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WHAT DOES A GENERIC MARKOV OPERATOR LOOK LIKE?

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To the memory of O. A. Ladyzhenskaya

Abstract. Generic (i.e., forming an everywhere dense massive subset) classes of Markov operators in the space $L^2(X, \mu)$ with a finite continuous measure are considered. In a canonical way, each Markov operator is associated with a multivalued measure-preserving transformation (i.e., a polymorphism), and also with a stationary Markov chain; therefore, one can also talk of generic polymorphisms and generic Markov chains. Not only had the generic nature of the properties discussed in the paper been unclear before this research, but even the very existence of Markov operators that enjoy these properties in full or partly was known. The most important result is that the class of totally nondeterministic nonmixing operators is generic. A number of problems is posed; there is some hope that generic Markov operators will find applications in various fields, including statistical hydrodynamics.

I was fortunate to be friends with O. A., especially in the 1970s, and some time I will write about this. In the late 1960s and 1970s, she was very interested in dynamical systems, and this was an additional motive for our contacts. Working on the Hopf equation, she arrived at the necessity of considering multivalued (Markov) mappings and suggested that we start a joint research on multivalued solutions of equations. Our work resulted in a series of papers; see [4, 5]. We also had grandiose projects for further research, for example, working on metric hydrodynamics, but these were never realized. Approximately at the same time, I started to develop general (multivalued) dynamics ([1]), and recently, after a long interval, I have returned to this subject. In this paper, dedicated to the unforgettable O. A., I continue this topic.

§1. Markov operators

1.1. Definitions.

Definition 1. A *Markov operator* in the Hilbert space $L^2(X, \mu)$ of complex-valued square-integrable functions on a Lebesgue–Rokhlin space (X, μ) with a continuous normalized measure μ is a continuous linear operator V satisfying the following conditions:

1) V is a contraction: $\|V\| \le 1$ (in the operator norm);
2) $V\mathit{1} = V^*\mathit{1} = \mathit{1}$, where $\mathit{1}$ is the function identically equal to one;
3) V preserves the nonnegativity of functions: Vf is nonnegative whenever $f \in L^2(X, \mu)$ is nonnegative.

Note that condition 1) follows from 2) and 3), and the second condition in 2) follows from the others.

In short, a Markov operator is a unity-preserving positive contraction.

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In the same way we can define a Markov operator V from one space $L^2(X,\mu)$ to another space $L^2(Y,\nu)$.

1.2. Three languages. A geometric analog of a Markov operator $V : L^2(X,\mu) \to L^2(Y,\nu)$ is a *polymorphism*, i.e., a measure-preserving multivalued mapping of the space (Y,ν) into the space (X,μ) (for a detailed exposition, see [1, 2]).

Each Markov operator is uniquely mod 0 generated by a polymorphism, and conversely, for each class of mod 0 coinciding polymorphisms there is a Markov operator canonically associated with it. The correspondence between Markov operators and polymorphisms extends the classical correspondence between unitary positive unity-preserving operators and measure-preserving transformations (the Koopman correspondence) to the case of Markov operators.

This correspondence can easily be explained by using the helpful intermediate notion of a bistochastic measure. A bistochastic measure ν on the space $X \times X$ is a measure whose projections to the first and the second coordinate (the marginal measures) coincide with a given measure μ. A bistochastic measure is the generalized kernel of a Markov operator. A bijection between the set of Markov operators (V) and the set of bistochastic measures (ν) is given by the formula

$$\nu(A \times B) = \langle V\chi_A, \chi_B \rangle;$$

here A and B are measurable subsets of X with characteristic functions χ_A and χ_B, respectively, and ν is a bistochastic measure on $X \times X$, defined on the σ-field that is the square of the original σ-field on (X,μ). It is easy to prove that this formula determines a bijection between the Markov operators and the bistochastic measures; of course, ν can be singular with respect to the product measure $\mu \times \mu$, and the conditional measures can be singular with respect to μ.

In these terms, the polymorphism corresponding to a Markov operator V is the mapping that associates with μ-almost every point $x \in X$ some measure on X, namely, the conditional measure $\mu_V^x = \mu^x$ on the space X regarded as an element of the partition of the space $(X \times X, \nu)$ into the preimages of points under the projection from $X \times X$ to the first coordinate; the conjugate polymorphism associates with a point $x \in X$ the conditional measure μ_x on the space X regarded as an element of the partition of the space $(X \times X, \nu)$ into the preimages of points under the projection to the second coordinate. By the general Rokhlin theorem, the conditional measures exist for almost all elements of a measurable partition.

