Алгебра и анализ
Том. 17 (2005), вып. 4

St. Petersburg Math. J.
Vol. 17 (2006), No. 4, Pages 635–640
S 1061-0022(06)00923-X
Article electronically published on May 3, 2006

A THEOREM ON INTERSECTION WITH A k-DIMENSIONAL BARYCENTER

V. A. MALYSHEV

Abstract. For a multidimensional analog of the barycenter (which is a certain union of intervals), a multidimensional analog of the following statement is proved: if a continuous mapping maps each face of a simplex into itself, then the image of the mapping meets the barycenter.

Introduction

This paper originated from the following witty problem:

Given $k+1$ boards of pairwise different lengths, is it possible to obtain $n+1$ equal boards and k smaller boards by making n cuts?

This problem about boards has substantial generalizations.

Consider a polynomial

$$f(t) = (t - a_1)\cdots(t - a_k)(t - x_1)\cdots(t - x_n)$$

on the interval $[0,1]$. We view the roots $0 < a_1 < \cdots < a_k < 1$ as fixed and the roots $0 \le x_1 \le \cdots \le x_n \le 1$ as free. The collection of free and fixed roots splits the interval $[0,1]$ into $n+k+1$ parts. On each of these subintervals, we can consider the maximal deviation of $|f|$, the area of the subgraph of $|f|$, and the length of the graph of f. We give a geometric proof of the following statements.

For any k fixed roots, there exist n free roots such that $n+1$ maximal deviations are equal and the other k are smaller.

For any k fixed roots, there exist n free roots such that $n+1$ areas are equal and the other k are smaller.

For any k fixed roots, there exist n free roots such that $n+1$ lengths are equal and the other k are smaller.

For $k = 0$ we meet the Chebyshev polynomials in the first statement!

§1. A problem about intersection

Consider the unit simplex

$$\Delta^n = \{\lambda \in \mathbb{R}_+^{n+1} : \lambda_0 + \cdots + \lambda_n = 1\}$$

of dimension $n \ge 1$. The point

$$\operatorname{bar} \Delta^n = \frac{1}{n+1}\big(\underbrace{1, 1, \ldots, 1}_{n+1}\big)$$

is called the *barycenter* of Δ^n. We denote by

$$\Delta_p^n = \{\lambda \in \Delta^n : \lambda_p = 0\}$$

2000 *Mathematics Subject Classification.* Primary 26C10, 57Q15, 41A50.
Key words and phrases. Barycenter.

the faces of Δ^n. It is well known that the boundary

$$\partial\Delta^n = \bigcup_{p=0}^{n} \Delta_p^n$$

is not a retract of the simplex Δ^n (see [1]).

For simplicity, we use the following notation. Consider a topological space X with subspaces $X_1, \dots, X_m$ and a topological space Y with subspaces $Y_1, \dots, Y_m$. The notation

$$F : (X; X_1, \dots, X_m) \longrightarrow (Y; Y_1, \dots, Y_m)$$

means that $F : X \longrightarrow Y$ is a continuous mapping such that $F(X_i) \subset Y_i$ for all $1 \le i \le m$.

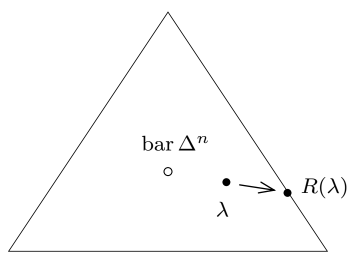
Lemma. *For any continuous mapping*

$$F : (\Delta^n; \Delta_0^n, \dots, \Delta_n^n) \longrightarrow (\Delta^n; \Delta_0^n, \dots, \Delta_n^n)$$

we have

$$F(\Delta^n) \cap \{\operatorname{bar} \Delta^n\} \neq \varnothing.$$

Proof. In the sequel, we denote a singleton $\{x\}$ simply by x; this will not lead to confusion. Arguing by contradiction, we assume that $F(\Delta^n) \subset \Delta^n \setminus \operatorname{bar} \Delta^n$. We construct a standard retraction $R : \Delta^n \setminus \operatorname{bar} \Delta^n \longrightarrow \partial\Delta^n$ as follows. Let $\lambda \in \Delta^n \setminus \operatorname{bar} \Delta^n$. We draw a ray from $\operatorname{bar} \Delta^n$ through λ and define $R(\lambda)$ as the intersection of this ray with the boundary $\partial\Delta^n$.



