugate function is established. All estimates are realized by linear methods of approximation, and they remain sharp even if the left-hand side is replaced by the best approximation. They refine the classical Akhiezer–Kreĭn–Favard type inequalities. We also establish inequalities where the right-hand side is a linear combination of moduli of continuity with increasing orders. Estimates for approximations of periodic functions by trigonometric polynomials and of almost-periodic functions by generalized trigonometric polynomials are special cases of the inequalities obtained. Before, such inequalities were known only for classes of differentiable periodic functions.

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In what follows, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$ are the sets of reals, integers, nonnegative integers, and positive integers, respectively; $CB(\mathbb{R})$ is the space of bounded continuous functions on $\mathbb{R}$ and $C$ is the space of continuous $2\pi$-periodic functions, with the uniform norm $\|\cdot\| = \|\cdot\|_\infty$; $L(\mathbb{R}) = L_1(\mathbb{R})$ is the space of functions integrable on $\mathbb{R}$; $L = L_1$ is the space of $2\pi$-periodic functions integrable on the period, with the integral norm $\|\cdot\|_1$; unless implied otherwise by the context, function spaces may be real or complex; $\widetilde{f}$ is the function trigonometrically conjugate to $f$ (see, e.g., [1, §79]); $CB^{(r)}(\mathbb{R})$, $\widetilde{CB}^{(r)}(\mathbb{R})$, $C^{(r)}$, and $\widetilde{C}^{(r)}$ are the classes of functions $f$ belonging to $C(\mathbb{R})$ or $C$ and such that $f^{(r)}$ or $\widetilde{f}^{(r)}$ exists and is continuous and bounded, respectively; $\delta_t^r(f)$ is the central difference, $\omega_r(f,h)$ is the $r$th-order modulus of continuity of a function $f$ with step $h$, $E_n(f)$ is the uniform best approximation of $f$ in the set $\mathcal{T}_{2n-1}$ of trigonometric polynomials of order at most $n-1$, $A_\sigma(f)$ and $A_{\sigma-0}(f)$ are the uniform best approximations of $f$ in the set $\mathbf{E}_\sigma$ ($\mathbf{E}_{\sigma-0}$) of entire functions of type not exceeding (less than) $\sigma > 0$; finally,

$$\mathcal{K}_r = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k(r+1)}}{(2k+1)^{r+1}}, \quad \widetilde{\mathcal{K}}_r = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{kr}}{(2k+1)^{r+1}}.$$

Below, we review some known estimates (sharp in the uniform and the integral norm) for approximation of a function in terms of norms and moduli of continuity for derivatives of the function and of its conjugate.

In 1937, Favard [2] and Akhiezer and Kreĭn [3] constructed a linear approximation method $X_{nr} : L \to \mathcal{T}_{2n-1}$ ($r, n \in \mathbb{N}$) such that

$$\|f - X_{nr}(f)\| \le \frac{\mathcal{K}_r}{n^r}\|f^{(r)}\| \tag{1.1}$$

for every $f \in C^{(r)}$ and proved that on the class $C^{(r)}$ the constant $\mathcal{K}_r$ cannot be reduced, even if the left-hand side is replaced by $E_n(f)$. Moreover, also in [3], a linear approximation method $\widetilde{X}_{nr} : L \to \mathcal{T}_{2n-1}$ was constructed such that

$$\|f - \widetilde{X}_{nr}(f)\| \le \frac{\widetilde{\mathcal{K}}_r}{n^r}\|\widetilde{f}^{(r)}\| \tag{1.2}$$

for every $f \in \widetilde{C}^{(r)}$, and it was proved that on the class $\widetilde{C}^{(r)}$ the constant $\widetilde{\mathcal{K}}_r$ cannot be reduced even if the left-hand side is replaced by $E_n(f)$. Later on, analogs of (1.1) and (1.2) were obtained for various convolution classes of periodic functions (these classes are generalizations of $C^{(r)}$ and $\widetilde{C}^{(r)}$); we mention the papers [4]–[12] among other publications on this topic. Dzyadyk's paper [8] finished a series of papers devoted to relations of type (1.1) and (1.2) for fractional derivatives. Akhiezer [4], Kreĭn [6], and Nguen Tkhi Thk'eu Khoa [9, 10] studied similar inequalities for differential operators. Nikol′skiĭ [11] established estimates similar to (1.1) and (1.2) for the norm in the space $L$.

In [13], Kreĭn obtained analogs of (1.1) and (1.2) for approximation by entire functions of exponential type in function classes on the real line that are defined with the help of differential operators $\mathcal{P}(D)$ ($D$ is the operator of differentiation) with constant coefficients and noticed that the previous results for periodic functions follow from his theorems. For $\sigma$ sufficiently large (namely, for $\sigma > 3^{\varkappa-1}\mu$, where $\mu$ is the maximal imaginary part for the roots of the polynomial $\mathcal{P}$, and $\varkappa$ is the number of pairs of the nonreal roots of $\mathcal{P}$), he proved the inequality

$$A_{\sigma-0}(f) \le \mathcal{K}(\sigma, \mathcal{P})\|\mathcal{P}(D)f\| \tag{1.3}$$

with a sharp constant $\mathcal{K}(\sigma, \mathcal{P})$. In the case where the polynomial $\mathcal{P}$ has no pure imaginary roots and no zero roots, he presented linear operators that realize the constant in (1.3) (up to an arbitrary positive $\varepsilon$). Moreover, if $f$ is almost-periodic (in particular, periodic), then so is the approximating function, and, moreover, its exponents belong to $f$.

In [14], Sz.-Nagy specified some sufficient conditions on the kernel of the convolution operator (in terms of the Fourier transform of the convolution kernel) under which a sharp estimate of type (1.3) is valid for approximation of convolutions (see Lemma 6 below). In particular, for all $\sigma > 0$ he obtained the sharp inequality

$$A_{\sigma-0}(f) \le \frac{\mathcal{K}_r}{\sigma^r}\|f^{(r)}\| \tag{1.4}$$

and constructed a linear operator $X_{\sigma r}$ with values in $\mathbf{E}_\sigma$ whose deviation admits the same estimate. The results of Sz.-Nagy also imply the sharp inequality

$$A_{\sigma-0}(f) \le \frac{\widetilde{\mathcal{K}}_r}{\sigma^r}\|\widetilde{f}^{(r)}\| \tag{1.5}$$

and a similar estimate for the deviation of the linear operator $\widetilde{X}_{\sigma r}$ with values in $\mathbf{E}_\sigma$. Inequalities (1.4) and (1.5) remain sharp even if their left-hand sides are replaced by $A_\sigma(f)$. Results related to approximation of functions with bounded derivatives and of analytic functions (including inequalities (1.4) and (1.5)) can be found in the book [1], where the upper estimates were extended to the spaces $L_p(\mathbb{R})$ and $L_p$. In Khvostenko's paper [15], sharp estimates of the type (1.3) were obtained for convolutions with the kernels that are even rational fractions of a special type (actually, they are the kernels of some differential operators).

Relation (1.1) was refined in the case of odd $r$ by Zhuk [16] ($r = 1$) and by Ligun [17] ($r > 1$); they established the Jackson inequality with a sharp constant:

$$\|f - X_{nr}(f)\| \le \frac{\mathcal{K}_r}{2n^r}\omega_1\left(f^{(r)}, \frac{\pi}{n}\right) \tag{1.6}$$

for every $f \in C^{(r)}$. In [18], Gromov proved the sharp inequality

$$A_{\sigma-0}(f) \le \frac{\mathcal{K}_r}{2\sigma^r}\omega_1\left(f^{(r)}, \frac{\pi}{\sigma}\right) \tag{1.7}$$

($r$ is odd, $f \in CB^{(r)}(\mathbb{R})$) for approximation by entire functions of exponential type, together with its analog in the integral metric and also inequalities for the deviation of the operator $X_{\sigma r}$. Later, Babenko and Gromov [19] observed that inequality (1.7) is sharp (even if its left-hand side is replaced by $A_\sigma(f)$) in the integral metric. Estimates (1.6) and (1.7) are sharp in the uniform and the integral metric because the sharp inequalities (1.1) and (1.4) follow from them. For $r$ even, Ligun [20] proved that, for the real-valued functions $f \in C^{(r)}$, in the uniform metric we have

$$E_n(f) \le \frac{\mathcal{K}_r}{2n^r}\omega_1\left(f^{(r)}, \frac{2\pi}{n}\right).$$

In [21, 22], Merlina refined inequalities (1.4) and (1.7) in the uniform and the integral metric:

$$A_\sigma(f) \le \left(\frac{\pi}{\sigma}\right)^r \left\{A_{r0}\|f^{(r)}\| + \sum_{\nu=1}^{m-1} A_{r\nu}\|\delta^\nu_{\frac{\pi}{\sigma}}(f^{(r)})\| + \left(\frac{\mathcal{K}_r}{\pi^r} - \sum_{\nu=0}^{m-1} 2^\nu A_{r\nu}\right)2^{-m}\|\delta^m_{\frac{\pi}{\sigma}}(f^{(r)})\|\right\} \tag{1.8}$$

for all $r \in \mathbb{N}$. If $r$ is odd, then $\|f^{(r)}\|$ can be replaced by $\frac{1}{2}\omega_1\left(f^{(r)}, \frac{\pi}{\sigma}\right)$. Here $m \in \mathbb{N}$, $\sigma > 0$, and the $A_{r\nu}$ are constants constructed explicitly. Inequalities of type (1.8) for periodic functions were obtained by Zhuk earlier in [23]. Results of type (1.6) and (1.8) can be found in [24, Chapters 4 and 8], where upper estimates were established for a wide class of seminorms.

In [25] the quantities $\sup_{f\in C^{(r)}} \frac{E_n(f)}{\omega_1(f^{(r)},h)}$ were found for $h=\frac{\pi}{\alpha n}$, where $\alpha$ and $r$ are odd positive integers. As to the Jackson-type inequalities for derivatives of the conjugate function, Vinogradov and Zhuk [26] proved that, for even $r$ and real-valued functions $f$, in the uniform metric we have

$$\|f-\widetilde{X}_{nr}(f)\| \le \frac{\widetilde{\mathcal{K}}_r}{2n^r}\omega_1\left(\widetilde{f}^{(r)}, \frac{2\pi}{n}\right).$$

In the same paper the quantity $\sup_{f\in C^{(r)}} \frac{\|f-X_{nr}(f)\|}{\omega_1\left(f^{(r-1)},\frac{\pi}{n}\right)}$ $\left(n, \frac{r}{2}\in\mathbb{N}\right)$ was evaluated.

In the present paper we develop a new method of proving sharp Jackson-type inequalities for the approximation of convolution classes by linear methods. We consider convolutions with even or odd kernels $G\in L(\mathbb{R})$, whose Fourier transform for $y\ge y_0$ admits representation in the form

$$a(G,y)=\int_0^{+\infty} e^{-y^2u}\,d\Phi(u) \quad\text{or}\quad b(G,y)=\int_0^{+\infty} e^{-y^2u}\,d\Psi(u),$$

where $\Phi$ and $\Psi$ are functions monotone increasing on $(0,+\infty)$. Many classical kernels (those of Poisson, of heat conduction, of differential operators) satisfy these conditions.

For odd kernels, approximations of convolutions are estimated in terms of the modulus of continuity, and for even kernels in terms of a linear combination of a seminorm and the modulus of continuity. In particular, we prove a sharp Jackson-type inequality for the derivative of the conjugate function,

$$\|f-\widetilde{X}_{\sigma r}(f)\| \le \frac{\widetilde{\mathcal{K}}_r}{2\sigma^r}\omega_1\left(\widetilde{f}^{(r)}, \frac{\pi}{\sigma}\right) \tag{1.9}$$

($r$ is even), and sharp Jackson-type inequalities for differential operators applied to the function itself or to its conjugate. Inequalities for approximations of almost-periodic and periodic functions are special cases of the inequalities mentioned above. In particular, for periodic functions $f\in\widetilde{C}^{(r)}$, inequality (1.9) takes the form

$$\|f-\widetilde{X}_{nr}(f)\| \le \frac{\widetilde{\mathcal{K}}_r}{2n^r}\omega_1\left(\widetilde{f}^{(r)}, \frac{\pi}{n}\right).$$

In the case where the step is equal to $\frac{\pi}{\sigma}$ $(\frac{\pi}{n})$, our results refine the Akhiezer–Kreĭn–Favard type inequalities (1.1)–(1.5) and admit further refinement, taking the form of an estimate by linear combinations of moduli of continuity, as in (1.8). These estimates are valid for a wide class of spaces whose seminorm is shift-invariant and is dominated by the uniform norm. For the uniform and the integral norm, the inequalities are sharp, and they remain sharp if the left-hand side is replaced by the best approximation. In the case of an odd kernel we also establish approximation estimates by moduli of continuity with step $\frac{\pi}{\alpha\sigma}$, where $\alpha$ is an odd positive integer; these estimates are sharp for the uniform norm.

In what follows, in addition to the notation introduced earlier, $C(E)$, $C^{(r)}(E)$, and $L(E)$ are the sets of functions that are continuous, $r$ times continuously differentiable, and integrable on the set $E$, respectively; $UCB(\mathbb{R})$ is the space of functions uniformly continuous and bounded on $\mathbb{R}$; if $1\le p<\infty$, then $L_p(\mathbb{R})$ is the space of measurable functions $f$ that are integrable on $\mathbb{R}$ with exponent $p$, $L_p$ is the space of measurable and $2\pi$-periodic functions $f$ that are integrable on any period with exponent $p$, the norm in these spaces is defined by $\|f\|_p=\left(\int_E|f|^p\right)^{1/p}$, where $E=\mathbb{R}$ or $[-\pi,\pi]$, respectively; $L_\infty(\mathbb{R})$ and $L_\infty$ are the spaces of measurable functions $f$ that are essentially bounded on $\mathbb{R}$, with the norm

$$\|f\|_\infty=\operatorname*{ess\,sup}_{x\in\mathbb{R}}|f(x)|;$$

$W_p^{(r)}(\mathbb{R})$ and $\widetilde{W}_p^{(r)}(\mathbb{R})$ are the sets of functions $f$ such that $f^{(r-1)}$ ($\widetilde{f}^{(r-1)}$) is absolutely continuous on every segment and $f^{(r)}$ ($\widetilde{f}^{(r)}$) belongs to $L_p(\mathbb{R})$.

Let $\mathfrak{M}$ be a closed subspace of $L_p(\mathbb{R})$ $(1 \le p < \infty)$ or of $UCB(\mathbb{R})$ $(p = \infty)$, and let $P$ be a seminorm defined on $\mathfrak{M}$. We say that the space $(\mathfrak{M}, P)$ is of class $\mathcal{B}$ if the following conditions are fulfilled:

1) the space is shift-invariant, i.e., for every $f \in \mathfrak{M}$ and $h \in \mathbb{R}$ we have $f(\cdot + h) \in \mathfrak{M}$ and $P(f(\cdot + h)) = P(f)$;

2) there exists a constant $B$ such that $P(f) \le B\|f\|_p$ for every $f \in \mathfrak{M}$.

As examples of spaces of class $\mathcal{B}$ we mention $\big(UCB(\mathbb{R}), \|\cdot\|_\infty\big)$, $(L_p(\mathbb{R}), \|\cdot\|_p)$ $(1 \le p < \infty)$, the spaces of periodic functions $\big(C, \|\cdot\|_p\big)$ $(1 \le p \le \infty)$, and also more general spaces of uniformly continuous almost-periodic functions (see [27]) with exponents belonging to a fixed set, with various norms (the uniform norm, the norms of Stepanov, Weyl, Bezikovich).

The $r$th-order central differences $(r \in \mathbb{Z}_+)$ and the moduli of continuity of a function $f$ are defined by

$$\delta_t^r(f, x) = \sum_{k=0}^{r} (-1)^k C_r^k f\left(x + \frac{rt}{2} - kt\right),$$
$$\omega_r(f, h)_P = \sup_{0 \le t \le h} P\big(\delta_t^r(f)\big);$$

next,

$$A_\sigma(f)_P = \inf_{\substack{g \in \mathbf{E}_\sigma \\ f - g \in \mathfrak{M}}} P(f - g)$$

is the best approximation of a function $f$ by entire functions of type not exceeding $\sigma$ with respect to a seminorm $P$ ($\inf \varnothing = +\infty$); the quantity $A_{\sigma-0}(f)_P$ is defined similarly. The index $p$ attached to the best approximation, to the modulus of continuity, or to similar objects means that $P(f) = \|f\|_p$. We mention that if $(\mathfrak{M}, P) \in \mathcal{B}$, then the set $\mathfrak{M}$ with one of the seminorms $\omega_r(\cdot, h)_P$, $A_\sigma(\cdot)_P$, $A_{\sigma-0}(\cdot)_P$ is also a space of class $\mathcal{B}$.

The functions are assumed to be extended to the points of removable break by continuity; in other cases the symbol $\frac{0}{0}$ is understood as 0. A prime attached to the sum symbol means that the initial term is divided by two, and a small circle means that the term with the zero index is omitted.

Let $a_k(f)$, $b_k(f)$, $c_k(f)$ be the Fourier coefficients of a function $f \in L$, and let

$$a(f, y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(t) \cos yt \, dt,$$
$$b(f, y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(t) \sin yt \, dt,$$
$$c(f, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) e^{-iyt} \, dt$$

be the Fourier transforms of a function $f \in L(\mathbb{R})$. We make several simple remarks concerning the Fourier transforms. Clearly,

$$c(f) = \frac{a(f) - ib(f)}{2}, \quad a(f, y) = c(f, y) + c(f, -y), \quad b(f, y) = i\big(c(f, y) - c(f, -y)\big).$$

Since the parity of the complex Fourier transform of a function coincides with the parity of the function itself, we have $a(f) = 2c(f)$ if $f$ is even, and $b(f) = 2ic(f)$ if $f$ is odd.

By analogy, if a function $c$ is defined on a symmetric (with respect to zero) subset of $\mathbb{R}$, we put

$$a(x) = c(x) + c(-x), \quad b(x) = i(c(x) - c(-x)); \tag{1.10}$$

then $c(x) = \frac{a(x)-ib(x)}{2}$. We have $a(x) = 2c(x)$ if $c$ is even, and $b(x) = 2ic(x)$ if $c$ is odd.

The convolution of two functions $f$ and $g$ is defined by

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x-t)g(t)\,dt,$$

and with such normalization we have $c(f * g) = c(f)c(g)$. In the sequel,

$$\begin{aligned}
\Pi_u(t) &= \sum_{k=-\infty}^{\infty} e^{-k^2u}e^{ikt},\\
\widetilde{\Pi}_u(t) &= \sum_{k=-\infty}^{\infty} (-i\operatorname{sgn} k)e^{-k^2u}e^{ikt},\\
W_u(t) &= \int_{-\infty}^{+\infty} e^{-y^2u}e^{ity}\,dy = \frac{\sqrt{\pi}}{\sqrt{u}}e^{-\frac{t^2}{4u}},\\
\widetilde{W}_u(t) &= \int_{-\infty}^{+\infty} (-i\operatorname{sgn} y)e^{-y^2u}e^{ity}\,dy
\end{aligned}$$

are the periodic and the nonperiodic kernels of heat conduction and the conjugate kernels of heat conduction.

