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GENERATING BOREL SETS BY BALLS

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ABSTRACT. It is proved that an arbitrary infinite-dimensional Banach space with basis admits an equivalent norm such that any Borel set can be obtained from balls by taking complements and countable disjoint unions. For reflexive spaces, the new norm can be chosen arbitrarily close to the initial norm.

INTRODUCTION

Let X be a separable Banach space. The following question will be discussed: is it possible to obtain an arbitrary Borel set from balls by using the operations of taking complements and at most countable disjoint unions? We start with some definitions (see, e.g., [1] or [2]).

Definition 1. A nonempty family D of subsets of a set A is called a *Dynkin system* if D is closed under taking complements and at most countable disjoint unions.[1]

Definition 2. A nonempty family M of subsets of a set A is called a *monotone system* if M is closed under taking at most countable disjoint unions and countable monotone limits (i.e., increasing unions and decreasing intersections).

It is easy to show that any Dynkin system is a monotone system; see [1].

Consider the minimal Dynkin system $D(X)$ that contains all balls in X. All elements of $D(X)$ are Borel sets. Therefore, our main question can be reformulated as follows: is it true that $D(X)$ coincides with the Borel σ-algebra $\mathcal{B}(X)$ of the space X?

A similar question is meaningful for the minimal monotone system $M(X)$ that contains all balls. Clearly, if $M(X) = \mathcal{B}(X)$, then $D(X) = \mathcal{B}(X)$. Furthermore, we can refine the question, using only balls belonging to some fixed class (for example, only "small" balls, i.e., those with radius not exceeding 1, or, by contrast, only "large" balls with radius at least 1). In what follows, the balls we consider are assumed to be open.

A detailed survey of the history and results concerning the problem under discussion, including proofs, can be found in the book [1] and the paper [2]. We say a few words about history. For the first time, the question concerning the coincidence of the minimal Dynkin system generated by balls and the Borel σ-algebra for separable Banach spaces was raised at the 3rd Conference on Topology and Measure (1980), in connection with the problem of coincidence of two finite Borel measures taking the same values at all balls of the space. Later on, these two questions were separated from each other.

For finite-dimensional spaces, the problem of generating the Borel sets by small balls was solved completely (in the affirmative).

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[1]Such systems were considered and studied much earlier by W. Sierpiński and some other mathematicians after him. The author thanks Professor Z. Lipecki for this remark.

Theorem A. *Let X be a finite-dimensional Banach space, and let C be a family of balls such that for every $x \in X$ and every $r_0 > 0$ there exists $r \in (0, r_0)$ such that $B(x,r) \in C$ ($B(x,r)$ is the open ball centered at x and of radius r). Then the minimal monotone system containing C coincides with the Borel σ-algebra of X.*

(This remarkable theorem is Corollary 8.4 in [1].)

In [1], Theorem A was obtained as a consequence of a general statement whose proof occupies several pages and refers the reader to some strong abstract covering theorems. The history of this result is rather rich in events. In 1988, Olejček [3] proved that in $\mathbb{R}^n$ the minimal Dynkin system containing all balls coincides with the Borel σ-algebra for $n = 2$, and in 1995, in [4], he proved that the same is true for $n = 3$. Later, Jackson and Mauldin in [5], and also Zelený in [6] independently extended this result to $\mathbb{R}^n$ with an arbitrary n. In fact, the proof given by Jackson and Mauldin works in the case of an arbitrary finite-dimensional space (not only Euclidean), and M. Zelený used only monotone systems.

For infinite-dimensional Banach spaces the situation is completely different. In [2] it was proved that the minimal Dynkin system containing all balls in the Hilbert space l_2 does not coincide with the Borel σ-algebra.

Theorem B. *There exists a Borel subset of l_2 that is not generated by balls (with the help of taking complements and at most countable disjoint unions).*

Nevertheless, in some infinite-dimensional spaces the situation is the same as in finite-dimensional spaces. For instance, in c_0 the small balls generate the Borel sets, and the same is true for the large balls. (Apparently, this fact, together with a simple proof, should be regarded as well known.) The following natural questions arise:

(1) What are the infinite-dimensional Banach spaces, besides c_0, in which the balls generate all Borel sets?

(2) Is it true that every separable Banach space admits an equivalent norm for which the balls generate the Borel sets? In other words, is the question we consider purely topological, or is the geometry of the unit ball also important?

As a result of misunderstanding, in the paper [2] some strong statements in this direction were announced and ascribed to the present author.

Now some weaker versions of those statements are proved indeed; the proofs are presented below.

The paper is organized as follows.

In §1 we give an example of a "nice" space (in the sense of question (1)): the minimal Dynkin system containing the large balls in $C[0,1]$ coincides with the Borel σ-algebra of $C[0,1]$ (Proposition 1).

In §3 we present a simple construction allowing us to "correct" slightly the unit ball B in the spaces l_p ($1 \leq p < \infty$). As a result, we obtain an equivalent ball B_0 such that the new large balls (the sets $RB_0 + x$, $R \geq 1$, $x \in l_p$) generate the Borel sets. The construction is described in the proof of Proposition 2 and admits much freedom, which motivates seeking generalizations. Such generalizations can be found in §§4, 5 (unfortunately, the transparentness of the initial construction is lost there).

Theorem 1 (in §4) says that if a Banach space possesses a Schauder basis, then this space admits an equivalent norm whose large balls generate the Borel sets.

Unfortunately, the norm obtained in Theorem 1 may fail to be close to the original norm (our correction of the ball is far from being delicate).

In §5 we try to refine Theorem 1. In Theorem 2 we prove that in the case of a reflexive Banach space with basis, the "nice" ball of Theorem 1 can be chosen as close to the original ball B as we wish.

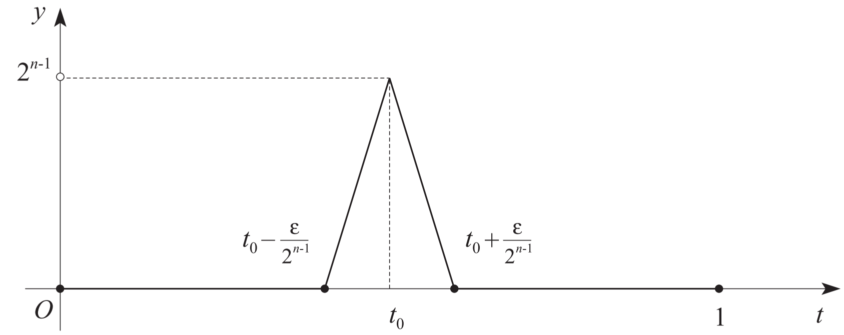

FIGURE 1.

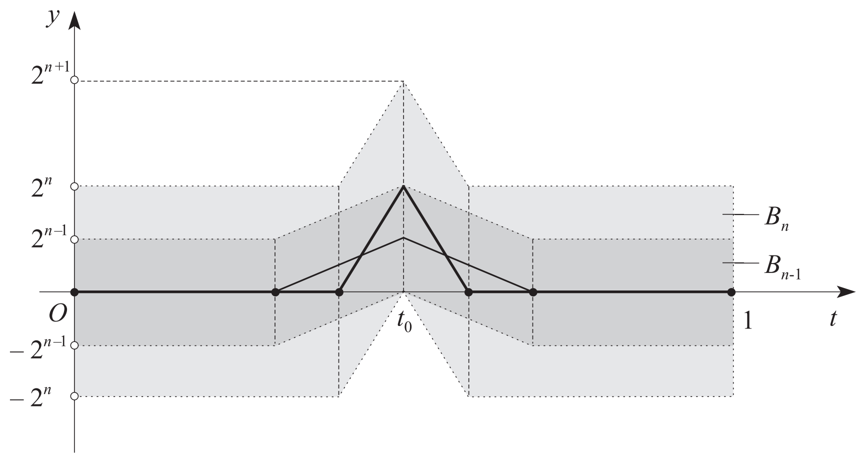

FIGURE 2.

The author wants to express her sincere gratitude to Professor J. Tišer for useful discussions and to the referee for advice that helped to improve the structure of the paper.

§1. THE LARGE BALLS OF $C[0,1]$ GENERATE ALL BOREL SETS

Proposition 1. *Let D be the minimal Dynkin system of subsets of $C[0,1]$ containing all balls with radius $r \geq 1$. Then D coincides with the Borel σ-algebra.*

Proof. Step 1. We fix a point $t_0 \in (0,1)$ and a number $\varepsilon > 0$, $\varepsilon < \min\{1-t_0, t_0\}$. Consider a sequence of balls $B_n = B(x_n, r_n)$ such that $r_n = 2^{n-1}$ and the centers $x_n \in C[0,1]$ coincide with the functions whose graphs are depicted in Figure 1.