The assumption $F(\Delta^n) \subset \Delta^n \setminus \operatorname{bar} \Delta^n$ allows us to define a mapping $f = R \circ F$ of the simplex Δ^n onto the boundary $\partial\Delta^n$. Since the mapping

$$f|_{\partial\Delta^n} : \partial\Delta^n \longrightarrow \partial\Delta^n$$

takes each face of Δ^n into itself, this mapping is homotopic to the identity mapping $\mathrm{id} : \partial\Delta^n \longrightarrow \partial\Delta^n$. The corresponding homotopy has the form

$$h(\lambda, \tau) = (1 - \tau) f(\lambda) + \tau\lambda.$$

Since $(\Delta^n, \partial\Delta^n)$ is a Borsuk pair (see [1]), the homotopy

$$h : \partial\Delta^n \times [0, 1] \longrightarrow \partial\Delta^n$$

extends up to a homotopy

$$H : \Delta^n \times [0, 1] \longrightarrow \partial\Delta^n.$$

Then the mapping

$$r(\lambda) = H(\lambda, 1)$$

is a retraction of the simplex Δ^n onto the boundary $\partial\Delta^n$, a contradiction. $\square$

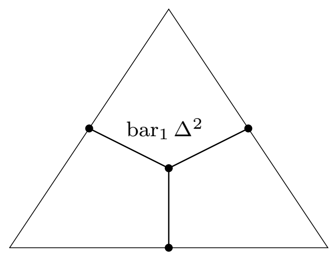
In order to generalize this lemma, we use induction to introduce the notion of the k-*dimensional barycenter*. The 0-dimensional barycenter is defined by the rule

$$\operatorname{bar}_0 \Delta^n = \operatorname{bar} \Delta^n.$$

Then we define the k-dimensional barycenter as the union of joins

$$\operatorname{bar}_k \Delta^n = \bigcup_{p=0}^{n} \operatorname{bar}_0 \Delta^n * \operatorname{bar}_{k-1} \Delta^n_p.$$

In other words, the k-dimensional barycenter $\operatorname{bar}_k \Delta^n$ is the union of the intervals that connect the 0-dimensional barycenter $\operatorname{bar}_0 \Delta^n$ with all points of the $(k-1)$-dimensional barycenters $\operatorname{bar}_{k-1} \Delta^n_p$ of all faces of the simplex Δ^n. For example, the one-dimensional barycenter $\operatorname{bar}_1 \Delta^2$ consists of three intervals.



For the simplex Δ^{n+k}, the points of the k-dimensional barycenter have the following property: if a point belongs to the k-dimensional barycenter $\operatorname{bar}_k \Delta^{n+k}$, then some $n+1$ coordinates of this point are equal and the other k are smaller.

This extremal property of the k-dimensional barycenter $\operatorname{bar}_k \Delta^{n+k}$ makes the following problem meaningful: under what conditions on X and $X_0, \dots, X_{n+k}$ can we guarantee that for any continuous mapping

$$F : (X; X_0, \dots, X_{n+k}) \longrightarrow (\Delta^{n+k}; \Delta^n_0, \dots, \Delta^{n+k}_{n+k})$$

we have

$$F(X) \cap \operatorname{bar}_k \Delta^{n+k} \neq \varnothing?$$

In what follows we assume that X is a cell space of dimension n and $X_0, \dots, X_{n+k}$ are cell subspaces of dimension $n-1$. For $k=0$, the above lemma solves the problem.