## §2. Auxiliary results

**Lemma 1.** *Suppose the following*: $1 \le p \le \infty$; $\mathfrak{M}$ *is a closed subspace of the space* $L_p(\mathbb{R})$; $P$ *is a seminorm defined on* $\mathfrak{M}$; *there exists a constant* $B$ *such that* $P(f) \le B\|f\|_p$ *for every* $f \in \mathfrak{M}$; $-\infty < a < b < +\infty$; $F : [a,b] \to \mathfrak{M}$ *is a continuous mapping from* $[a,b]$ *to* $L_p(\mathbb{R})$; $g = \int_a^b F$. *Then* $g \in \mathfrak{M}$ *and*

$$P(g) \le \int_a^b P(F(t))\,dt.$$

*Proof.* For $n \in \mathbb{N}$, we put $h = \frac{b-a}{n}$, $t_k = a + kh$, $g_n = \sum_{k=1}^n F(t_k)h$. Clearly, $g_n \in \mathfrak{M}$. Fix $\varepsilon > 0$. By the uniform continuity of $F$, there exists $\delta > 0$ such that if $t', t'' \in [a,b]$ and $|t' - t''| < \delta$, then $\|F(t') - F(t'')\|_p < \varepsilon$. If $n$ is sufficiently large, then $h < \delta$, and the generalized Minkowski inequality yields

$$\begin{aligned}
\|g - g_n\|_p &= \left\| \sum_{k=1}^n \int_{t_{k-1}}^{t_k} (F(t) - F(t_k))\,dt \right\|_p\\
&\le \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|F(t) - F(t_k)\|_p\,dt \le \varepsilon.
\end{aligned}$$

This means that $g_n$ converges to $g$ in $L_p(\mathbb{R})$. Then $g \in \mathfrak{M}$ by the completeness of $\mathfrak{M}$. By the triangle inequality,

$$P(g_n) \le \sum_{k=1}^n P(F(t_k))h. \tag{2.1}$$

Since

$$|P(g_n) - P(g)| \le P(g_n - g) \le B\|g_n - g\|_p,$$

we have $P(g_n) \xrightarrow[n\to\infty]{} P(g)$. The right-hand side in (2.1) is an integral sum for the integral $\int_a^b P(F(t))\,dt$, where the integrand is continuous because

$$|P(F(t_1)) - P(F(t_2))| \le P(F(t_1) - F(t_2)) \le B\|F(t_1) - F(t_2)\|_p$$

and $F$ is continuous. Therefore, the integral sums converge to the integral. Passage to the limit in (2.1) finishes the proof. $\square$

**Corollary 1.** *Under the assumptions of Lemma* 1, *suppose that* $K \in L[a,b]$ *and put* $h = \int_a^b F(t)K(t)\,dt$. *Then* $h \in \mathfrak{M}$ *and*

$$P(h) \le \int_a^b P(F(t))|K(t)|\,dt.$$

*Proof.* Let $\{K_n\} \subset C[a,b]$ be such that $\|K - K_n\|_1 \to 0$, and let $h_n = \int_a^b F(t)K_n(t)\,dt$. Since the mappings $FK_n$ satisfy the assumptions of Lemma 1, we have $h_n \in \mathfrak{M}$ and

$$P(h_n) \le \int_a^b P(F(t))|K_n(t)|\,dt. \tag{2.2}$$

Since

$$\|h - h_n\|_p \le \int_a^b \|F(K - K_n)\|_p = \|F\|_p \|K - K_n\|_1 \to 0,$$

the sequence $h_n$ converges to $h$ in $L_p(\mathbb{R})$ as $n \to \infty$. Thus, $h \in \mathfrak{M}$. It remains to pass to the limit in (2.2). $\square$

**Corollary 2.** *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $G \in L(\mathbb{R})$, *and* $\varphi \in \mathfrak{M}$. *Then* $\varphi * G \in \mathfrak{M}$ *and* $P(\varphi * G) \le \frac{1}{2\pi}\|G\|_1 P(\varphi)$.

*Proof.* Since the shift is continuous in $L_p(\mathbb{R})$ $(1 \le p < \infty)$ and in $UCB(\mathbb{R})$ $(p = \infty)$, the mapping $F(t) = \varphi(\cdot - t)$ acts continuously from $\mathbb{R}$ to $L_p(\mathbb{R})$ with the corresponding $p$. For every $a \in (0, +\infty)$ we put $h_a = \int_{-a}^a \varphi(\cdot - t)G(t)\,dt$. Since the space $(\mathfrak{M}, P)$ is shift invariant, Corollary 1 shows that $h_a \in \mathfrak{M}$ and

$$P(h_a) \le \int_{-a}^a P(\varphi(\cdot - t))|G(t)|\,dt \le \|G\|_1 P(\varphi).$$

It remains to pass to the limit as $a \to +\infty$ and use the relation

$$\|h - h_a\|_p \le \left(\int_{\mathbb{R}\setminus[-a,a]} |G|\right)\|\varphi\|_p \to 0. \qquad \square$$

*Remark* 1. If $P(\cdot) = \|\cdot\|_\infty$ and $\varphi \in L_\infty(\mathbb{R})$, then the inequality of Corollary 2 is valid without the uniform continuity assumption about $\varphi$.

**Lemma 2.** *Suppose that* $\{f_k\}_{k=0}^\infty$ *is a sequence of functions integrable on a set* $E$ *with respect to a measure* $\mu$, *and that the sequence* $\{f_k\}$ *is monotone decreasing and converges to* 0 *on* $E$. *Then*

$$\int_E \sum_{k=0}^\infty (-1)^k f_k\,d\mu = \sum_{k=0}^\infty (-1)^k \int_E f_k\,d\mu.$$

*Proof.* By the Lebesgue dominated convergence theorem, we have $\int_E f_k\,d\mu \to 0$. Grouping the terms (which tend to zero) pairwise, integrating the nonnegative series term by term (we use the Levi theorem), and regrouping again, we obtain

$$\begin{aligned}\int_E \sum_{k=0}^\infty (-1)^k f_k\,d\mu &= \int_E \sum_{m=0}^\infty (f_{2m} - f_{2m+1})\,d\mu \\ &= \sum_{m=0}^\infty \int_E (f_{2m} - f_{2m+1})\,d\mu = \sum_{k=0}^\infty (-1)^k \int_E f_k\,d\mu. \qquad \square\end{aligned}$$

**Lemma 3.** *For $p, \gamma > 0$ and $q, \lambda \geq 0$, the following identities are true*:

$$e^{-\gamma q} = \frac{\gamma}{2\sqrt{\pi}} \int_0^{+\infty} e^{-q^2 s} e^{-\frac{\gamma^2}{4s}} \frac{ds}{s^{3/2}}, \tag{2.3}$$

$$\frac{e^{-\lambda p}}{\sqrt{p}} = \frac{1}{\sqrt{\pi}} \int_\lambda^{+\infty} \frac{e^{-pu}}{\sqrt{u-\lambda}}\, du, \tag{2.4}$$

$$\begin{aligned} \frac{pe^{-\lambda p} - qe^{-\lambda q}}{\sqrt{p}(q-p)} &= \frac{1}{\sqrt{\pi}} \int_0^\lambda e^{-pu} \int_0^u \frac{qe^{q(v-\lambda)}}{\sqrt{u-v}}\, dvdu \\ &\quad + \frac{1}{\sqrt{\pi}} \int_\lambda^{+\infty} e^{-pu} \Big( -\frac{1}{\sqrt{u-\lambda}} + \int_0^\lambda \frac{qe^{q(v-\lambda)}}{\sqrt{u-v}}\, dv \Big) du, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \frac{\sqrt{p}e^{-\lambda p} - \sqrt{q}e^{-\lambda q}}{q-p} &= \sqrt{q} \int_0^\lambda e^{-pu} e^{q(u-\lambda)}\, du \\ &\quad - \frac{1}{\sqrt{\pi}} \int_\lambda^{+\infty} e^{-pu} \left( \frac{1}{\sqrt{u-\lambda}} - \int_{u-\lambda}^{+\infty} \frac{qe^{q(u-v-\lambda)}}{\sqrt{v}}\, dv \right) du. \end{aligned} \tag{2.6}$$

Identities (2.3) and (2.4) are well known (see, e.g., [28, formulas 4.5 (28) and 4.3 (5)]). Identities (2.5) and (2.6) are proved by direct calculation.

We need the following (known) properties of functions related to the heat conduction kernel.

**Lemma 4.** 1. *For all $u > 0$, the function $\Pi_u$ is positive, and its derivative is negative on $(0, \pi)$.*

2. *The functions*

$$g_1(u) = \frac{1}{2} \frac{\partial}{\partial u} \Pi_u(\pi) = \sum_{k=1}^\infty (-1)^{k-1} k^2 e^{-k^2 u},$$

$$g_2(u) = \sum_{k=0}^\infty (-1)^k (2k+1) e^{-(2k+1)^2 u},$$

$$g_3(u) = \sum_{k=1}^\infty (-1)^{k-1} k e^{-k^2 u}$$

*are positive on $(0, +\infty)$.*

3. *The function $\widetilde{\Pi}_u$ is positive on $(0, \pi)$ for all $u > 0$.*

*Proof.* 1. We set $q = e^{-u}$, $H = \prod_{k=1}^\infty (1 - q^{2k})$. The representations of theta-functions in the form of infinite products (see [29, formulas 6.181.1 and 6.181.3]) show that

$$\Pi_u(t) = H \prod_{k=1}^\infty (1 + 2q^{2k-1} \cos t + q^{4k-2}), \tag{2.7}$$

$$\sum_{k=0}^\infty q^{\frac{(2k+1)^2}{4}} (-1)^k \sin{(2k+1)t} = H q^{1/4} \sin t \prod_{k=1}^\infty (1 - 2q^{2k} \cos 2t + q^{4k}). \tag{2.8}$$

Statement 1 is a consequence of (2.7).

2. To prove that $g_1$ is positive, it suffices to put $t = \pi$ in (2.7),

$$\Pi_u(\pi) = \prod_{k=1}^\infty \big( (1 - q^{2k})(1 - 2q^{2k-1} + q^{4k-2}) \big),$$

and to observe that for $q \in (0, 1)$ all factors are positive and their derivatives with respect to $q$ are negative. To check the positivity of $g_2$, it suffices to differentiate identity (2.8) with respect to $t$ at the point $t = 0$. The positivity of $g_3$ follows from that of $g_1$ and formula (2.4):

$$\sum_{k=1}^{\infty}(-1)^{k-1}ke^{-k^2u} = \frac{1}{\sqrt{\pi}}\int_0^{+\infty}\sum_{k=1}^{\infty}(-1)^{k-1}k^2e^{-k^2(u+v)}\frac{dv}{\sqrt{v}} > 0.$$

Termwise integration is justified by the Lebesgue dominated convergence theorem, because

$$\frac{1}{\sqrt{\pi}}\int_0^{+\infty}\sum_{k=1}^{\infty}k^2e^{-k^2(u+v)}\frac{dv}{\sqrt{v}} = \sum_{k=1}^{\infty}ke^{-k^2u} < +\infty.$$

3. By (2.4) with $p = k^2$ and statement 1, we have

$$\frac{1}{2}\widetilde{\Pi}_u(t) = -\frac{\partial}{\partial t}\sum_{k=1}^{\infty}e^{-k^2u}\frac{\cos kt}{k} = -\frac{\partial}{\partial t}\sum_{k=1}^{\infty}\frac{1}{\sqrt{\pi}}\int_0^{+\infty}e^{-k^2(u+y)}\frac{dy}{\sqrt{y}}\cos kt$$
$$= -\frac{\partial}{\partial t}\frac{1}{\sqrt{\pi}}\int_0^{+\infty}\Pi_{u+y}(t)\frac{dy}{\sqrt{y}} > 0$$

for all $t \in (0, \pi)$. $\square$

## §3. The Kreĭn kernels

Let $\sigma > 0$. We consider the approximation of classes of convolutions with real kernel $G \in L(\mathbb{R})$,

$$f = T + \varphi * G, \tag{3.1}$$

by entire functions of type not exceeding (less than) $\sigma$. The function $\varphi$ belongs to a certain fixed space: $CB(\mathbb{R})$, $L(\mathbb{R})$, a space of periodic functions, etc.; we may also require that $\varphi$ be orthogonal to the space $\mathbf{E}_{\sigma_1}$ ($\sigma_1 \le \sigma$ or $\sigma_1 < \sigma$). $T$ is an arbitrary function belonging to $\mathbf{E}_{\sigma_2}$ ($\sigma_2 \le \sigma$ or $\sigma_2 < \sigma$). The results of the paper do not depend on the parameters $\sigma_1$ and $\sigma_2$, i.e., if we change the class under consideration by adding an entire function $T$ and by imposing the orthogonality condition on $\varphi$, then the inequalities for the functions in this class will not change. Therefore, in what follows we do not mention the parameters $\sigma_1$ and $\sigma_2$ in the notation and statements.

In particular, the classes $CB^{(r)}(\mathbb{R})$ and $\widetilde{CB}^{(r)}(\mathbb{R})$ have the form (3.1) (see [1, pp. 238–241]).

Suppose that the kernel $G$ is even or odd, $G \in L(\mathbb{R}) \cap C(\mathbb{R} \setminus \{0\})$, and $G(t) = O(t^{-2})$ as $t \to \infty$. Let $\varkappa$ be the parity of the kernel $G$, i.e., $\varkappa = 0$ if $G$ is even and $\varkappa = 1$ if $G$ is odd. We denote (see [1, pp. 199–203])

$$L_\sigma(G, z) = \frac{\sin\left(\sigma z - \frac{\pi}{2}(1-\varkappa)\right)}{\sigma}\sum_{k=-\infty}^{\infty}(-1)^k\frac{G\left(\frac{(2k+1-\varkappa)\pi}{2\sigma}\right)}{z - \frac{(2k+1-\varkappa)\pi}{2\sigma}}.$$

Then $L_\sigma(G)$ is an even or odd function that belongs to $\mathbf{E}_\sigma \cap L(\mathbb{R})$ and interpolates the function $G$ at the points $\frac{(2k+1-\varkappa)\pi}{2\sigma}$ ($k \in \mathbb{Z}$), i.e., at the zeros of the functions $\sin \sigma t$ or $\cos \sigma t$ in accordance with its parity. (The value of an odd function $G$ at the point zero is equal to zero, and if $G$ is even, then its value at the point zero is irrelevant.) The function $L_\sigma(G)$ is expressed in terms of its Fourier transform as follows:

$$L_\sigma(G, t) = \int_{-\sigma}^{\sigma} c(L_\sigma, y)e^{ity}\,dy, \tag{3.2}$$

$$c(L_\sigma, y) = \sum_{k=-\infty}^{\infty} e^{ik\pi(1-\varkappa)}c(G, y + 2k\sigma), \quad |y| \le \sigma. \tag{3.3}$$

For these formulas to be valid, it suffices to require that the series on the right-hand side of (3.3) converges uniformly on $[0,\sigma]$ and that, after multiplying by $e^{i\frac{\pi(1-\varkappa)}{2\sigma}y}$, its sum admits expansion in a Fourier series (see [1]).

We denote
$$X_{\sigma,G}(f) = T + \varphi * L_\sigma(G),$$
where $f$ and $\varphi$ are functions related to each other as in (3.1),
$$\mathcal{K}_{\sigma,G} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(t)\,\mathrm{sgn}\sin\sigma t\,dt = \frac{2}{\pi}\sum_{\nu=0}^{\infty}\frac{b(G,(2\nu+1)\sigma)}{2\nu+1}$$
if $G$ is odd, and
$$\mathcal{K}_{\sigma,G} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(t)\,\mathrm{sgn}\cos\sigma t\,dt = \frac{2}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{2\nu+1}a(G,(2\nu+1)\sigma)$$
if $G$ is even.

*Remark* 2. Clearly, $X_{\sigma,G}(f) \in \mathbf{E}_\sigma$. In some cases (e.g., if $T=0$ and $c(G)$ never vanishes) it is possible to represent $X_{\sigma,G}$ as a convolution operator:
$$X_{\sigma,G}(f) = f * Q_{\sigma,G}, \quad c(Q_{\sigma,G},y) = \frac{c(L_\sigma(G),y)}{c(G,y)}.$$
By this formula, $X_{\sigma,G}(f)$ extends to some function classes wider than those allowed by formula (3.1).

Let $T=0$. If the function $\varphi$ has period $\frac{2\pi}{\rho}$ ($\rho>0$), then $X_{\sigma,G}(f)$ is a trigonometric polynomial of order less than $\frac{\sigma}{\rho}$. In particular, if $\rho \geq \sigma$, then $X_{\sigma,G}(f)$ is a constant:
$$X_{\sigma,G}(f) = \frac{M}{2\pi}\int_{-\pi}^{\pi}\varphi\left(\frac{t}{\rho}\right)dt.$$
If the function $\varphi$ is almost-periodic, then $X_{\sigma,G}(f)$ is also an almost-periodic function whose exponents belong to $\varphi$.

**Lemma 5.** *Suppose that a function $G$ is even or odd, $G \in L(\mathbb{R}) \cap C(\mathbb{R}\setminus\{0\})$, $G(t) = O(t^{-2})$ as $t\to\infty$. If $\sigma>0$ and*
$$(G(t) - L_\sigma(G,t))\cos\sigma t \geq 0 \quad \text{or} \quad (G(t) - L_\sigma(G,t))\sin\sigma t \geq 0$$
(*respectively*) *for almost every $t\in\mathbb{R}$, then*
$$\mathcal{K}_{\sigma,G} = \frac{1}{2\pi}A_\sigma(G)_1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|G - L_\sigma(G)|.$$
*If, moreover, $(\mathfrak{M},P)\in\mathcal{B}$, $\varphi\in\mathfrak{M}$, and the functions $f$ and $\varphi$ are related as in* (3.1), *then*
$$P(f - X_{\sigma,G}(f)) \leq \mathcal{K}_{\sigma,G}P(\varphi), \tag{3.4}$$
$$A_\sigma(f)_P \leq \mathcal{K}_{\sigma,G}A_\sigma(\varphi)_P, \tag{3.5}$$
$$A_{\sigma-0}(f)_P \leq \mathcal{K}_{\sigma,G}A_{\sigma-0}(\varphi)_P, \tag{3.6}$$
$$A_{\sigma-0}(f)_P \leq \mathcal{K}_{\sigma,G}P(\varphi), \tag{3.7}$$
$$A_\sigma(f)_P \leq \mathcal{K}_{\sigma,G}P(\varphi) \tag{3.8}$$
(*in* (3.6) *and* (3.7) *it is assumed that $T\in\mathbf{E}_{\sigma-0}$*). *In the case of the spaces $(CB(\mathbb{R}),\|\cdot\|_\infty)$ and $(L(\mathbb{R}),\|\cdot\|_1)$, the constant $\mathcal{K}_{\sigma,G}$ in* (3.4)–(3.8) *cannot be reduced. In the case of $\frac{2\pi}{\sigma}$-periodic functions with the uniform or the integral norm, the constant in* (3.4), (3.6), *and* (3.7) *also cannot be reduced.*

The identities for $\mathcal{K}_{\sigma,G}$ and inequalities (3.4) and (3.7) for the spaces $L_p(\mathbb{R})$ and $CB(\mathbb{R})$ are contained in [1, §§87, 100]. (The general statements in [1] involve the condition $G \in C(\mathbb{R})$, but this condition can be lifted; in [1, §101], where the class $\widetilde{CB}^{(1)}(\mathbb{R})$ was considered, the statement under discussion is applied, without any reservations, to a kernel discontinuous at the point zero.) Inequality (3.5) (for uniform approximations in classes of differentiable functions) can be found in [30]. No inequalities for the spaces of class $\mathcal{B}$ were formulated earlier in an explicit form, but their proofs do not require new ideas. Inequality (3.4) follows from the identity

$$f - X_{\sigma,G}(f) = \varphi * (G - L_\sigma(G))$$

and the possibility of moving the seminorm under the integral sign (Corollary 2). Inequality (3.5) follows straightforwardly from (3.1) or (3.4) if we take $A_\sigma(\cdot)_P$ as a seminorm and use the identity $A_\sigma(f - X_{\sigma,G}(f))_P = A_\sigma(f)_P$. Inequality (3.6) follows by passing to the limit as $\rho \to \sigma - 0$ from inequalities of the form (3.5) for $A_\rho$ ($\rho \in (0,\sigma)$), because $A_{\sigma-0}(G)_1 = A_\sigma(G)_1$ (see [1]). Relations (3.7) and (3.8) are consequences of (3.6).