Clearly, $B_1 \subset B_2 \subset \cdots$ (see Figure 2). Therefore, $\bigcup_{n=1}^{\infty} B_n \in D$. But $\bigcup_{n=1}^{\infty} B_n = \{x \in C[0,1] : x(t_0) > 0\}$. Similar arguments show that for all $t \in (0,1)$ and all $q \in \mathbb{R}$ the set

$$A(q,t) \stackrel{\text{def}}{=} \{x \in C[0,1] : x(t) > q\}$$

belongs to D. Next, let $\overline{A}(q,t)$ denote the set $\{x \in C[0,1] : x(t) \geq q\}$. Then $\overline{A}(q,t) = \bigcap_{n=1}^{\infty} A(q - \frac{1}{n}, t)$. Since the sequence $\{A(q - \frac{1}{n}, t)\}_{n=1}^{\infty}$ is monotone decreasing, we have $\overline{A}(q,t) \in D$. Observe that if $A \in D$, $B \in D$, and $A \subset B$, then $B \setminus A \in D$. Now, denoting

$$A(q_1, q_2, t) \stackrel{\text{def}}{=} \{x \in C[0,1] : q_1 \leq x(t) \leq q_2\},$$

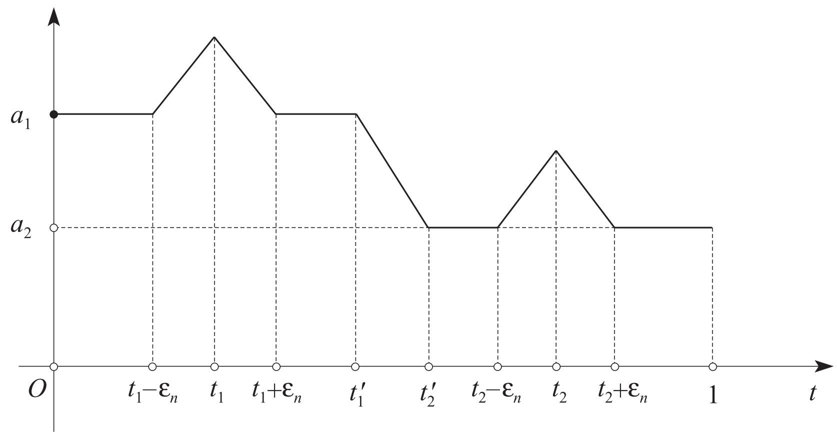

FIGURE 3.

we see that

$$A(q_1, q_2, t) = \bar{A}(q_1, t) \setminus A(q_2, t) \in D.$$

Thus,

$$\{x \in C[0,1] : q_1 \le x(t) \le q_2\} \in D. \tag{1}$$

Step 2. In (1), we can take any finite set $\{t_1, \dots, t_n\} \subset (0,1)$ instead of one point t:

$$\{x \in C[0,1] : x(t_i) \in [a_i, b_i], i = 1, \dots, n\} \in D \tag{2}$$

for any $t_1, t_2, \dots, t_n \in (0,1)$ and any $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$.

We explain how to pass from one point to two points. We fix t_1, $t_2 \in (0,1)$, $a_1 \le b_1$, $a_2 \le b_2$ (if $a_1 > b_1$ or $a_2 > b_2$, then the set in question is empty, and $\varnothing \in D$). As the centers of balls we take broken lines with two suitably high peaks at t_1 and t_2 (see Figure 3). As before, we see that the set $E_1 \stackrel{\text{def}}{=} \{x \in C[0,1] : x(t_1) > a_1,\ x(t_2) > a_2\}$ belongs to D.

Repeating the arguments of Step 1, we see that

$$\overline{E}_1 = \{x \in C[0,1] : x(t_1) \ge a_1, x(t_2) \ge a_2\} \in D$$

and

$$\overline{E}_1' = \{x \in C[0,1] : x(t_1) > b_1, x(t_2) \ge a_2\} \in D.$$

Then $E_2 \stackrel{\text{def}}{=} \{x \in C([0,1] : a_1 \le x(t_1) \le b_1, x(t_2) \ge a_2\} = \overline{E}_1 \setminus \overline{E}_1' \in D$.

Next, $\{x : a_1 \le x(t_1) \le b_1,\ x(t_2) \ge b_2 + \frac{1}{n}\} \in D$, and this sequence is monotone increasing, so that its union

$$E_3 = \{x : a_1 \le x(t_1) \le b_1, x(t_2) > b_2\}$$

belongs to D. Taking the difference of the sets E_2 and E_3, we conclude that

$$\{x : x(t_1) \in [a_1, b_1], x(t_2) \in [a_2, b_2]\} \in D,$$

as required. Similar arguments give (2) by induction. □

Step 3. For fixed n, we put $t_k = \frac{k}{2^n}$, $k = 1, 2, \dots, 2^n - 1$. Then (2) takes the following form:

$$\Big\{x \in C[0,1] : a_k \le x\Big(\frac{k}{2^n}\Big) \le b_k,\ \ k = 1, \dots, 2^n - 1\Big\} \in D \tag{3}$$

for all $n \in \mathbb{N}$ and all a_k, b_k with $a_k \le b_k$, $k = 1, \dots, 2^n - 1$. Suppose f, $g \in C[0,1]$, $f \le g$. For $n \in \mathbb{N}$, in (3) we take $a_k = f\big(\frac{k}{2^n}\big)$, $b_k = g\big(\frac{k}{2^n}\big)$, $k = 1, 2, \dots, 2^n - 1$. Let

$$C_n = \Big\{x \in C[0,1] : f\Big(\frac{k}{2^n}\Big) \le x\Big(\frac{k}{2^n}\Big) \le g\Big(\frac{k}{2^n}\Big),\ \ k = 1, \dots, 2^n - 1\Big\}.$$

Then $C_1 \supset C_2 \supset \cdots$, and $C_n \in D$, $n \in \mathbb{N}$. Therefore, the set $C = \bigcap_{n=1}^{\infty} C_n$ belongs to D. But $C = \{x \in C[0,1] : f \le x \le g\}$.

The result of the first 3 steps is the following:

$$\{x \in C[0,1] : f \le x \le g\} \in D \tag{4}$$

for all $f, g \in C[0,1]$ with $f \le g$.

So, all order intervals (and, in particular, all balls with radius less than 1) belong to D.

Step 4. Let $\overline{B}(x,r)$ be the closed ball with center $x \in C[0,1]$ and radius $r > 0$. Observe that for any $x_1, x_2, \dots, x_n \in C[0,1]$ and any $r_1, \dots, r_n > 0$ the set $\bigcap_{k=1}^{n} \overline{B}(x_k, r_k)$ is equal to $\{x \in C[0,1] : f \le x \le g\}$, where

$$f = \max\{x_1 - r_1, x_2 - r_2, \dots, x_n - r_n\}, \quad g = \min\{x_1 + r_1, x_2 + r_2, \dots, x_n + r_n\}$$

(in particular, it can be empty). Therefore, $\bigcap_{k=1}^{n} \overline{B}(x_k, r_k) \in D$, by (4).

Thus, the intersections of finite families of closed balls in $C[0,1]$ belong to D.

Step 5. Observe that if the finite intersections of balls belong to D, then the same is true for the finite unions of balls: $\bigcup_{k=1}^{n} \overline{B}(x_k, r_k) \in D$ for all $x_i \in X$, $r_i > 0$. (For two balls we have $\overline{B}_1 \cup \overline{B}_2 = \overline{B}_1 \cup (\overline{B}_2 \setminus (\overline{B}_1 \cap \overline{B}_2)) \in D$. In the general case the argument is similar: the inclusion-exclusion formula works.)

To finish the proof of Proposition 1, now it suffices to check the following statement, which will be used repeatedly in the sequel.

Lemma 1. *If (X, ρ) is a separable metric space and M is the minimal monotone system in X containing all closed balls and all unions of finite families of balls, then the system M coincides with the Borel σ-algebra of (X, ρ).*

Proof. First, if a monotone system contains the finite unions of balls, then it also contains the countable unions: $\bigcup_{n=1}^{\infty} B_n = \bigcup_{N=1}^{\infty} \big(\bigcup_{n=1}^{N} \overline{B}_n\big)$ is the limit of a monotone increasing sequence of subsets of M.

By the Lindelöf theorem, any open set in the separable space (X, ρ) can be represented as the union of a finite or countable family of closed balls. Thus, the system M contains all open sets.