§2. A theorem about intersection

We consider the n-dimensional simplex

$$\nabla^n = \{x \in \mathbb{R}^n : 0 \leq x_1 \leq \cdots \leq x_n \leq 1\}$$

of vectors with monotone increasing coordinates and introduce the mapping

$$x \longmapsto x_{\leq}$$

that rearranges the coordinates of x in increasing order. For example,

$$(1, 4, 1, 2)_{\leq} = (1, 1, 2, 4).$$

We fix points

$$0 < a_1 < \cdots < a_k < 1,$$

and define functions $y_0(x)$, $y_1(x)$, $\dots$, $y_{n+k+1}(x)$ by the rule

$$\big(y_0(x), y_1(x), \dots, y_{n+k+1}(x)\big) = \big(0, a_1, \dots, a_k, x_1, \dots, x_n, 1\big)_{\leq}.$$

Put

$$\nabla^n_p = \{x \in \nabla^n : y_p(x) = y_{p+1}(x)\}.$$

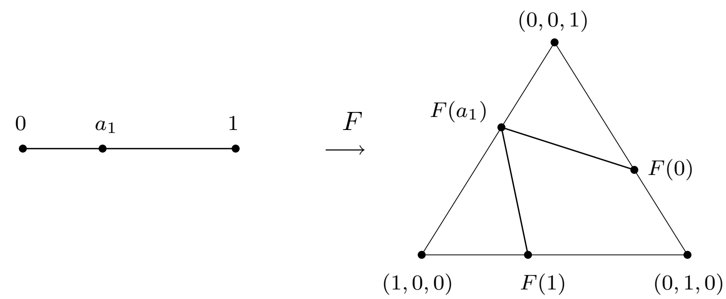
Any continuous mapping

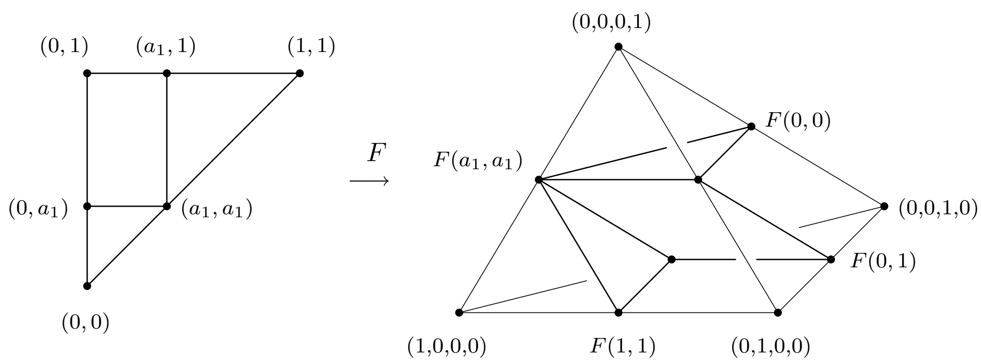
$$F : \left(\nabla^n; \nabla^n_0, \dots, \nabla^n_{n+k}\right) \longrightarrow \left(\Delta^{n+k}; \Delta^{n+k}_0, \dots, \Delta^{n+k}_{n+k}\right)$$

will be called a *bound mapping.*

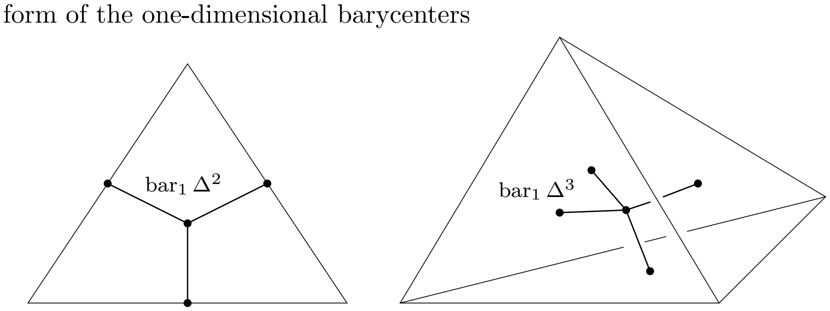
If $F : \nabla^n \longrightarrow \Delta^{n+k}$ is a bound mapping, then, with each interval $[y_p(x), y_{p+1}(x)]$, a nonnegative value $F_p(x)$ is associated in such a way that the value $F_p(x)$ vanishes when the interval $[y_p(x), y_{p+1}(x)]$ degenerates into a point. Examples of bound mappings were mentioned in the Introduction.