A statement about sharpness on subclasses of $L_\infty(\mathbb{R})$ (under slightly different conditions) is contained in [1, §100], where the kernels satisfying the assumptions of Lemma 5 were called the M. G. Kreĭn kernels. The sharpness of inequality (3.8) (and, consequently, of all preceding inequalities) can be proved with the help of the $\frac{2\pi}{\rho}$-periodic functions $\varphi_\rho(t) = \operatorname{sgn}\sin\left(\rho t + \frac{\pi(1-\varkappa)}{2}\right)$ ($\rho > \sigma$), by passing to the limit as $\rho \to \sigma + 0$ and using the fact that $\varphi_\rho * G \to \varphi_\sigma * G$ by the Lebesgue dominated convergence theorem. For the $\frac{2\pi}{\sigma}$-periodic functions $\varphi_\sigma$, equality occurs in inequalities (3.4), (3.6) and (3.7). The inability to improve upon the inequalities on the set of continuous functions is deduced in a standard way from that on $L_\infty(\mathbb{R})$ (for instance, with the help of approximation of the functions $\varphi_\rho$ by their Fejér integrals). The sharpness of the inequalities in the case of the integral norm follows from Remark 3 below and the duality relations

$$\begin{aligned}\sup_{\|\varphi\|_1 \le 1} A_\sigma(\varphi * G)_1 &= \sup_{\|\varphi\|_1 \le 1} \sup_{\substack{g \perp \mathbf{E}_\sigma \\ \|g\|_\infty \le 1}} \int_{-\infty}^{+\infty} (\varphi * G) g \\ &= \sup_{\substack{g \perp \mathbf{E}_\sigma \\ \|g\|_\infty \le 1}} \|g * G\|_\infty = \mathcal{K}_{\sigma,G}\end{aligned}$$

(see [31, §1.4]).

*Remark* 3. If $T = 0$ and $\varphi \perp \mathbf{E}_\sigma$, then $X_{\sigma,G}(f) = 0$. Thus, for this class of functions the left-hand side in (3.8) can be replaced by $P(f)$. For the norm in $L_\infty(\mathbb{R})$, sharpness is proved with the help of the same functions $\varphi_\rho$.

In what follows we only formulate inequalities of the form (3.4).

**Lemma 6** (see [14] and [1, Section 88])**.** *Suppose $G \in L(\mathbb{R})$, $\sigma > 0$.*

1. *If $G$ is even, $a(G) \in C^{(3)}(\mathbb{R})$, and $a(G)$ is three times monotone (i.e., $(-1)^r a^{(r)}(G,y) \ge 0$ for $0 \le r \le 3$ and $y \ge \sigma$), then*

$$\begin{aligned}L_\sigma(G,t) &= \int_0^\sigma \left( \sum_{\nu=-\infty}^{\infty} (-1)^\nu a(G, 2\nu\sigma + y) \right) \cos ty\, dy, \\ G(t) - L_\sigma(G,t) &= 2\cos\sigma t\, C_{\sigma,G}(t),\end{aligned}$$

*where*

$$C_{\sigma,G}(t) = \int_0^{+\infty} \left( \sum_{\nu=0}^{\infty} (-1)^\nu a(G, (2\nu+1)\sigma + y) \right) \cos ty\, dy \ge 0.$$

2. *If $G$ is odd, $b(G) \in C^{(2)}(\mathbb{R})$, and $b(G)$ is two times monotone* (*that is*, $(-1)^r b^{(r)}(G,y) \geq 0$ *for* $0 \leq r \leq 2$ *and* $y \geq \sigma$), *then*

$$L_\sigma(G,t) = \int_0^\sigma \left( \sum_{\nu=-\infty}^{\infty} b(G, 2\nu\sigma + y) \right) \sin ty \, dt,$$

$$G(t) - L_\sigma(G,t) = 2 \sin \sigma t \, S_{\sigma,G}(t),$$

*where*

$$S_{\sigma,G}(t) = \lim_{j\to\infty} \int_0^{+\infty} \left( \sum_{\nu=0}^{j} b(G, (2\nu+1)\sigma + y) \right) \cos ty \, dy \geq 0.$$

*In both cases the conclusion of Lemma* 5 *is true.*

In the periodic case, analogs of Lemma 6 and estimates (3.4) and (3.7), together with a statement about sharpness, were established by Sz.-Nagy (see [7] and [29, 2.11.5 and 2.13.32]); analogs of (3.6) were obtained by Sun Yung-sheng [12] and analogs of (3.4) and (3.7) in the case of the integral norm by Nikol′skiĭ [11]. Analogs of spaces of class $\mathcal{B}$ for the periodic case were introduced by Zhuk and Natanson [32]; analogs of inequalities (3.4), (3.6), and (3.7) for the classes $C^{(r)}$ can be found in [24, formulas (168)–(170)].

*Remark* 4. In a standard way (for instance, with the help of approximation of the function $\varphi$ by its Fejér integral), inequalities of type (3.4)–(3.8) can be carried over from sets of continuous functions to $L_\infty(\mathbb{R})$ and $L_p$ $(1 \leq p \leq \infty)$ with the seminorms $\|\cdot\|_p$, $\omega_m(\cdot,h)_p$, $A_\sigma(\cdot)_p$, and $A_{\sigma-0}(\cdot)_p$. Moreover, for the convolution classes of the form

$$f = T + \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(x-t) \, dg(t)$$

($g$ has bounded variation on $\mathbb{R}$ and satisfies $g(0) = 0$, $g(x+) + g(x-) = 2g(x)$), we have the sharp estimate

$$A_\sigma(f)_1 \leq \mathcal{K}_{\sigma,G} \|g\|_V,$$

where $\|g\|_V$ is the variation of $g$ on $\mathbb{R}$. An inequality similar to (3.4), with the operator $X_{\sigma,G}$ suitably modified, is also valid. These statements are analogs of the results of [11] for the periodic case.

In what follows, we shall not dwell on the possibility of extending our estimates to wider function classes, restricting ourselves to statements for spaces of class $\mathcal{B}$.

## §4. Absolutely monotone functions and Fourier transforms

Suppose $y_0 > 0$, $r \in \{1,2\}$. We denote by $AM_c^r(y_0)$ and $AM_s^r(y_0)$ the sets of even and odd (respectively) functions $c$ defined at least on $\mathbb{R} \setminus (-y_0, y_0)$ and such that, for all $y \geq y_0$,

$$a(y) = \int_0^{+\infty} e^{-y^r u} \, d\Phi(u), \tag{4.1}$$

$$b(y) = \int_0^{+\infty} e^{-y^r u} \, d\Psi(u) \tag{4.2}$$

respectively (see the agreement (1.10)); here $\Phi$ and $\Psi$ are monotone increasing functions on $(0,+\infty)$ such that the integrals $\int_0^{+\infty} e^{-y_0^r u} \, d\Phi(u)$ and $\int_0^{+\infty} e^{-y_0^r u} \, d\Psi(u)$ are finite. We also write $AM^r(y_0) = AM_c^r(y_0) \cup AM_s^r(y_0)$. The notation $AM$ is explained by the fact that the functions belonging to $AM^1$ are absolutely monotone, i.e., their consecutive derivatives have alternating signs:

$$(-1)^n a^{(n)}(y) \geq 0 \quad (n \in \mathbb{Z}_+, y > y_0).$$

Formula (4.1) gives the general form of an absolutely monotone function (not equal to a nonzero constant) of the argument $y^r$ on the ray $(y_0, +\infty)$ [33, §14].

We denote by $\widehat{AM}_c^r(y_0)$ and $\widehat{AM}_s^r(y_0)$ ($y_0 > 0$, $r \in \{1,2\}$) the sets of even and odd (respectively) functions $G$ belonging to $L(\mathbb{R})$ and such that the Fourier transform $c(G)$ belongs to $AM_c^r(y_0)$ or to $AM_s^r(y_0)$; we put $\widehat{AM}^r(y_0) = \widehat{AM}_c^r(y_0) \cup \widehat{AM}_s^r(y_0)$. Thus, for $y \geq y_0$ the Fourier transforms $a(G)$ or $b(G)$ of the functions $G$ belonging to $\widehat{AM}^r(y_0)$ are expressed by formulas (4.1) or (4.2). In what follows, formulas (4.1) and (4.2) for $r = 1$ and $r = 2$ will be referred to as formulas (4.1.1), (4.1.2), (4.2.1), and (4.2.2).

Now we list several properties of functions of classes $AM^r(y_0)$ and $\widehat{AM}^r(y_0)$.

**AM1.** *Formula* (2.3) *implies that* $AM_c^1(y_0) \subset AM_c^2(y_0)$, $AM_s^1(y_0) \subset AM_s^2(y_0)$, *and similarly for the classes* $\widehat{AM}$.

**AM2.** *The derivative of a function of class* $AM_c^2(y_0)$ (*or of a function of class* $AM_s^2(y_0)$, *multiplied by* $i$) *is nonpositive on* $(y_0, +\infty)$, *tends to zero at infinity, and is integrable on* $[y_0, +\infty)$.

*Proof.* For definiteness, we prove the statement in the case where the function $a$ is even. Obviously, formula (4.1.2) implies that $a$ is monotone decreasing. Therefore,

$$\int_{y_0}^{+\infty} |a'| = a(y_0) < +\infty.$$

We check that $a'(y) \xrightarrow[y \to +\infty]{} 0$. For all $y > y_0$, by the inequality $xe^{-\gamma x} \leq \frac{1}{\gamma e}$ ($x, \gamma > 0$), we have

$$0 < -\left(e^{-y^2 u}\right)'_y = 2yue^{-y^2 u} \leq \frac{2y}{e(y^2 - y_0^2)} e^{-y_0^2 u},$$

$$0 \leq -a'(y) \leq \frac{2y}{e(y^2 - y_0^2)} a(y_0) \xrightarrow[y \to +\infty]{} 0,$$

which completes the proof. $\square$

**AM3.** *The second derivative of a function of class* $AM^2(y_0)$ *is integrable on* $[y_1, +\infty)$ *for all* $y_1 > y_0$.

*Proof.* The improper integral $\int_{y_1}^{\to +\infty} a'' = -a'(y_1)$ converges by property AM2 (because $a'(y) \xrightarrow[y \to +\infty]{} 0$). Next, we have $a(y) = \alpha(y^2)$ with $\alpha \in AM^1(y_0^2)$. Then

$$a'(y) = 2y\alpha'(y^2), \quad a''(y) = 2\alpha'(y^2) + 4y^2\alpha''(y^2).$$

We write

$$f_1(y) = 2\alpha'(y^2) = \frac{a'(y)}{y}, \quad f_2(y) = 4y^2\alpha''(y^2).$$

By AM2, the function $f_1$ is positive and integrable on $[y_1, +\infty)$. The function $f_2$ is nonnegative on $(y_0, +\infty)$. Moreover, $f_2 \in L[y_1, +\infty)$ because

$$\int_{y_1}^{\to +\infty} f_2 = \int_{y_1}^{\to +\infty} a'' - \int_{y_1}^{+\infty} f_1 < +\infty.$$

Then $a''$ is also integrable as the sum of two integrable functions. $\square$

**AM4.** *If* $G \in \widehat{AM}^2(y_0)$ *and* $c(G) \in C^{(2)}(\mathbb{R})$, *then*

$$G(t) = \int_{\to -\infty}^{\to +\infty} c(G, y) e^{ity} \, dy, \tag{4.3}$$

$G \in C(\mathbb{R} \setminus \{0\})$, *and* $G(t) = O(t^{-2})$ *as* $t \to \infty$.

*Proof.* The Fourier formula (4.3) is valid for $G$, because the function $a(G)$, respectively $b(G)$, tends to zero monotonically. Since the integral on the right-hand side of (4.3) converges uniformly on every closed ray not containing zero, we have $G \in C(\mathbb{R} \setminus \{0\})$. Integrating by parts twice and using AM2 and AM3, we see that $G(t) = O(t^{-2})$ as $t \to \infty$. $\square$

*Remark* 5. An absolutely monotone function $b$ can tend to zero slowly, so that the function

$$G(t) = \int_0^{\to +\infty} b(y) \sin ty \, dy$$

may fail to be integrable in the vicinity of the point zero.

**AM5.** *If* $G \in \widehat{AM}^2(y_0)$, $c(G) \in C^{(2)}(\mathbb{R})$, *and* $\sigma > 0$, *then formulas* (3.2) *and* (3.3) *are valid for* $L_\sigma(G)$.

*Proof.* For brevity, we shall write $a$, $b$, and $c$ in place of $a(G)$, $b(G)$, and $c(G)$.

Suppose the function $G$ is odd. Then the right-hand side of (3.3) is equal to

$$H_1(y) = \sum_{k=-\infty}^{\infty} c(y + 2k\sigma) = \frac{1}{2i}\Bigl(b(y) - \sum_{k=1}^{\infty}\bigl(b(2k\sigma - y) - b(2k\sigma + y)\bigr)\Bigr). \tag{4.4}$$

The inequality

$$|b(2k\sigma - y) - b(2k\sigma + y)| \le \int_{(2k-1)\sigma}^{(2k+1)\sigma} |b'|,$$

which is true for all $y \in [-\sigma, \sigma]$, and the integrability of $b'$ (property AM2) show that the series on the right-hand side of (4.4) converges uniformly. Therefore, the function $H_1$ is continuous on $[-\sigma, \sigma]$. For $k$ sufficiently large, the terms of the series are monotone decreasing in $y \in [0, \sigma]$. Consequently, $H_1$ has bounded variation on $[-\sigma, \sigma]$, which allows us to expand it in a Fourier series.

Suppose the function $G$ is even. Then the right-hand side of (3.3) is equal to

$$\begin{aligned} H_2(y) &= \sum_{k=-\infty}^{\infty} (-1)^k c(y + 2k\sigma) \\ &= \frac{1}{2}\Bigl(a(y) - \sum_{k=1}^{\infty} (-1)^{k-1}\bigl(a(2k\sigma - y) + a(2k\sigma + y)\bigr)\Bigr). \end{aligned} \tag{4.5}$$

By the Leibniz test, the series on the right-hand side of (4.5) converges uniformly in $y \in [-\sigma, \sigma]$; therefore, $H_2$ is continuous. Grouping the terms of the series,

$$-\sum_{k=1}^{\infty} (-1)^{k-1} a(2k\sigma \pm y) = \sum_{n=1}^{\infty}\bigl(a(4n\sigma \pm y) - a((4n-2)\sigma \pm y)\bigr), \tag{4.6}$$

and using the inequality

$$\bigl|a'(4n\sigma \pm y) - a'((4n-2)\sigma \pm y)\bigr| \le \int_{(4n-3)\sigma}^{(4n+1)\sigma} |a''|,$$

which is true for all $y \in [-\sigma, \sigma]$, and the integrability of $a''$ (property AM3), we conclude that the differentiated series on the right in (4.6) converges uniformly on $[-\sigma, \sigma]$. This shows that the function $e^{i\frac{\pi}{2\sigma}y} H_2(y)$ can be expanded in a Fourier series. $\square$

**Lemma 7.** *Suppose that* $y_0 > 0$, $G$ *is a function in* $\widehat{AM}_c^2(y_0)$ *or* $\widehat{AM}_s^2(y_0)$, *its Fourier transform is expressed by* (4.1.2) *or* (4.2.2), $c(G) \in C^{(2)}(\mathbb{R})$, *and* $\sigma \ge y_0$. *Then,*

$$G(t) - L_\sigma(G, t) = 2 \cos \sigma t \, C_{\sigma,G}(t)$$

*or*

$$G(t) - L_\sigma(G,t) = 2\sin\sigma t\, S_{\sigma,G}(t),$$

*respectively, where the functions*

$$C_{\sigma,G}(t) = \int_0^{+\infty}\int_0^{+\infty}\sum_{\nu=0}^{\infty}(-1)^\nu e^{-((2\nu+1)\sigma+y)^2\lambda}\cos ty\, dyd\Phi(\lambda),$$
$$S_{\sigma,G}(t) = \int_0^{+\infty}\int_0^{+\infty}\sum_{\nu=0}^{\infty} e^{-((2\nu+1)\sigma+y)^2\lambda}\cos ty\, dyd\Psi(\lambda)$$

*are positive if* $\Phi$ *and* $\Psi$ *are nonconstant. Also, the conclusion of Lemma* 5 *is valid.*

*Proof.* We note that, by property AM4, $L_\sigma(G)$ is well defined under the assumptions of the lemma, and that, by property AM4, we can use the expressions (3.2) and (3.3) for $L_\sigma(G)$. First, let $a(G,y) = e^{-\lambda y^2}$, $b(G,y) = e^{-\lambda y^2}$ $(\lambda, y > 0)$, i.e., $G = \frac{1}{2}W_\lambda$ or $G = \frac{1}{2}\widetilde{W}_\lambda$. We denote

$$C_{\sigma\lambda}(t) = \int_0^{+\infty}\sum_{\nu=0}^{\infty}(-1)^\nu e^{-((2\nu+1)\sigma+y)^2\lambda}\cos ty\, dy,$$
$$S_{\sigma\lambda}(t) = \int_0^{+\infty}\sum_{\nu=0}^{\infty} e^{-((2\nu+1)\sigma+y)^2\lambda}\cos ty\, dy.$$

Multiplying these integrals by $2\cos\sigma t$ and $2\sin\sigma t$ and using (3.2) and (3.3), we get

$$\begin{aligned} 2\cos\sigma t\, C_{\sigma\lambda}(t) &= \int_0^{+\infty}\sum_{\nu=0}^{\infty}(-1)^\nu e^{-((2\nu+1)\sigma+y)^2\lambda}(\cos t(\sigma+y) + \cos t(\sigma-y))\, dy \\ &= \int_\sigma^{+\infty}\sum_{\nu=0}^{\infty}(-1)^\nu e^{-(2\nu\sigma+y)^2\lambda}\cos ty\, dy \\ &\quad - \int_{-\sigma}^{+\infty}\sum_{\nu=1}^{\infty}(-1)^\nu e^{-(2\nu\sigma+y)^2\lambda}\cos ty\, dy \\ &= \int_\sigma^{+\infty} e^{-y^2\lambda}\cos ty\, dy - \int_{-\sigma}^{\sigma}\sum_{\nu=1}^{\infty}(-1)^\nu e^{-(2\nu\sigma+y)^2\lambda}\cos ty\, dy \\ &= \frac{1}{2}W_\lambda(t) - \frac{1}{2}L_\sigma(W_\lambda, t); \end{aligned} \tag{4.7}$$

$$\begin{aligned} 2\sin\sigma t\, S_{\sigma\lambda}(t) &= \int_0^{+\infty}\sum_{\nu=0}^{\infty} e^{-((2\nu+1)\sigma+y)^2\lambda}(\sin t(\sigma+y) + \sin t(\sigma-y))\, dy \\ &= \int_\sigma^{+\infty}\sum_{\nu=0}^{\infty} e^{-(2\nu\sigma+y)^2\lambda}\sin ty\, dy - \int_{-\sigma}^{+\infty}\sum_{\nu=1}^{\infty} e^{-(2\nu\sigma+y)^2\lambda}\sin ty\, dy \\ &= \int_\sigma^{+\infty} e^{-y^2\lambda}\sin ty\, dy - \int_{-\sigma}^{\sigma}\sum_{\nu=1}^{\infty} e^{-(2\nu\sigma+y)^2\lambda}\sin ty\, dy \\ &= \frac{1}{2}\widetilde{W}_\lambda(t) - \frac{1}{2}L_\sigma(\widetilde{W}_\lambda, t). \end{aligned} \tag{4.8}$$

By formula (2.3) with $q = y$ and $\gamma = 2(2\nu+1)\lambda\sigma$, we have

$$(4.9)\quad \begin{aligned} C_{\sigma\lambda}(t) = \frac{\sigma\lambda}{\sqrt{\pi}} \int_0^{+\infty} \left( \int_0^{+\infty} e^{-y^2(s+\lambda)} \cos ty\, dy \right) \\ \times \left( \sum_{\nu=0}^{\infty} (-1)^\nu (2\nu+1) e^{-(2\nu+1)^2(\frac{\lambda^2\sigma^2}{s}+\sigma^2\lambda)} \right) \frac{ds}{s^{3/2}}, \end{aligned}$$

$$(4.10)\quad \begin{aligned} S_{\sigma\lambda}(t) = \frac{\sigma\lambda}{\sqrt{\pi}} \int_0^{+\infty} \left( \int_0^{+\infty} e^{-y^2(s+\lambda)} \cos ty\, dy \right) \\ \times \left( \sum_{\nu=0}^{\infty} (2\nu+1) e^{-(2\nu+1)^2(\frac{\lambda^2\sigma^2}{s}+\sigma^2\lambda)} \right) \frac{ds}{s^{3/2}}. \end{aligned}$$

Termwise integration is justified by the Fubini theorem, because

$$\begin{aligned} &\frac{\sigma\lambda}{\sqrt{\pi}} \int_0^{+\infty} \left( \int_0^{+\infty} e^{-y^2(s+\lambda)}\, dy \right) \left( \sum_{\nu=0}^{\infty} (2\nu+1) e^{-(2\nu+1)^2(\frac{\lambda^2\sigma^2}{s}+\sigma^2\lambda)} \right) \frac{ds}{s^{3/2}} \\ &= \int_0^{+\infty} \sum_{\nu=0}^{\infty} e^{-((2\nu+1)\sigma+y)^2\lambda}\, dy < +\infty. \end{aligned}$$

Formulas (4.9) and (4.10) and Lemma 4 show that the functions $C_{\sigma\lambda}$ and $S_{\sigma\lambda}$ are positive.