To complete the proof of Lemma 1, we need to show that any monotone system containing all open sets contains all Borel sets. This fact can be found, e.g., in [1, Subsection 8.4]. For completeness, we present a proof, for which the author thanks the referee.

Containing all open sets, the system M contains all G_δ-sets and, therefore, all sets that are G_δ and F_σ simultaneously. Since such sets form an algebra, the minimal monotone system containing them is the Borel σ-algebra. This completes the proof of Lemma 1 and, with it, of Proposition 1. □

Remark. The above arguments can be repeated almost word for word to obtain a similar property for the space c_0: the large balls in c_0 also generate all Borel sets.

§2. Some auxiliary notions and facts

Suppose that X is a Banach space with a normalized basis $\{e_n\}_1^\infty$. We set $X_n \stackrel{\text{def}}{=} \operatorname{span}\{e_1,\dots,e_n\}$, denote by P_n the canonical projection onto X_n, and put $L_n = \overline{\operatorname{span}}\{e_{n+1},e_{n+2},\dots\} = \operatorname{Ker} P_n$.

Definition 3. A set of the form $C = P_n^{-1}(A) = A + L_n$, where $A \subset X_n$ is a Borel set, is called a *cylinder* with base A. If A is a cone, then the cylinder with base A is called a *wedge*.

Let M_c be the minimal monotone system containing all cylinders.

Lemma 2. *The system M_c coincides with the Borel σ-algebra of X.*

Proof. By Lemma 1, it suffices to show that M_c contains all finite unions of closed balls. Since the situation is purely topological, we may assume that the basis $\{e_n\}_1^\infty$ is monotone.

For simplicity, we consider only two balls $B_1 = B(x,r_1)$ and $B_2 = B(y,r_2)$. Let $F = B_1 \cup B_2$ and $F_n = P_n^{-1}(P_n(B_1 \cup B_2))$.

We must show that $F \in M_c$. It is clear that the family $\{F_n\}$ is monotone decreasing and that all cylinders F_n belong to M_c. Therefore, $\bigcap_{n=1}^\infty F_n \in M_c$. We shall check that $F = \bigcap_{n=1}^\infty F_n$.

Fix $z = (z_1,z_2,\dots) \in F$, $k \in N$. The definition of F_k implies that $z \in F_k$. Conversely, suppose $z = (z_1,z_2,\dots) \in \bigcap_{k=1}^\infty F_k$. We want to show that $z \in F$. If not, then $z \notin B_1$ and $z \notin B_2$, i.e., $\|z-x\| > r_1$ and $\|z-y\| > r_2$. For $x = (x_1,x_2,\dots)$ we have $\|(z_1-x_1,z_2-x_2,\dots)\| = r_1+\varepsilon$ for some $\varepsilon > 0$. Then, for some k, we have $\|(z_1-x_1,\dots,z_k-x_k,0,0,\dots)\| > r_1$. Using the monotonicity of the basis, we see that

$$\|(z_1-x_1,\dots,z_k-x_k,\alpha_{k+1}-x_{k+1},\dots)\| > r_1$$

for all $\alpha_{k+1},\alpha_{k+2},\dots$. This means that for arbitrary $\alpha_{k+1},\alpha_{k+2},\dots$ the point $(z_1,z_2,\dots, z_k,\alpha_{k+1},\alpha_{k+2},\dots)$ does not belong to B_1. Similarly, the condition $z \notin B_2$ implies that for some $m \in \mathbb{N}$ any point of the form $(z_1,z_2,\dots,z_m,\beta_{m+1},\beta_{m+2},\dots)$ lies off B_2. Assuming that $m \ge k$, we see that the points of $P_m^{-1}(P_m z)$ do not belong to $B_1 \cup B_2$, i.e., $z \notin F_m$. This contradiction finishes the proof of Lemma 2. □

Remark. Let C_n denote the class $\{P_n^{-1}(A) : A \subset X_n,\ A \text{ is Borel}\}$ of all cylinders with base of dimension not exceeding n. Let $\{n_k\}$ be a monotone sequence of integers tending to infinity. Since $C_{n_k} \supset C_m$ for $m \le n_k$, the minimal monotone system containing all sets of all C_{n_k} coincides with M_c. Thus, this new system coincides with the Borel σ-algebra.

So, the cylinders generate all Borel sets. Now we shall show that, instead of cylinders, it suffices to use only wedges of a special kind.

Definition 4. Let $\{y_1,\dots,y_n\}$ be a linearly independent system of points in X_n. The cone with vertex 0 spanned by $\operatorname{co}\{y_1,\dots,y_n\}$, and also all shifts of this cone, will be called *nice cones* in X_n. A wedge is called a *nice wedge* of dimension n if its base is a nice cone in X_n.

Lemma 3. *Let K_2, $K_3,\dots$ be nice cones in the spaces $X_2,X_3,\dots$, and let $\widetilde{K}_2$, $\widetilde{K}_3,\dots$ be wedges with bases K_2, $K_3,\dots$, respectively. Then the minimal Dynkin system that contains the wedges $\widetilde{K}_2$, $\widetilde{K}_3,\dots$ and their shifts $\widetilde{K}_n + z_n$, $z_n \in X_n$, coincides with the Borel σ-algebra.*

Proof. It suffices to show that, for each $n = 2,3,\dots$,

(∗) if a Dynkin system D in X_n contains all shifts of the cone K_n, then D contains all Borel sets of X_n.

Suppose the cone K_n is generated by a system $\{y_1, \dots, y_n\}$. We regard this system as a basis of X_n, and the cone K_n as the positive hyperoctant $P = \{(x_1, \dots, x_n) : x_k \geq 0, k = 1, \dots, n\}$.

We show that all shifts of the cubes

$$L_r = \{(x_1, x_2, \dots, x_n) : 0 \leq x_k \leq r\}$$

are contained in D.

Indeed, taking the union of the monotone increasing system of hyperoctants $P_k = P + \frac{1}{k}y_1 = \{(x_1, \dots, x_n) : x_1 \geq \frac{1}{k}, x_2 \geq 0, \dots, x_n \geq 0\}$, we see that the set

$$\widetilde{P}^{(1)} = \{(x_1, \dots, x_n) : x_1 > 0, x_2 \geq 0, \dots, x_n \geq 0\}$$

belongs to D. Therefore, the set $P \setminus (\widetilde{P}^{(1)} + ry_1) = \{(x_1, \dots, x_n) : 0 \leq x_1 \leq r,\ x_2 \geq 0, \dots, x_n \geq 0\}$ belongs to D.

Repeating these arguments for the other coordinates, we obtain $L_r \in D$. Obviously, all shifts of the cube L_r (the sets $L_r + z, z \in X_n$) also belong to D. But we know that in the finite-dimensional space l_n^∞ every Dynkin system containing all small balls contains all Borel sets. The lemma is proved. □

So, in order to generate the Borel sets starting with balls, it suffices to generate some nice wedge of dimension n for every n; moreover, it suffices to do this for n running through some sequence $(n_k), n_k \to \infty$.

Precisely this will be done in the proof of Proposition 2 and Theorems 1 and 2 below. Namely, the original ball will be modified so that the new ball acquires the form of a nice wedge in small neighborhoods of certain points of X_n. Then we can "blow" the new ball up so as to get an entire nice wedge. More precisely, to generate a nice wedge of dimension n, we require that the new ball B_0 satisfy the following conditions (K_n):

(K_n) There exists a vector $x_n \in X_n$, $\|x_n\| = 1$, such that
1) $B_0(-kx_n, k) \subset B_0(-(k+1)x_n, k+1)$ for $k = 1, 2, \dots$;
2) $\bigcup_{k=1}^\infty \overline{B}_0(-kx_n, k)$ is a nice wedge with base in X_n.

§3. A "GOOD" NORM CLOSE TO THE STANDARD NORM IN l_p

Proposition 2. *Let $B = B(0,1)$ be the unit ball of l_p, $1 \leq p < \infty$, and let ε be a positive number. There exists an equivalent norm with unit ball $B_0 = B_0(0,1)$ such that*

1) $\operatorname{dist}(B, B_0) < \varepsilon$ *(in the Hausdorff metric)*;

2) *the minimal Dynkin system D containing all balls $B_0(x, R)$, $x \in l_p$, $R \geq 1$, coincides with the Borel σ-algebra of l_p.*

Remark. Proposition 2 is a special case of Theorem 1 (to be proved in §4). However, here we give a sketch of the proof, because this proof is quite simple and clear, follows the same leading idea, and is not overloaded with many technical details that arise unavoidably in the proof of Theorem 1.