Each Markov operator determines a random stationary Markov chain (discrete-time process) as follows: its state space is the initial Lebesgue space (X,μ), the measure μ being invariant for this Markov chain, and the transition probability $P(\cdot,x)$, $x \in X$, is determined by the kernel of the Markov operator, more precisely, by the polymorphism that, with almost every point x of the space (X,μ), associates the conditional measure μ^x on X (see above); the transition probabilities are precisely these conditional measures. Conversely, each stationary Markov chain determines a bistochastic measure, namely, the two-dimensional distribution of two adjacent states, and hence a Markov operator and a polymorphism.

Thus, we have three equivalent languages: **the language of Markov operators, the language of polymorphisms and bistochastic measures, and the language of stationary Markov chains** with continuous state space. See [1] for the details. In this paper, we shall mainly use the language of Markov operators and sometimes, when it is helpful, provide explanations in the other two languages.

1.3. Structures. The Markov operators in the Hilbert space $L^2(X,\mu)$ form a semigroup with respect to multiplication; this semigroup has an identity element (the identity operator), a zero element (the one-dimensional orthogonal projection θ to the subspace of constants), and an involution (the operator conjugation $*$). It is also a convex compact topological semigroup in the weak topology in the algebra of all continuous operators. Indeed, it is easy to verify that the class of Markov operators is closed under the operations mentioned above. All these structures are also defined on the set of polymorphisms of the space (X,μ) (or on the set of bistochastic measures), and the correspondence "polymorphism — Markov operator" is an antiisomorphism of semigroups that preserves these structures (but reverses the arrows); see [1].

Since a Lebesgue–Rokhlin space with continuous measure is unique up to a metric isomorphism (it is isomorphic to the interval with the Lebesgue measure), the compact space of Markov operators is also unique in the same sense. Denote it by $\mathcal{P}$. The subgroup of invertible elements of this semigroup is precisely the subgroup of positive unitary unity-preserving operators, i.e., the group of (mod 0 classes of) all measure-preserving mappings of the Lebesgue space into itself. Below we give a definition of the compact space $\mathcal{P}$ and all structures on it in approximation terms; this definition is independent of operator theory.

1.4. Approximation lemma. Consider matrices of order n with nonnegative entries and with row and column sums equal to 1 (bistochastic matrices); they form a compact convex semigroup (under matrix multiplication), with involution (transposition), zero element (the matrix with all entries equal to n^{-1}), and identity element (the identity matrix). Let $\mathcal{P}_n$ denote the compact convex space of such matrices; its dimension is equal to $(n-1)^2$; $\mathcal{P}_n$ is the compact space of Markov operators on a finite space with the uniform measure.

For positive integers n, m, $k>1$ with $n=mk$, we partition the rows and columns of the matrices in $\mathcal{P}_n$ into blocks of order k. Thus, the matrices in $\mathcal{P}_n$ acquire the structure of block matrices with blocks of order k. Consider the natural projection

$$\pi_{n,m} : \mathcal{P}_n \to \mathcal{P}_m$$

that replaces each matrix block with the sum of the elements of this block divided by k. The following obvious result, which is however important for our further considerations, provides another definition of our main object.

Lemma 1. *The compact space $\mathcal{P}$ of bistochastic measures, regarded with all structures defined above* (*the structure of a compact space, of a semigroup, etc.*), *is the inverse* (*projective*) *limit of the spaces $\mathcal{P}_n$ with respect to the partially ordered set of projections* $(\pi_{n,m}, m|n)$:

$$\mathcal{P} = \varprojlim_{n,m} (\mathcal{P}_n, \pi_{n,m}).$$

It is convenient to restrict ourselves only to the powers of a single number (for instance, $n=2^s$, $s=1,2,\dots$) and consider the limit along a linearly ordered set. The proof is straightforward. This construction can easily be interpreted in terms of weak approximation of operators, but in what follows we shall use this lemma in a slightly different way.

§2. Classes of Markov operators

2.1. Generic classes and the list of properties. The following definition is a specialization of the well-known terminology.

Definition 2. We say that a class of Markov operators is *generic* or *forms a set of second category* if this class, regarded as a subset of the set $\mathcal{P}$ of all Markov operators, contains an everywhere dense G_δ-set (= intersection of countably many open sets). A property is said to be generic if the class of operators satisfying this property is generic.