In the general case, we integrate identities (4.7) and (4.8) over $\Phi(\lambda)$ and $\Psi(\lambda)$, respectively. For definiteness, while justifying the interchange of the order of operations, we restrict ourselves to the case where $G$ is odd; if $G$ is even, the proof is similar. The second mean value theorem shows that, for all $N > \sigma$ and $t \neq 0$, we have

$$\left| \int_\sigma^N e^{-y^2\lambda} \sin ty\, dy \right| \le \frac{2e^{-\sigma^2\lambda}}{|t|}.$$

The right-hand side of this inequality is integrable over $\Psi(\lambda)$. Substituting the expression for $b(G, y)$ from (4.2.2) and using the Lebesgue dominated convergence theorem, we obtain

$$(4.11)\quad \int_\sigma^{+\infty} b(G, y) \sin ty\, dy = \int_0^{+\infty} \int_\sigma^{+\infty} e^{-y^2\lambda} \sin ty\, dy d\Psi(\lambda).$$

For $t = 0$ identity (4.11) is trivial. We may apply the Fubini theorem to the second term in the next to the last line in (4.8), because

$$\begin{aligned} &\int_{-\sigma}^{\sigma} \sum_{\nu=1}^{\infty} e^{-(2\nu\sigma+y)^2\lambda} \sin ty\, dy \\ &\quad = \int_0^{\sigma} \sum_{\nu=1}^{\infty} \left( e^{-(2\nu\sigma+y)^2\lambda} - e^{-(2\nu\sigma-y)^2\lambda} \right) \sin ty\, dy, \\ &\int_0^{+\infty} \int_0^{\sigma} \sum_{\nu=1}^{\infty} \left| e^{-(2\nu\sigma+y)^2\lambda} - e^{-(2\nu\sigma-y)^2\lambda} \right| dy d\Psi(\lambda) \\ &\quad \le \sigma \int_0^{+\infty} e^{-\sigma^2\lambda}\, d\Psi(\lambda) < +\infty. \end{aligned}$$

Therefore,

$$\int_{-\sigma}^{\sigma} \sum_{\nu=1}^{\infty} b(G, 2\nu\sigma + y) \sin ty\, dy = \int_0^{+\infty} \int_{-\sigma}^{\sigma} \sum_{\nu=1}^{\infty} e^{(2\nu\sigma+y)^2\lambda} \sin ty\, dy d\Psi(\lambda).$$

Consequently,

$$G(t) - L_\sigma(G,t) = \frac{1}{2}\int_0^{+\infty} \left(\widetilde{W}_\lambda(t) - L_\sigma(\widetilde{W}_\lambda, t)\right) d\Psi(\lambda).$$

Substituting the expression (4.8) for the integrand, we get the desired formula for $S_{\sigma,G}(t)$. The positivity of $S_{\sigma,G}$ follows from that of $S_{\sigma\lambda}$.

Thus, the assumptions of Lemma 5 are fulfilled, and so is its conclusion. □

*Remark* 6. The condition that the Fourier transform should be three (two) times monotone is fulfilled for the functions of class $\widehat{AM}^1$, but may fail for functions of class $\widehat{AM}^2$.

*Remark* 7. In Lemma 7, if the Fourier transforms $a(G)$ and $b(G)$ of the function $G$ of class $\widehat{AM}^1(y_0)$ are expressed by formulas (4.1.1) and (4.2.1), then for $G$ even we have

$$\mathcal{K}_{\sigma,G} = \frac{2}{\pi}\int_0^{+\infty} \arctan e^{-\sigma u}\, d\Phi(u),$$
$$C_{\sigma,G}(t) = \frac{1}{2}\int_0^{+\infty} \frac{u}{t^2+u^2}\frac{d\Phi(u)}{\cosh \sigma u},$$

and if $G$ is odd, then

$$\mathcal{K}_{\sigma,G} = \frac{1}{\pi}\int_0^{+\infty} \ln\frac{1+e^{-\sigma u}}{1-e^{-\sigma u}}\, d\Psi(u),$$
$$S_{\sigma,G}(t) = \frac{1}{2}\int_0^{+\infty} \frac{u}{t^2+u^2}\frac{d\Psi(u)}{\sinh \sigma u}.$$

To prove this, it suffices to substitute the expressions for the Fourier transforms in (4.1.1) and (4.2.1) in Lemma 6 and integrate termwise.

Obviously, these formulas imply that of the functions $C_{\sigma,G}$ and $S_{\sigma,G}$ are positive.

**Lemma 8.** 1. *If $F \in L$, $F$ is even, and*

$$a_k(F) = \int_0^{+\infty} e^{-k^2 u}\, d\Phi(u) \quad (k \in \mathbb{N}),$$

*where $\Phi$ is a monotone increasing function on $(0,+\infty)$, then $F$ is nonnegative and monotone decreasing on $(0,\pi]$, and*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |F - F(\pi)| = \sum_{k=1}^{\infty}(-1)^{k-1}a_k(F) = \frac{a_0(F)}{2} - F(\pi).$$

2. *If $F \in L$, $F$ is odd, and*

$$b_k(F) = \int_0^{+\infty} e^{-k^2 u}\, d\Psi(u) \quad (k \in \mathbb{N}),$$

*where $\Psi$ is a function monotone increasing on $(0,+\infty)$, then $F$ is nonnegative on $(0,\pi)$, and*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |F| = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{b_{2k+1}(F)}{2k+1}.$$

*Proof.* 1. The fact that $F$ is monotone decreasing follows from the similar property of the heat conduction kernel and the formula

$$F(t) - \frac{a_0(F)}{2} = \sum_{k=1}^{\infty} \int_0^{+\infty} e^{-k^2 u}\, d\Phi(u) \cos kt$$
$$= \int_0^{+\infty} \sum_{k=1}^{\infty} e^{-k^2 u} \cos kt\, d\Phi(u) = \int_0^{+\infty} \frac{\Pi_u(t) - 1}{2}\, d\Phi(u).$$

Termwise integration is justified by the Lebesgue dominated convergence theorem, because

$$\left| \sum_{k=1}^{N} e^{-k^2 u} \cos kt \right| \leq \frac{e^{-u}}{\sin(t/2)}$$

for all $N \in \mathbb{N}$, $u > 0$ and $t \in (0, \pi]$, which can be proved with the help of the Abel transformation.

The identity for the integral, claimed in the lemma, obviously follows from the fact that $F$ is monotone decreasing.

2. The nonnegativity of $F$ on $(0, \pi)$ follows from statement 3 of Lemma 4 and the relation

$$F(t) = \frac{1}{2} \int_0^{+\infty} \widetilde{\Pi}_u(t)\, d\Psi(u).$$

Now it remains to integrate the Fourier series termwise. □

*Remark* 8. It can be proved that under the assumptions of statement 2 of Lemma 8, for the odd trigonometric polynomial $\ell_n(F)$ of order at most $n-1$ that interpolates $F$ at the nodes $\frac{k\pi}{n}$ $(k \in \mathbb{Z})$ we have

$$\sin nt \big( F(t) - \ell_n(F, t) \big) \geq 0.$$

Statement 2 is obtained by putting $n = 1$. This remark can be used for direct inspection of the approximation for periodic functions.

*Remark* 9. By applying the Abel transformation four times, we can prove that in statement 1 of Lemma 8 the function $F$ will be monotone decreasing if the sequence of the Fourier coefficients is four times monotone.

## §5. Decomposition of kernels and the construction of approximating operators

Let $h, y_0 > 0$. We want to represent the kernel $G$ in the form

$$G = K_h + \delta_h^1(H_h) + M_h, \tag{5.1}$$

where the functions on the right-hand side are integrable on $\mathbb{R}$, the kernel $K_h$ is compactly supported, $K_h(t) = 0$ if $|t| > \frac{h}{2}$, and $M_h \in \mathbf{E}_{y_0}$. Then the convolution is written as

$$\varphi * G = \varphi * K_h + \delta_h^1(\varphi) * H_h + \varphi * M_h.$$

The third term belongs to $\mathbf{E}_{y_0}$ and can be included for $\sigma \geq y_0$ in the approximating aggregate. The approximation of the second term can be estimated by the modulus of continuity (even by the seminorm of the difference) of the function $\varphi$ with step $h$ (for instance, as in Lemmas 5 or 7). In general, the first term can be estimated by the seminorm of $\varphi$, but if the kernel $K_h$ is odd, then this term can be estimated by the modulus of continuity of $\varphi$. We also require that $K_h$ be continuous at the points $\pm\frac{h}{2}$ (in the sequel, this condition will ensure the preservation of the sign of $K_h$ and the sharpness of the estimate for convolutions with such a kernel).

First, we try to guess the functions occurring in (5.1), without taking care of justifying the operations to be made. Later, we shall specify a class of kernels for which these operations are legal.

We write the Fourier expansion of $K_h$ on the segment $[-h/2, h/2]$:

$$K_h(t) = \sum_{s=-\infty}^{\infty} \lambda_{sh} e^{i\frac{2\pi s}{h}t}, \quad |t| \le \frac{h}{2}.$$

Then the Fourier transform of $K_h$ looks like the following:

$$\begin{aligned} c(K_h, y) &= \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \lambda_{sh} \int_{-h/2}^{h/2} e^{i\frac{2\pi s}{h}t} e^{-iyt}\, dt \\ &= \frac{h}{2\pi} \sum_{s=-\infty}^{\infty} \lambda_{sh} \frac{\sin\left(\left(\frac{2\pi s}{h} - y\right)\frac{h}{2}\right)}{\left(\frac{2\pi s}{h} - y\right)\frac{h}{2}} = \frac{h}{2\pi} \sum_{s=-\infty}^{\infty} \lambda_{sh} \frac{(-1)^{s-1}\sin\frac{hy}{2}}{\pi s - \frac{hy}{2}} \end{aligned}$$

(for $\pi s = \frac{hy}{2}$ the fraction is assumed to be equal to 1, which corresponds to its continuity with respect to $y$). By (5.1) and the well-known expansion of the cosecant into partial fractions

$$\frac{1}{\sin a\pi} = \frac{1}{\pi} \sum_{q=-\infty}^{\infty} \frac{(-1)^{q-1}}{q-a}, \tag{5.2}$$

for $|y| \ge y_0$ we have

$$c(\delta_h^1(H_h), y) = c(G, y) - c(K_h, y) = \sum_{s=-\infty}^{\infty} \left(c(G, y) - \frac{h}{2\pi}\lambda_{sh}\right) \frac{(-1)^{s-1}\sin\frac{hy}{2}}{\pi s - \frac{hy}{2}}.$$

If $\frac{2\pi s}{h} = y$, $|y| \ge y_0$, we must put $\frac{h}{2\pi}\lambda_{sh} = c(G, y)$. Let $c_0(y) = c(G, y)$ for $|y| \ge y_0$, and let

$$\begin{aligned} c_0(0) &= \sum_{s=-\infty}^{\infty}{}^{\circ} (-1)^{s-1} c\left(G, \frac{2\pi s}{h}\right), \\ \frac{h}{2\pi}\lambda_{sh} &= c_0\left(\frac{2\pi s}{h}\right), \quad s \in \mathbb{Z}. \end{aligned}$$

Then $\lambda_0 = \sum_{s=-\infty}^{\infty}{}^{\circ} (-1)^{s-1}\lambda_{sh}$, and $K_h\left(\pm\frac{h}{2}\right) = 0$. Now the Fourier transform of the function $H_h$ can be recovered by (5.1):

$$\begin{aligned} c(K_h, y) &= \sum_{s=-\infty}^{\infty} c_0\left(\frac{2\pi s}{h}\right) \frac{(-1)^{s-1}\sin\frac{hy}{2}}{\pi s - \frac{hy}{2}}, \quad y \in \mathbb{R}, \\ c(H_h, y) &= \frac{1}{2i} \sum_{s=-\infty}^{\infty} \left(c(G, y) - c_0\left(\frac{2\pi s}{h}\right)\right) \frac{(-1)^{s-1}}{\pi s - \frac{hy}{2}}, \quad |y| \ge y_0 \end{aligned}$$

(for $\pi s = \frac{hy}{2}$ the value of the $s$th term is defined by continuity). The values $c(G, y)$ and $c(H_h, y)$ for $|y| < y_0$ do not matter here; they will contribute to the term $M_h$.

The decomposition process of type (5.1) can be applied to the kernel $H_h$ and continued further.

Now we pass to formal considerations.

Suppose $y_0 > 0$, $c_0 \in AM^2(y_0)$, and $0 < h < \frac{2\pi}{y_0}$. We put $c_{h0}(y) = c_0(y)$ ($|y| \geq y_0$) and

$$
\begin{aligned}
c_{h\nu}(y) &= \frac{1}{2i} \sum_{s=-\infty}^{\infty} \left( c_{h,\nu-1}(y) - c_{h,\nu-1}\left(\frac{2\pi s}{h}\right) \right) \frac{(-1)^{s-1}}{\pi s - hy/2}, \quad |y| \geq y_0, \ \nu \in \mathbb{N}, \\
c_{h\nu}(0) &= \sum_{s=-\infty}^{\infty}{}^{\circ} (-1)^{s-1} c_{h\nu}\left(\frac{2\pi s}{h}\right), \quad \nu \in \mathbb{Z}_+, \\
\lambda_{sh\nu} &= \frac{2\pi}{h} c_{h\nu}\left(\frac{2\pi s}{h}\right), \quad s \in \mathbb{Z}, \ \nu \in \mathbb{Z}_+, \\
\varkappa_{h\nu}(y) &= \sum_{s=-\infty}^{\infty} c_{h\nu}\left(\frac{2\pi s}{h}\right) \frac{(-1)^{s-1} \sin \frac{hy}{2}}{\pi s - hy/2}, \quad y \in \mathbb{R}, \ \nu \in \mathbb{Z}_+, \\
K_{h\nu}(t) &= \begin{cases} \sum_{s=-\infty}^{\infty} \lambda_{sh\nu} e^{i\frac{2\pi s}{h}t}, & |t| \leq \frac{h}{2}, \\ 0, & |t| > \frac{h}{2}, \end{cases} \quad \nu \in \mathbb{Z}_+.
\end{aligned}
$$

We introduce the functions $a_{h\nu}$ and $b_{h\nu}$ in accordance with the general agreement (1.10).

If the function $c_0$ is odd,

$$c_0(y) = \frac{1}{2i} b_0(y),$$

then the function $c_1$ is even, and for $|y| \geq y_0$ we have

$$
\begin{aligned}
-a_{h1}(y) = -2c_{h1}(y) &= \sum_{\nu=-\infty}^{\infty}{}^{\circ} \left( b_0(y) - b_0\left(\frac{2\pi\nu}{h}\right) \right) \frac{(-1)^{\nu-1}}{2\pi\nu - hy} + \frac{b_0(y)}{hy} \\
&= \frac{2}{h} \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \frac{y b_0(y) - \frac{2\pi\nu}{h} b_0\left(\frac{2\pi\nu}{h}\right)}{\left(\frac{2\pi\nu}{h}\right)^2 - y^2} + \frac{b_0(y)}{hy}.
\end{aligned}
\tag{5.3}
$$

If the function $c_0$ is even,

$$c_0(y) = \frac{1}{2} a_0(y),$$

then the function $c_1$ is odd, and for $|y| \geq y_0$ we have

$$
\begin{aligned}
b_{h1}(y) &= 2ic_{h1}(y) \\
&= \sum_{\nu=-\infty}^{\infty}{}^{\circ} \left( a_0(y) - a_0\left(\frac{2\pi\nu}{h}\right) \right) \frac{(-1)^{\nu-1}}{2\pi\nu - hy} \\
&\quad + \left( a_0(y) - 2\sum_{\nu=1}^{\infty} (-1)^{\nu-1} a_0\left(\frac{2\pi\nu}{h}\right) \right) \frac{1}{hy} \\
&= \frac{2}{h} \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \frac{y^2 a_0(y) - \left(\frac{2\pi\nu}{h}\right)^2 a_0\left(\frac{2\pi\nu}{h}\right)}{y\left(\left(\frac{2\pi\nu}{h}\right)^2 - y^2\right)} + \frac{a_0(y)}{hy}.
\end{aligned}
\tag{5.4}
$$

**Lemma 9.** *Suppose $y_0 > 0$, $0 < h < \frac{2\pi}{y_0}$.*

1. *If $c_0 \in AM_c^2(y_0)$ and $a_0(y)$ is expressed by formula (4.1.2) for $y \geq y_0$, then $c_{h1} \in AM_s^2(y_0)$, and for $y \geq y_0$ we have*

$$b_{h1}(y) = \int_0^{+\infty} \Psi_1(u) e^{-y^2 u}\, du,$$

*where*

$$\Psi_1(u) = \frac{1}{h}\int_0^{+\infty} \psi_1(u,\lambda)\,d\Phi(\lambda),$$
$$\psi_1(u,\lambda) = \frac{2}{\sqrt{\pi}}\int_0^{\min\{u,\lambda\}} \sum_{\nu=1}^{\infty}(-1)^{\nu-1}\left(\frac{2\pi\nu}{h}\right)^2 e^{\left(\frac{2\pi\nu}{h}\right)^2(v-\lambda)}\frac{dv}{\sqrt{u-v}}.$$

2. *If* $c_0 \in AM_s^2(y_0)$ *and* $b_0(y)$ *is expressed by formula* (4.2.2) *for* $y \ge y_0$, *then* $-c_{h1} \in AM_c^2(y_0)$, *and for* $y \ge y_0$ *we have*

$$-a_{h1}(y) = \int_0^{+\infty} \Phi_1(u)e^{-y^2u}\,du,$$

*where*

$$\Phi_1(u) = \frac{1}{h}\int_0^{+\infty} \varphi_1(u,\lambda)\,d\Psi(\lambda),$$
$$\varphi_1(u,\lambda) = \begin{cases} 2\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\frac{2\pi\nu}{h}e^{(\frac{2\pi\nu}{h})^2(u-\lambda)}, & 0<u<\lambda,\\ \frac{2}{\sqrt{\pi}}\int_{u-\lambda}^{+\infty}\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\left(\frac{2\pi\nu}{h}\right)^2 e^{(\frac{2\pi\nu}{h})^2(u-v-\lambda)}\frac{dv}{\sqrt{v}}, & u>\lambda.\end{cases}$$

*Proof.* The parity of the function $c_{h1}$ has already been discussed. First, let

$$a_0(y) = e^{-\lambda y^2}, \quad b_0(y) = (\operatorname{sgn} y)e^{-\lambda y^2}$$

($\lambda > 0$). We deduce formulas for $a_{h1}(y)$ and $b_{h1}(y)$ with $y \ge y_0$. By (5.3) and (5.4),

$$-ha_{h1}(y) = 2\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\frac{ye^{-\lambda y^2} - \frac{2\pi\nu}{h}e^{-\lambda\left(\frac{2\pi\nu}{h}\right)^2}}{\left(\frac{2\pi\nu}{h}\right)^2 - y^2} + \frac{e^{-\lambda y^2}}{y}, \tag{5.5}$$

$$hb_{h1}(y) = 2\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\frac{y^2e^{-\lambda y^2} - \left(\frac{2\pi\nu}{h}\right)^2 e^{-\lambda\left(\frac{2\pi\nu}{h}\right)^2}}{y\left(\left(\frac{2\pi\nu}{h}\right)^2 - y^2\right)} + \frac{e^{-\lambda y^2}}{y}. \tag{5.6}$$