Proof. The construction of the unit ball B_0 can be described as follows. We fix $\varepsilon > 0$, assuming that ε is sufficiently small. The ball B_0 will be obtained as the monotone intersection of some intermediate balls B_n, $n = 2, 3, \dots$, which will be constructed by induction, $B_0 = \bigcap_{n=2}^\infty B_n$, $B \supset B_2 \supset B_3 \supset \cdots$. For the standard basis $\{e_k\}_{k=1}^\infty$ of l_p, we denote $X_n = \operatorname{span}\{e_1, \dots, e_n\}$.

Base of induction $(n = 2)$**.** In the two-dimensional space X_2 we draw the line parallel to X_1 and passing through the point $(1 - \varepsilon)e_2$. This line intersects the oval $B \cap X_2$ at points M and N (see Figure 4). Let A be the point $\left(1 - \frac{\varepsilon}{2}\right)e_2$, let K_2 be the angle $\angle MAN$, and let K_2^∞ be the cylinder with the base K_2.

We put $B_2 = K_2^\infty \cap (-K_2^\infty) \cap B$.

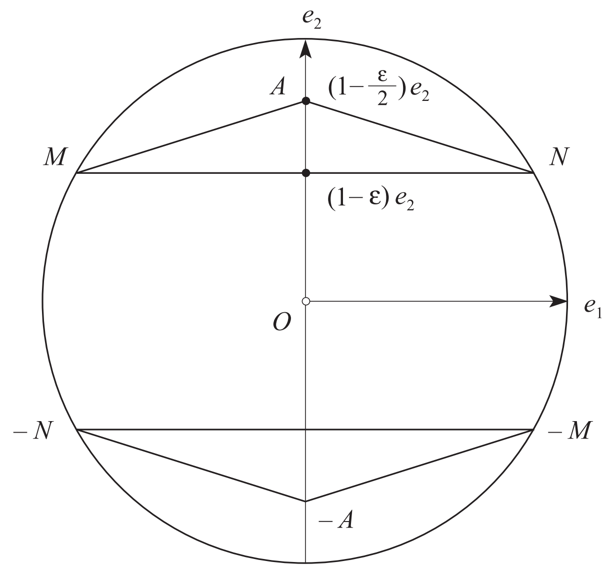

FIGURE 4.

Step of induction. Suppose that the balls B_2, $B_3, \dots, B_{n-1}$ have already been constructed, $B \supset B_2 \supset B_3 \supset \cdots \supset B_{n-1}$.

In the space X_n we take the hyperplane parallel to X_n and passing through the point $(1-\varepsilon)e_n$. Let S denote the intersection of this hyperplane and ∂B_{n-1}.

We choose n points $y_1, \dots, y_n$ on S in such a way that, for some $\delta_n > 0$ with $\delta_n < \varepsilon$,

1) the system $\{(1-\varepsilon+\delta_n)e_n - y_k\}_{k=1}^n$ is linearly independent;

2) the ball $B(0, 1-\varepsilon)$ is contained in the cone with the vertex $(1-\varepsilon+\delta_n)e_n$ spanned by the system $\{y_1, \dots, y_n\}$.

Observe that in the space l_2 the points $y_1, \dots, y_n$ can be taken at the vertices of a regular simplex.

For the present, let us agree that the existence of such a system $\{y_1, \dots, y_n\}$ is obvious. Below, in the proof of Theorem 1, this fact will be proved.

Let K_n denote the cone with the vertex $(1-\varepsilon+\delta_n)e_n$ spanned by the system $\{y_1, \dots, y_n\}$. The wedge with the base K_n is denoted by K_n^∞.

We put

$$B_n \overset{\text{def}}{=} K_n^\infty \cap (-K_n^\infty) \cap B_{n-1}.$$

Observe that the ball B_n is obtained from B_{n-1} by "grinding" in some neighborhood U_n of the point e_n. If ε is sufficiently small, then the neighborhoods U_n and U_m are disjoint for $m \neq n$. Thus, the "grindings" at different steps do not affect each other.

So, we have constructed the ball $B_0 = \bigcap_{n=2}^\infty B_n$. Obviously, by construction we have $B_0 \subset B$ and $\operatorname{dist}(B_0, B) < \varepsilon$. Now we verify the main property of B_0 (property (K_n) for $n = 2, 3, \dots$). We fix $n \in \mathbb{N}$, $n \geq 2$, and put $x_n = (1-\varepsilon+\delta_n)e_n$ for $n \geq 3$, $x_2 = \left(1-\frac{\varepsilon}{2}\right)e_2$. Denoting the new norm by $\|\cdot\|_0$, we have $\|x_n\|_0 = 1$.

Lemma 4. *For the point x_n and the ball B_0, conditions (K_n) are fulfilled:*

1) $B_0(-kx_n, k) \subset B_0(-(k+1)x_n, k+1)$, $k = 1, 2, \dots,$ *and*

2) $\bigcup_{k=1}^\infty \overline{B_0}(-kx_n, k) = (K_n^\infty - x_n)$ *is a nice wedge with the base* $K_n - x_n$.

Proof. 1) We fix $k \in \mathbb{N}$ and consider the homothety with center 0 and coefficient $\frac{k+1}{k}$. The image of $B_0(-kx_n, k)$ under this homothety is the ball $B_0(-(k+1)x_n, k+1)$. Since

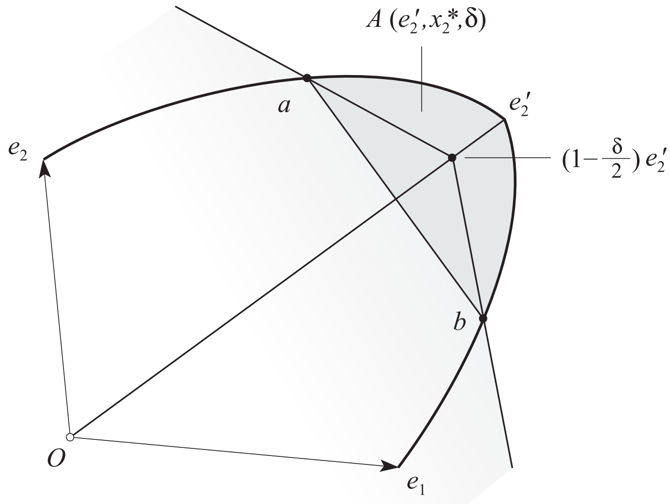

FIGURE 5.

$0 \in B_0(-kx_n, k)$ and the set $B_0(-kx_n, k)$ is convex, we have

$$B_0(-(k+1)x_n, k+1) \supset B_0(-kx_n, k).$$

2) Observe that $\bigcup_{k=1}^{\infty} B_0(-kx_n, k)$ is the cone with vertex 0 spanned by the ball $B_0(-x_n, 1)$. In some neighborhood U of the point zero, the ball $B_0(-x_n, 1)$ coincides with the wedge $K_n^{\infty} - x_n$, because $B_0(-x_n, 1) \cap U = (K_n^{\infty} - x_n) \cap U$. Therefore,

$$\bigcup_{k=1}^{\infty} B_0(-kx_n, k) = K_n^{\infty} - x_n. \qquad \square$$

By Lemma 4, the limit of the monotone increasing sequence $\{B_0(-kxn, k)\}_{k=1}^{\infty}$ of balls, which belongs to the Dynkin system D, coincides with the wedge $K_n^{\infty} - x_n$. Thus, for every $n \in \mathbb{N} \setminus \{1\}$ we have a nice wedge of dimension n lying in D. Now, to finish the proof of Proposition 2, it remains to apply Lemma 3.

§4. THE EXISTENCE OF A "GOOD" EQUIVALENT NORM IN A SPACE WITH BASIS

Theorem 1. *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space with a normalized basis $\{e_n\}_1^{\infty}$. Then X admits an equivalent norm $\|\cdot\|_0$ with unit ball B_0 such that the minimal Dynkin system D containing the large balls $RB_0 + x$ $(x \in X, R \geq 1)$ coincides with the Borel σ-algebra of $(X, \|\cdot\|)$.*

If the basis $\{e_n\}_1^{\infty}$ is monotone, then for every $\varepsilon > 0$ the norm $\|\cdot\|_0$ can be chosen to be e^{ε}-equivalent to the norm $\|\cdot\|$.

Proof. Let $\{e_n\}_1^{\infty}$ be a monotone basis, let B be the unit ball in $(X, \|\cdot\|)$, and let $\varepsilon > 0$. We are going to generalize the construction described in the proof of Proposition 2. As before, we construct balls $B_2, B_3, \ldots$ such that $B \supset B_2 \supset B_3 \supset \cdots$ by induction and put $B_0 = \bigcap_{n=2}^{\infty} B_n$. As always, $X_n = \operatorname{span}\{e_1, \ldots, e_n\}$.