We shall be describing generic classes of Markov operators and, consequently, generic classes of polymorphisms, bistochastic measures, and stationary Markov chains. As often happens, the generic classes we are going to consider have been studied little, and some of the seemingly paradoxical generic properties of Markov chains given below were not even known until recently.

We shall be interested in the following properties of Markov operators.

Definition 3. A Markov operator V in the space $L^2(X,\mu)$ is said to be

0) *ergodic* if it has no nonconstant invariant vector; recall that the spectrum of a Markov operator lies in the unit circle;

1) *mixing* (respectively, *comixing*) if $V^n \to \theta$ (respectively, $V^{*n} \to \theta$) as $n \to +\infty$ (recall that θ is the orthogonal projection to the subspace of constants);

2) *totally nonisometric* (respectively, *totally noncoisometric*) if the operator V (respectively, the conjugate operator V^*) is not isometric on any closed invariant subring (sublattice) in $L^2(X,\mu)$ except that of constants (the subrings or sublattices in $L^2(X,\mu)$ are linear subspaces consisting of functions that are constant on the elements of some measurable partition of (X,μ));

3) *dense* if its kernel (= the preimage of the zero) is zero and the image of the space $L^2(X,\mu)$ is a dense linear subspace in $L^2(X,\mu)$; in other words, if the kernel of V and the kernel of the conjugate operator V^* (= the cokernel of V) are trivial;

4) *extremal* if V (and hence V^*) is an extreme point of the compact convex space $\mathcal{P}$ of Markov operators;

5) *indecomposable* if there is no measurable subset $A \subset X$ of positive μ-measure and no measurable subset $B \subset X \times X$ of positive ν-measure such that the images of the characteristic function χ_B under the operator V and the conjugate operator V^* are positive and strictly less than 1 almost everywhere on A.

The properties 0)–2) are *operator-dynamical*; 3) is *measure-geometric*; 4) and 5) are *properly geometric*.

2.2. Analysis of properties. Let us comment on the above definitions.

1. The term and notion of a mixing appeared in the theory of dynamical systems; here it means that the shift in the space of trajectories of the corresponding stationary Markov chain is a mixing in the sense of that theory. In the case of Markov chains *with finite state space*, this property is equivalent to a much stronger property of a chain — the triviality of the tail σ-field at minus infinity, which means that the σ-field $\bigcap_{n=0}^{\infty} \mathcal{A}_{-\infty}^{-n}$ of measurable subsets in the space of two-sided trajectories of a Markov chain consists of two elements: the class of zero-measure sets and the class of the entire space. This property has many other names and many other formulations (Kolmogorov regularity, 0–2 law, etc.; see [9]). In some cases, there are well-known conditions of mixing; for example, an aperiodic chain with finitely or countably many states and, more generally, an aperiodic chain satisfying the Harris condition (see [9]) are mixing. The notions of mixing and comixing are in general position.

In the general theory of contractions in Hilbert spaces (see [7]), four classes of contractions are considered, depending on whether or not the sequence of positive powers of the operator or the conjugate operator tends weakly to zero. Depending on which of the four variants occurs, the notation $C_{0,0}$, $C_{0,1}$, $C_{1,0}$, $C_{1,1}$ is used. In this notation, we can say that the class of nonmixing and noncomixing Markov operators, which is most important for our purposes, lies in $C_{1,1}$.

2. It is obvious that the mixing Markov operators are totally nonisometric, and the comixing ones are totally noncoisometric. The question arises, as to whether the converse is true, that is, whether every totally nonisometric Markov operator is mixing. One of the main points of the general theory of contractions (see [7]) is that for contractions this is not the case; in other words, there exists a totally nonisometric (respectively, noncoisometric) contraction such that the sequence of positive (respectively, negative) powers does not tend to zero. It turns out that in the theory of Markov operators the situation is similar: there exist nonmixing and noncomixing Markov operators that are still totally nonisometric and noncoisometric. In order to understand the paradoxical nature of this situation, we reformulate the condition of being totally nonisometric in geometric terms.

Definition 4. A stationary Markov chain with state space X and invariant measure μ is said to be *totally nondeterministic* if there is no measurable partition ξ of the space (X, μ) such that the transition operator acts deterministically on its blocks, i.e., sends a block to a block.

Lemma 2. *A Markov chain is totally nondeterministic if and only if the corresponding Markov operator is totally nonisometric.*

In the theory of chains with finitely many states, the property of being totally nondeterministic is known as "the absence of subclasses" or "aperiodicity," etc.; see [9]. In [1], for certain reasons, the corresponding polymorphisms were called *simple*. For chains with countably many states and, more generally, for chains satisfying the Harris condition (see [9]), the condition of being aperiodic, i.e., totally nonisometric, is equivalent to mixing.