Denoting $p = y^2$, $q = \left(\frac{2\pi\nu}{h}\right)^2$ and substituting (2.4)–(2.6) in (5.5) and (5.6), we obtain

$$\begin{aligned} &-\sqrt{\pi}ha_{h1}(y)\\ &\quad= 2\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\Bigg(\sqrt{\pi q}\int_0^{\lambda} e^{-pu}e^{q(u-\lambda)}\,du\\ &\qquad\qquad - \int_{\lambda}^{+\infty} e^{-pu}\left(\frac{1}{\sqrt{u-\lambda}} - \int_{u-\lambda}^{+\infty}\frac{qe^{q(u-v-\lambda)}}{\sqrt{v}}\,dv\right)du\Bigg) + \int_{\lambda}^{+\infty}\frac{e^{-pu}}{\sqrt{u-\lambda}}\,du,\\ &\sqrt{\pi}hb_{h1}(y)\\ &\quad= 2\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\Bigg(\int_0^{\lambda} e^{-pu}\int_0^{u}\frac{qe^{q(v-\lambda)}}{\sqrt{u-v}}\,dvdu\\ &\qquad\qquad + \int_{\lambda}^{+\infty} e^{-pu}\left(-\frac{1}{\sqrt{u-\lambda}} + \int_0^{\lambda}\frac{qe^{q(v-\lambda)}}{\sqrt{u-v}}\,dv\right)du\Bigg) + \int_{\lambda}^{+\infty}\frac{e^{-pu}}{\sqrt{u-\lambda}}\,du. \end{aligned}$$

We are going to interchange the order of summation and integration and justify this. We have

$$\begin{aligned}
&\sqrt{\pi} h b_{h1}(y) \\
&\quad = \lim_{\alpha\to 0+} 2\sum_{\nu=1}^{\infty}(-1)^{\nu-1}e^{-q\alpha} \\
&\qquad\qquad \times \biggl(\int_0^{\lambda} e^{-pu}\int_0^u \frac{qe^{q(v-\lambda)}}{\sqrt{u-v}}\,dvdu \\
&\qquad\qquad\quad + \int_{\lambda}^{+\infty} e^{-pu}\left(-\frac{1}{\sqrt{u-\lambda}}+\int_0^{\lambda}\frac{qe^{q(v-\lambda)}}{\sqrt{u-v}}\,dv\right)du\biggr) + \int_{\lambda}^{+\infty}\frac{e^{-pu}}{\sqrt{u-\lambda}}\,du.
\end{aligned}$$

Since $\lim_{\alpha\to 0+} 2\sum_{\nu=1}^{\infty}(-1)^{\nu-1}e^{-q\alpha}=1$, we can write

$$\sqrt{\pi} h b_{h1}(y) = \lim_{\alpha\to 0+} 2\sum_{\nu=1}^{\infty}(-1)^{\nu-1}e^{-q\alpha}\int_0^{+\infty} e^{-pu}\int_0^{\min\{u,\lambda\}}\frac{qe^{q(v-\lambda)}}{\sqrt{u-v}}\,dvdu.$$

For all positive $\alpha$, summation and integration can be interchanged by the Lebesgue dominated convergence theorem, because

$$\begin{gathered}
\left|\sum_{\nu=1}^{N}(-1)^{\nu-1}e^{-q\alpha}e^{-pu}\frac{qe^{q(v-\lambda)}}{\sqrt{u-v}}\right| \le \sum_{\nu=1}^{\infty}e^{-q\alpha}e^{-pu}\frac{qe^{q(v-\lambda)}}{\sqrt{u-v}} = F_1(u,v), \\
\int_0^{+\infty}\int_0^{\min\{u,\lambda\}} F_1(u,v)\,dvdu < +\infty.
\end{gathered}$$

Consequently,

$$\sqrt{\pi} h b_{h1}(y) = \lim_{\alpha\to 0+}\int_0^{+\infty} e^{-pu}\int_0^{\min\{u,\lambda\}} 2\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\frac{qe^{q(v-\lambda-\alpha)}}{\sqrt{u-v}}\,dvdu.$$

It remains to pass to the limit with respect to $\alpha$ under the symbols of integrals and the sum. By statement 2 of Lemma 4, the function

$$F_2(\tau) = 2\sum_{\nu=1}^{\infty}(-1)^{\nu-1}qe^{-q\tau}$$

is positive on $(0,+\infty)$. Moreover, it is continuous and has finite limits at zero (see [34]) and at $+\infty$; therefore, on $(0,+\infty)$ it is bounded by some constant $M$. Since for all positive $\alpha$ the integrand in the last integral above is dominated by the integrable function $\frac{Me^{-pu}}{\sqrt{u-v}}$, the Lebesgue theorem allows us to pass to the limit under the integrals, and since $F_2$ is continuous, we may do the same under the symbol of the sum.

Thus, we have proved the identity

$$\frac{2}{h}\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\frac{y^2e^{-\lambda y^2}-\left(\frac{2\pi\nu}{h}\right)^2 e^{-\lambda\left(\frac{2\pi\nu}{h}\right)^2}}{y\left(\left(\frac{2\pi\nu}{h}\right)^2-y^2\right)} + \frac{e^{-\lambda y^2}}{hy} = \frac{1}{h}\int_0^{+\infty}\psi_1(u,\lambda)e^{-y^2u}\,du. \tag{5.7}$$

The identity

$$\frac{2}{h}\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\frac{ye^{-\lambda y^2}-\frac{2\pi\nu}{h}e^{-\lambda\left(\frac{2\pi\nu}{h}\right)^2}}{\left(\frac{2\pi\nu}{h}\right)^2-y^2} + \frac{e^{-\lambda y^2}}{hy} = \frac{1}{h}\int_0^{+\infty}\varphi_1(u,\lambda)e^{-y^2u}\,du \tag{5.8}$$

is proved similarly. Statement 2 of Lemma 4 shows that the functions $\varphi_1(u,\lambda)$ and $\psi_1(u,\lambda)$ are positive.

In order to complete the proof of the lemma in the general case, where

$$a_0(y) = \int_0^{+\infty} e^{-\lambda y^2}\,d\Phi(\lambda), \quad b_0(y) = (\operatorname{sgn} y)\int_0^{+\infty} e^{-\lambda y^2}\,d\Psi(\lambda)$$

(the functions $\Phi$ and $\Psi$ are monotone increasing on $(0,+\infty)$), it remains to integrate identities (5.7) and (5.8) over $\Phi(\lambda)$ and $\Psi(\lambda)$, respectively, and to interchange the order of operations on both sides. On the right-hand side, this interchange is justified by the Tonelli theorem. On the left-hand side, the termwise integration is justified by Lemma 2, because the general term of the series without the factor $(-1)^{\nu-1}$ can be represented as the difference of two sequences of functions monotone decreasing to zero and integrable with respect to the measures $d\Phi$ and $d\Psi$: we have

$$\frac{pe^{-\lambda p} - qe^{-\lambda q}}{\sqrt{p}\,(q-p)} = \sqrt{p}\,\frac{e^{-\lambda p} - e^{-\lambda q}}{q-p} - \frac{e^{-\lambda q}}{\sqrt{p}},$$

$$\frac{\sqrt{p}\,e^{-\lambda p} - \sqrt{q}\,e^{-\lambda q}}{q-p} = \sqrt{p}\,\frac{e^{-\lambda p} - e^{-\lambda q}}{q-p} - \frac{e^{-\lambda q}}{\sqrt{p}+\sqrt{q}}$$

for (5.7) and (5.8), respectively. □

A statement similar to Lemma 9 is true also for the classes $AM^1(y_0)$.

**Lemma 10.** *Suppose* $y_0 > 0$, $0 < h < \frac{2\pi}{y_0}$.

1. *If* $c_0 \in AM_c^1(y_0)$ *and* $a_0(y)$ *is expressed by formula* (4.1.1) *for* $y \ge y_0$, *then* $c_{h1} \in AM_s^1(y_0)$, *and for* $y \ge y_0$ *we have*

$$b_{h1}(y) = \int_0^{+\infty} \Psi_1(u) e^{-yu}\,du,$$

*where*

$$\Psi_1(u) = \frac{1}{h}\int_0^{+\infty} \left(\frac{1}{e^{\frac{2\pi}{h}(\lambda+u)}+1} + \frac{1}{e^{\frac{2\pi}{h}(\lambda-u)}+1} - \frac{1}{e^{\frac{2\pi}{h}\lambda}+1}\right) d\Phi(\lambda).$$

2. *If* $c_0 \in AM_s^1(y_0)$ *and* $b_0(y)$ *is expressed by formula* (4.2.1) *for* $y \ge y_0$, *then* $-c_{h1} \in AM_c^1(y_0)$, *and for* $y \ge y_0$ *we have*

$$-a_{h1}(y) = \int_0^{+\infty} \Phi_1(u) e^{-yu}\,du,$$

*where*

$$\Phi_1(u) = \frac{1}{h}\int_0^{+\infty} \left(\frac{1}{e^{\frac{2\pi}{h}(\lambda-u)}+1} - \frac{1}{e^{\frac{2\pi}{h}(\lambda+u)}+1}\right) d\Psi(\lambda).$$

The proof of Lemma 10 is similar to that of Lemma 9 but is simpler technically. We shall not use Lemma 10 in what follows.

Now we formulate several properties of the functions defined above.

**C1.** *The functions* $c_{h\nu}$ *and* $K_{h\nu}$ *have the same parity as* $c_0$ *if* $\nu$ *is even, and their parity is opposite to that of* $c_0$ *if* $\nu$ *is odd.*

This follows from Lemma 9 by induction.

**C2.** *The series that define* $c_{h\nu}(y)$ *for* $|y| \ge y_0$ *and for* $y = 0$ *converge for all* $\nu$, *and the functions* $c_{h\nu}$ *(up to a sign) belong to* $AM^2(y_0)$. *The series that define* $\varkappa_{h\nu}(y)$ *converge as well. The series defining* $K_{h\nu}(t)$ *on* $[-h/2, h/2]$ *converges on* $[-h/2, h/2]$ *everywhere except, possibly, the point* $t = 0$.

The convergence of the series for $c_{h\nu}(y)$ and the fact that $\pm c_{h\nu} \in AM^2(y_0)$ follow from Lemma 9 by induction. The series for $K_{h\nu}(t)$ is (up to a sign) a series in sines or in cosines, with coefficients monotonically decreasing to zero, and therefore converges on $[-h/2, h/2]$ everywhere except, possibly, the point $t = 0$.

**C3.**

$$\lambda_{0h\nu} = \sum_{s=-\infty}^{\infty}{}^{\circ} (-1)^{s-1} \lambda_{sh\nu} \quad (\nu \in \mathbb{Z}_+).$$

This follows immediately from the definitions of $c_{h\nu}(0)$ and $\lambda_{sh\nu}$.

**C4.** *For $|y| \geq y_0$ and $m \in \mathbb{Z}_+$ we have*

$$c_{h0}(y) - \sum_{\nu=0}^{m-1} \left(2i \sin \frac{hy}{2}\right)^{\nu} \varkappa_{h\nu}(y) = \left(2i \sin \frac{hy}{2}\right)^{m} c_{hm}(y). \tag{5.9}$$

*Proof.* We prove property C4 by induction. The case of $m = 0$ is trivial. The induction step from $m-1$ to $m$ looks like the following:

$$\begin{aligned}
&c_{h0}(y) - \sum_{\nu=0}^{m-1} \left(2i \sin \frac{hy}{2}\right)^{\nu} \varkappa_{h\nu}(y) \\
&\quad = \left(2i \sin \frac{hy}{2}\right)^{m-1} (c_{h,m-1}(y) - \varkappa_{h,m-1}(y)) \\
&\quad = \left(2i \sin \frac{hy}{2}\right)^{m-1} \\
&\quad\quad \times \left(c_{h,m-1}(y) \sum_{s=-\infty}^{\infty} \frac{(-1)^{s-1} \sin \frac{hy}{2}}{\pi s - hy/2} - \sum_{s=-\infty}^{\infty} c_{h,m-1}\left(\frac{2\pi s}{h}\right) \frac{(-1)^{s-1} \sin \frac{hy}{2}}{\pi s - hy/2}\right) \\
&\quad = \left(2i \sin \frac{hy}{2}\right)^{m} c_{hm}(y).
\end{aligned}$$

We have used the expansion (5.2). □

*Suppose $G \in \widehat{AM}^2(y_0)$, $c(G) \in C^{(2)}(\mathbb{R})$, and $c(G, y) = c_0(y)$ for every $y$ with $|y| \geq y_0$.* By property AM4, these assumptions imply that $G \in C(\mathbb{R} \setminus \{0\})$, $G(t) = O(t^{-2})$ as $t \to \infty$.

The following two statements will be proved simultaneously by induction on $m$.

**C5.** *For each $m \in \mathbb{Z}_+$, there exists a function $G_{hm} \in L(\mathbb{R})$ having the same parity as $c_{hm}$ and such that $c(G_{hm}) \in C^{(2)}(\mathbb{R})$ and $c(G_{hm}, y) = c_{hm}(y)$ for every $y$ with $|y| \geq y_0$.*

**C6.** *The function $K_{hm}$ is integrable on $\mathbb{R}$ for each $m \in \mathbb{Z}_+$; the series that defines it on $[-h/2, h/2]$ is its Fourier series on this segment, and*

$$c(K_{hm}, y) = \varkappa_{hm}(y), \quad y \in \mathbb{R}, \ m \in \mathbb{Z}_+.$$

*Proof.* First, we note that, for each fixed $m$, C6 follows from C5. It is known (see, e.g., [35]) that if the coefficients of a trigonometric series are the values at equidistant points of the Fourier transform of a function integrable on the line, then this series is the Fourier series of an integrable periodic function. Consequently, $K_{hm} \in L(\mathbb{R})$. Multiplying the series by $e^{-iyt}$ and integrating termwise, we obtain

$$\begin{aligned} c(K_{hm}, y) &= \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \lambda_{shm} \int_{-h/2}^{h/2} e^{i\frac{2\pi s}{h}t} e^{-iyt}\, dt \\ &= \frac{h}{2\pi} \sum_{\nu=-\infty}^{\infty} \lambda_{shm} \frac{\sin\left(\left(\frac{2\pi s}{h} - y\right)\frac{h}{2}\right)}{\left(\frac{2\pi s}{h} - y\right)\frac{h}{2}} = \frac{h}{2\pi} \sum_{s=-\infty}^{\infty} \lambda_{shm} \frac{(-1)^{s-1} \sin\frac{hy}{2}}{\pi s - \frac{hy}{2}} = \varkappa_{hm}(y) \end{aligned}$$

(for $\pi s = \frac{hy}{2}$ the fraction is assumed to be equal to 1, which corresponds to continuity with respect to $y$).

The case where $m = 0$ serves as the base of induction. Statement C5 is assumed to be fulfilled: $G_{h0} = G$, so that statement C6 is also fulfilled.

Suppose statements C5 and C6 are true for all numbers smaller than $m$ ($m \in \mathbb{N}$); we shall prove that C5 (and, consequently, C6) is true for $m$ as well.

By (5.9), we have

$$c\Big(G - \sum_{\nu=0}^{m-1} \delta_h^{\nu}(K_{h\nu}), y\Big) = \left(2i \sin\frac{hy}{2}\right)^m c_{hm}(y), \quad |y| \geq y_0.$$

If $\psi_{hm} \in C^{(2)}(\mathbb{R})$ and $\psi_{hm}(y) = c_{hm}(y)$ for $|y| \geq y_0$, then

$$c\Big(G - \sum_{\nu=0}^{m-1} \delta_h^{\nu}(K_{h\nu}), y\Big) - \left(2i \sin\frac{hy}{2}\right)^m \psi_{hm}(y) \tag{5.10}$$

is a compactly supported function of class $C^{(2)}(\mathbb{R})$; therefore, it is the Fourier transform of an entire function $M_{hm}$ of class $\mathbf{E}_{y_0}$, and moreover, $M_{hm}(t) = O(t^{-2})$ as $t \to \infty$. We put

$$G_{hm}(t) = \int_{\to -\infty}^{\to +\infty} \psi_{hm}(y) e^{ity}\, dy. \tag{5.11}$$

Then $G_{hm} \in C(\mathbb{R} \setminus \{0\})$ and $G_{hm}(t) = O(t^{-2})$. Since the function $\delta_h^m(G_{hm})$ is integrable on $\mathbb{R}$ by (5.10) and (5.11), so is the function $G_{hm}$, as required. $\square$

Thus, we have obtained the following decomposition.

**C7.** *For every $m \in \mathbb{N}$ we have*

$$G = \sum_{\nu=0}^{m-1} \delta_h^{\nu}(K_{h\nu}) + \delta_h^m(G_{hm}) + M_{hm}, \tag{5.12}$$

*and the function $f$ related to $\varphi$ by* (3.1) *is represented in the form*

$$f = \mathcal{P}_{hm}(f) + \sum_{\nu=0}^{m-1} \delta_h^{\nu}(\varphi) * K_{h\nu}, \tag{5.13}$$

*where*

$$\mathcal{P}_{hm}(f) = T + \varphi * M_{hm} + \delta_h^m(\varphi) * G_{hm}.$$

*Moreover, $\varphi * M_{hm} \in \mathbf{E}_{y_0}$.*

Let $\sigma \geq y_0$. We put

$$U_{\sigma hm}(f) = U_{\sigma hm,G}(f) = T + \varphi * M_{hm} + \delta_h^m(\varphi) * L_\sigma(G_{hm}),$$

$$A_{h\nu} = \begin{cases} \frac{2}{\pi}\left| \sum_{s=0}^\infty \frac{1}{2s+1} b_{h\nu}\left(\frac{2\pi(2s+1)}{h}\right)\right| \\ \quad \text{if the parity of } \nu \text{ is opposite to that of } G, \\ \left|\sum_{s=1}^\infty (-1)^{s-1} a_{h\nu}\left(\frac{2\pi s}{h}\right)\right| \\ \quad \text{if the parity of } \nu \text{ coincides with that of } G, \end{cases}$$

$$B_{\sigma hm} = \begin{cases} \frac{2}{\pi}\left| \sum_{s=0}^\infty \frac{1}{2s+1} b_{hm}((2s+1)\sigma)\right| \\ \quad \text{if the parity of } m \text{ is opposite to that of } G, \\ \frac{2}{\pi}\left|\sum_{s=0}^\infty \frac{(-1)^s}{2s+1} a_{hm}((2s+1)\sigma)\right| \\ \quad \text{if the parity of } m \text{ coincides with that of } G. \end{cases}$$

We do not indicate the dependence of the constants on $G$ in the notation: $A_{h\nu} = A_{h\nu,G}$, $B_{\sigma hm} = B_{\sigma hm,G}$. It is clear that $U_{\sigma hm,G}(f) \in \mathbf{E}_\sigma$ and

$$\mathcal{P}_{hm}(f) - U_{\sigma hm}(f) = \delta_h^m(\varphi) * (G_{hm} - L_\sigma(G_{hm})). \tag{5.14}$$

A remark similar to Remark 2 about the operators $X_{\sigma,G}$ is valid also for the operators $U_{\sigma hm,G}$.

## §6. Jackson-type inequalities

**Lemma 11.** *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $y_0 > 0$, $G \in \widehat{AM}^2(y_0)$, $c(G) \in C^{(2)}(\mathbb{R})$, *functions* $f$ *and* $\varphi$ *are related as in* (3.1), $m \in \mathbb{N}$, *and* $0 < h < \frac{2\pi}{y_0}$. *Then*

$$P(f - \mathcal{P}_{hm}(f)) \leq \sum_{\nu=0}^{m-1} A_{h\nu} P(\delta_h^\nu(\varphi)). \tag{6.1}$$

*If, in addition, the kernel* $G$ *is odd, then*

$$P(f - \mathcal{P}_{hm}(f)) \leq \frac{A_{h0}}{2}\omega_1(\varphi, h)_P + \sum_{\nu=1}^{m-1} A_{h\nu} P(\delta_h^\nu(\varphi)).$$

*Proof.* Formula (5.13) implies an inequality of type (6.1) with the coefficients

$$\alpha_{h\nu} = \frac{1}{2\pi}\int_{-h/2}^{h/2} |K_{h\nu}|$$

in place of $A_{h\nu}$. We prove that $\alpha_{h\nu} = A_{h\nu}$. Let $\varkappa$ be the parity of the kernel $G$, and let $\varepsilon_\nu = (-1)^{\frac{(\nu+\varkappa-1)(\nu+\varkappa)}{2}}$.