Bound mappings $F : \nabla^1 \longrightarrow \Delta^2$ and $F : \nabla^2 \longrightarrow \Delta^3$ act in the following way:





The form of the one-dimensional barycenters



suggests the following statement.

Theorem. *For any continuous mapping*

$$F : \left(\nabla^n; \nabla^n_0, \dots, \nabla^n_{n+k}\right) \longrightarrow \left(\Delta^{n+k}; \Delta^{n+k}_0, \dots, \Delta^{n+k}_{n+k}\right)$$

we have

$$F(\nabla^n) \cap \operatorname{bar}_k(\Delta^{n+k}) \neq \varnothing.$$

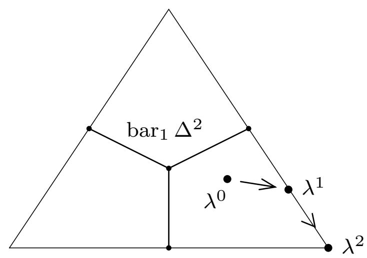
Proof. For $k = 0$ this was proved in the lemma in §1. Let $k \geq 1$ and suppose that a bound mapping $F : \nabla^n \longrightarrow \Delta^{n+k}$ satisfies

$$F(\nabla^n) \subset \Delta^{n+k} \setminus \operatorname{bar}_k \Delta^{n+k}.$$

Let $\operatorname{ske}_{n-1}\Delta^{n+k}$ denote the union of the $(n-1)$-dimensional faces of the simplex Δ^{n+k}. We shall construct a retraction

$$R : \Delta^{n+k} \setminus \operatorname{bar}_k \Delta^{n+k} \longrightarrow \operatorname{ske}_{n-1}\Delta^{n+k}.$$

Let $\lambda^0 \in \Delta^{n+k} \setminus \operatorname{bar}_k \Delta^{n+k}$. We draw the ray from the point $\operatorname{bar}\Delta^{n+k}$ through the point λ^0 and define λ^1 as the intersection of this ray with the boundary of the simplex Δ^{n+k}. Then $\lambda^1 \in \Delta^{n+k}_{p_1}$, where $0 \le p_1 \le n+k$. We see that $\lambda^1 \in \Delta^{n+k}_{p_1} \setminus \operatorname{bar}_{k-1}\Delta^{n+k}_{p_1}$. Next we draw the ray from the point $\operatorname{bar}\Delta^{n+k}_{p_1}$ through the point λ^1 and define λ^2 as the intersection of this ray with the boundary of the simplex $\Delta^{n+k}_{p_1}$.



After such $k+1$ displacements of the initial point λ^0, we get a sequence λ^0, λ^1, ..., λ^k, λ^{k+1} in the simplex Δ^{n+k}. Now, the mapping R defined by the rule $R : \lambda^0 \longmapsto \lambda^{k+1}$ is the required retraction.

The inclusion $F(\nabla^n) \subset \Delta^{n+k} \setminus \operatorname{bar}_k \Delta^{n+k}$ allows us to introduce the mapping $f = R \circ F$ of the simplex Δ^{n+k} onto the skeleton $\operatorname{ske}_{n-1}\Delta^{n+k}$. Since the retraction R maps each face of Δ^{n+k} into itself, the mapping $f : \nabla^n \longrightarrow \Delta^{n+k}$ is bound.