However, for general Markov chains this is not the case: there exist totally nonisometric and nonmixing Markov operators, i.e., totally nondeterministic and nonmixing Markov processes. The first example of this type is due to M. Rosenblatt [6]. For other examples, see [1, 3]; in these examples the transition probabilities are singular with respect to the invariant measure and their behavior is rather complicated; the behavior of the powers of the Markov operator is also quite nontrivial. Here we do not describe these examples, referring the reader to the papers mentioned above, but we shall prove that they are generic.

3. Property 3) needs no comments; it means that the operator may have no bounded inverse, yet the inverse operator exists on an everywhere dense subset, and the same is true for the conjugate operator. However, it is worth explaining why this property is important. We say that a Markov operator V is a *quasiimage* (in [7], it was called a *quasiaffinitet*) of a Markov operator W if there exists a dense Markov operator L such that

$$LV = WL.$$

If V is a quasiimage of W and W is a quasiimage of V (i.e., there exists a dense Markov operator M such that $MW = VM$), then we say that the operators V and W are *quasisimilar*. Quasisimilarity is an equivalence relation; it would not have been of any interest if we did not require that the intertwining operators L and M should be dense. In the theory of contractions there is a number of important results on quasisimilarity (see [7]), but it seems that for Markov operators this notion has never been introduced and studied. The main problem, which we do not discuss here, is *when are two unitary Markov operators quasisimilar and what unitary Markov operators can be quasisimilar to totally nonisometric Markov operators.* These problems are extremely important for the theory of dynamical systems and statistical physics in connection with the discussion of irreversibility (see the references in [2, 3]).

4, 5. The notions of extremality and indecomposability are of a completely different nature. The fact that a Markov operator is a nontrivial convex combination of other Markov operators means that the corresponding Markov shift is a *skew product over a Bernoulli shift*; in other words, it is a random walk over the trajectories of Markov components. In particular, if the Markov operators occurring in the convex combination are unitary, then we have a *random walk over the trajectories of deterministic transformations with invariant measure* or a so-called random dynamical system. Precisely this is the case for Markov chains with finitely many states and the uniform measure, because, by the Birkhoff–von Neumann theorem, the extreme points of the polyhedron of bistochastic matrices are permutation matrices. In the case of general bistochastic measures, extremal Markov operators are not necessarily unitary; moreover, the conditional measures can even be continuous (see below). From a geometric point of view, bistochastic measures were studied by many authors; see, e.g., [8, 2] for nontrivial examples of extremal polymorphisms and Markov operators, and for further references. An obvious necessary condition for extremality is as follows: *there is no set of constant width strictly between* 0 *and* 1 *with respect to both projections*, i.e., there is no measurable set of intermediate measure such that the images of the characteristic function of this set under the Markov operator and its conjugate are constant functions. We may go further and introduce the notion of indecomposability (see above). If an operator is indecomposable, then it cannot be represented as a convex combination of other Markov operators, even with nonconstant (depending on the point) coefficients that are not equal to 0 or 1 on sets of positive measure. In this case, the shift in the space of trajectories of the corresponding Markov process cannot be represented as a random walk over the trajectories of any Markov shifts with probabilities depending on the point and different from 0 and 1 almost everywhere. It turns out that even this condition, which is much stronger than the usual extremality, determines a generic class of Markov operators. The indecomposability means that there is *no subset of nontrivial width with respect to both projections over the entire space or at least over a set of positive measure*. It is not difficult to deduce that indecomposability implies extremality, but the converse is not true. A remarkable characteristic property of every indecomposable bistochastic measure ν is that in the space $L^2(X\times X,\nu)$ every function can be approximated by functions of the form $f(x)+f(y)$; in other words, *there is no nonzero function that has zero expectation with respect to both subalgebras.*

§3. The main theorem

3.1. Formulation. We consider a Lebesgue–Rokhlin space (X,μ) with continuous measure. All Markov operators act in the space $L^2(X,\mu)$ of square-integrable complex-valued functions. Recall that a Markov operator is ergodic if it has no nonconstant invariant vector. Below, by singularity we mean singularity with respect to the measure μ.