If the parity of $\nu$ coincides with that of $G$, then the kernel $K_{h\nu}$ is even. By Lemma 9, the sequence

$$\left\{\varepsilon_\nu \frac{2\pi}{h} a\left(G_{h\nu}, \frac{2\pi s}{h}\right)\right\}_{s=1}^\infty$$

of the Fourier coefficients of the kernel $\varepsilon_\nu K_{h\nu}$ on the segment $[-h/2, h/2]$ can be represented in the form (4.1.2) with a monotone increasing function $\Phi$. By statement 1 of Lemma 8 applied to the function

$$F(t) = \varepsilon_\nu K_{h\nu}\left(\frac{h}{2\pi}t\right),$$

the kernel preserves its sign, whence

$$\alpha_{h\nu} = \left|\frac{1}{2\pi}\int_{-h/2}^{h/2} K_{h\nu}\right| = \left|\sum_{s=1}^{\infty}(-1)^{s-1}a\left(G_{h\nu}, \frac{2\pi s}{h}\right)\right| = A_{h\nu}.$$

If the parity of $\nu$ is opposite to that of $G$, then the kernel $K_{h\nu}$ is odd. By Lemma 9, the sequence

$$\left\{\varepsilon_\nu \frac{2\pi}{h} b\left(G_{h\nu}, \frac{2\pi s}{h}\right)\right\}_{s=1}^{\infty}$$

of the Fourier coefficients of the kernel $\varepsilon_\nu K_{h\nu}$ on the segment $[-h/2, h/2]$ can be represented in the form (4.2.2) with a monotone increasing function $\Psi$. By statement 2 of Lemma 8, the kernel preserves its sign, whence

$$\alpha_{h\nu} = \frac{1}{\pi}\left|\int_0^{h/2} K_{h\nu}\right| = \frac{2}{\pi}\left|\sum_{s=0}^{\infty}\frac{1}{2s+1}b\left(G_{h\nu}, \frac{2\pi(2s+1)}{h}\right)\right| = A_{h\nu}.$$

Moreover, in the case of an odd kernel $G$ the initial term can be written as

$$(\varphi * K_{h0})(x)\frac{1}{2\pi}\int_{-h/2}^{h/2}\varphi(x-t)K_{h0}(t)\,dt = -\frac{1}{2\pi}\int_0^{h/2}\delta^1_{2t}(\varphi, x)K_{h0}(t)\,dt \tag{6.2}$$

and estimated by the modulus of continuity:

$$P(\varphi * K_{h0}) \le \frac{A_{h0}}{2}\omega_1(\varphi, h)_P. \qquad \square$$

**Lemma 12.** *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $y_0 > 0$, $G \in \widehat{AM}^2(y_0)$, $c(G) \in C^{(2)}(\mathbb{R})$, *functions* $f$ *and* $\varphi$ *are related as in* (3.1), $m \in \mathbb{N}$, $0 < h < \frac{2\pi}{y_0}$, *and* $\sigma \ge y_0$. *Then*

$$P(\mathcal{P}_{hm}(f) - U_{\sigma hm}(f)) \le B_{\sigma hm} P(\delta_h^m(\varphi)). \tag{6.3}$$

*Proof.* By Lemma 9, the kernel $\varepsilon_m G_{hm}$ belongs to $\widehat{AM}^2(y_0)$. Applying (5.14) and Lemma 7, we obtain inequality (6.3) with the constant $\mathcal{K}_{\sigma,\varepsilon_m G_{hm}} = B_{\sigma hm}$. $\square$

*Remark* 10. Lemmas 5, 9, 11, and 12 show that the sign of the expression under the modulus symbol in the definition of $A_{h\nu}$ and $B_{\sigma hm}$ is equal to $\varepsilon_\nu$ and $\varepsilon_m$, respectively. Also,

$$\operatorname{sgn} K_{h\nu}(t) = \operatorname{sgn}\sin\left(\frac{\pi t}{h} + \frac{(1-\varkappa-\nu)\pi}{2}\right), \quad |t| < \frac{h}{2},$$
$$\operatorname{sgn}\big(G_{hm}(t) - L_\sigma(G_{hm}, t)\big) = \operatorname{sgn}\sin\left(\sigma t + \frac{(1-\varkappa-m)\pi}{2}\right).$$

**Theorem 1.** *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $y_0 > 0$, $G \in \widehat{AM}^2(y_0)$, $c(G) \in C^{(2)}(\mathbb{R})$, *functions* $f$ *and* $\varphi$ *are related as in* (3.1), $m \in \mathbb{N}$, $0 < h < \frac{2\pi}{y_0}$, *and* $\sigma \ge y_0$. *Then*

$$P(f - U_{\sigma hm}(f)) \le \sum_{\nu=0}^{m-1} A_{h\nu} P(\delta_h^\nu(\varphi)) + B_{\sigma hm} P(\delta_h^m(\varphi)),$$
$$P(f - U_{\sigma hm}(f)) \le \sum_{\nu=0}^{m-1} A_{h\nu}\omega_\nu(\varphi, h)_P + B_{\sigma hm}\omega_m(\varphi, h)_P.$$

*If, moreover, the kernel $G$ is odd, then*

$$P(f - U_{\sigma hm}(f)) \le A_{h0}\frac{\omega_1(\varphi,h)_P}{2} + \sum_{\nu=1}^{m-1} A_{h\nu}P(\delta_h^\nu(\varphi)) + B_{\sigma hm}P(\delta_h^m(\varphi)),$$

$$P(f - U_{\sigma hm}(f)) \le A_{h0}\frac{\omega_1(\varphi,h)_P}{2} + \sum_{\nu=1}^{m-1} A_{h\nu}\omega_\nu(\varphi,h)_P + B_{\sigma hm}\omega_m(\varphi,h)_P.$$

*Proof.* Since

$$f - U_{\sigma hm}(f) = f - \mathcal{P}_{hm}(f) + \mathcal{P}_{hm}(f) - U_{\sigma hm}(f), \tag{6.4}$$

Theorem 1 follows from Lemmas 11 and 12. □

*Remark* 11. We have the identity

$$\begin{aligned} f(x) - U_{\sigma hm}(f,x) &= \sum_{\nu=0}^{m-1}\frac{1}{2\pi}\int_{-h/2}^{h/2}\delta_h^\nu(\varphi,x-t)K_{h\nu}(t)\,dt \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\delta_h^m(\varphi,x-t)\big(G_{hm}(t) - L_\sigma(G_{hm},t)\big)\,dt. \end{aligned}\tag{6.5}$$

If the kernel $G$ is odd, then we can write

$$\begin{aligned} f(x) - U_{\sigma hm}(f,x) &= -\frac{1}{2\pi}\int_0^{h/2}\delta_{2t}^1(\varphi,x)K_{h0}(t)\,dt \\ &\quad + \sum_{\nu=1}^{m-1}\frac{1}{2\pi}\int_{-h/2}^{h/2}\delta_h^\nu(\varphi,x-t)K_{h\nu}(t)\,dt \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\delta_h^m(\varphi,x-t)\big(G_{hm}(t) - L_\sigma(G_{hm},t)\big)\,dt. \end{aligned}\tag{6.6}$$

These identities are proved by combining formulas (5.13), (5.14), (6.2), and (6.4).

**Corollary 3.** *Suppose that $(\mathfrak{M},P)\in\mathcal{B}$, $\varphi\in\mathfrak{M}$, $y_0>0$, $G\in\widehat{AM}_s^2(y_0)$, $c(G)\in C^{(2)}(\mathbb{R})$, functions $f$ and $\varphi$ are related as in* (3.1), *$0<h<\frac{2\pi}{y_0}$, and $\sigma\ge y_0$. Then*

$$P(f - U_{\sigma h1}(f)) \le \left(\frac{A_{h0}}{2} + B_{\sigma h1}\right)\omega_1(\varphi,h)_P.$$

For the proof it suffices to put $m=1$ in Theorem 1.

Now we specify some cases where the inequalities obtained above are sharp.

**Theorem 2.** *Suppose that $y_0>0$, $G\in\widehat{AM}_s^2(y_0)$, $c(G)\in C^{(2)}(\mathbb{R})$, functions $f$ and $\varphi$ are related as in* (3.1), *$\alpha$ is an odd positive integer, and $\sigma\ge y_0$. Then*

$$\sup_{\varphi\in CB(\mathbb{R})}\frac{A_{\sigma-0}(f)_\infty}{\omega_1(\varphi,\frac{\pi}{\alpha\sigma})_\infty} = \sup_{\varphi\in CB(\mathbb{R})}\frac{\|f - U_{\sigma,\frac{\pi}{\alpha\sigma},1}(f)\|_\infty}{\omega_1(\varphi,\frac{\pi}{\alpha\sigma})_\infty} = \frac{A_{\frac{\pi}{\alpha\sigma},0}}{2} + B_{\sigma,\frac{\pi}{\alpha\sigma},1}.$$

*The smallest upper bounds do not change if they are taken over the set $L_\infty(\mathbb{R})$ and also if we restrict ourselves to $\frac{2\pi}{\sigma}$-periodic functions with zero mean value.*

*Proof.* The estimates of the upper bounds from above were obtained in Theorem 1. It remains to establish estimates from below. In [25], the $\frac{2\pi}{\sigma}$-periodic odd functions $\varphi_{\sigma\alpha}\in L_\infty(\mathbb{R})$ were considered,

$$\varphi_{\sigma\alpha}(t) = \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{(2\nu+1)\sin\frac{(2\nu+1)\pi}{2\alpha}}\sin(2\nu+1)\sigma t$$

(for $\sigma \in \mathbb{N}$, which is not important). The extremal properties of the function $\varphi_{\sigma 1}(t) = \operatorname{sgn}\sin\sigma t$ are well known. These functions satisfy

$$\delta^1_{\frac{\pi}{\alpha\sigma}}(\varphi_{\sigma\alpha}, t) = 2\operatorname{sgn}\cos\sigma t, \quad \delta^1_u(\varphi_{\sigma\alpha}, 0) = 2 \text{ for } 0 < u \le \frac{\pi}{\alpha\sigma},$$

and $\omega_1\left(\varphi_{\sigma\alpha}, \frac{\pi}{\alpha\sigma}\right)_\infty = 2$. Let $f_{\sigma\alpha} = \varphi_{\sigma\alpha} * G$. Since the functions $\varphi_{\sigma\alpha}$ have zero mean and are $\frac{2\pi}{\sigma}$-periodic, we have $U_{\sigma,\frac{\pi}{\alpha\sigma},1}(f_{\sigma\alpha}) = 0$. We write identity (6.6) for these functions with $m = 1$ and at the points $x = 0$ and $x = \frac{\pi}{\sigma}$:

$$\begin{aligned} f_{\sigma\alpha}(0) &= -f_{\sigma\alpha}\left(\frac{\pi}{\sigma}\right) \\ &= -\frac{1}{2\pi}\int_0^{\frac{\pi}{2\alpha\sigma}} \delta^1_{2t}(\varphi_{\sigma\alpha}, 0) K_{\frac{\pi}{\alpha\sigma},0}(t)\, dt \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \delta^1_{\frac{\pi}{\alpha\sigma}}(\varphi_{\sigma\alpha}, -t)\left(G_{\frac{\pi}{\alpha\sigma},1}(t) - L_\sigma\left(G_{\frac{\pi}{\alpha\sigma},1}, t\right)\right). \end{aligned}$$

Taking the signs of the kernels into account, we obtain

$$\begin{aligned} -f_{\sigma\alpha}(0) = f_{\sigma\alpha}\left(\frac{\pi}{\sigma}\right) &= 2\left(\frac{A_{\frac{\pi}{\alpha\sigma},0}}{2} + B_{\sigma,\frac{\pi}{\alpha\sigma},1}\right) \\ &= \left(\frac{A_{\frac{\pi}{\alpha\sigma},0}}{2} + B_{\sigma,\frac{\pi}{\alpha\sigma},1}\right)\omega_1\left(\varphi_{\sigma\alpha}, \frac{\pi}{\alpha\sigma}\right)_\infty. \end{aligned} \tag{6.7}$$

Since $f_{\sigma\alpha}$ is $\frac{2\pi}{\sigma}$-periodic, it follows (see [1, §99]) that $A_{\sigma-0}(f_{\sigma\alpha})_\infty$ is equal to the best approximation of the function $f_{\sigma\alpha}$ by constant functions. Using the de la Vallée-Poussin theorem on the lower estimate for the best approximation, from (6.7) we immediately deduce the inequality

$$A_{\sigma-0}(f_{\sigma\alpha})_\infty \ge \left(\frac{A_{\frac{\pi}{\alpha\sigma},0}}{2} + B_{\sigma,\frac{\pi}{\alpha\sigma},1}\right)\omega_1\left(\varphi_{\sigma\alpha}, \frac{\pi}{\alpha\sigma}\right)_\infty.$$

For the deviation of the operator $U_{\sigma,\frac{\pi}{\alpha\sigma},1}$ the lower estimate is obvious, because

$$\|f_{\sigma\alpha} - U_{\sigma,\frac{\pi}{\alpha\sigma},1}(f_{\sigma\alpha})\|_\infty = \|f_{\sigma\alpha}\|_\infty \ge A_{\sigma-0}(f_{\sigma\alpha})_\infty.$$

This proves the required estimates from below for the functions $\varphi \in L_\infty(\mathbb{R})$.

Suppose that

$$A_{\sigma-0}(f)_\infty \le K\omega_1(\varphi, h)_\infty \tag{6.8}$$

for all functions belonging to $CB(\mathbb{R})$ (or even only for all entire functions of exponential type that are bounded on $\mathbb{R}$); let $\varphi_\tau$ be the Fejér integral of the function $\varphi$, and let $f_\tau = T + \varphi_\tau * G$. Then for every $\varphi \in L_\infty(\mathbb{R})$ we have

$$A_{\sigma-0}(f_\tau)_\infty \le K\omega_1(\varphi_\tau, h)_\infty \le K\omega_1(\varphi, h)_\infty.$$

Letting $\tau$ on the left-hand side tend to infinity, we see that inequality (6.8) is fulfilled for every $\varphi \in L_\infty(\mathbb{R})$. This argument proves that the least upper bound for the approximations $A_{\sigma-0}$ will not change if we take it over the functions $\varphi$ belonging to $L_\infty(\mathbb{R})$. The statement about the deviation of the operator $U_{\sigma,\frac{\pi}{\alpha\sigma},1}$ is proved similarly. □

*Remark* 12. Under the assumptions of Theorem 2, we have

$$\frac{A_{\frac{\pi}{\alpha\sigma},0}}{2} + B_{\sigma,\frac{\pi}{\alpha\sigma},1} = \frac{1}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu b(G,(2\nu+1)\sigma)}{(2\nu+1)\sin\frac{(2\nu+1)\pi}{2\alpha}}.$$

Indeed, in the proof of Theorem 2 it was established that the constant is equal to $-\frac{f_{\sigma\alpha}(0)}{2}$. Convolving $\varphi_{\sigma\alpha}$ with $G$, we get the series on the right-hand side (similar calculations were done in more detail in [1]).

**Lemma 13.** *Suppose $y_0 > 0$, $G \in \widehat{AM}^2(y_0)$, $c(G) \in C^{(2)}(\mathbb{R})$, $m \in \mathbb{N}$, and $\sigma \geq y_0$. Then*

$$U_{\sigma,\frac{\pi}{\sigma},m,G} = X_{\sigma,G}.$$

*Proof.* Let $f$ and $\varphi$ be functions related as in (3.1). Without loss of generality we may assume that $T = 0$. By the definition of the operator $U_{\sigma,\frac{\pi}{\sigma},m,G}$, we have

$$U_{\sigma,\frac{\pi}{\sigma},m,G}(f) = \varphi * M_{\frac{\pi}{\sigma},m} + \delta^m_{\frac{\pi}{\sigma}}(\varphi) * L_\sigma(G_{\frac{\pi}{\sigma},m}) = \varphi * \Big( M_{\frac{\pi}{\sigma},m} + \delta^m_{\frac{\pi}{\sigma}}\big(L_\sigma\big(G_{\frac{\pi}{\sigma},m}\big)\big)\Big).$$

But $\delta^m_{\frac{\pi}{\sigma}}\big(L_\sigma\big(G_{\frac{\pi}{\sigma},m}\big)\big)$ is an entire function that belongs to $\mathbf{E}_\sigma \cap L(\mathbb{R})$, has the same parity as $G$, and takes the same values at the points $\frac{(2k+1-\varkappa)\pi}{2\sigma}$ as the function $\delta^m_{\frac{\pi}{\sigma}}\big(G_{\frac{\pi}{\sigma},m}\big)$. Since such an interpolating function is unique (see [30]), we have

$$\delta^m_{\frac{\pi}{\sigma}}\big(L_\sigma\big(G_{\frac{\pi}{\sigma},m}\big)\big) = L_\sigma\big(\delta^m_{\frac{\pi}{\sigma}}\big(G_{\frac{\pi}{\sigma},m}\big)\big).$$

Also, $L_\sigma(M_{\frac{\pi}{\sigma},m}) = M_{\frac{\pi}{\sigma},m}$. Next, identity (5.12) implies

$$M_{\frac{\pi}{\sigma},m} + \delta^m_{\frac{\pi}{\sigma}}\big(L_\sigma\big(G_{\frac{\pi}{\sigma},m}\big)\big) = L_\sigma\Big(G - \sum_{\nu=0}^{m-1} \delta^\nu_{\frac{\pi}{\sigma}}\big(K_{\frac{\pi}{\sigma},\nu}\big)\Big).$$

Since $K_{\frac{\pi}{\sigma},\nu}\big(\frac{(2k+1)\pi}{2\sigma}\big) = 0$ for all $k \in \mathbb{Z}$, the functions $\delta^\nu_{\frac{\pi}{\sigma}}\big(K_{\frac{\pi}{\sigma},\nu}\big)$ vanish at the interpolation points. Consequently,

$$L_\sigma\Big(G - \sum_{\nu=0}^{m-1} \delta^\nu_{\frac{\pi}{\sigma}}\big(K_{\frac{\pi}{\sigma},\nu}\big)\Big) = L_\sigma(G),$$

which finishes the proof of the lemma:

$$U_{\sigma,\frac{\pi}{\sigma},m,G}(f) = \varphi * L_\sigma(G) = X_{\sigma,G}(f). \qquad \square$$

**Corollary 4.** *Suppose that $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $y_0 > 0$, $G \in \widehat{AM}^2_s(y_0)$, $c(G) \in C^{(2)}(\mathbb{R})$, functions $f$ and $\varphi$ are related as in* (3.1), *and $\sigma \geq y_0$. Then*

$$P(f - X_{\sigma,G}(f)) \leq \frac{\mathcal{K}_{\sigma,G}}{2}\omega_1\left(\varphi, \frac{\pi}{\sigma}\right)_P.$$

**Corollary 5.** *Suppose that $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $y_0 > 0$, $G \in \widehat{AM}^2(y_0)$, $c(G) \in C^{(2)}(\mathbb{R})$, functions $f$ and $\varphi$ are related as in* (3.1), *$m \in \mathbb{N}$, and $\sigma \geq y_0$. Then*

$$P(f - X_{\sigma,G}(f)) \leq \sum_{\nu=0}^{m-1} A_{\frac{\pi}{\sigma},\nu} P(\delta^\nu_{\frac{\pi}{\sigma}}(\varphi)) + \Big(\mathcal{K}_{\sigma,G} - \sum_{\nu=0}^{m-1} 2^\nu A_{\frac{\pi}{\sigma},\nu}\Big) 2^{-m} P(\delta^m_{\frac{\pi}{\sigma}}(\varphi)),$$

$$P(f - X_{\sigma,G}(f)) \leq \sum_{\nu=0}^{m-1} A_{\frac{\pi}{\sigma},\nu}\omega_\nu\left(\varphi, \frac{\pi}{\sigma}\right)_P + \Big(\mathcal{K}_{\sigma,G} - \sum_{\nu=0}^{m-1} 2^\nu A_{\frac{\pi}{\sigma},\nu}\Big) 2^{-m}\omega_m\left(\varphi, \frac{\pi}{\sigma}\right)_P.$$