Base of induction ($n = 2$). In the two-dimensional space X_2, we consider the ball $B^2 = B \cap X_2$.

By [7, Corollary 5.18], every weakly compact set in a Banach space is the closure of the convex hull of its strictly exposed points. (Recall that a point e_2' on the unit sphere is said to be *strictly exposed* for the unit ball if there exists a functional x_2^* such that

$\|x_2^*\| = 1$ and $1 = x_2^*(e_2') > x_2^*(y)$ for all y with $\|y\| = 1$, $y \neq e_2'$. Moreover, it is required that $\operatorname{diam}\{x \in B : x_2^*(x) > 1 - \delta\} \xrightarrow[\delta \to 0]{} 0$.)

In ∂B^2 we can find a strictly exposed point e_2', $\|e_2'\| = 1$, such that $e_2' = \alpha e_1 + \beta e_2$ with $\beta > 1 - \varepsilon$. (Such a point does exist. Indeed, assuming that every strictly exposed point has a coefficient β not exceeding $1 - \varepsilon$ in the representation $e = \alpha e_1 + \beta e_2$, we see that the point e_2 admits no approximation by convex combinations of strictly exposed points.)

Let x_2^* be a functional occurring in the definition of a strictly exposed point: $1 = \|x_2^*\| = x_2^*(e_2') > x_2^*(y)$ for all $y \in \partial B^2$, $y \neq e_2'$. We find δ, $0 < \delta < 1 - e^{-\frac{\varepsilon}{8}}$, such that the inequality $x_2^*(x) > 1 - \delta$ for $x \in B^2$ implies that $\|x - e_2'\| < \varepsilon$. Consider the slice $A_2 = A(e_2', x_2^*, \delta) = \{x \in B^2 : x_2^*(x) > 1 - \delta\}$ (see Figure 5). Then for all $x \in A_2$ we have $\|x - e_2'\| < \varepsilon$. The intersection of the line $\{x : x_2^*(x) = 1 - \delta\}$ and the unit sphere S^2 consists of points a and b. Let K_2 denote the cone with the vertex $e_2' \cdot \left(1 - \frac{\delta}{2}\right)$ spanned by the segment $[a, b]$, and let K_2^∞ be the wedge with the base K_2. As before, we put

$$B_2 = K_2^\infty \cap (-K_2^\infty) \cap B.$$

Step of induction. Suppose the balls $B_2, B_3, \dots, B_{n-1}$ have already been constructed, $B \supset B_2 \supset B_3 \supset \cdots \supset B_{n-1}$. We consider the intersection B_{n-1}^n of the ball B_{n-1} and the space X_n. On the sphere of the n-dimensional ball B_{n-1}^n, we find a strictly exposed point $e_n' = (\alpha_1, \dots, \alpha_n)$ with $1 - \varepsilon < \alpha_n$; let x_n^* be a functional, $\|x_n^*\| = 1$, that attains its strict maximum on B_{n-1}^n at the point e_n'. Next, we take $\delta > 0$, $\delta \leq 1 - e^{-\frac{\varepsilon}{2^{n+1}}}$, such that the inequality $x_n^*(x) > 1 - \delta$ for $x \in B_{n-1}^n$ implies that $\|x - e_n'\| < \varepsilon$. As above, the diameter of the slice $A_n = A(e_n', x_n^*, \delta)$ is less than ε.

The next step is to choose a number $\sigma > 0$ with $\sigma < \delta$ and points $y_1, \dots, y_n$ in the intersection of the sphere S_{n-1}^n and the hyperplane $\{x : x_n^*(x) = 1 - \delta\}$ in such a way that

(5) the points $\{(1 - \delta + \sigma)e_n' - y_k\}_{k=1}^n$ are linearly independent,

and

(6) the cone K_n with the vertex $(1 - \delta + \sigma)e_n'$ spanned by $\operatorname{co}\{y_1, \dots, y_n\}$ includes the ball $e^{-\varepsilon/2^{n+1}} \cdot B_{n-1}^n$.

□

The following two lemmas show that this can be done indeed.

Lemma 5. *Let X_n be an n-dimensional Banach space, let $B = B(0,1)$ be its unit ball, let $e \in \partial B$ be a strictly exposed point of B, and let $f \in X_n^*$ be a functional that attains its strict maximum on B at the point e. We put $Y_n = \operatorname{Ker} f$ and $F = (Y_n + (1 - \delta)e) \cap \partial B$, where $\delta \in (0, 1)$. Suppose that $y_1, \dots, y_n \in F$ and $(1 - \delta)e \in \operatorname{int} \operatorname{co}\{y_1, \dots, y_n\}$. Then there exists $\sigma > 0$ such that the cone with the vertex $(1 - \delta + \sigma)e$ spanned by $\operatorname{co}\{y_1, \dots, y_{n-1}\}$ includes the ball $(1 - \delta)B$.*

Proof. Since $(1 - \delta)e \in \operatorname{int} \operatorname{co}\{y_1, \dots, y_n\}$, for some $r_0 > 0$ the intersection of the Euclidean ball $B_2((1-\delta)e, r_0)$ with the hyperplane $(1-\delta)e + Y_n$ is included in $\operatorname{co}\{y_1, \dots, y_n\}$. Observe that $\Gamma = Y_n + (1 - \delta)e$ is the support plane to the ball $B(0, 1 - \delta)$ at the point $(1 - \delta)e$, and the point $(1 - \delta)e$ is a strictly exposed point of that ball. Therefore, Γ intersects the ball $B(0, 1 - \delta)$ at a unique point. Thus, for some $\alpha > 0$, the distance between the finite-dimensional Euclidean sphere $\partial B_2((1 - \delta)e, r_0) \cap ((1 - \delta)e + Y_n)$ and $\partial B(0, 1 - \delta)$ is at least α.

As σ, we take any number in the interval $(0, \alpha)$. Then, obviously, the cone with the vertex $(1-\delta+\sigma)e$ spanned by $B_2((1-\delta)e, r_0) \cap ((1-\delta)e + Y_n)$ includes the ball $B(0, 1-\delta)$.

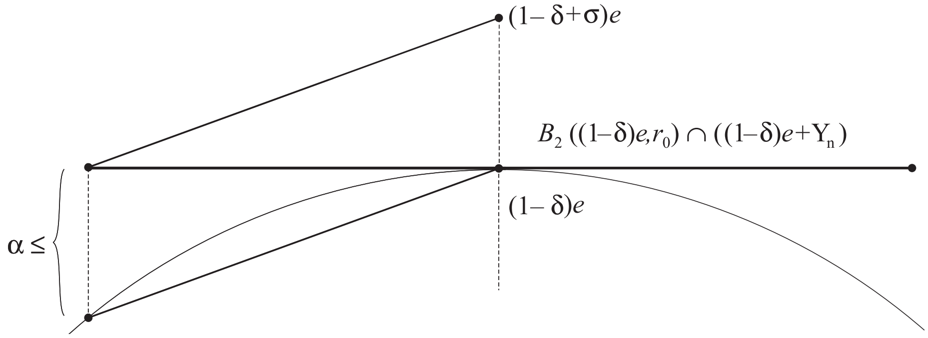

FIGURE 6.

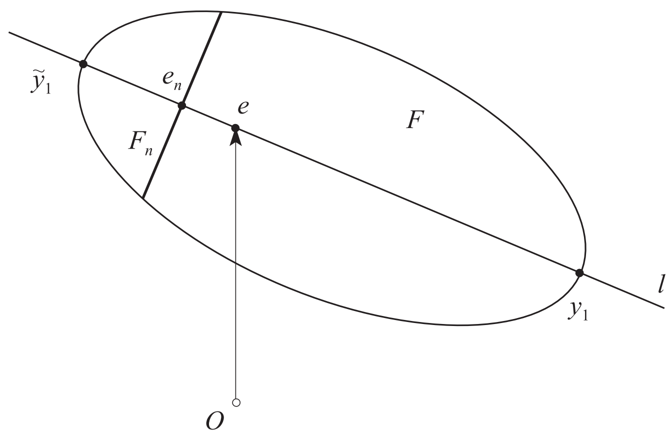

FIGURE 7.