Theorem 1. *A generic Markov operator enjoys the following properties*:

1) *its spectrum has no discrete component* (*in the orthogonal complement to the subspace of constants*); *in particular*, *it is ergodic*;
2) *it is neither mixing nor comixing*;
3) *it is totally nonisometric and totally noncoisometric*;
4) *it is dense*;
5) *it is extremal and indecomposable*;
6) *almost all of its transition probabilities are continuous and singular.*

Remark. Most papers on the theory of Markov chains deal with the cases of either absolutely continuous or discrete transition probabilities (for example, the Doeblin condition, the Harris condition, etc.). In these cases, it is difficult to discover most important and generic effects, such as the absence of mixing for totally nondeterministic operators, as well as other generic properties.

3.2. Proof. Since the intersection of finitely or countably many generic classes in a complete metrizable separable space is generic (the Baire theorem), it suffices to prove that each class is generic by itself. Furthermore, all "coproperties" are similar to the corresponding properties for the conjugate operator, and hence they are generic provided that the original properties are generic.

We start with observing that the group of positive unitary operators is everywhere dense in $\mathcal{P}$; this follows from the approximation lemma and the following simple fact: every rational bistochastic matrix of order n all entries of which have denominator N is the projection ($\pi_{Nn,n}$) of some permutation matrix of order nN (see [2]). This implies that the sets of operators satisfying properties 1, 2, 4, 5 are everywhere dense, because a generic measure-preserving automorphism is ergodic, has simple continuous spectrum ([10]), is nonmixing (since it is deterministic), extremal, and, of course, indecomposable, and dense. The fact that the set of totally nonisometric operators is everywhere dense and even satisfies the Baire property (G_δ) also follows from the lemma, but in this case we should use other matrices, namely, irreducible ones: irreducible bistochastic matrices are generic even in the finite-dimensional case, and the projections preserve irreducibility.

We verify that the sets of operators satisfying the remaining properties are G_δ-sets. The fact that property 6 is generic can also be seen from the lemma, because the existence of a discrete component in the conditional measures can be written in terms of approximating bistochastic matrices (see [2] for the details). The extreme points of every compact convex set form a G_δ-set (see [11]). The G_δ-condition for indecomposable bistochastic measures was proved in [2]. For properties 2 and 4, the G_δ-condition is trivial. Finally, this condition for property 1 is satisfied in the algebra of all bounded operators: the set of operators that have no discrete component in the spectrum is a G_δ-set; and so is its intersection with the set of Markov operators.

3.3. Remarks, problems, conjectures. 1. As often happens, it is easier to prove that a property is generic than to construct an explicit example of a generic object. The deepest problem, which has a nontrivial solution (the construction of a Markov operator that simultaneously satisfies properties 2 and 3, i.e., is nonmixing and totally nonisometric) was considered in the author's recent paper [1]. In that paper, a relationship was established between such examples in the hyperbolic theory of dynamical systems. This led to a new characterization of the K-systems and to the following conjecture, which we state here without specifying the details (this would require giving new definitions and will be done elsewhere).

A generic polymorphism is a singular random perturbation of a Kolmogorov automorphism. Accordingly, a generic Markov operator is a singular perturbation of a unitary positive operator conjugate to a K-automorphism.

It seems possible to construct a Markov operator (polymorphism) that simultaneously satisfies all properties 1–6 by specializing these new examples.

2. It is natural to ask whether the class of Markov operators such that the shift in the space of trajectories of the corresponding Markov chain is a K-automorphism is generic. We emphasize that, as observed above, for a generic Markov operator, the Markov generator is not a K-generator, because there is no mixing. However, in all known examples, there exists another (non-Markov) K-generator.

The next question: what Markov operators generate a shift that is (isomorphic to) a Bernoulli shift? Unfortunately, Ornstein's technique (the $\bar{d}$-metric) is not suitable for studying processes with continual state space. The famous Kalikow's example [13] of a non-Bernoulli and even non-loosely-Bernoulli automorphism (a random walk over (T, T^{-1}), where T is a Bernoulli shift) demonstrates the wide possibilities of the natural Markov generators. A close and, apparently, difficult question is whether every K-automorphism has a Markov K-generator.

3. There is an acute problem concerning the definition of the entropy of a Markov operator or a polymorphism. There are different suggestions, and, probably, there should be "different" entropies corresponding to different properties of polymorphisms. One definition was introduced, not quite distinctly, in [12]; this entropy is positive for generic polymorphisms of a finite space (bistochastic matrices). Another definition of the entropy of Markov operators, as well as further references, can be found in [14]. In ergodic theory, the generic value of the Kolmogorov entropy is zero. The same question for polymorphisms and any of the entropies is open.