We define a continuous mapping $g : \nabla^n \longrightarrow \Delta^n$ by the rule

$$g_q(x) = \sum_{p:[y_p(x),y_{p+1}(x)]\subset[x_q,x_{q+1}]} f_p(x),$$

where $x_0 = 0$ and $x_{n+1} = 1$. Then the mapping g is bound,

$$g : \left(\nabla^n; \nabla^n_0, \dots, \nabla^n_n\right) \longrightarrow \left(\Delta^n; \Delta^n_0, \dots, \Delta^n_n\right).$$

Since $f(\nabla^n) \subset \operatorname{ske}_{n-1}\Delta^{n+k}$ and any point of $\operatorname{ske}_{n-1}\Delta^{n+k}$ has at most n nonzero coordinates, we have $g(\nabla^n) \subset \partial\Delta^n$. This contradicts the case of $k = 0$. The theorem is proved. $\square$

§3. Corollaries

Useful consequences of the above theorem arise in problems concerning polynomials

$$f(t) = (t-a_1)\cdots(t-a_k)(t-x_1)\cdots(t-x_n)$$

with fixed roots $0 < a_1 < \cdots < a_k < 1$ and free roots $0 \le x_1 \le \cdots \le x_n \le 1$. In the problem of maximal deviations, a bound mapping $F : \nabla^n \longrightarrow \Delta^{n+k}$ can be defined by the rule

$$F_p(x) = \frac{1}{\sum_{q=0}^{n+k} \|f\|_{C[y_q(x),y_{q+1}(x)]}} \|f\|_{C[y_p(x),y_{p+1}(x)]},$$

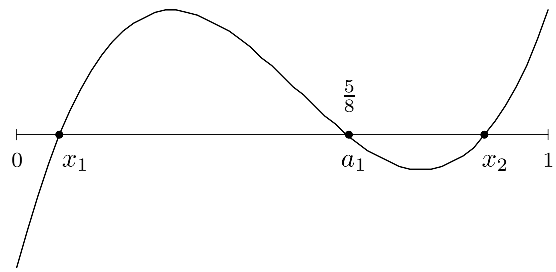
and in other problems such mappings are constructed similarly.

Observe that the question of the uniqueness of the extremal polynomials is equivalent to the question of the number of points in the intersection $F(\nabla^n) \cap \operatorname{bar}_k(\Delta^{n+k})$. The disposition of the image $F(\nabla^n)$ in the simplex Δ^{n+k} and the form of the k-dimensional barycenter $\operatorname{bar}_k(\Delta^{n+k})$ do not exclude the existence of several points.

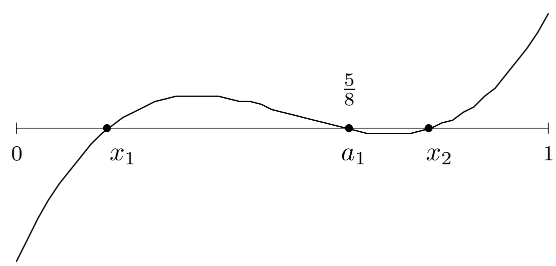
We give two examples. For $k = 1$ and $n = 2$ the polynomial

$$f(t) = (t - a_1)(t - x_1)(t - x_2)$$

has one fixed and two free roots. For a proper choice of the free roots x_1 and x_2, the polynomial



has 3 equal maximal deviations, and the polynomial



has 3 equal areas of the subgraph.

References

[1] V. A. Rokhlin and D. B. Fuks, *Beginner's course in topology. Geometric chapters*, Nauka, Moscow, 1977; English transl., Springer-Verlag, Berlin, 1984. MR0645388 (58:31080); MR0759162 (86a:57001)

Rybinsk State Aviation Technology Academy, Rybinsk, Russia
E-mail address: wmal@ryb.adm.yar.ru

Received 10/SEP/2004

Translated by THE AUTHOR