Thereby, this ball is included in the cone with the vertex $(1-\delta+\sigma)e$ and spanned by $\operatorname{co}\{y_1,\ldots,y_{n-1}\}$ (Figure 6). □

Lemma 6. *Let X_n be an n-dimensional Banach space, let $Y \subset X$ be its $(n-1)$-dimensional subspace, and let $e \in X_n \setminus Y$. If F is a closed, bounded, and convex subset of $Y+e$, where $e \in \operatorname{int} F$ (in the topology of $Y+e$), then there exist points $y_1,\ldots,y_n \in \partial F$ such that the system $\{y_1,\ldots,y_n\}$ is linearly independent and $e \in \operatorname{int}\operatorname{co}\{y_1,\ldots,y_n\}$.*

Proof. We use induction on the dimension n. If $n=2$, then $F=[a,b]$. We put $y_1=a$, $y_2=b$.

To pass from n to $n+1$, first we take a point $\tilde{y}_1$ on the boundary of F and strictly separate it from e by a functional f. Let $\operatorname{Ker} f = Z$, and let $Z \cap F = F_n$. Then F_n is a bounded convex closed set (Figure 7). We draw the line l passing through the points $\tilde{y}_1$ and e and put

$$l \cap F_n \stackrel{\text{def}}{=} e_n, \quad l \cap \partial F = \{\tilde{y}_1, y_1\}.$$

We have constructed the point y_1. The required linearly independent points $y_2,\ldots,y_{n+1}$ are constructed on ∂F_n with the help of the induction hypothesis in such a way that $e_n \in \operatorname{int}\operatorname{co}\{y_2,\ldots,\ y_{n+1}\}$. Then, obviously, $e \in \operatorname{int}\operatorname{co}\{y_1,\ldots,y_{n+1}\}$, and the system $\{y_1,\ldots,y_{n+1}\}$ is linearly independent. The lemma is proved.

Lemmas 5 and 6 allow us to find a number $\sigma > 0$ and points $y_1,\ldots,y_n$ satisfying conditions (5) and (6). Indeed, we only need to check that if a system $\gamma = \{y_1,\ldots,y_n\}$

is linearly independent and $e \in \operatorname{co}\{y_1, \dots, y_n\}$, then for every $\alpha \neq 0$ the system $\gamma' = \{y_1 - (1+\alpha)e, y_2 - (1+\alpha)e, \dots, y_n - (1+\alpha)e\}$ is also linearly independent. Suppose $e = \sum_{i=1}^n \alpha_i y_i$, $\alpha_i \geq 0$, $\sum_{i=1}^n \alpha_i = 1$. If the system γ' is linearly dependent, then for some numbers $\beta_1, \dots, \beta_n$ with $\sum_{i=1}^n |\beta_i| \neq 0$ we have $\sum_{i=1}^n (y_i - (1+\alpha)e)\beta_i = 0$. Then $\sum_{i=1}^n \beta_i y_i = (1+\alpha)\sum_{i=1}^n \beta_i e = (1+\alpha)\sum_{j=1}^n \beta_j \sum_{i=1}^n \alpha_i y_i = \sum_{i=1}^n (1+\alpha)\big(\sum_{j=1}^n \beta_j\big)\alpha_i y_i$, whence $\sum_{i=1}^n \big(\beta_i - (1+\alpha)\big(\sum_{j=1}^n \beta_j\big)\alpha_i\big) y_i = 0$. Since the system γ is linearly independent, this means that $\beta_i = (1+\alpha)\big(\sum_{j=1}^n \beta_j\big)\alpha_i$ for $i = 1, \dots, n$, i.e., $\beta_i = c\alpha_i$ with $c = (1+\alpha)\sum_{j=1}^n \beta_j$. But then $\sum_{i=1}^n \beta_i = c\sum_{i=1}^n \alpha_i = c = \sum_{i=1}^n \beta_i \cdot (1+\alpha)$. We see that either $\alpha = 0$, which is impossible, or $\sum_{i=1}^n \beta_i = 0$, which yields $\sum_{i=1}^n \beta_i y_i = 0$, contradicting the linear independence of γ.

So, we have chosen a number $\sigma > 0$ and points $y_1, \dots, y_n$ such that conditions (5) and (6) are fulfilled.

As before, let K_n be the cone with vertex $(1-\delta+\sigma)e_n'$ and spanned by $\operatorname{co}\{y_1, \dots, y_n\}$, and let K_n^∞ be the wedge with the base K_n. We put $B_n = K_n^\infty \cap (-K_n^\infty) \cap B_{n-1}$. Finally, we form $B_0 = \bigcap_{k=2}^\infty B_k$; this completes the construction of the desired unit ball.

All the required properties of B_0 except for the inclusion $B_0 \supset e^{-\varepsilon}B$ can be checked precisely as in the proof of Proposition 2. (Observe that if ε is sufficiently small, then the slice A_n does not intersect the ball B_{n-1}^{n-1}. Therefore, the transformations of the ball at the nth step do not affect the preceding transformations, which were made in X_{n-1}.)

We check that $B_0 \supset e^{-\varepsilon}B$. By Lemma 5, we have $B_n^n \supset e^{-\frac{\varepsilon}{2^{n+1}}} B_{n-1}^n$. We show that also

$$B_n \supset e^{-\frac{\varepsilon}{2^{n+1}}} B_{n-1}. \tag{7}$$

Let $x \in e^{-\frac{\varepsilon}{2^{n+1}}} B_{n-1}$, $x = (x_1, x_2, \dots, x_{n-1}, x_n, \dots)$. Then $x \in B_{n-1}$. By construction, for the norm $\|\cdot\|_{n-1}$ with the unit ball B_{n-1} we always have

$$\|(z_1, z_2, \dots, z_n, \dots)\|_{n-1} \geq \|(z_1, \dots, z_n, 0, 0, \dots)\|_{n-1}.$$

This implies that $(x_1, \dots, x_n) \in e^{-\frac{\varepsilon}{2^{n-1}}} B_{n-1}^n \subset B_n^n$. Thus, $x \in B_{n-1}$ and $(x_1, \dots, x_n) \in B_n^n$. By construction, this means that $x \in B_n$. Now we have

$$\begin{aligned} B_n &\supset e^{-\frac{\varepsilon}{2^{n+1}}} B_{n-1} \supset e^{-\frac{\varepsilon}{2^{n+1}}} \cdot e^{-\frac{\varepsilon}{2^n}} B_{n-2} \supset \cdots \\ &\supset e^{-\frac{\varepsilon}{2^{n+1}}} \cdot e^{-\frac{\varepsilon}{2^n}} \cdots e^{-\frac{\varepsilon}{2^4}} B_2 \supset e^{-\frac{\varepsilon}{2^{n+1}}} \cdot e^{-\frac{\varepsilon}{2^n}} \cdots e^{-\frac{\varepsilon}{2^4}} \cdot e^{-\frac{\varepsilon}{2^3}} B \\ &\supset e^{-\frac{\varepsilon}{8}\cdot(1+\frac{1}{2}+\cdots)} B = e^{-\frac{\varepsilon}{4}} B \supset e^{-\varepsilon} B, \end{aligned}$$

as required. Theorem 1 is proved. □

§5. The "good" norms are dense in the case of a reflexive space with basis

Theorem 2. *Let $(X, \|\cdot\|)$ be a reflexive infinite-dimensional Banach space with a normalized basis $\{e_n\}_{n=1}^\infty$. Let $B = B(0,1)$ be the unit ball of X, and let ε be a positive number. Then X admits an equivalent norm $\|\cdot\|_0$ with unit ball B_0 such that*

1) $\operatorname{dist}(B_0, B) < \varepsilon$ (*in the Hausdorff metric*);

2) *the minimal Dynkin system D containing all large balls $RB_0 + x$ ($x \in X$, $R \geq 1$) coincides with the Borel σ-algebra.*

Thus, for every reflexive Banach space with basis, the family of "good" balls is dense (in the Hausdorff metric) in the space of convex bounded symmetric sets.