*If, moreover, the kernel $G$ is odd, then*

$$P(f - X_{\sigma,G}(f)) \le A_{\frac{\pi}{\sigma},0} \frac{\omega_1\left(\varphi, \frac{\pi}{\sigma}\right)_P}{2} + \sum_{\nu=1}^{m-1} A_{\frac{\pi}{\sigma},\nu} P(\delta^{\nu}_{\frac{\pi}{\sigma}}(\varphi))$$
$$+ \Big(\mathcal{K}_{\sigma,G} - \sum_{\nu=0}^{m-1} 2^{\nu} A_{\frac{\pi}{\sigma},\nu}\Big) 2^{-m} P(\delta^{m}_{\frac{\pi}{\sigma}}(\varphi)),$$
$$P(f - X_{\sigma,G}(f)) \le A_{\frac{\pi}{\sigma},0} \frac{\omega_1\left(\varphi, \frac{\pi}{\sigma}\right)_P}{2} + \sum_{\nu=1}^{m-1} A_{\frac{\pi}{\sigma},\nu} \omega_{\nu}\left(\varphi, \frac{\pi}{\sigma}\right)_P$$
$$+ \Big(\mathcal{K}_{\sigma,G} - \sum_{\nu=0}^{m-1} 2^{\nu} A_{\frac{\pi}{\sigma},\nu}\Big) 2^{-m} \omega_m\left(\varphi, \frac{\pi}{\sigma}\right)_P.$$

*Proof.* By Lemma 13, it suffices to check the identity

$$B_{\sigma,\frac{\pi}{\sigma},m} = 2^{-m}\Big(\mathcal{K}_{\sigma,G} - \sum_{\nu=0}^{m-1} 2^{\nu} A_{\frac{\pi}{\sigma},\nu}\Big).$$

We substitute the function $f_{\sigma 1}$, $h = \frac{\pi}{\sigma}$, and $x = \frac{(1-\varkappa)\pi}{2\sigma}$ ($\varkappa$ is the parity of the kernel $G$) in (6.5). For this function we have

$$f_{\sigma 1}\left(\frac{(1-\varkappa)\pi}{2\sigma}\right) = (-1)^{\varkappa}\mathcal{K}_{\sigma,G}, \quad X_{\sigma,G}(f_{\sigma 1}) = 0,$$
$$\delta^{\nu}_{\frac{\pi}{\sigma}}(\varphi_{\sigma 1}, t) = 2^{\nu} \operatorname{sgn} \sin\left(\sigma t + \frac{\nu\pi}{2}\right).$$

Taking the signs of the kernels (specified in Remark 10) into account, we get

$$(-1)^{\varkappa}\mathcal{K}_{\sigma,G} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \delta^{m}_{\frac{\pi}{\sigma}}\Big(\varphi, \frac{(1-\varkappa)\pi}{2\sigma} - t\Big) G_{\frac{\pi}{\sigma},m}(t)\,dt$$
$$+ \sum_{\nu=0}^{m-1} \frac{1}{2\pi}\int_{-\frac{\pi}{2\sigma}}^{\frac{\pi}{2\sigma}} \delta^{\nu}_{\frac{\pi}{\sigma}}\Big(\varphi, \frac{(1-\varkappa)\pi}{2\sigma} - t\Big) K_{\frac{\pi}{\sigma},\nu}(t)\,dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2^m \operatorname{sgn}\sin\left(-\sigma t + \frac{(m+1-\varkappa)\pi}{2}\right) G_{\frac{\pi}{\sigma},m}(t)\,dt$$
$$+ \sum_{\nu=0}^{m-1} \frac{1}{2\pi}\int_{-\frac{\pi}{2\sigma}}^{\frac{\pi}{2\sigma}} 2^{\nu} \operatorname{sgn}\sin\left(-\sigma t + \frac{(\nu+1-\varkappa)\pi}{2}\right) K_{\frac{\pi}{\sigma},\nu}(t)\,dt$$
$$= (-1)^{\varkappa}\Big(2^m B_{\sigma,\frac{\pi}{\sigma},m} + \sum_{\nu=0}^{m-1} 2^{\nu} A_{\frac{\pi}{\sigma},\nu}\Big),$$

which completes the proof. □

*Remark* 13. From what has been proven above it follows that

$$\sum_{\nu=0}^{\infty} 2^{\nu} A_{\frac{\pi}{\sigma},\nu} \le \mathcal{K}_{\sigma,G}.$$

Corollaries 4 and 5 improve the Akhiezer–Kreĭn–Favard type inequalities (see Lemma 7), which are formal consequences of Corollary 5 for $m = 0$. Thus, in the spaces $(CB(\mathbb{R}), \|\cdot\|_{\infty})$ and $(L(\mathbb{R}), \|\cdot\|_1)$ the inequalities remain sharp even if their left-hand sides are replaced by $A_{\sigma}(f)_p$ $(p = 1, \infty)$. For the corresponding spaces of periodic functions, the inequalities are sharp even if their left-hand sides are replaced by $A_{\sigma-0}(f)_p$.

We denote

$$\eta_\nu(\varphi, h)_P = 2^{-\nu} P\big(\delta_h^\nu(\varphi)\big), \quad \zeta_\nu(\varphi, h)_P = 2^{-\nu}\omega_\nu(\varphi, h)_P.$$

Since the sequences $\big\{\eta_\nu(\varphi, h)_P\big\}_{\nu=0}^\infty$ and $\big\{\zeta_\nu(\varphi, h)_P\big\}_{\nu=0}^\infty$ are monotone decreasing, the limits

$$\eta_\infty(\varphi, h)_P = \lim_{\nu\to\infty} \eta_\nu(\varphi, h)_P, \quad \zeta_\infty(\varphi, h)_P = \lim_{\nu\to\infty} \zeta_\nu(\varphi, h)_P$$

exist. It also follows that the right-hand sides of the inequalities in Corollary 5 are monotone decreasing as functions of $m$. Therefore, the best estimate is obtained in the limit as $m \to \infty$.

**Corollary 6.** *Suppose that* $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $y_0 > 0$, $G \in \widehat{AM}^2(y_0)$, $c(G) \in C^{(2)}(\mathbb{R})$, *functions* $f$ *and* $\varphi$ *are related as in* (3.1), *and* $\sigma \geq y_0$. *Then*

$$P(f - X_{\sigma,G}(f)) \leq \sum_{\nu=0}^\infty 2^\nu A_{\frac{\pi}{\sigma},\nu} \eta_\nu \left(\varphi, \frac{\pi}{\sigma}\right)_P + \Big(\mathcal{K}_{\sigma,G} - \sum_{\nu=0}^\infty 2^\nu A_{\frac{\pi}{\sigma},\nu}\Big) \eta_\infty \left(\varphi, \frac{\pi}{\sigma}\right)_P,$$

$$P(f - X_{\sigma,G}(f)) \leq \sum_{\nu=0}^\infty 2^\nu A_{\frac{\pi}{\sigma},\nu} \zeta_\nu \left(\varphi, \frac{\pi}{\sigma}\right)_P + \Big(\mathcal{K}_{\sigma,G} - \sum_{\nu=0}^\infty 2^\nu A_{\frac{\pi}{\sigma},\nu}\Big) \zeta_\infty \left(\varphi, \frac{\pi}{\sigma}\right)_P.$$

*If, moreover, the kernel* $G$ *is odd, then*

$$\begin{aligned} P(f - X_{\sigma,G}(f)) &\leq A_{\frac{\pi}{\sigma},0} \zeta_1 \left(\varphi, \frac{\pi}{\sigma}\right) + \sum_{\nu=1}^\infty 2^\nu A_{\frac{\pi}{\sigma},\nu} \eta_\nu \left(\varphi, \frac{\pi}{\sigma}\right)_P \\ &\quad + \Big(\mathcal{K}_{\sigma,G} - \sum_{\nu=0}^\infty 2^\nu A_{\frac{\pi}{\sigma},\nu}\Big) \eta_\infty \left(\varphi, \frac{\pi}{\sigma}\right)_P, \\ P(f - X_{\sigma,G}(f)) &\leq A_{\frac{\pi}{\sigma},0} \zeta_1 \left(\varphi, \frac{\pi}{\sigma}\right) + \sum_{\nu=1}^\infty 2^\nu A_{\frac{\pi}{\sigma},\nu} \zeta_\nu \left(\varphi, \frac{\pi}{\sigma}\right)_P \\ &\quad + \Big(\mathcal{K}_{\sigma,G} - \sum_{\nu=0}^\infty 2^\nu A_{\frac{\pi}{\sigma},\nu}\Big) \zeta_\infty \left(\varphi, \frac{\pi}{\sigma}\right)_P. \end{aligned}$$

## §7. Application of general theorems to specific kernels

When writing constants explicitly, we restrict ourselves to the case where $m = 1$. We recall that

$$P(f - U_{\sigma h1,G}(f)) \leq A_{h0} P(\varphi) + B_{\sigma h1} \omega_1(\varphi, h)_P,$$

and that if the kernel $G$ is odd, then

$$P(f - U_{\sigma h1,G}(f)) \leq \left(\frac{A_{h0}}{2} + B_{\sigma h1}\right) \omega_1(\varphi, h)_P.$$

In order to obtain further corollaries, we need to verify that the kernels belong to $\widehat{AM}^2$ and to substitute the constants $A_{h0}$ and $B_{\sigma h1}$ in the inequalities of Theorem 1; the Fourier transforms of the functions $G_{hm}$ are expressed by formulas (5.3) and (5.4). It is convenient to write the step of the modulus of continuity in the form $\frac{\pi}{\alpha\sigma}$, $\alpha > 0$. The condition $h < \frac{2\pi}{y_0}$ becomes $\alpha > \frac{y_0}{2\sigma}$. For $\alpha = 1$ the representation of the constant simplifies as in Corollaries 3 and 4, and the inequalities improve (3.4)–(3.8).

**1.** Let $r, y_0 > 0$, let $I_{ry_0}$ be an even function belonging to $L(\mathbb{R})$ and such that $c(I_{ry_0}, y) = \frac{1}{y^r}$ for $y \geq y_0$, and let $c(I_{ry_0}) \in C^{(2)}(\mathbb{R})$. The identity

$$\frac{1}{y^r} = \frac{1}{\Gamma(r)} \int_0^{+\infty} e^{-yu} u^{r-1}\, du \tag{7.1}$$

shows that the kernel $I_{ry_0}$ belongs to $\widehat{AM}^1_c(y_0)$.

**Corollary 7.** *Suppose* $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $r, y_0 > 0$, $\sigma \geq y_0$, $T \in \mathbf{E}_{y_0}$, $\alpha > \frac{y_0}{2\sigma}$, *and*

$$f = \varphi * I_{ry_0} + T. \tag{7.2}$$

*Then*

$$\begin{aligned} &P(f - U_{\sigma, \frac{\pi}{\alpha\sigma}, 1, I_{ry_0}}(f)) \\ &\quad \leq \left(\frac{2}{(2\alpha\sigma)^r} \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s^r}\right) P(\varphi) \\ &\qquad + \frac{4\alpha}{\pi^2\sigma^r} \sum_{s=0}^{\infty} \left(2 \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \frac{(2s+1)^{2-r} - (2\alpha\nu)^{2-r}}{(2s+1)^2\big((2\alpha\nu)^2 - (2s+1)^2\big)} + \frac{1}{(2s+1)^{r+2}}\right) \omega_1\left(\varphi, \frac{\pi}{\alpha\sigma}\right)_P. \end{aligned}$$

*In particular,*

$$\begin{aligned} &P(f - X_{\sigma, I_{ry_0}}(f)) \\ &\quad \leq \left(\frac{1}{2^{r-1}\sigma^r} \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s^r}\right) P(\varphi) \\ &\qquad + \frac{1}{\sigma^r} \left(\frac{2}{\pi} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)^{r+1}} - \frac{1}{2^r} \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s^r}\right) \omega_1\left(\varphi, \frac{\pi}{\sigma}\right)_P. \end{aligned} \tag{7.3}$$

Let $r \in \mathbb{N}$. If $r$ is even, then (see [1, §§101, 102]) for every $y_0 > 0$ every function $f$ belonging to $CB^{(r)}(\mathbb{R})$ or to $W_p^{(r)}(\mathbb{R})$ can be represented in the form (7.2) with $\varphi = (-1)^{r/2} f^{(r)}$. If $r$ is odd, then for every $y_0 > 0$ every function $f$ belonging to $\widetilde{CB}^{(r)}(\mathbb{R})$ or to $\widetilde{W}_p^{(r)}(\mathbb{R})$ can be represented in the form (7.2) with $\varphi = (-1)^{(r+1)/2} \widetilde{f}^{(r)}$. Thus, for these function classes the estimates in question are valid for all $\sigma > 0$.

For $r$ even, (7.3) improves inequalities (1.1) and (1.4), and for $r$ odd, (7.3) improves (1.2) and (1.5),

**2.** Let $r, y_0 > 0$, let $\widetilde{I}_{ry_0}$ be an odd function belonging to $L(\mathbb{R})$ and such that $c(\widetilde{I}_{ry_0}, y) = \frac{1}{iy^r}$ for $y \geq y_0$, and let $c(\widetilde{I}_{ry_0}) \in C^{(2)}(\mathbb{R})$. By (7.1), the kernel $\widetilde{I}_{ry_0}$ belongs to $\widehat{AM}^1_s(y_0)$.

**Corollary 8.** *Suppose* $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $r, y_0 > 0$, $\sigma \geq y_0$, $T \in \mathbf{E}_{y_0}$, $\alpha > \frac{y_0}{2\sigma}$, *and*

$$f = \varphi * \widetilde{I}_{ry_0} + T. \tag{7.4}$$

*Then*

$$\begin{aligned} &P(f - U_{\sigma, \frac{\pi}{\alpha\sigma}, 1, \widetilde{I}_{ry_0}}(f)) \\ &\quad \leq \Bigg\{\frac{2}{\pi(2\alpha\sigma)^r} \sum_{s=0}^{\infty} \frac{1}{(2s+1)^{r+1}} \\ &\qquad + \frac{4\alpha}{\pi^2\sigma^r} \sum_{s=0}^{\infty} (-1)^s \left(2 \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \frac{(2s+1)^{1-r} - (2\alpha\nu)^{1-r}}{(2s+1)\big((2\alpha\nu)^2 - (2s+1)^2\big)} + \frac{1}{(2s+1)^{r+2}}\right)\Bigg\} \\ &\quad \times \omega_1\left(\varphi, \frac{\pi}{\alpha\sigma}\right)_P. \end{aligned}$$

*In particular,*

$$P(f - X_{\sigma,\widetilde{I}_{ry_0}}(f)) \leq \left(\frac{2}{\pi\sigma^r}\sum_{\nu=0}^{\infty}\frac{1}{(2\nu+1)^{r+1}}\right)\omega_1\left(\varphi, \frac{\pi}{\sigma}\right)_P. \tag{7.5}$$

Let $r \in \mathbb{N}$. If $r$ is odd, then (see [1, §§101, 102]) for every $y_0 > 0$ every function $f$ belonging to $CB^{(r)}(\mathbb{R})$ or to $W_p^{(r)}(\mathbb{R})$ can be represented in the form (7.4) with $\varphi = (-1)^{(r-1)/2} f^{(r)}$. If $r$ is even, then for every $y_0 > 0$ every function $f$ belonging to $\widetilde{CB}^{(r)}(\mathbb{R})$ or to $\widetilde{W}_p^{(r)}(\mathbb{R})$ can be represented in the form (7.4) with $\varphi = (-1)^{r/2}\widetilde{f}^{(r)}$. Thus, for these function classes the estimates in question are valid for all $\sigma > 0$.

It is easily seen that, in examples 1 and 2, for the classes $CB^{(r)}(\mathbb{R})$ and $\widetilde{CB}^{(r)}(\mathbb{R})$ the operators do not depend on the choice of $y_0$ in the representations (7.2) and (7.4) and coincide with the convolution operators $X_{\sigma r}$ and $\widetilde{X}_{\sigma r}$.

For $r$ odd, inequality (7.5) turns into (1.6) and (1.7) and improves inequalities (1.1) and (1.4); see the Introduction. For $r$ even, inequality (7.5) reads as follows:

$$P(f - \widetilde{X}_{\sigma r}(f)) \leq \frac{\widetilde{\mathcal{K}}_r}{2\sigma^r}\omega_1\left(\widetilde{f}^{(r)}, \frac{\pi}{\sigma}\right)_P$$

and improves inequalities (1.2) and (1.5). Corollaries 7 and 8 for the classes $C^{(r)}$ were obtained in [25].

*Remark* 14. By comparison of Fourier transforms, it can be proved that for the classes $CB^{(r)}$ the representation (5.13) with $m = 1$ coincides with a version of the Euler–Maclaurin formula (see [24]):

$$\begin{aligned} f = \frac{1}{h}\int_{-h/2}^{h/2} f + \sum_{l=1}^{r-1}\frac{\beta_l(1/2)}{l!}h^{l-1}\delta_h^1(f^{(l-1)}) \\ - \frac{h^r}{r!}\int_{-1/2}^{1/2}(\beta_r(u) - \beta_r(1/2))f^{(r)}(\cdot - uh)\,dt. \end{aligned}$$

Here the $\beta_r$ are the 1-periodic functions that coincide with the Bernoulli polynomials on $(0,1)$. Since the values $\beta_l(1/2)$ are equal to zero for $l$ odd, for the constant $B_{\sigma h1}$ we have

$$B_{\sigma h1} = \sum_{l=0}^{r-1}\frac{|\beta_l(1/2)|}{l!}h^{l-1}\mathcal{K}_{r+l-1}.$$

If $m \geq 2$, the representation (5.13) coincides with the iterations of the Euler–Maclaurin formula that were used by V. V. Zhuk and N. I. Merlina. The constants $B_{\sigma hm}$ can be expressed in the form of linear combinations of the constants $\mathcal{K}_\nu$ (see [24, Chapter 4]).

**3.** For $\lambda > 0$, let

$$P_\lambda(t) = \int_{-\infty}^{+\infty} e^{-\lambda|y|}e^{ity}\,dy = \frac{2\lambda}{\lambda^2 + t^2}$$

be the Poisson kernel. Clearly, $P_\lambda \in \widehat{AM}_c^1(y_0)$ for all $y_0 > 0$. The function $c(P_\lambda)$ does not belong to $C^{(2)}(\mathbb{R})$, but for every $y_0 > 0$ we can "correct" $c(P_\lambda)$ on $(-y_0, y_0)$ so as to obtain a function of class $C^{(2)}(\mathbb{R})$; the new function is the Fourier transform of the kernel $P_{\lambda y_0}$, which belongs to $\widehat{AM}_c^1(y_0)$ and satisfies the conditions of the theorems of the preceding section. The Poisson integral of $\varphi$ can be written in the form

$$f = \varphi * P_\lambda = \varphi * P_{\lambda y_0} + T$$

with $T \in \mathbf{E}_{y_0}$. Since $y_0$ is arbitrary, the estimates are valid for all $\sigma > 0$.