4. Every polymorphism (Markov operator) generates certain equivalence relations on (X, μ). One of them is the partition into orbits: two points x and y lie in the same orbit if there exist positive integers n, m such that the conditional measures $\mu^x_{V^n}$ and $\mu^y_{V^m}$ are not singular (their mutual densities are not identically zero nor identically infinite). Another equivalence relation is the transitive envelope of the nonsingularity relation for conditional measures: $x \sim y$ if there exists a positive integer k and a finite chain of points $x_0 = x, x_1, \dots, x_k = y$ such that the conditional measures $\mu^{x_i}_V$ and $\mu^{x_{i+1}}_V$ are not singular for $i = 0, \dots, k-1$. What is the generic behavior of these equivalence relations? It seems that the class of Markov operators for which these equivalence relations are ergodic is generic.

5. We do not dwell on other generic properties of Markov operators, only indicating one important link to the theory of groupoids, and one more problem. Consider a Markov operator V, the conjugate operator V^*, and the semigroup spanned by these two operators. Let $T = \sum_w c_w w(V, V^*)$, where $w(\cdot,\cdot)$ ranges over all finite words in the alphabet $\{V, V^*\}$ and the c_w are positive coefficients decaying sufficiently fast and with total sum equal to 1. The operator T is again a Markov operator related to the measurable groupoid generated by the initial operator, more precisely, by the first equivalence relation mentioned above, the partition of the space (X, μ) into orbits. It is very important to find out when this partition is hyperfinite (in another terminology, tame) and whether this case is generic; most likely, it is not.

6. On the other hand, with every Markov operator we can associate the C^*-algebra generated by this operator, the conjugate operator, and the operators of multiplication by some class of bounded measurable functions. Such algebras generalize the notion of a skew product, and the study of their properties (for example, amenability, simplicity, etc.) will give new examples of C^*-algebras.

7. Finally, the last question is also related to the theory of C^*-algebras. Of great interest is the C^*-algebra generated by all Markov operators. It seems that is has never been considered. This algebra is not separable and does not coincide with the algebra of all operators. Presumably, it coincides with the algebra of all operators that preserve the order-bounded sets in L^2, and its elements have a natural integral representation with a kernel that may be nonpositive. It must play the same role in the theory of Markov operators and dynamical systems that the algebra of all operators plays in general operator theory.

8. One particular class of polymorphisms is of special interest, namely, the class of so-called *algebraic polymorphisms*, i.e., correspondences on compact Abelian groups with

the Haar measure. A typical example is an algebraic polymorphism of the circle, i.e., the uniform measure on the one-dimensional cycle of the two-dimensional torus determined by the equation $u^p = v^q$, where p, q are positive integers and u, v are coordinates on the torus. These examples will be considered from different viewpoints in a joint paper by the author and K. Schmidt, which is now in preparation.

§4. Comments

4.1. Relation to previous work. In the joint papers with O. A. Ladyzhenskaya [4, 5] mentioned in the Introduction, we considered applications of theorems on measurable selection. In order to apply general theorems, we needed some *a priori* estimates. More or less simultaneously, several papers in the same direction appeared — Foiaş, Temam, and others — with similar results. The common feature of all of these papers was that they regarded multivaluedness out of the context of Markov operators, i.e., the set of solutions was associated with Cauchy data, rather than a measure on them. The theory of polymorphisms and Markov operators can have more subtle applications and simulate more complicated phenomena than theorems on measurable selection. However, for that it is necessary to develop the theory of one-parameter semigroups of Markov operators and polymorphisms and their Lie generators. Apparently, the above results on generic properties can be generalized to the case of such semigroups, but this is still to be done. It can be expected that generic semigroups of polymorphisms will also find applications in hydrodynamics, as we once discussed with O. A. I have been keeping the memory of our discussions through all these years.

4.2. Acknowledgments. I repeatedly discussed the theory of Markov operators with M. I. Gordin, who put forward important ideas relating hyperbolic theory to probability theory; he also gave me several useful references. Many years ago, D. Z. Arov drew my attention to the tentative relationship between the theory of polymorphisms and the Foiaş–Sz.-Nagy theory of contractions, and at the same time, B. Rubshtein pointed out M. Rosenblatt's example to me. With V. A. Kaimanovich I discussed links to the theory of boundaries, and with L. A. Khalfin (1932–1998) we talked about possible links to physics. To all of them I am deeply grateful.

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