Proof. We start with the construction of small spherical segments (slices of the form $B \cap \Gamma$, where Γ is a half-space), in which the subsequent "grinding" of the ball B will be made. □

Step 1. Since the space X is reflexive, its ball B is weakly compact and, thus, is the closure of the convex hull of its strictly exposed points. Let $x_1 = (t_1^{(1)}, t_2^{(1)}, \dots) \in B$ be a strictly exposed point, and let f_1, $\|f_1\| = 1$, be a functional attaining its strict maximum on B at the point x_1. We find a number $\delta_1 < \frac{\varepsilon}{4}$ such that, for $x \in B$,

$$(1_1) \qquad f_1(x) > 1 - \delta_1 \implies \|x - x_1\| < \frac{\varepsilon}{8}.$$

Being a basis of a reflexive space X, the system $\{e_n\}_{n=1}^{\infty}$ is shrinking (this means that $\|f|_{\operatorname{span}\{e_{n+1}, e_{n+2}, \dots\}}\| \xrightarrow[n\to\infty]{} 0$ for all $f \in X^*$). Let M_1 be the basis constant of $\{e_n\}_{n=1}^{\infty}$, and let N_1 be such that

$$(2_1) \qquad \begin{cases} \|f_1|_{\operatorname{span}\{e_{N_1+1}, \dots\}}\| < \frac{\varepsilon}{4(1+M_1+M_1^2\varepsilon)}, \\ \|(0, \dots, 0, t_{N_1+1}^{(1)}, t_{N_1+2}^{(1)}, \dots)\| < \frac{\varepsilon}{8}. \end{cases}$$

Now we modify the basis $\{e_n\}_{n=1}^{\infty}$ somewhat: we want to replace the vectors $e_1, \dots, e_{N_1}$ with vectors $e'_1, \dots, e'_{N_1}$ in such a way that $x_1 \in \operatorname{span}\{e'_1, \dots, e'_{N_1}\}$ and

$$(3_1) \qquad \sum_{i=1}^{N_1} \|e_i - e'_i\| < \frac{\varepsilon}{4}.$$

We explain how to do this. Denoting $(t_1^{(1)}, \dots, t_{N_1}^{(1)}) = \bar{x}_1$, $(0, \dots, 0, t_{N_1+1, \dots}^{(1)}) = \tilde{x}_1$, we have $x_1 = \bar{x}_1 + \tilde{x}_1$. Using (2_1), we get $\|\tilde{x}_1\| < \frac{\varepsilon}{8}$. If ε is small, then $\|x_1\| = 1 \implies \|\bar{x}_1\| \geq 1 - \frac{\varepsilon}{8} \implies |t_1^{(1)}| + \dots + |t_{N_1}^{(1)}| \geq 1 - \frac{\varepsilon}{8} \geq \frac{1}{2}$. Put $e'_i = e_i + \frac{\operatorname{sgn} t_i^{(1)}}{|t_1^{(1)}| + \dots + |t_{N_1}^{(1)}|} \tilde{x}_1$. Then

$$\sum_{i=1}^{N_1} t_i^{(1)} e'_i = \sum_{i=1}^{N_1} t_i^{(1)} e_i + \frac{1}{|t_1^{(1)}| + \dots + |t_{N_1}^{(1)}|} \tilde{x}_1 \sum_{i=1}^{N_1} t_i \cdot \operatorname{sgn} t_i^{(1)} = \bar{x}_1 + \tilde{x}_1 = x_1,$$

so that $x_1 \in \operatorname{span}\{e'_1, \dots, e'_{N_1}\}$, and

$$\sum_{i=1}^{N_1} \|e_i - e'_i\| = \|\tilde{x}_1\| \cdot \frac{1}{|t_1^{(1)}| + \dots + |t_{N_1}^{(1)}|} < \frac{\varepsilon}{8} \cdot 2 = \frac{\varepsilon}{4}.$$

Thus, both required conditions are satisfied.

Step 2. We recall the Kreĭn–Milman–Rutman theorem (see [8, p. 246]). If $\{e_n\}_1^{\infty}$ is a normalized basis of X with basis constant K and $\{x_n\}_1^{\infty}$ is a sequence in X satisfying $\sum_{n=1}^{\infty} \|e_n - x_n\| < \frac{1}{2K}$, then $\{x_n\}_1^{\infty}$ is a basis of X equivalent to $\{e_n\}_1^{\infty}$.

So, if $\varepsilon > 0$ is sufficiently small, then the system $\{e'_1, \dots, e'_{N_1}, e_{N_1+1}, e_{N_1+2, \dots}\}$ is a basis of X. In what follows we shall work with this latter basis. Let M_2 be its basis constant. Then $\operatorname{dist}(e_{N_1+1}, \operatorname{span}\{e'_1, \dots, e'_{N_1}\}) \geq \frac{1}{2M_2}$. Since the point e_{N_1+1} lies in the closure of the convex hull of the strictly exposed points of B, there exists a strictly exposed point $x_2 = (t_1^{(2)}, t_2^{(2)}, \dots)$ (i.e., $x_2 = \sum_{i=1}^{N_1} t_i^{(2)} e'_i + \sum_{i=N_1+1}^{\infty} t_i^{(2)} e_i$) such that

$$(*_2) \qquad \operatorname{dist}(x_2, \operatorname{span}\{e'_1, \dots, e'_{N_1}\}) \geq \frac{1}{4M_2}.$$

Let f_2 be a functional of norm 1 that attains its strict maximum on B at the point x_2. This functional gives us arbitrarily small slices: for some δ_2 with $0 < \delta_2 < \frac{\varepsilon}{2}$, for $x \in B$ we have

$$(1_2) \qquad f_2(x) > 1 - \delta_2 \implies \|x - x_2\| < \min\Big(\frac{1}{4M_2} - \frac{\varepsilon}{4}, \frac{\varepsilon}{8}\Big).$$

Let N_2 be a number for which

$$(2_2)\qquad \begin{cases} \|f_2|_{\operatorname{span}\{e_{N_2+1,\dots,}\}}\| < \frac{\varepsilon}{4(1+M_1+M_1^2\varepsilon)}, \\ \|(0,\dots,0,t^{(2)}_{N_2+1},\dots)\| < \min\left(\frac{\varepsilon}{16\cdot 4M_2}, \frac{1}{8M_2}\right). \end{cases}$$

As above, we modify the vectors $\{e_{N_1+1},\dots,e_{N_2}\}$ so as to get vectors $\{e'_{N_1+1},\dots,e'_{N_2}\}$ such that $x_2 \in \operatorname{span}\{e'_1,\dots,e'_{N_2}\}$ and

$$(3_2)\qquad \sum_{i=N_1+1}^{N_2} \|e_i - e'_i\| < \frac{\varepsilon}{8}.$$

(By $(*_2)$ and (2_2), we now have $|t^{(2)}_{N_1+1}| + |t^{(2)}_{N_1+2}| + \cdots + |t^{(2)}_{N_2}| \geq \frac{1}{4M_2} - \frac{1}{8M_2} = \frac{1}{8M_2}$.) Conditions (1_1) and (1_2) imply that the slice $\{x \in B : f_2(x) > 1-\delta_2\}$ does not intersect the slice $\{x \in B : f_1(x) > 1-\delta_1\}$ constructed at Step 1. Indeed, suppose x belongs to the latter slice and y belongs to the former. Then

$$\begin{aligned} \|x-y\| &\geq \|x_2 - x_1\| - \|x_1 - x\| - \|y_2 - y\| \\ &\geq \|x_2 - x_1\| - \left(\frac{1}{4M_2} - \frac{\varepsilon}{4}\right) - \frac{\varepsilon}{8} \overset{(*_2)}{\geq} \frac{1}{4M_2} - \frac{1}{4M_2} + \frac{\varepsilon}{4} - \frac{\varepsilon}{8} = \frac{\varepsilon}{8} > 0, \end{aligned}$$

whence $x \neq y$.

We continue the process described above. At the nth step, we deal with a basis $\{e'_1,\dots,e'_{N_{n-1}},e_{N_{n-1}+1},\dots\}$ with basis constant M_n. We find a strictly exposed point $x_n = (t^{(n)}_1, t^{(n)}_2,\dots)$ satisfying

$$(*_{n-1})\qquad \operatorname{dist}(x_n, \operatorname{span}\{e'_1,\dots,e'_{N_{n-1}}\}) \geq \frac{1}{4M_n},$$

the corresponding functional f_n, and numbers N_n and $\delta_n < \frac{\varepsilon}{8}$ such that, for $x \in B$,

$$(1_n)\qquad f_n(x) > 1-\delta_n \implies \|x - x_n\| < \min\left(\frac{1}{4M_n} - \frac{\varepsilon}{4}, \frac{\varepsilon}{8}\right),$$

and

$$(2_n)\qquad \begin{cases} \|f_n|_{\operatorname{span}\{e_{N_{n+1}},e_{N_{n+2}},\dots\}}\| < \frac{\varepsilon}{4(1+M_1+M_1^2\varepsilon)}, \\ \|(0,\dots,0,t^{(n)}_{N_n+1},\dots)\| < \min\left(\frac{\varepsilon}{4M_n\cdot 2^{n+2}}, \frac{1}{8M_n}\right). \end{cases}$$

Then we modify the part $\{e_{N_{n-1}+1},\dots,e_{N_n}\}$ of the original basis so as to get a system $\{e'_{N_{n-1}+1},\dots,e'_{N_n}\}$ such that $x_n \in \operatorname{span}\{e'_1,\dots,e'_{N_n}\}$ and

$$(3_n)\qquad \sum_{i=N_{n-1}+1}^{N_n} \|e_i - e'_i\| < \frac{\varepsilon}{2^{n+1}}$$

(this is done precisely as at Steps 1 and 2, by using inequalities $(*_n)$ and (2_n)).