**Corollary 9.** *Suppose* $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $\lambda, \alpha, \sigma > 0$, *and* $f = \varphi * P_\lambda$. *Then*

$$\begin{aligned} &P(f - U_{\sigma, \frac{\pi}{\alpha\sigma}, 1, P_\lambda}(f)) \\ &\quad \le \frac{2}{e^{2\lambda\alpha\sigma}+1} P(\varphi) \\ &\qquad + \left( \frac{8\alpha}{\pi^2} \sum_{s=0}^{\infty} \frac{1}{(2s+1)^2} \sideset{}{'}\sum_{\nu=0}^{\infty} (-1)^{\nu-1} \frac{(2s+1)^2 e^{-\lambda(2s+1)\sigma} - (2\alpha\nu)^2 e^{-2\lambda\alpha\nu\sigma}}{(2\alpha\nu)^2 - (2s+1)^2} \right) \omega_1 \left(\varphi, \frac{\pi}{\alpha\sigma}\right)_P . \end{aligned}$$

*In particular,*

$$P(f - X_{\sigma, P_\lambda}(f)) \le \frac{2}{e^{2\lambda\sigma}+1} P(\varphi) + \left( \frac{2}{\pi} \arctan e^{-\lambda\sigma} - \frac{1}{e^{2\lambda\sigma}+1} \right) \omega_1 \left(\varphi, \frac{\pi}{\sigma}\right)_P . \tag{7.6}$$

**4.** For $\lambda > 0$, let

$$\widetilde{P}_\lambda(t) = \int_{-\infty}^{+\infty} (-i \operatorname{sgn} y) e^{-\lambda|y|} e^{ity}\, dy = \frac{2t}{\lambda^2 + t^2}$$

be the conjugate Poisson kernel. The kernel $\widetilde{P}_\lambda$ belongs to $L_p(\mathbb{R})$ for all $p > 1$, but does not belong to $L(\mathbb{R})$. The conjugate Poisson integral is defined by the formula $f = \varphi * \widetilde{P}_\lambda$ for every $\varphi \in L_p(\mathbb{R})$ with $p < \infty$. For every $y_0 > 0$ we can "correct" $c(\widetilde{P}_\lambda)$ on $(-y_0, y_0)$ up to a function of class $C^{(2)}(\mathbb{R})$; the new function is the Fourier transform of the kernel $\widetilde{P}_{\lambda y_0}$, which belongs to $\widehat{AM}^{1}_{s}(y_0)$ and satisfies the conditions of the theorems of the preceding section. The conjugate Poisson integral can be written as

$$f = \varphi * \widetilde{P}_{\lambda y_0} + T \tag{7.7}$$

with $T \in \mathbf{E}_{y_0}$. In this form it is also defined for the functions $\varphi$ belonging to $L_\infty(\mathbb{R})$. The operator does not depend on the choice of $y_0$; since $y_0$ is arbitrary, the estimates are valid for all $\sigma > 0$. In Corollary 10, to avoid introducing the fictitious parameter $y_0$, we write the convolution in the form $f = \varphi * \widetilde{P}_\lambda$ and interpret this as identity (7.7) if necessary. The same refers to Corollaries 12 and 14 below.

**Corollary 10.** *Suppose* $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $\lambda, \alpha, \sigma > 0$, *and* $f = \varphi * \widetilde{P}_\lambda$. *Then*

$$\begin{aligned} &P(f - U_{\sigma, \frac{\pi}{\alpha\sigma}, 1, \widetilde{P}_\lambda}(f)) \\ &\quad \le \left\{ \frac{1}{\pi} \ln \frac{1 + e^{-2\lambda\alpha\sigma}}{1 - e^{-2\lambda\alpha\sigma}} \right. \\ &\qquad \left. + \frac{8\alpha}{\pi^2} \sum_{s=0}^{\infty} \frac{(-1)^s}{2s+1} \sideset{}{'}\sum_{\nu=0}^{\infty} (-1)^{\nu-1} \frac{(2s+1) e^{-\lambda(2s+1)\sigma} - 2\alpha\nu e^{-2\lambda\alpha\nu\sigma}}{(2\alpha\nu)^2 - (2s+1)^2} \right\} \omega_1 \left(\varphi, \frac{\pi}{\alpha\sigma}\right)_P . \end{aligned}$$

*In particular,*

$$P(f - X_{\sigma, \widetilde{P}_\lambda}(f)) \le \frac{1}{\pi} \ln \frac{1 + e^{-\lambda\sigma}}{1 - e^{-\lambda\sigma}} \omega_1 \left(\varphi, \frac{\pi}{\sigma}\right)_P . \tag{7.8}$$

For periodic functions, inequalities (7.6) and (7.8) improve the inequalities

$$\begin{aligned} P(f - X_{n, P_\lambda}(f)) &\le \frac{4}{\pi} \arctan e^{-\lambda n} P(\varphi), \\ P(f - X_{n, \widetilde{P}_\lambda}(f)) &\le \frac{2}{\pi} \ln \frac{1 + e^{-\lambda n}}{1 - e^{-\lambda n}} P(\varphi), \end{aligned}$$

which were established in the case of the uniform norm by Kreĭn [6] and Sz.-Nagy [7].

**5.** Let $\lambda > 0$. Clearly, $W_\lambda \in \widehat{AM}^2_c(y_0)$ for all $y_0 > 0$. The convolution $f = \varphi * W_\lambda$ is the Weierstrass integral of the function $\varphi$. Since $y_0$ is arbitrary, the estimates are valid for all $\sigma > 0$.

**Corollary 11.** *Suppose* $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $\lambda, \alpha, \sigma > 0$, *and* $f = \varphi * W_\lambda$. *Then*

$$\begin{aligned} &P(f - U_{\sigma,\frac{\pi}{\alpha\sigma},1,W_\lambda}(f)) \\ &\quad\le \left(2\sum_{s=1}^{\infty}(-1)^{s-1}e^{-\lambda(2\alpha s\sigma)^2}\right)P(\varphi) \\ &\qquad + \left(\frac{8\alpha}{\pi^2}\sum_{s=0}^{\infty}\frac{1}{(2s+1)^2}\sum_{\nu=0}^{\infty}{}'(-1)^{\nu-1}\frac{(2s+1)^2e^{-\lambda((2s+1)\sigma)^2}-(2\alpha\nu)^2e^{-\lambda(2\alpha\nu\sigma)^2}}{(2\alpha\nu)^2-(2s+1)^2}\right)\omega_1\left(\varphi, \frac{\pi}{\alpha\sigma}\right)_P. \end{aligned}$$

*In particular,*

$$\begin{aligned} &P(f - X_{\sigma,W_\lambda}(f)) \\ &\quad\le \left(2\sum_{s=1}^{\infty}(-1)^{s-1}e^{-\lambda(2s\sigma)^2}\right)P(\varphi) \\ &\qquad + \left(\frac{2}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{2\nu+1}e^{-\lambda((2\nu+1)\sigma)^2} - \sum_{s=1}^{\infty}(-1)^{s-1}e^{-\lambda(2s\sigma)^2}\right)\omega_1\left(\varphi, \frac{\pi}{\sigma}\right)_P. \end{aligned} \tag{7.9}$$

Inequality (7.9) improves the Akhiezer inequality for periodic functions (see [5]).

**6.** Let $\lambda > 0$. The convolution $f = \varphi * \widetilde{W}_\lambda$ is the conjugate Weierstrass integral of the function $\varphi$. As in example 4, for every $y_0 > 0$ we can "correct" $c(\widetilde{W}_\lambda)$ to get the kernel $\widetilde{W}_{\lambda y_0}$, which belongs to $\widehat{AM}^2_s(y_0)$. The estimates are valid for all $\sigma > 0$.

**Corollary 12.** *Suppose* $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $\lambda, \alpha, \sigma > 0$, *and* $f = \varphi * \widetilde{W}_\lambda$. *Then*

$$\begin{aligned} &P(f - U_{\sigma,\frac{\pi}{\alpha\sigma},1,\widetilde{W}_\lambda}(f)) \\ &\quad\le \left\{\frac{2}{\pi}\sum_{s=0}^{\infty}\frac{e^{-\lambda(2\alpha(2s+1)\sigma)^2}}{2s+1}\right. \\ &\qquad\left. + \frac{8\alpha}{\pi^2}\sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1}\sum_{\nu=0}^{\infty}{}'(-1)^{\nu-1}\frac{(2s+1)e^{-\lambda((2s+1)\sigma)^2}-2\alpha\nu e^{-\lambda(2\alpha\nu\sigma)^2}}{(2\alpha\nu)^2-(2s+1)^2}\right\} \\ &\quad\times\omega_1\left(\varphi, \frac{\pi}{\alpha\sigma}\right)_P. \end{aligned}$$

*In particular,*

$$P(f - X_{\sigma,\widetilde{W}_\lambda}(f)) \le \left(\frac{2}{\pi}\sum_{\nu=0}^{\infty}\frac{e^{-\lambda((2\nu+1)\sigma)^2}}{2\nu+1}\right)\omega_1\left(\varphi, \frac{\pi}{\sigma}\right)_P.$$

**7.** For $\lambda > 0$, let

$$\Theta_\lambda(t) = \int_{-\infty}^{+\infty}\frac{e^{ity}}{\cosh\lambda y}\,dy = \frac{\pi}{\lambda\cosh\frac{\pi t}{2\lambda}}.$$

The kernel $\Theta_\lambda$ arises in the description of classes of analytic functions in the band $\{z : |\operatorname{Im} z| < \lambda\}$ (see [1, §110]). Using formula (2.3), we get

$$\begin{aligned} \frac{1}{\cosh\lambda y} &= \frac{2e^{-\lambda y}}{1+e^{-2\lambda y}} = 2\sum_{l=0}^{\infty}(-1)^l e^{-(2l+1)\lambda y} \\ &= \frac{\lambda}{\sqrt{\pi}}\int_0^{+\infty}e^{-y^2u}\sum_{l=0}^{\infty}(-1)^l(2l+1)e^{-\frac{(2l+1)^2\lambda^2}{4u}}\frac{du}{u^{3/2}}. \end{aligned} \tag{7.10}$$

Statement 2 of Lemma 4 shows that the integrand is positive. Therefore, $\Theta_\lambda \in \widehat{AM}^2_c(y_0)$ for all $y_0 > 0$.

**Corollary 13.** *Suppose* $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $\lambda, \alpha, \sigma > 0$, *and* $f = \varphi * \Theta_\lambda$. *Then*

$$\begin{aligned} &P(f - U_{\sigma, \frac{\pi}{\alpha\sigma}, 1, \Theta_\lambda}(f)) \\ &\quad \le \left(2\sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{\cosh 2\lambda\alpha s\sigma}\right) P(\varphi) \\ &\qquad + \left(\frac{8\alpha}{\pi^2}\sum_{s=0}^{\infty}\frac{1}{(2s+1)^2}\sum_{\nu=0}^{\infty}{}' (-1)^{\nu-1}\frac{\frac{(2s+1)^2}{\cosh\lambda(2s+1)\sigma} - \frac{(2\alpha\nu)^2}{\cosh 2\lambda\alpha\nu\sigma}}{(2\alpha\nu)^2 - (2s+1)^2}\right)\omega_1\left(\varphi, \frac{\pi}{\alpha\sigma}\right)_P. \end{aligned}$$

*In particular*,

$$\begin{aligned} &P(f - X_{\sigma, \Theta_\lambda}(f)) \\ &\quad \le \left(2\sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\cosh 2\lambda s\sigma}\right)P(\varphi) \\ &\qquad + \left(\frac{2}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{(2\nu+1)\cosh\lambda(2\nu+1)\sigma} - \sum_{s=1}^{\infty}\frac{(-1)^{s-1}}{\cosh 2\lambda s\sigma}\right)\omega_1\left(\varphi, \frac{\pi}{\sigma}\right)_P. \end{aligned} \tag{7.11}$$

Inequality (7.11) improves the Akhiezer inequalities for periodic and nonperiodic functions (see [5] and [1, §110, D85]).

**8.** For $\lambda > 0$, let

$$\widetilde{\Theta}_\lambda(t) = \int_{-\infty}^{\infty}(-i\operatorname{sgn} y)\frac{e^{ity}}{\cosh\lambda y}\,dy.$$

Formula (7.10) implies that for every $y_0 > 0$ the "corrected" kernel $\widetilde{\Theta}_{\lambda y_0}$ belongs to $\widehat{AM}^2_s(y_0)$.

**Corollary 14.** *Suppose* $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $\lambda, \alpha, \sigma > 0$, *and* $f = \varphi * \widetilde{\Theta}_\lambda$. *Then*

$$\begin{aligned} &P(f - U_{\sigma, \frac{\pi}{\alpha\sigma}, 1, \widetilde{\Theta}_\lambda}(f)) \\ &\quad \le \left\{\frac{2}{\pi}\sum_{s=0}^{\infty}\frac{1}{(2s+1)\cosh 2\lambda\alpha(2s+1)\sigma}\right. \\ &\qquad \left. + \frac{8\alpha}{\pi^2}\sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1}\sum_{\nu=0}^{\infty}{}'(-1)^{\nu-1}\frac{\frac{2s+1}{\cosh\lambda(2s+1)\sigma} - \frac{2\alpha\nu}{\cosh 2\lambda\alpha\nu\sigma}}{(2\alpha\nu)^2 - (2s+1)^2}\right\}\omega_1\left(\varphi, \frac{\pi}{\alpha\sigma}\right)_P. \end{aligned}$$

*In particular*,

$$P(f - X_{\sigma, \widetilde{\Theta}_\lambda}(f)) \le \left(\frac{2}{\pi}\sum_{\nu=0}^{\infty}\frac{1}{(2\nu+1)\cosh\lambda(2\nu+1)\sigma}\right)\omega_1\left(\varphi, \frac{\pi}{\sigma}\right)_P.$$

The next two examples generalize Examples 1 and 2. Here, we restrict ourselves to statements for $\alpha = 1$.

**9.** Suppose $N \in \mathbb{N}$, $0 < r_j \le 2$, $s_j > 0$, and $\gamma_j \in \mathbb{R}$ $(1 \le j \le N)$. We put $y^* = 0$ if all $\gamma_j$ are nonpositive and we put $y^* = \max_{j:\gamma_j>0}\gamma_j^{1/r_j}$ if at least one $\gamma_j$ is positive.

Suppose that $y_0 > y^*$, $Q_{y_0} \in L(\mathbb{R})$, $Q_{y_0}$ is even, $c(Q_{y_0}) \in C^{(2)}(\mathbb{R})$, and

$$c(Q_{y_0}, y) = \frac{1}{2}a_0(y) = \prod_{j=1}^{N}(y^{r_j} - \gamma_j)^{-s_j} \quad (y \ge y_0).$$

We shall prove that $a_0 \in AM^2(y_0)$. This is equivalent to the absolute monotonicity of the function $g(y) = a(\sqrt{y})$, i.e., to the fact that $g(y)$ can be written in the form (4.1.1). Since the product of absolutely monotone functions is absolutely monotone, it suffices to consider only one of the factors. Suppose $\rho \in (0,1)$, $\gamma \in \mathbb{R}$, $s > 0$, $F(x) = \frac{1}{(x-\gamma)^s}$, $\ell(y) = y^\rho$, and $J(y) = F(\ell(y))$. The function $F$ is absolutely monotone for $x > \gamma$ because $(-1)^m F^{(m)}(x) > 0$ for all $m \in \mathbb{Z}_+$; also, we can write an identity of type (4.1.1):

$$\frac{1}{(x-\gamma)^s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-xu} e^{\gamma u} u^{s-1}\, du.$$

For the function $\ell$ we have $(-1)^m \ell^{(m)}(y) < 0$ for all $y > 0$ and all $m \in \mathbb{N}$. Consequently, by [29, formula 0.430.2],

$$J^{(m)}(y) = \sum_{\substack{i_1,\dots,i_q \in \mathbb{N},\\ i_1+2i_2+\dots+qi_q=m}} \frac{m!}{i_1!\cdots i_q!} F^{(i_1+\dots+i_q)}(\ell(y)) \prod_{j=1}^{q} \left(\frac{\ell^{(j)}(y)}{j!}\right)^{i_j},$$

and we have $(-1)^m J^{(m)}(y) > 0$ for all $x > \gamma^{1/\rho}$ and all $m \in \mathbb{Z}_+$, which implies a representation of $J$ in the form (4.1.1). Thus, $Q_{y_0} \in \widehat{AM}^2_c(y_0)$.

**Corollary 15.** *Suppose* $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $y_0 > y^*$, $T \in \mathbf{E}_{y_0}$, $\sigma \geq y_0$, *and*

$$f = \varphi * Q_{y_0} + T. \tag{7.12}$$

*Then*

$$\begin{aligned} &P(f - X_{\sigma,Q_{y_0}}(f)) \\ &\quad\leq \left(2\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\prod_{j=1}^{N}\frac{1}{((2\sigma\nu)^{r_j}-\gamma_j)^{s_j}}\right)P(\varphi) \\ &\qquad+\left(\frac{2}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{2\nu+1}\prod_{j=1}^{N}\frac{1}{(((2\nu+1)\sigma)^{r_j}-\gamma_j)^{s_j}}-\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\prod_{j=1}^{N}\frac{1}{((2\sigma\nu)^{r_j}-\gamma_j)^{s_j}}\right) \\ &\qquad\times\omega_1\left(\varphi,\frac{\pi}{\sigma}\right)_P. \end{aligned} \tag{7.13}$$

Let $D$ denote the operator of differentiation, and let $s_j \in \mathbb{N}$. We consider the differential operators

$$\mathcal{L}(D) = \prod_{j=1}^{N}(D^2+\gamma_j)^{s_j}, \quad \mathcal{M}(D) = D\prod_{j=2}^{N}(D^2+\gamma_j)^{s_j}.$$

If $r_j = 2$ $(1 \leq j \leq N)$ and

$$\varphi = (-1)^{s_1+\dots+s_N}\mathcal{L}(D)f,$$

or $\gamma_1 = 0$, $r_1 = s_1 = 1$, $r_j = 2$ $(2 \leq j \leq N)$, and

$$\varphi = (-1)^{1+s_2+\dots+s_N}\mathcal{M}(D)\widetilde{f},$$

then $f$ is expressed in terms of $\varphi$ as in formula (7.12). This fact is proved in the same way as in [1] for the operators $D^r$ and $\widetilde{D}^r$.

**10.** Assume that $N \in \mathbb{N}$, $0 < r_j \leq 2$, $s_j > 0$, $\gamma_j \in \mathbb{R}$ $(1 \leq j \leq N)$, $y^*$ is defined as in example 9, $y_0 > y^*$, $\widetilde{Q}_{y_0} \in L(\mathbb{R})$, $\widetilde{Q}_{y_0}$ is odd, $c(\widetilde{Q}_{y_0}) \in C^{(2)}(\mathbb{R})$, and

$$ic(\widetilde{Q}_{y_0}, y) = \frac{1}{2}b_0(y) = \prod_{j=1}^{N}(y^{r_j}-\gamma_j)^{-s_j} \quad (y \geq y_0).$$

Then, as was shown in example 9, $\widetilde{Q}_{y_0} \in \widehat{AM}^2_s(y_0)$.

**Corollary 16.** *Suppose* $(\mathfrak{M}, P) \in \mathcal{B}$, $\varphi \in \mathfrak{M}$, $y_0 > y^*$, $T \in \mathbf{E}_{y_0}$, $\sigma \geq y_0$, *and*

$$f = \varphi * \widetilde{Q}_{y_0} + T. \tag{7.14}$$

*Then*

$$P(f - X_{\sigma,\widetilde{Q}_{y_0}}(f)) \leq \left(\frac{2}{\pi}\sum_{\nu=0}^{\infty}\frac{1}{2\nu+1}\prod_{j=1}^{N}\frac{1}{(((2\nu+1)\sigma)^{r_j} - \gamma_j)^{s_j}}\right)\omega_1\left(\varphi, \frac{\pi}{\sigma}\right)_P. \tag{7.15}$$

Let $s_j \in \mathbb{N}$. If $r_j = 2$ $(1 \leq j \leq N)$ and

$$\varphi = (-1)^{s_1+\cdots+s_N}\mathcal{L}(D)\widetilde{f},$$

or $\gamma_1 = 0$, $r_1 = s_1 = 1$, and $r_j = 2$ $(2 \leq j \leq N)$,

$$\varphi = (-1)^{s_2+\cdots+s_N}\mathcal{M}(D)f,$$

then $f$ is expressed in terms of $\varphi$ as in formula (7.14).

Inequalities (7.13) and (7.15) improve the inequalities of type (3.4) that were proved for periodic functions in the case of the uniform norm by Akhiezer in [4] and by Kreĭn in [6], and for nonperiodic functions by Kreĭn in [13] (see inequality (1.3) in the Introduction).

**11.** By using the fact that the product of absolutely monotone functions is absolutely monotone, new kernels belonging to $\widehat{AM}^2$ can be constructed with the help of convolution. In particular, this leads to estimates of the seminorm of the convolution $f = \varphi * G$ with $G$ of class $\widehat{AM}^2$ by moduli of continuity of the function $\varphi'$, to estimates of the seminorms of the Poisson and Weierstrass integrals of the function $f$, etc.

*Remark* 15. The heat conduction kernel $W_\lambda$ and the kernel $\Theta_\lambda$ occurring in example 7 do not increase oscillation. For such kernels some results similar to Lemma 6 are known (see, e.g., [36], where approximation of periodic functions was studied). At the same time, kernels of class $\widehat{AM}^2$ may increase oscillation.

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