By the Kreĭn–Milman–Rutman theorem, the resulting system $\{e'_n\}_{n=1}^{\infty}$ is a basis of X equivalent to $\{e_n\}$. Moreover, obviously, all intermediate basis constants M_n are bounded from above by some number M. (For small ε, this allows us to keep the inequalities (1_n) satisfied.) We also note that

$$\operatorname{span}\{e'_{N_k+1}, e'_{N_k+2},\dots\} = \operatorname{span}\{e_{N_k+1}, e_{N_k+2},\dots\}$$

for every k.

So, we have prepared slices $A_n = \{x \in B : f_n(x) > 1-\delta_n\}$. As at Step 2, it is easy to check that these slices are pairwise disjoint. Now we start "grinding" the ball B within A_n to obtain the required ball B_0 (we shall "grind" the part of B in A_n to get a part of a nice wedge with the vertex x_n). As always, the ball B_0 will be obtained as the intersection of balls B_n, $n = 1, 2, 3, \dots$, constructed by induction. We denote $\operatorname{span}\{e'_1, \dots, e'_k\}$ by X'_k and $B_n \cap X'_k$ by B^k_n.

We construct the ball B_1. Recall that M_2 is the basis constant of the basis $\{e'_1, \dots, e'_{N_1}, e_{N_1+1,\dots}\}$. In the space X'_{N_1}, which contains the point x_1, we consider two slices

$$A_1^{N_1} = \{x \in B^{N_1} : f_1(x) > 1 - \delta_1\},$$
$$\tilde{A}_1^{N_1} = \Big\{x \in B^{N_1} : f_1(x) > 1 - \frac{\delta_1}{2}\Big\}.$$

Let $\Gamma_1 = \{x \in X'_{N_1} : f_1(x) = 1 - \delta_1\}$ and $\widetilde{\Gamma}_1 = \{x \in X'_{N_1} : f_1(x) = 1 - \frac{\delta_1}{2}\}$.

The intersection of the hyperplane $\widetilde{\Gamma}_1$ and the ball B^{N_1} is a bounded closed set F. As in the proof of Theorem 1, we choose a number $\sigma_1 > 0$ and linearly independent points $\{y_1, \dots, y_{N_1}\}$ in F so that the cone K_1 with the vertex $(1 - \sigma_1)x_1$ spanned by $\operatorname{co}\{y_1, \dots, y_{N_1}\}$ contains the ball $\big(1 - \frac{\varepsilon}{4}\big)B^{N_1}$.

A natural modification of the proof of Lemma 5 allows us to ensure the following additional condition:

$$\partial K_1 \cap \Gamma_1 \text{ lies outside of the ball } (3 + M_1)B^{N_1}. \tag{9}$$

As always, we denote by K_1^∞ the wedge with the base K_1:

$$K_1^\infty = \{(z_1, z_2, \dots) : (z_1, \dots, z_{N_1}, 0, \dots) \in K_1\}$$

(here the coordinates correspond to the basis $\{e'_1, \dots, e'_{N_1}, e_{N_1+1}, \dots\}$).

We show that

$$K_1^\infty \supset \Big(1 - \frac{\varepsilon}{2}\Big)B \tag{10}$$

and

$$K_1^\infty \supset B_0 \setminus A_1. \tag{11}$$

To check (10), we take $z \in \big(1 - \frac{\varepsilon}{2}\big)B$, $z = \sum_{i=1}^{N_1} z_i e'_i + \sum_{i=N_1+1}^{\infty} z_i e_i$, and denote $\bar{z}' = \sum_{i=1}^{N_1} z_i e'_i$, $\tilde{z} = \sum_{i=N_1+1}^{\infty} z_i e_i$, $\bar{z} = \sum_{i=1}^{N_1} z_i e_i$. We need to verify that $z \in K_1^\infty$, i.e., $\bar{z}' \in K_1$. Assuming that this is not true, we consider two cases:

1) $f_1(\bar{z}') \geq 1 - \delta_1$ and $\bar{z}' \notin K_1$;
2) $f_1(\bar{z}') < 1 - \delta_1$ and $\bar{z}' \notin K_1$.

In the first case we have

$$\begin{aligned} f_1(z) &= f_1(\bar{z}') + f_1(\tilde{z}) \geq 1 - \delta_1 + f_1(\tilde{z}) \geq 1 - \frac{\varepsilon}{4} + f_1(\tilde{z}) \\ &\overset{(2_1)}{\geq} 1 - \frac{\varepsilon}{4} - \frac{\varepsilon}{4(1 + M_1 + M_1^2\varepsilon)} \cdot \|\tilde{z}\| \\ &\geq 1 - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = 1 - \frac{\varepsilon}{2}, \end{aligned}$$

because

$$\begin{aligned} \|\tilde{z}\| &\leq \|z\| + \|\bar{z}'\| \leq \|z\| + \|\bar{z}\| + \|\bar{z}' - \bar{z}\| \\ &\leq \|z\| + M_1\|z\| + 2M_1\|\bar{z}\| \sum_{i=1}^{N_1} \|e_i - e'_i\| \\ &\leq \|z\|(1 + M_1) + 2M_1 \cdot \frac{\varepsilon}{2}\|\bar{z}\| \\ &\leq \|z\| \cdot (1 + M_1) + M_1\varepsilon \cdot M_1\|z\| = \|z\| \cdot (1 + M_1 + M_1^2\varepsilon) \\ &\leq 1 + M_1 + M_1^2\varepsilon. \end{aligned}$$

This contradicts the inequality $f(z_1) \leq \|z\| < 1 - \frac{\varepsilon}{2}$. Thus, case 1) is impossible.

In case 2), condition (9) implies that $\bar{z}' \notin (3+M_1)B^{N_1}$, i.e., $\|\bar{z}'\| \geq 3+M_1$. But then

$$\|z\| \geq \|\bar{z}'\| - \|\tilde{z}\| \geq 3 + M_1 - (1 + M_1 + M_1^2\varepsilon)\Big(1 - \frac{\varepsilon}{2}\Big) > 2 - M_1^2\varepsilon > 1$$

for sufficiently small ε (recall that above it was shown that $\|\tilde{z}\| \leq \|z\| \cdot (1 + M_1 + M_1^2\varepsilon)$).

Since we have a contradiction again, case 2) is also impossible. This proves (10).

The inclusion (11) was actually verified when we analyzed case 2).

As usual, we put

$$B_1 = K_1^\infty \cap (-K_1^\infty) \cap B.$$

Now, we perform similar constructions in the ball $B_1^{N_2}$. The inclusion (11) means that all modifications of the ball were made only within the slice A_1. Since the slice A_2 appearing at the second step does not intersect A_1, the "grinding" to be made when we construct B_2 is independent of a similar process occurring in the construction of B_1. Also, observe that all modifications of the ball B are made only inside the spherical layer $B \setminus \big(1 - \frac{\varepsilon}{2}\big)B$. Continuing the construction, we obtain balls $B_1 \supset B_2 \supset \cdots$ and put $B_0 = \bigcap_{n=1}^\infty B_n$. Obviously, $B_0 \supset (1-\varepsilon)B$. It remains to show that the balls $\{RB_0 + x : x \in X, R \geq 1\}$ generate the Borel σ-algebra. Applying Lemma 4 to each point $(1-\sigma_n)x_n$ in X'_{N_n}, we see that all shifts of the nice wedge K_n^∞ belong to the Dynkin system D. By the proof of Lemma 3 (see $(*)$), for all k and all Borel sets $A \subset X'_{N_k}$ the set

$$A' \overset{\text{def}}{=} \Big\{x \in X : x = \sum_{n=1}^{N_k} \alpha_n e'_n + \sum_{n=N_k+1}^{\infty} \alpha_n e_n, \sum_{n=1}^{N_k} \alpha_n e'_n \in A\Big\}$$

belongs to D.

But

$$A' = \Big\{x \in X : x = \sum_{n=1}^{N_k} \alpha_n e'_n + \sum_{n=N_{k+1}}^{\infty} \alpha_n e'_n,\ \sum_{n=1}^{N_k} \alpha_n, e'_n \in A\Big\}.$$

Thus, all the sets A' are cylinders relative to the basis $\{e'_n\}_{n=1}^\infty$. By the remark after Lemma 2, D coincides with the Borel σ-algebra.

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